A COURSE OF PLANE GEOMETRY
FOR ADVANCED STUDENTS

PART I.
PREFACE

The present volume is the First Part of a Course in Plane Geometry which, it is hoped, will be found suitable for the higher mathematical divisions of Schools, and for undergraduates attending lectures on Geometry during their first year of residence. It therefore covers the Schedule of Pure Geometry required for the first part of the Cambridge Mathematical Tripos.

Part I. deals with the properties of the straight line, triangle, quadrilateral and quadrangle, and circle.

Part II., which will be published shortly, is concerned with the application of cross-ratio, involution, projection, and reciprocation, to the properties of the conic; together with an account of the meaning, validity, and advantage of the use of imaginary elements, viewed from their analytical source.

The following considerations have guided the development of the present course:

1. It is only by constant practice in rigorous work that the many novel ideas in advanced geometry can be appreciated and assimilated. Very numerous exercises have therefore been attached to each set of standard theorems. Some of these are of a simple nature, being partly mere reproductions of the bookwork in different forms; while others will be found hard enough to test the capacities of the best students.

2. In many places, it is desirable to compare analytical and geometrical methods, as for example in the treatment of inversion, coaxal circles, and polar properties. In such cases, the analysis has been included in the text in preference to relegating it to an appendix, where it is apt to be overlooked.

3. The connection between Geometry and Statics, supplied by the theory of Vectors, and the graphical interpretation of complex
number which it yields, afford cogent arguments for including some exposition of vector quantities in a regular school course. A chapter has therefore been inserted which introduces the subject on somewhat new lines.

(4) Some familiarity may well be acquired with the Principle of Duality long before the process is studied in detail under reciprocation. Numerous examples present themselves in elementary theorems which pave the way to a better final understanding of the general principle.

(5) Interest is added to the study of Geometry, if some indication is given of its historical development. Accordingly, most chapters have been prefaced by brief historical notes on the subject matter they contain.

The author wishes to acknowledge his indebtedness to Mr. A. E. Broomfield and Mr. G. M. Bell for the valuable assistance received from them.

The Editor of the Educational Times has kindly authorised the use of riders which have appeared in the columns of that journal, and the Syndics of the Cambridge University Press have courteously allowed questions to be taken from the Entrance Scholarship and University examination papers.

The historical information is derived mainly from the writings of Professor Cayley, Sir R. Ball, Mr. W. W. Rouse Ball, Professor Casey, and Dr. Taylor.

The author will welcome any corrections or criticisms tending to the improvement of the book.

October, 1908.

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CHAPTER I.

SIMILAR FIGURES.

When constructing a map of a town, or obtaining a numerical solution to a problem involving heights or distances by a drawing to scale, an appeal is made to the theory of similar figures. The picture thrown on a screen by a magic lantern is another example of a figure similar in every respect to the slide from which it is formed.

It is not surprising to find in the early history of geometry some acquaintance with properties of this kind. Thales of Miletus (640-550 B.C.) enunciated the theory of similar triangles, and Euclid (350-275 B.C.) included in his Elements a more systematic treatment, which was no doubt largely derived from the work of the three preceding centuries.

Definition.
Polygons which are equiangular and have their corresponding sides proportional are called similar.

THEOREM 1.

If two similar polygons are so placed that their corresponding sides are parallel, then the lines joining corresponding vertices are concurrent.

Let $ABCD \ldots \ X$ and $AB'C'D' \ldots \ X'$ be the polygons

and let $\frac{AB}{BC} = \frac{BC}{CD} = \ldots = k.$

Produce $AA'$ and $BB'$ to meet at $O.$ Join $OC$ and let it cut $B'C'$ at $C'_1.$

D.G. A
By similar triangles, \( \frac{BA}{OB} = \frac{BC}{OC} \)

but \( \frac{BC}{OC} = \frac{BC'}{OC'} \)

\( \therefore \frac{BA}{OB} = \frac{BC'}{OC'} \)

\( \therefore C_i \) coincides with \( C \),

\( \therefore C' \) lies on \( OC \); and similarly \( D' \) lies on \( OD \), etc.,

\( \therefore AA', BB', CC', \ldots XX' \) intersect at \( O \).

Q.E.D.

![Diagram](image)

In the figure of this theorem, the two polygons have been so drawn that the directions \( A \to B, A' \to B' \) are the same. If however these directions are opposite (Fig. 2) the theorem still holds good, and may be proved in precisely the same way.

**Corollary.**

If \( P \) is any point on one of the sides of the polygon \( AB \ldots X \), and if \( OP \) meets the corresponding side of \( A'B' \ldots X' \) at \( P' \), then \( \frac{OP'}{OP} = k \).

1. Prove this Corollary.

---

**THEOREM 2.**

If \( P \) is a variable point on the side of a given polygon \( ABC \ldots X \) and if \( O \) is any other given point, and if a point \( P \) is taken on \( OP \)

such that \( \frac{OP'}{OP} = k \) (a constant), then as \( P \) traces out the perimeter of \( ABC \ldots X \), \( P' \) traces out the perimeter of a polygon whose sides are parallel and proportional to the sides of \( ABC \ldots X \), and the ratio is \( k \).

2. Prove Theorem 2. Notice that there are two cases according as \( P' \) is on the same or opposite side of \( O \) as \( P \).

**Definitions.**

(a) The polygon traced out by \( P' \) in this theorem is said to be similar and similarly situated to the polygon traced out by \( P \); and \( O \) is called their centre of similitude. Or, to use a shorter expression, the polygons are said to be homothetic.

(b) If \( O \) is a fixed point and if \( P \) traces out a given curve \( S \), and if \( P' \) is a point on the line \( OP \) such that \( \frac{OP'}{OP} = k \) (a constant), then \( P \) traces out a curve \( S' \) which is said to be homothetic to \( S \); and \( P, P' \) are called corresponding points. If \( P' \) traces out \( S' \) in the same direction that \( P \) traces out \( S \), they are called directly homothetic; but if in opposite directions, they are called inversely homothetic.

The results stated in Ex. 3–5 are of great importance. The proofs are however so simple that they are left to the reader.

3. Prove that the curve homothetic to a straight line must be a straight line.

4. (1) If \( P, Q \) are two points corresponding to \( P', Q' \), prove that \( PQ, P'O' \) are parallel.

5. (2) If \( P \) and \( P' \) are corresponding points on two homothetic curves, prove that the tangents at \( P, P' \) are parallel.

6. If two triangles are homothetic, prove that: (1) the centres of their circumscribing circles, (2) the centres of their inscribed circles, are corresponding points.

"w.r.t. will be used as an abbreviation for ‘within respect to.’"
7. Prove that the curve homothetic to a circle is another circle and that their centres are corresponding points.

8. $P, Q$ are two points on the perimeter of a polygon $S'$, which correspond to the points $P, Q$ on a homothetic polygon $S$. $O$ is the centre of similitude, prove that the ratio of the areas of $S$ and $S'$ equals the ratio of the areas of the triangles $OPQ, OP'Q'$.

9. Two plans of the same estate, on different scales, are placed, one wholly on the other, so that a north and south line in one lies on a north and south line in the other, prove that there is one point of contact between the maps at which the same place is represented in each map.

**THEOREM 3.**

$O$ is any fixed point. $P$ is a variable point on a given circle; $P'$ is a point on $OP$ such that $\frac{OP}{OP'} = \frac{a}{b}$ (a constant), then the locus of $P'$ is a circle.

![Diagram](image)

Let $C$ be the centre of the given circle.

Join $OC$; draw $PC$ parallel to $PC$ to meet $OC$ at $C'$.

By parallels $\frac{OC}{OC'} = \frac{PC}{PC'} = \frac{OP}{OP'} = \frac{a}{b}$.

But $OC, PC$ are constants.

\[ \therefore \text{OC', PC' are constants.} \]

\[ \therefore \text{C' is a fixed point and P' lies on a fixed circle, centre C'.} \]

Q.E.D.

The proof of the theorem remains unaltered when $O$ lies inside the given circle, or when $P$ lies on $PO$ or $OP$ produced. Notice that the centres of the two circles are corresponding points.

**THEOREM 4.**

$A, B$ are the centres of two circles of radii $a, b$; $AB$ is divided externally at $O$ and internally at $O_1$ in the ratio of the radii $\frac{OA}{O_1}$.

\[ \therefore \text{OA} = \frac{OA_1}{O_1B} \cdot \frac{a}{b} \]

then the circles are homothetic w.r.t. both $O$ and $O_1$, and corresponding points lie on the extremities of parallel radii.

![Diagram](image)

$AP$ is any radius of the first circle; $BP$ and $BP_1'$ are the radii of the second circle parallel to $AP$.

Then $\frac{PA}{PB} = \frac{OA}{OB}$; $\frac{PA}{PB} = \frac{PA_1}{PB_1'}$ and angle $PAO = P'BO$ by parallels;

\[ \therefore \text{the triangles PAO, P'BO are similar;} \]

\[ \therefore \text{P' lies along OP and OP'' = P'B = \frac{b}{a}} \]

\[ \therefore \text{the circles are homothetic w.r.t. O.} \text{ [Th. 3].} \]

Also $P, P'$ are corresponding points and lie on the extremities of parallel radii.

Precisely the same argument, applied to the triangles $O_1AP, O_1BP_1'$, proves that the circles are homothetic w.r.t. $O_1$, and that corresponding points $P, P'$ lie on the extremities of parallel radii.

Q.E.D.

Notice that the circles are directly homothetic w.r.t. $O$ and inversely homothetic w.r.t. $O_1$; the centres being corresponding points in each case.

The second part of Theorem 4 may be stated in another way:

If any line through $O$ or $O_1$ meets the circle, centre $A$, at $P$ and the circle, centre $B$, at $P'$ and $Q$, the radius $AP$ is parallel to one of the radii $BP', BQ'$.
10. A is a fixed point on a given circle: a variable chord $AP$ is produced to $Q$ so that $\frac{AP}{AQ}$ is constant, find the locus of $Q$.

11. Prove that the common tangents to two circles pass through one or other of the two points w.r.t. which the circles are homothetic.

12. The angles of the triangle $APQ$ are given; $A$ is fixed; $P$ moves on a given circle, prove that the locus of $Q$ is a circle.

13. Two circles are homothetic w.r.t. $O$; a variable line through $O$ cuts the circles in two pairs of corresponding points $P$, $P'$; $Q$, $Q'$ where $P$, $Q$ are on the same circle. Prove that $OP \cdot OQ = OQ' \cdot OP' =$ constant.

14. $A$ is a fixed point; $P$ is a variable point on a fixed circle centre $C$; the line bisecting the angle $ACP$ meets $AP$ at $Q$; find the locus of $Q$.

15. $APQ$ is a variable triangle; $A$ is fixed, $P$ moves on a fixed line $CD$, $AP$ meets a fixed line parallel to $CD$ at $R$; if $PQ = AR$ and if the angle $APQ$ is constant, find the locus of $Q$.

It is possible to describe a figure homothetic to a given figure, and drawn to any convenient scale, by means of an instrument, called a pantograph.

![Diagram](image)

In the figure, rods are represented by straight lines. Two equal rods $OA$, $AP$ are jointed together at $A$, and the end-point $O$ is fixed. But $OQ$ can turn freely about $O$. Two other rods $PB$, $PCK$ are jointed together at $P$, and are pinned to $OA$, $AP$ respectively at $B$, $C$, so that $BP = BO$ and $BP : CA$ is a parallelogram.

If $P$ is now made to describe any curve $S$, it can be proved that $P'$ describes a homothetic curve $S'$.

The proof is left to the reader.

[Show that (1) $O$, $P$, $P'$ are collinear, (2) $\frac{OP}{OQ} = \frac{OB}{OA} =$ constant.]

---

17. SIMILAR FIGURES

In order to obtain any required scale, it is only necessary to adjust the clamp at the position of $P$ for which the ratio $OB : OA$ represents the given scale. If the figure is to be enlarged, it is clear that $P'$ must be made to trace out the given figure.

Certain constructions may be performed by the use of homothetic figures. This is best illustrated by examples.

EXAMPLE I.

In a given triangle $ABC$, to inscribe a triangle having its sides parallel to those of a given triangle $PQR$.

Draw any line parallel to $QR$ cutting $AC$, $AB$ at $M$, $N$.

Draw $ML$, $NL$ parallel to $QP$, $RP$ to meet at $L$.

Through $L$ draw $LM$, $LN$ parallel to $LM$, $LN$ to meet $AC$, $AB$ at $M'$, $N'$.

Then $LMN'$ is the required triangle.

18. Prove the construction given in Example I.

Another method may be applied to this type of problem. It is not so elegant, but perhaps it brings out more clearly the properties of the centre of similitude. Fundamentally however the two methods are identical.

In the previous construction, through $P$, $Q$, $R$ draw lines $YZ$, $ZX$, $XY$ parallel to $BC$, $CA$, $AB$ to form the triangle $XYZ$; then $AX$, $BY$, $CZ$ are concurrent, at $O$ say [Theorem I]. Produce $OP$, $OQ$, $OR$ to meet $BC$, $CA$, $AB$ at $L'$, $M'$, $N'$; then $LMN'$ is the required triangle.

19. Prove this alternative method of construction.

20. Apply this method to construct a square having one side along a side of a given triangle, and the other two corners of the square on the remaining sides of the triangle.
EXAMPLE II.

To inscribe a square in a given triangle.

$ABC$ is the given triangle.

On $BC$ describe a square $BHKC$, externally

to the triangle.

Join $AH$, $AK$ and let them cut $BC$ at $H$, $K$.

Draw $H'K'$, $KC'$ perpendicular to $BC$ to cut $AB$, $AC$ at $B'$, $C'$.

Then $B'H'KC'$ is the required square.

[Fig. 7.]

20. Prove the construction in Example II.

21. If $AD$ is an altitude of the triangle $ABC$, describe a square on $AD$, regard $B$ as the centre of similitude, and hence perform the construction of Example II.

THEOREM 5.

A triangle is given in species (i.e., its angles are given), one vertex is fixed, another lies on a given circle, then the locus of the third vertex is a circle.

[Fig. 8.]

$ABC$ is one position of the triangle; $A$ is fixed, $B$ lies on a given circle, centre $O$.

SPECIAL FIGURES

On $AC$ take a point $B'$ such that $AB = AB'$.

On $AB'$ describe a triangle $AOC$, directly similar and

$\therefore$ congruent to the triangle $AOB$.

Since $OAB = OAB'$, $OAO = OAO' = \angle B'AB = \text{constant}$; also

$AOA = AO \cdot \text{constant}$;

$\therefore$ $O_1$ is a fixed point.

But $O_1B = OB = \text{constant}$;

$\therefore$ $B_1$ describes a circle.

Now $\frac{AC}{AB} = \frac{AC}{AB'} = \text{constant}$, since triangle $ABC$ is given in species.

$\therefore$ the locus of $C$ is a circle [Th. 3].

Q.E.D.

If the circle, centre $O_1$, is rotated rigidly about $A$ through the constant angle $BAC$, it is clear that $B_1$ always lies on the new position of this circle. The construction, given above, places this however on a more logical basis.

22. Two given lines intersect at an inaccessible point $H$, $A$ is a given point, construct the line $AH$.

23. In a given triangle, inscribe an equilateral triangle with one side parallel to a given line.

24. $OB$, $OC$ are two given radii of a given circle. Describe a square so that two of its vertices lie on $OB$, $OC$ and the other two on the arc $BC$.

25. $ABCD$ is a quadrilateral such that $BA = BC$ and $DA = DC$; inscribe a square in the quadrilateral.

26. $ABCD$ is a square; construct a semicircle to touch $BA$, $BC$ and to have the ends of its base on $DA$, $DC$.

27. Construct a circle to touch two given lines and to pass through a given point.

28. $A$, $B$, $C$, $D$ are the mid points of the sides $BC$, $CA$, $AB$ of a triangle $ABC$; $G$, $H$, $O$ are the centroid, orthocentre and circumcentre of the triangle $ABC$; prove that

$(1)$ $G$ is the centre of similitude of the triangles $ABC$, $A'BC$;

$(2)$ $O$ is the orthocentre of the triangle $A'BC$; and hence

$(3)$ $G$, $H$, $O$ are collinear; and find the value of the ratio $\frac{GH}{GO}$.

29. A triangle $ABC$ is given in species; $A$ is fixed, $B$ moves on a fixed straight line; find the locus of $C$. 
30. Two triangles $ABC, A'B'C'$ are directly similar (but corresponding sides are not parallel). $BC$ meets $B'C'$ at $D$; the circles $BB', CC'$ meet again at $O$; prove that, by a rotation about $O$, the triangle $A'B'C'$ can be made homothetic to the triangle $ABC$.

31. $A, B$ are two fixed points on a circle; $P$ is a variable point on the circle; $Q$ is a point on $PA$ such that $PQ = PB$; prove that the locus of $Q$ is a circle.

32. If $S$ and $S'$ are two curves, each homothetic to a third curve $\Sigma$ (with different centres), prove that $S$ is homothetic to $S'$.

33. $P$ is a variable point on a fixed line $AB$; $AC$ is another fixed line; a circle is drawn to touch $AB, AC$ and the circle on $AP$ as diameter; find the locus of the point of contact of the two circles.

34. $A, B$ are two fixed points on a fixed circle $\Sigma$, $P$ is a variable point of $\Sigma$; $Q$ is taken on $BP$ so that $\frac{BQ}{AP}$ is constant; find the locus of $Q$.

CHAPTER II.

RATIO THEOREMS.

In the present chapter will be found a few useful theorems on ratios and their application to areas.

Theorem 9 is due to Apollonius, who lectured at Alexandria in the third century B.C. He is said to have been of a concial and jealous disposition, but little else of a personal nature is known except that his nickname at the university was $\psi$, possibly from the name or number of his lecture-room. His great work was a systematic treatise on conic sections.

Theorem 10 is usually ascribed to Ptolemy ($\sigma 150$ A.D.), but it was probably discovered by Hipparchus nearly three centuries earlier. From this theorem may be deduced all the elementary trigonometrical formulae. This is particularly interesting, because Hipparchus may be regarded as the originator of the science of Trigonometry, which he required for astronomical purposes.

He was the first to systematise astronomical knowledge by compiling a star-catalogue; but his name is immortalised by his discovery of the phenomenon, known as the precession of the equinoxes—a motion so slow that its period is about 25,000 years. Ptolemy stands out as a unique figure in the history of astronomy. From the second to the sixteenth century, i.e. till the era of Copernicus, Galileo, Tycho Brahe, Kepler and Newton, the science of astronomy was embodied in Ptolemy's great work, *The Almagest*. His real discoveries, that the earth was spherical and poised in space, and his erroneous beliefs, that the earth was fixed at the centre of a gigantic celestial sphere, rotating daily, to which the stars were attached, and that the paths of the planets were explained by a system of superposed circles, held equal undisputed sway for thirteen or fourteen centuries.
THEOREM 6.

Two triangles $ABC$, $ABD$ have their vertices $C$ and $D$ on opposite sides of a common base $AB$. If $CD$ meets $AB$ (or $AB$ produced) at $H$; then

$$\frac{CH}{HD} \propto \frac{\triangle ACB}{\triangle ADB}$$

Q.E.D.

1. Prove that Theorem 6 still holds if $C$ and $D$ are on the same side of the common base $AB$.

THEOREM 7.

If in the triangles $ABC$, $DEF$, the angles $BAC$, $DEF$ are equal; then

$$\frac{\triangle ABC}{\triangle DEF} = \frac{AB}{AC} \propto \frac{DE}{EF}$$

Q.E.D.

Draw $BH$, $EK$ perpendicular to $AC$, $DF$.

RATIO THEOREMS

Then the triangles $ABH$, $DEK$ are similar, i.e.,

$$\triangle ABH \propto \triangle DEK$$

and $\hat{BHA} = \hat{EDK}$, right angles.

$$\therefore \frac{BH}{AB} \propto \frac{EK}{DE}$$

and

$$\triangle ABC \propto \frac{1}{2} BH \cdot AC = AB \cdot AC$$

$$\triangle DEF \propto \frac{1}{2} EK \cdot DF = DE \cdot DF$$

Q.E.D.

Notice that this theorem is an immediate consequence of the trigonometrical formula $\triangle ABC = \frac{1}{2} AB \cdot AC \sin BAC$.

2. $O$ is a point inside the triangle $ABC$; $AO$, $BO$, $CO$ meet $BC$, $CA$, $AB$ at $P$, $Q$, $R$; prove that

$$\therefore \frac{\triangle BPA}{\triangle CPA} \propto \frac{BP}{CP}$$

and

$$\frac{\triangle CPA}{\triangle QOA} \propto \frac{CQ}{QA} \times \frac{AP}{PA}$$

where $b$, $q$, $r$ are given constants.

3. With the figure of Ex. 2, prove that

$$\frac{\triangle BPA}{\triangle CPA} \propto \frac{BP}{CP}$$

and

$$\frac{\triangle CPA}{\triangle QOA} \propto \frac{CQ}{QA} \times \frac{AP}{PA}$$

4. $O$ is a point on the line bisecting the angle $BAC$ of the triangle $ABC$, prove that

$$\triangle BAO \propto \triangle BAC$$

5. Find a point $O$ inside the triangle $ABC$ such that

$$\triangle AOB : \triangle BOC : \triangle COA = \phi : \gamma : \tau$$

6. Points $P$, $Q$, $R$ are taken on the sides $BC$, $CA$, $AB$ of the triangle $ABC$ so that $BP : PC = CQ : QA = AR : RB = 1 : 2$; prove that

$$\triangle PQR \propto \triangle ABC$$

7. Points $P$, $Q$ are taken on the sides $BC$, $CA$ of the triangle $ABC$ so that $BP = 2PC$, $CQ = 3QA$; $AP$ meets $BQ$ at $O$; calculate the ratio of the areas of the quadrilateral $PQOC$ and the triangle $ABC$. Use Ex. 2.

8. Points $P$, $Q$, $R$ are taken on $BC$, $CA$, $AB$ so that $BP : PC = 1$, $CQ : QA = 2$, $AR : RB = 3$; the lines $AP$, $BQ$, $CR$ form the triangle $LMN$; calculate the ratio $\triangle LMN : \triangle ABC$.

9. In the triangles $ABC$, $DEF$, if the angles $BAC$, $EDF$ are supplementary, prove that

$$\frac{\triangle ABC}{\triangle DEF} \propto \frac{AB}{AC} \propto \frac{DE}{DF}$$

10. The diagonals $AC$, $BD$ of a cyclic quadrilateral $ABCD$ meet at $O$; prove that

$$\frac{AB \cdot BC}{BO} = \frac{AD \cdot DC}{OD}$$

11. Two circles touch internally at $O$; a straight line $ABCD$ cuts the outer at $A$, $D$ and the inner at $B$, $C$; prove that

$$AB : CD = OA : OB : OC : OD.$$
THEOREM 8.

If two straight lines PRQ, ACB cut three concurrent straight lines OPA, ORC, OQB; then \( \frac{PR}{OR} \cdot \frac{AC}{OB} = \frac{QP}{OC} \cdot \frac{OP}{QA} \).

The theorem is still true if OB lies between OA and OC; or if the line PRQ cuts some or all of the lines OA, OC, OB when produced backwards through O.

12. With the notation of Theorem 8, prove that
\[
\frac{PQ}{QR} \cdot \frac{AB}{OP} \cdot OR = \frac{AC}{OB}.
\]

13. The medians AA', CC' of the triangle ABC meet at G, use Theorem 8 to prove that CG = \( \frac{1}{3} \) BC.

14. P is any point on AB, the median AA' of the triangle ABC cuts PC at H, prove that \( \frac{PH}{HC} = \frac{AP}{AB} \).

THEOREM 9.

A, B are two fixed points; P is a moving point such that \( \frac{PA}{PB} \) is constant; then P lies on a fixed circle.

Let \( \frac{PA}{PB} = \lambda \); produce AP to Q.

Divide AB internally at H and externally at K in the ratio \( \lambda \).
The Converse Theorem is as follows:
If \( AB \) is divided internally at \( H \) and externally at \( K \) in the same ratio (say \( \lambda \)), and if \( P \) is any point on the circle on \( HK \) as diameter, then \( \frac{PA}{PB} = \lambda \).

Draw \( PB' \), so that \( \angle P'B'H = \angle AP'H \) to meet \( HK \) at \( B' \).

Since \( \frac{AH}{HB} = \frac{AK}{BK} \); \( \therefore \frac{AH}{HB} = \frac{AK}{BK} \).

Now \( PH \) bisects \( \angle AP'B' \), but \( HPK \) is a right angle being in a semicircle;
\( \therefore PK \) bisects \( \angle AP'B' \) externally;
\( \therefore \frac{AH}{HB} = \frac{AK}{BK} \); \( \therefore \frac{HB}{HB'} = \frac{BK}{BK'} \);
\( \therefore \frac{HB'}{HB} = \frac{KB}{KB'} \) or \( \frac{HE}{HB'} = \frac{HE}{HK} \);
\( \therefore HP \) bisects \( \angle AP'B' \);
\( \therefore \frac{PA}{AH} = \frac{PB}{HK} = \lambda. \)

Q.E.D.

The circle in Theorem 9 which is the locus of \( P \) is called the circle of Apollonius.

A simpler proof of the converse of Theorem 9 will be found in the chapter on Harmonic Ranges, page 164. See also Ex. 18 (4).

21. \( ABC \) is a given triangle, find a point \( P \) such that \( PA : PB : PC = 1 : 2 : 4 \).

22. \( ABCD \) are four collinear points. Find a point \( P \) at which \( AB, BC, CD \) subtend equal angles.

23. Given the base, vertical angle and the ratio of the other sides of a triangle, construct it.

24. Given the base, one of the base angles and the ratio of the other sides of a triangle, construct it.

25. \( AB \) is a chord of a circle perpendicular to the diameter \( DE \) which it meets at \( H \); the tangent at \( A \) meets \( DE \) at \( T \), prove that \( TE \cdot TD = TH \).

26. \( A, B \) are two given points; \( AP, BQ \) are parallel chords of a variable circle through \( A, B \). If \( \frac{AP}{BQ} = \text{constant} \), prove that the locus of \( P \) is a circle.

27. Find the locus of a point at which two given circles subtend equal angles.

28. \( A, B \) are two fixed points, \( P \) is a variable point, such that \( \frac{AP}{PB} = \lambda \). For any particular value of \( \lambda \), \( P \) therefore describes a circle. Describe the changes in the position and magnitude of this circle as \( \lambda \) increases from 0 to 1 and then increases without limit.

29. With the notation of Ex. 28, if \( D \) is the mid point of \( AB \), and if \( X \) is the centre of the circle described by \( P \), prove that

(i) its radius = \( \frac{\lambda - 1}{\lambda} \cdot AB \);
(ii) \( DX = \frac{\lambda + 1}{\lambda} \cdot DB \);
(iii) the length of the tangent from \( D \) to the circle is independent of the value of \( \lambda \).

30. \( A, B \) are two fixed points; a point \( P \) moves so that \( \frac{AP}{PB} \) is constant; prove that when \( P \) is moving directly towards \( A \), the angle \( PBA \) is a right angle.

31. \( A, B, C \) are three collinear points; a straight line \( AP \) revolves about \( A \); \( P \) is a point on it such that \( BP \), \( CP \) are equally inclined to \( AP \), prove that the locus of \( P \) is a circle through \( A \).
32. \( ABA'B' \) are any four points; find a point \( C \) such that the triangles \( CAB, CMB' \) are similar, equal angles being denoted by corresponding letters.

33. If two figures are similar find a point \( O \) such that if one figure is rigidly rotated about \( O \) it can be made homothetic to the other figure. [Use Ex. 32.]

34. The internal and external bisectors of the angles of a triangle are drawn and form intercepts on the opposite sides. Three circles are described having these three intercepts as diameters, prove that they have two common points.

**THEOREM 10.** [Ptolemy's Theorem.]

If \( ABCD \) is a cyclic quadrilateral, then
\[
AB \cdot CD + AD \cdot BC = AC \cdot BD.
\]

![Diagram](image)

Draw \( AH \) so that \( \triangle DAB = \triangle BAC \), and let it meet \( BD \) at \( H \). The triangles \( ADB, ABC \) are similar,
\[
\frac{DAH}{BAC} = \frac{ADH}{ACB} = \frac{BD}{AC}.
\]

Also the triangles \( ADC, ABD \) are similar,
\[
\frac{DCA}{BAH} = \frac{AC}{BH}.
\]

and
\[
\frac{DAH}{BAC} = \frac{AC}{DH}.
\]

35. \( P \) is a point on the minor arc \( BC \) of the circumference of an equilateral triangle \( ABC \); prove that \( PA = PB + PC \).

36. Show that Pythagoras' theorem on the relations connecting the sides of a right-angled triangle is a special case of Ptolemy's theorem.

37. In the triangle \( ABC, AB = AC \); the altitude \( AD \) of the triangle meets the circumference at \( P \); prove that \( AP \cdot BC = AB \cdot BP \).

38. \( C \) is the mid point of the minor arc \( AB \) of a circle; \( P \) is a variable point on the major arc, prove that \( PA + PB \) is proportional to \( PC \).

39. \( P \) is a point on the minor arc \( AB \) of the circumference of the square \( ABCD \), prove that
\[
\frac{PA + PC}{PD} = \frac{PB + PC}{PD}.
\]

40. \( P \) is a point on the minor arc \( AB \) of the circumference of the regular pentagon \( ABCDE \), prove that
\[
\frac{PA + PD - PE}{PB + PE} = \frac{PC}{PD}.
\]

41. \( P \) is a point on the minor arc \( AB \) of the circumference of the regular hexagon \( ABCDEF \), prove that
\[
PE + PD = PA + PB + PC + PE.
\]

42. \( ABCD \) is an isosceles trapezium; \( AB, CD \) being the parallel sides, prove that \( AC^2 - AD^2 = AB \cdot DC \).

43. If the quadrilateral \( ABCD \) is not cyclic, prove that
\[
AD \cdot CB + AB \cdot CD > AC \cdot DB.
\]

[Draw \( AH \) and \( DH \) so that angle \( DAH = BAC \) and angle \( ADH = BAC \).]

44. \( P \) is a point on the minor arc \( AB \) of the circumference of the regular pentagon \( ABCDE \), prove that
\[
PA + PB + PD = PC + PE.
\]
CHAPTER III.

ON LINES AND CIRCLES CONNECTED WITH A TRIANGLE.

It is convenient to discuss the properties of the triangle in two sections. In the first, we shall establish the existence of certain points, lines and circles which are associated with every triangle; and in the second will be found relations of a more metrical character.

Most of the theorems of this chapter were probably known to Euclid. Theorem 25 however is ascribed to Robert Simson (1687-1758) who was a professor at Glasgow University. He undertook a complete revision of Euclid's elements, correcting and supplementing it, and all recent editions of the text are based on his work.

Theorem 18 was first enunciated by an Englishman, named Benjamin Bovean, in 1804: it seems, however, to have been obtained independently by Brianchon.

NOTATION.

It is convenient to denote particular points connected with the triangle \(ABC\) by definite letters.

The following notation will be adopted in future, unless otherwise stated:

- \(a, b, c\) denote the lengths of \(BC, CA, AB\).
- \(s = \frac{1}{2}(a+b+c)\).
- \(A', B', C'\) are the mid points of \(BC, CA, AB\).
- \(D, E, F\) are the feet of the perpendiculars from \(A, B, C\) to \(BC, CA, AB\).
- \(I\) is the incentre.
- \(r\) is the radius of the incircle.

\(J_p, J_o, J_b\) are the centres of the circles escribed to \(BC, CA, AB\).
\(r_p, r_o, r_b\) are the radii of these circles.
\(O\) is the circumcentre.
\(R\) is the radius of the circumcircle.
\(H\) is the orthocentre.
\(G\) is the centroid.
\(N\) is the nine-point centre.
\(X, Y, Z\) are the points of contact of the incircle with \(BC, CA, AB\).
\(X_p, Y_p, Z_p\) are the points of contact of the excircle, centre \(J_p\), with \(BC, CA, AB\), and similarly for \(X_o, Y_o, Z_o\) and \(X_b, Y_b, Z_b\).

THEOREM 11.

The perpendicular bisectors of the sides of a triangle are concurrent.

![Perpendicular Bisectors](image)

Draw the perpendicular bisectors of \(AB, AC\) to meet at \(O\). Every point on the perpendicular bisector of \(AB\) is equidistant from \(A\) and \(B\);
\[\therefore OA = OB.\]
Similarly, since \(O\) lies on the perpendicular bisector of \(AC\),
\[OA = OC;\]
\[\therefore OB = OC;\]
\[\therefore O\] lies on the perpendicular bisector of \(BC\);
\[\therefore\] the three perpendicular bisectors of the sides meet at \(O\).

Q.E.D.

Corollary.

Since \(OA = OB = OC\), a circle centre \(O\), radius \(OA\), passes through \(B\) and \(C\) and \(\therefore\) circumscribes the triangle \(ABC\).

Definition.

The point \(O\) is therefore called the circumcentre of the triangle \(ABC\).
1. Prove that $\hat{BOA} = \hat{BAC}$.

2. If $ABCD$ is a parallelogram, prove that the circumcentres of the triangles $ABG, ADC$ are equidistant from $AC$.

3. $L$ is a point in the base $BC$ of an isosceles triangle $ABC$; $P, Q$ are the circumcentres of the triangles $ABL, ALC$, prove that $PL$ is parallel to $AQ$.

4. Find a point $P$ such that the feet of the perpendiculars from $P$ to the sides of the triangle $ABC$ are the angular points of an equilateral triangle.

5. Prove that the perpendicular bisector of $EF$ bisects $BC$.

6. If $A, B, C$ are three collinear points, and if $K$ is any other point, prove that the circumcentres of the triangles $KBC, KAC, KAB$ are collinear with $K$.

7. The diagonals $AG, BD$ of a quadrilateral $ABCD$ meet at $K$, prove that the circumcentres of the triangles $KAB, KBC, KCD, KDA$ are the vertices of a parallelogram.

8. Equilateral triangles are described externally on the sides of a given triangle, prove that their circumcircles meet at a point, and that their circumcentres are the vertices of an equilateral triangle.

9. $AB, \overline{AB}$ are two lines of equal length in a plane; find a point $K$ such that the triangles $KAB, KAB'$ are congruent.

10. $P, Q, R$ are any points on the sides $BC, CA, AB$ of a triangle, prove that the circumcircles of the triangles $AP, BQ, CR$ are concurrent.

**THEOREM 12.**

The altitudes of a triangle are concurrent.

**FIG. 12.**

Let $\triangle ABC$ be the triangle.

Draw $AD, BE, CF$ perpendicular to $BC, CA, AB$.

Through $A, B, C$ draw lines parallel to $BC, CA, AB$ to form the triangle $PQR$.

---

Since $ABCD$ is a parallelogram, $KA = KB$.
Since $ABCD$ is a parallelogram, $AQ = QC$.

Similarly, $QC = CP$ and $PB = BR$.

Also, since $RQ$ is parallel to $BC$, $AD$ is perpendicular to $RQ$.

Similarly, $CF, BE$ are the perpendicular bisectors of $QP, PR$.

$: AD, BE, CF$ are concurrent. [Th. 11.]

**Q.E.D.**

**Definition.**

The point $H$ at which the altitudes concur is called the orthocentre of the triangle; and the triangle $DEF$ (see Fig. 16) is called the pencil triangle of the triangle $ABC$.

11. Prove that $\hat{AHE} = \hat{ABC}$.

12. Prove that the triangles $AEF, ABC$ are similar.

13. Prove that the triangles $AEF, BDF$ are similar.

14. Prove that $\triangle ABC, \triangle DEF$ are similar.

15. Prove that $\triangle ABC, \triangle DEF$ are similar.

16. Prove that $\overline{EF} = \overline{ED}$.

17. Prove that $\overline{EF} = 180° - 2 \overline{AC}$.

18. Prove that $\overline{BC} + \overline{AB} = 180°$.

19. If $S$ is the orthocentre of the triangle $PQR$, prove that $P$ is the orthocentre of the triangle $QRS$.

20. Given the base $BC$ and the angle $\hat{BAC}$ of the triangle $ABC$, find the locus of the orthocentre and construct its position when $A$ is very close to $B$.

21. $DP, DQ$ are the perpendiculars from $D$ to $AC, AB$; prove that $C, P, Q, B$ are concyclic and that $\hat{PBC} = \hat{QCD}$.

22. Prove that the circumcentre of the triangle $ABC$ is the orthocentre of the triangle $ABC$.

23. $P$ is a point on the base $BC$ of an isosceles triangle $ABC$; $K, L$ are the orthocentres of the triangles $ABP, APC$. Prove that $PKL$ is an isosceles triangle.

24. Prove that $OA, OB, OC$ are perpendicular to the sides of the pedal triangle of the triangle $ABC$.

**Definition.**

The line joining a vertex of a triangle to the mid point of the opposite side is called a median of the triangle.
THEOREM 13.

The three medians of a triangle are concurrent; and their point of intersection is a point of trisection of each median.

Let \( BB', CC' \) be two medians intersecting at \( G \).

Join \( AG \) and produce to \( K \), so that \( AG = GK \), and let it cut \( BC \) at \( L \). Join \( BK, CK \).

Since \( AC' = CB \) and \( AG = GK \), \( CG \parallel BK \).

Since \( AG = GK \) and \( AB' = BC' \), \( GB' \parallel KC \);

\( BGCK \) is a parallelogram.

\( BL = LC \) and \( GL = LK \);

\( AL \) is a median;

\( \therefore \) the three medians concur at \( G \).

Further

\[ GL = \frac{1}{3} GK = \frac{1}{3} AG \;
\]

\( \therefore GL = \frac{1}{3} AL \);

and similarly \( GB' = \frac{1}{3} BB' \) and \( GC' = \frac{1}{3} CC' \).

Q.E.D.

Alternative Method.

A simpler proof of Theorem 13 may be obtained by using similar triangles.

\( B'C' \parallel BC \) (see Fig. 17);

\( \therefore \) the triangles \( BGC', BG'C' \) are similar.

\( GS = GS, BG = B'C' \);

\( \therefore \frac{CG}{GC'} = \frac{BG}{GB'} = \frac{BC}{B'C'} = \frac{1}{3}; \)

\( BB' \) cuts \( CC' \) at the point of trisection of \( CC' \) nearest to \( C \).

Similarly \( AA' \) cuts \( CC' \) at this same point.

Q.E.D.

Definition.

The point at which the medians concur is called the centroid of the triangle.

25. Construct a triangle given the lengths of the three medians.

26. Construct a triangle given the lengths of two sides and one median.

[Two Cases.]

27. If \( B'C \) meets \( AA' \) at \( P \), calculate the ratios \( \frac{PG}{PA}, \frac{PG}{PA'}, \frac{AA'}{AA} \).

28. Prove that a triangle can be drawn with its sides equal and parallel to \( AA', BB', CC' \), and that its area is \( \frac{1}{4} \) the triangle \( ABC \).

29. In a tetrahedron \( ABCD \), the plane angles at each of three corners add up to two right angles, prove that the opposite edges are equal.

[Cut down the edges \( AB, AC, AD \) and fold it out flat. This appeared in the Mathematical Gazette.]

30. If \( GO \) meets \( AD \) at \( K \), prove that \( KG = \frac{1}{2} GO \). [Use similar triangles \( GOX', GKA \).] Hence prove that \( BE, CF \) pass through \( K \), so that \( K \) is the orthocentre. Hence prove that the orthocentre, centroid and circumcentre are collinear, and that the distance of the orthocentre from \( A \) equals \( 2A' \).

31. \( \triangle AMB, \triangle APQ \) are the squares described externally on the sides \( AB, AC \) of the triangle \( ABC \); prove that one median of the triangle \( LAM \) is an altitude of the triangle \( ABC \).

32. If \( K \) is the midpoint of \( EF \), prove that \( \angle AKE = A\hat{A}B \).

THEOREM 14.

The internal bisectors of the angles of a triangle are concurrent.

Draw the internal bisectors of the angles \( ABC, ACB \) to meet at \( I \). From \( I \) draw \( IX, IY, IZ \) perpendicular to \( BC, CA, AB \).
Now every point on the line bisecting the angle between $BA$, $BC$ is equidistant from $BA$, $BC$;
\[ \therefore IX = IZ. \]

But $I$ also lies on the bisector of the angle $BCA$;
\[ \therefore IX = IY; \]
\[ \therefore IY = IZ; \]
\[ \therefore I \text{ lies on the bisector of the angle } BAC; \]
\[ \therefore \text{ the internal bisectors concur at } I. \]

Corollary.
Since $IX = IY = IZ$ and since these lines are perpendicular to the sides of the triangle, $I$ is the centre of the circle inscribed in the triangle $ABC$.

Definition.
(1) The point of concurrency of the internal bisectors of a triangle is called the \textbf{incentre} of the triangle; and the circle, centre $I$, which touches the sides is called the \textbf{incircle} of the triangle.

(2) If a circle touches $BC$ and touches $AB$ produced and $AC$ produced it is called an \textbf{excircle} of the triangle or a circle inscribed to $BC$; and its centre is called an \textbf{excentre} of the triangle.

**Theorem 15.**

(1) The vertices of the triangle formed by the external bisectors of the angles of the triangle $ABC$ are the excentres of the triangle.

(2) The line joining an excentre to the incentre passes through a vertex of the triangle.

---

**Theorem 16.**

The incircle of the triangle $ABC$ touches $BC$, $CA$, $AB$ at $X$, $Y$, $Z$, and the circle inscribed to $BC$ touches $BC$, $CA$, $AB$ at $X_1$, $Y_1$, $Z_1$, then (1) $AZ = AY = s - a$;

(2) $AZ_1 = AY_1 = s$;

(3) $BX_1 = CX$;

where $BC = a$, $CA = b$, $AB = c$, and $s = \frac{1}{2}(a + b + c)$.

(1) $2s = a + b + c$

\[ = a + AY + XC + AZ + ZF \]

\[ = a + AY + XC + AE + BX \]

\[ = a + 2AY + BC \]

\[ = 2s + 2AY; \]

\[ \therefore AY = s - a. \]

Q.E.D.
THEOREM 17.

If the area of the triangle \(ABC\) equals \(\Delta\), then the radius \(r\) of the incircle equals \(\Delta/s\), and the radius \(r_1\) of the circle escribed to \(BC\) equals \(\Delta/(s-a)\).

\[
(2) \quad zAZ_1 = AZ_1 + AY_1 = AB + BZ_1 + AC + CY_1
\]

\[-AB + BX_1 + AC + Cx
\]

\[-c - BC + b
\]

\[= a + b + c = 2s;
\]

\[\therefore AZ_1 = s. \quad \text{Q.E.D.}
\]

\[
(3) \quad BX_1 = BZ_1 = AZ_1 - AB
\]

\[-s - c
\]

\[= CX \text{ by part } (1). \quad \text{Q.E.D.}
\]

33. Prove that (1) \(YY_1 = ZZ_1 = a\);

(2) \(XX_1 = c - b\);

(3) \(Z_1Z_2 = a + b\).

34. A polygon, of perimeter \(i\), circumscribes a circle of radius \(r\), prove that its area equals \(\pi r^2\).

35. From any point inside a regular polygon, perpendiculars are drawn to the sides of the polygon, prove that the sum of their lengths is constant.

36. Prove that the sum of the areas of the triangles \(I,AY, I, AZ\) is equal to the area of the triangle \(ABC\).

37. If \(BAC = 60^\circ\), prove that \(O, H, I, I_1, E, C\) lie on a circle.

38. Prove that \(\hat{A}O\) is half the difference of the angles \(ABC, ACM\).

39. \(M\) is the foot of the perpendicular from \(C\) to \(AI\), prove that the angles \(BMC, ACM\) are respectively equal to half the difference and sum of the angles \(ACB, ABC\).

40. The tangents at the ends of a chord \(AB\) of a circle meet at \(C\); prove that the incentre of the triangle \(ABC\) lies on the circle.

41. Prove that \(BIC = 90^\circ + \frac{A}{2}\), where \(A = \text{angle } BAC\).

42. Prove that the radius of the circumcircle of the triangle \(AYZ = \frac{IAA}{II}\).

43. \(P, Q\) are the feet of the perpendiculars from \(X, Y\) to \(YZ\) and \(XA\) respectively, prove that \(PQ\) is parallel to \(AB\).

44. Given the base and vertical angle of a triangle, find the locus of (1) the incentre; (2) the excentres.

45. If \(ABC = 90^\circ\), prove that \(2r = a + b - c\); and that \(r_5 = s\).

46. \(A_1, B_1, C_1\) are the excentres of the triangle \(ABC\); \(A_2, B_2, C_2\) are the excentres of the triangle \(A_1B_1C_1\) and so on; find the value to which \(A_1B_1C_1\) tends, as \(n\) becomes very large.
47. The incircle of the triangle $ABC$ touches $BC$, $CA$, $AB$ at $A_1$, $B_1$, $C_1$; the incircle of the triangle $A_1B_1C_1$ touches $B_1C_1$, $C_1A_1$, $A_1B_1$ at $A_2$, $B_2$, $C_2$, and so on; prove that $B_2A_2C_2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$.

What limit does $B_2A_2C_2$ tend towards, as $n$ becomes very large?

48. In the triangle $ABC$, the internal and external bisectors of the angles meet the opposite sides at $P$, $Q$, $R$; $p$, $q$, $r$ respectively. Prove that \[
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 3.
\]

49. A variable line $XY$ cuts two fixed lines $AB$, $CD$ at $X$, $Y$; find the locus of the point of intersection of the lines bisecting the angles $AXY$, $CYX$.

50. Prove that $H$ is the incentre of the pedal triangle of $ABC$.

51. $K$ and $L$ are the feet of the perpendiculars from $A$ to the lines bisecting internally the angles $ABC$, $ACB$; prove that $KL$ is parallel to $BC$.

52. From a given point, draw a line so as to form with two given lines a triangle of given perimeter.

53. $AH$ meets $BC$ produced at $P$, prove that $BP \cdot CP = AP^2 + BA \cdot AC$.

54. $K$ is the circumcentre of the triangle $I_1I_2I_3$; prove that $KI_1$ is perpendicular to $BC$.

55. Two sides of a triangle, of given perimeter, are given in position; find the envelope of the third side.

**THEOREM 18.**

$A'B'C'$ are the mid points of the sides of a triangle $ABC$;

\[\text{Fig. 22.}\]

$AD$, $BE$, $CF$ are the altitudes; $H$ is the orthocentre; $P$, $Q$, $R$ are the mid points of $HA$, $HB$, $HC$; then the nine points

**LINES AND CIRCLES**

$A'B'C'$, $DEF$, $PQR$ lie on a circle whose radius is equal to half that of the circumcircle of the triangle $ABC$.

Since $AP \parallel PH$ and $AC = CB$, $\therefore CP$ is parallel to $BH$.

Since $BC = CA$ and $BA' = AC'$, $\therefore A'C'$ is parallel to $AC$.

But $BH$ is perpendicular to $AC$, $\therefore CP$ is perpendicular to $A'C'$; $\therefore A'C'P$ is a right angle.

Similarly $A'B'P$ is a right angle.

Further $A'DP$ is a right angle;

$\therefore$ the circle on $AP$ as diameter passes through $E$, $C$, $D$;

$\therefore D$ and $P$ lie on the circle $A'B'C'$.

Similarly $E$, $Q$, $F$, $R$ lie on the circle $A'B'C'$;

$\therefore$ the nine points $A''B''C''$, $DEF$, $PQR$ lie on a circle.

Moreover, each side of the triangle $A'B'C'$ is half the corresponding side of the triangle $ABC$; and $\therefore$ the radius of the circle $A'B'C'$ is half that of the radius of the circle $ABC$. Q.E.D.

**Definition.**

The circle which passes through these nine points is called the nine-point circle of the triangle $ABC$, and its centre is called the nine-point centre.

**THEOREM 19.**

If $PQ$ is the diameter of the circumcircle of the triangle $ABC$ perpendicular to $BC$, then $AP$ and $AQ$ are the two bisectors of the angle $BAC$.

Since $PQ$ is a diameter perpendicular to $BC$, it bisects the arc $BPC$. 

[Image 25]
Since

\[ \overparen{BP} = \overparen{PC}, \]
\[ \overparen{BA} = \overparen{AC}; \]
\[ \therefore \overparen{AP} \text{ bisects } \overparen{BAC}. \]

But \( \overparen{AP} \) is a right angle, since \( \overparen{PQ} \) is a diameter;
\[ \therefore \overparen{AQ} \text{ is perpendicular to } \overparen{AP}; \]
\[ \therefore \overparen{AQ} \text{ is the other bisector of } \overparen{BAC}. \]
\[ \text{Q.E.D.} \]

**THEOREM 20.**

If \( O, I, H \) are the circumcentre, incentre and orthocentre of the triangle \( ABC \), then \( AI \) bisects the angle \( OAH \).

In Fig. 23, join \( AH \), which is perpendicular to \( BC \), and is \( \parallel \) to \( OP \). Also \( I \) lies on \( AP \), the bisector of \( BAC \).
\[ OA = OF, \text{ radii; } \therefore O\overparen{AP} = O\overparen{FA}. \]

But \( O\overparen{FA} = \overparen{PAH} \), alternate angles;
\[ \therefore O\overparen{AI} = \overparen{LAH}. \]
\[ \text{Q.E.D.} \]

56. If \( P \) is the mid-point of \( AH \), prove that (1) \( \overparen{APC} = \overparen{APC} \), and (2) \( PFC \) equals the difference of the angles \( BAC, ABC \).

57. Prove that the radius of the circumcircle of the pedal triangle equals \( \frac{1}{2} R \).

58. Prove that the tangent to the nine-point circle at the mid-point of the base is parallel to the tangent to the incircle from the point at which the bisector of the vertical angle meets the base.

59. Prove that \( I_1I_2 \) is bisected by the circumcircle of the triangle \( ABC \).

60. Prove that the triangles \( ABC, HBC \) have the same nine-point circle.

61. Prove that the nine-point circles of the triangles \( ABC, BCA, CA \) each touch the nine-point circle of the triangle \( ABC \).

62. If \( P, Q, R \) are the mid-points of \( HA, HB, HC \), prove that \( AP, BP, CR \) are equal and concurrent.

63. Prove that the centre \( N \) of the nine-point circle is at the mid-point of \( OI \).

64. Given the base \( BC \) and vertical angle of a triangle \( ABC \), prove that the locus of \( N \) is a circle centre \( A \).

65. \( A, B, C \) are three fixed points. Describe a square with one vertex at \( A \), so that the sides opposite to \( A \) pass through \( B \) and \( C \).

66. A sphere of radius 6 inches rests on three horizontal wires forming a plane triangle whose sides are 5 inches, 12 inches, 12 inches. Find the height of the top of the sphere above the plane of the wires.

67. A sphere is inscribed in a cube, edge \( a \); and a plane passing through the extremities of three edges which meet at a corner meets the sphere in a circle, diameter \( a \); prove that \( 3a^2 = a^2 \).

**THEOREM 21.**

If \( AII_1 \) meets the circumcircle of the triangle \( ABC \) at \( P \), then
\[ PI = PB = PC = PI. \]
THEOREM 22.

If \( AD, BE, CF \) are the altitudes of the triangle \( ABC \), then

1. The triangles \( AEF, ABC \) are similar,
2. \( AD \) bisects the angle \( FDE \).

![Diagram](image)

\( \angle BFC = 90^\circ = \angle BAC \), \( \therefore BFE \) is a cyclic quadrilateral;

\[ \angle AFE = \angle BFE \]

\[ \angle ADF = \angle BFC \]

\[ \therefore \text{the triangles } AEF, ABC \text{ are equiangular and } \therefore \text{ similar.} \]

Q.E.D.
Join \( CP \) and draw the altitude \( CHF \).

\[ \hat{P}CD \text{ or } \hat{P}CB - \hat{P}AB \text{ in the same segment} \]

\[ = 90^\circ - \hat{ABC} \]

\[ = \hat{P}CB, \text{ since } \hat{B}FC = 90^\circ; \]

\[ \therefore \text{ the triangles } \triangle HDC, \triangle PDC \text{ are congruent,} \]

for

\[ HD = \hat{P}D, \]

\[ HC = \hat{P}C, \]

\[ DC \text{ is common; } \]

\[ \therefore HD = DP. \]

Q.E.D.

84. Write out the proof of Theorem 23 when \( \triangle ABC \) is obtuse.

85. Prove \( AD, HD \parallel BD, DC \).

86. Prove that the circumcircles of the triangles \( \triangle ABC, \triangle HBC \) are equal.

87. Prove that the circumcircles of the triangles \( \triangle HBC, \triangle HCA, \triangle HAB \) are equal.

88. If \( Q, R \) are the circumcentres of the triangles \( \triangle CHA, \triangle ABH \), prove that \( BR \) and \( CQ \) are parallel.

89. If \( P, Q, R \) are the circumcentres of the triangles \( \triangle HBC, \triangle HCA, \triangle HAB \); prove that the triangles \( \triangle PQR, \triangle ABC \) are congruent; and that \( H \) is the circumcentre of \( PQR \).

90. Prove that the radius of the circle \( H1 \) is equal to the diameter of the circle \( \triangle ABC \).

91. \( AD, BE, CF \) meet the circumcircle of the triangle \( \triangle ABC \) at \( PQR \); prove that \( H \) is the incentre of \( PQR \).

92. \( BE, CF \) meet the circumcircle of the triangle \( \triangle ABC \) at \( Q, R \); prove that \( A \) is the centre of the circle \( \triangle HQR \).

93. \( ABCD \) is a cyclic quadrilateral. \( H, K \) are the orthocentres of the triangles \( \triangle ABC, \triangle ABD \); prove that \( A, B, H, K \) are concyclic.

THEOREM 24.

If \( AD \) is an altitude of the triangle \( \triangle ABC \), and if \( K \) is the radius of the circumcircle, then \( AB \cdot AC = 2R \cdot AD \).
100. Generalise Ex. 99 to apply to any polygon inscribed in a circle. [Draw tangents at its vertices to form a new polygon.]

101. \( OP, OQ \) are two variable chords of a fixed circle. \( O \) is a fixed point. If \( OP, OQ \) is constant, find the envelope of \( PQ \).

102. \( P \) is a point on the circumcircle of an equilateral triangle \( ABC \), and is situated on the minor arc \( AC \); \( PL, PM, PN \) are the perpendiculars to \( BC, CA, AB \); \( O \) is the circumcentre, prove that \( 4AO(PL+PM+PN)=3PB^2-PA^2-PC^2 \).

**Theorem 25.** [Simpson's Line.]

If \( P \) is any point on the circumcircle of the triangle \( ABC \), and if \( L, M, N \) are the feet of the perpendiculars from \( P \) to \( BC, CA, AB \), then \( L, M, N \) are collinear.

![Diagram of Theorem 25](image)

Join \( LN, LM, PB, PC \).

Since \( \hat{P}NB = 90^\circ = \hat{P}LB \), \( PLNB \) is a cyclic quadrilateral.

Since \( \hat{P}LC = 90^\circ = \hat{P}MC \), \( PLCM \) is a cyclic quadrilateral.

\( \therefore \hat{P}LM = \hat{P}CM \), since \( PLCM \) are concyclic,

\( \therefore \hat{P}BA \), since \( PCAB \) are concyclic,

\( = 180^\circ - \hat{P}LN \), since \( PLNB \) are concyclic;

\( \therefore \hat{P}LM + \hat{P}LN = 180^\circ \);

\( \therefore MLN \) is a straight line. Q.E.D.

**Definition.**

The line \( MLN \) is called the **Simpson Line** or the **pedal line** of

\( P \) w.r.t. the triangle \( ABC \).

A generalised form of Theorem 25 is given in Ex. 114.

III.

103. If the pedal line of \( P \) is parallel to \( AD \), prove that \( PA \) is parallel to \( BC \).

104. Find a point \( P \) having its pedal line parallel to a given line.

105. Prove that the pedal line of the point \( A \), which \( AD \) cuts the circumcircle is parallel to the tangent at \( A \) to the circumcircle.

106. \( KL \) is a diameter of the circumcircle of the triangle \( ABC \), prove that the pedal lines of \( K \) and \( L \) intersect at right angles on the nine-point circle.

107. The pedal line of a point \( P \) meets \( BC \) at \( L \) and \( AD \) at \( K \); prove that \( LW \) is parallel to \( PK \).

108. The perpendiculars from a point \( P \) on the circumcircle of the triangle \( ABC \) to \( BC, CA, AB \) cut the circle again at \( X, Y, Z \); prove that the triangles \( ABC, XYZ \) are congruent.

109. Find the three points, the pedal lines of which are perpendicular to the medians of the given triangle.

110. \( P, Q, R \) are the mid-points of the arcs \( BC, CA, AB \) of the circumcircle of the triangle \( ABC \); prove that the triangle formed by the pedal lines of \( P, Q, R \) w.r.t. the triangle \( ABC \) is similar to the triangle formed by joining the points of contact of \( ABC \) with its incircle.

111. \( P \) is a point on the circle circumscribing a quadrilateral \( ABCD \); if \( AB \) is parallel to \( CD \), prove that the feet of the perpendiculars from \( P \) to \( AC, AD, BC, BD \) are concyclic.

112. \( AD, BE, CF \) meet the circumcircle of the triangle \( ABC \) at \( P, Q, R \); prove that the pedal line of \( A \) w.r.t. the triangle \( PQR \) is parallel to \( BC \).

113. \( P, Q \) are any two points on the circumcircle of the triangle \( ABC \); prove that the angle between the pedal lines of \( P \) and \( Q \) equals \( \frac{1}{2} \hat{POQ} \).

114. If in Theorem 25, \( PL, PM, PN \) are drawn so that

\( \hat{PNB} = \hat{PBM} = \hat{PNC} \),

prove that \( L, M, N \) are collinear.

115. Prove that if \( P \) is such a point that the feet of the perpendiculars from \( P \) to the sides of a triangle are collinear, then \( P \) must lie on the circumcircle of the triangle. (Converse of Theorem 25)

116. Prove that the circumcircles of the four triangles formed by four straight lines have a common point. (Use Ex. 115.)

117. \( L, M, N \) are the feet of the perpendiculars from a point \( P \) on the circumcircle of the triangle \( ABC \) to the sides \( BC, CA, AB \). Prove that the triangles \( PLN, PAC \) are similar, and find the position of \( P \) for which \( ML = LN \).
118. With the notation of Ex. 117, prove that $PL \cdot MN, PM \cdot NL, PN \cdot LM$ are proportional to $BC, CA, AB$.

119. With the notation of Ex. 117, prove that of the ratios $BC, CA, AB$ $PC, PM, PN$, one of them is equal to the sum of the other two.

120. Prove that the pedal line of $P$ bisects $PH$.

121. Construct a triangle, given its pedal triangle.

122. With the notation of Fig. 28, if $PK$ is the perpendicular from $P$ to $MN$, prove that $PM \cdot PN = PA \cdot PK$.

123. Find a point $P$ inside the acute angled triangle $ABC$ such that the circumcircles of the triangles $APB, BPC, CPA$ are equal.

124. $P, Q, R$ are any points on the sides $BC, CA, AB$ of a triangle, prove that the circumcircles of the triangles $APB, BPC, CPA$ are the vertices of a triangle similar to $ABC$.

CHAPTER IV.

ON POINTS CONNECTED WITH A TRIANGLE.

The theorems of this chapter were enunciated at a much later date than the majority of those already considered. Theorems 26, 27, 29 are due to Euler (1707-1783), a Frenchman by birth, who spent most of his life at Berlin and St. Petersburg. He may be regarded as the first writer on modern analysis; but in addition to his numerous contributions to pure mathematics, there are a number of fundamental theorems in applied mathematics which are first found in his memoirs.

THEOREM 26.

The circumcentre, the centroid, and the orthocentre of a triangle are collinear.

Let $O$, $G$ be the circumcentre and centroid. Produce $OG$ to meet the altitude $AD$ at $K$.

The triangles $OGA'$, $KGA$ are similar, for $OA'$ is parallel to $AK$, since each is perpendicular to $BC$;

$\therefore \frac{OG}{OA'} = \frac{AG}{GA'} = 1$, since $G$ is the centroid;

$\therefore \frac{OG}{OA} = \frac{AK}{GA} = 1$.

$\therefore CK = 2OG$. 

![Diagram](Fig. 26)
Now if $OG$ is produced to meet the altitude $CF$ at $K'$, it follows in the same way that $GK' = 2OG$.

$\therefore GK = GK'$ and $K'$ coincides with $K$;

$\therefore AD$ and $CF$ meet at $K$;

$\therefore K$ is the orthocentre;

$\therefore$ the circumcentre, centroid, and orthocentre are collinear.

Q.E.D.

**THEOREM 27.**

If $O$, $G$, $H$, are the circumcentre, centroid, and orthocentre of the triangle $ABC$, and if $N$ is the nine-point centre; then

1. $HG = 2GO$;
2. $AH = 2OA'$;
3. $N$ is the mid-point of $OH$;

Since the triangles $OGA'$, $AGH$ are similar, [Th. 26.]

$$\frac{GH}{GA} = \frac{AH}{AG} = \frac{AH}{OA'}$$

But $GA = 2AG$, since $G$ is the centroid;

$\therefore GH = 2OG$ and $AH = 2OA'$.

Q.E.D.

The perpendicular bisector of $AD$ is parallel to $OA'$ and $HD$, and $\therefore$ bisects $OH$.

Similarly, the perpendicular bisector of $CP$ bisects $OH$;

$\therefore$ the perpendicular bisectors of $AD$ and $CP$ meet at the mid-point of $OH$.

But $AD$ and $CP$ are chords of the nine-point circle;

$\therefore$ the mid-point of $OH$ is the centre $N$ of the nine-point circle.

Lastly,

$$ON = \frac{1}{2}OH, \quad OG = \frac{1}{4}OH;$$

but

$$NG = \frac{1}{4}OH,$$

$$NH = \frac{1}{2}OH;$$

$\therefore OG : GN : NH = 2 : 1 : 3$.

Q.E.D.

**IV.] ON POINTS CONNECTED WITH A TRIANGLE 43**

1. The extremities of a straight line $PQ$ of constant length move on two fixed lines $AB$, $AC$; prove that the locus of the orthocentre of the triangle $APQ$ is a circle, centre $A$.

2. In the triangle $ABC$, prove that $AH + BC = 4AO$.

3. $AO$ meets $BC$ at $U$; prove that the line joining $D$ to the mid-point of $AU$ passes through $N$.

4. If $BAC = 45^\circ$, prove that $EF$ bisects $OH$.

5. Given the nine-point circle and one angular point of a triangle, prove that the locus of the orthocentre is a circle.


8. Given $A, B, N$, construct the triangle.

9. Prove that the circumcircle and the nine-point circle are homothetic w.r.t. either $H$ or $O$.

10. Prove that $HA'$ meets the circumcircle at the end $P$ of the diameter through $A$ of the circumcircle.

11. $GA', GD$ meet the circumcircle at $P, Q$, prove that $PQ$ is parallel to $AD$ and is equal to $2AD$.

12. $P$ is the mid-point of $AH$; $GP$ meets the circumcircle at $Q$, prove that $HQ$ is parallel to $AG$.

**THEOREM 28.**

The internal bisector of the angle $BAC$ meets the circumcircle of the triangle $ABC$ at $P$. If $M, N$ are the feet of the perpendiculars from $P$ to $AB$, $AC$,

$$AM \times AN = \frac{1}{4}(AB + AC),$$

$$MB \times CN = \frac{1}{4}(AB - AC).$$
Join $PB$, $PC$.

Since $P$, $B$, $A$, $C$ are concyclic,

$$PB + PC = 180^\circ;$$

$\therefore$ one of the angles $PB$, $PC$ is acute and the other is obtuse.

Suppose then (as in Fig. 31) that $PB$ is acute, then $M$ lies between $A$ and $B$ and $N$ lies on $AC$ produced.

[Why is this a necessary part of the proof?]

The triangles $AMP$, $ANP$ are congruent,

for

$$\hat{A}MP = 90^\circ = \hat{A}NP$$

and $AP$ is common;

$\therefore$ $AM = AN$ and $PM = PN$;

also the triangles $BMP$, $CNP$ are congruent,

for

$$\hat{B}MP = 90^\circ = \hat{C}NP$$

and $PB = PC$

since these chords subtend equal angles at the circumference,

and $PM = PN$;

$\therefore$ $MB = CN$;

$\therefore$ $AB = AM = AN = AC$;

$\therefore$ $AB + AC = AM + AN = 2AM$;

$\therefore$ $AM = AN = \frac{1}{2}(AB + AC)$.

Q.E.D.

13. Given the base and vertical angle of a triangle, and (1) the difference, or (2) the sum of the other two sides, construct it.

14. In the triangle $ABC$, $B\hat{A}C$ is fixed in magnitude and position, $AB + AC$ is constant, prove that the locus of $A'$ is a straight line perpendicular to $AI$.

15. If $AI$ meets the circumcircle at $Q$, and if $QR$ is drawn perpendicular to $AB$, prove that $AR$, $RB$ are respectively equal to half the difference and the sum of $AB$, $AC$.

16. $R$ is the mid-point of the minor arc $BC$ of the circumcircle of the triangle $ABC$; the circle on $AR$ as diameter meets $AB$ at $S$; prove that the tangent from $S$ to the incircle equals $\frac{1}{2}BC$.

17. With the notation of Theorem 28, prove that the tangents from $M$ to the incircle and excircle escribed to $BC$ are equal.

18. Theorem 29.

If $O$ and $I$ are the circumcentre and incentre of the triangle $ABC$, and if $K$, $r$ are the circumradius and inradius, then $OI^2 = R^2 - 2R \cdot r$.

Produce $AI$ to meet the circumcircle at $P$; produce $PO$ to meet the circumcircle at $Q$. Draw $OM$ perpendicular to $AP$; [it bisects $AP'$]. Draw $IZ$ perpendicular to $AB$.

Then

$$K^2 - OI^2 = OA^2 - OP^2$$

$$= OM^2 + MA^2 - (OM^2 + MP^2)$$

$$= MA^2 - MP^2$$

$$= (MA - MI)(MA + MI)$$

$$= AI \cdot IP$$, since $MA = PM$,

$$= AI \cdot PB$$, since $PI = PB$.

Now the triangles $AIZ$, $QPB$ are similar,

for $Q\hat{B}P = 90^\circ = \hat{A}IZ$ and $B\hat{O}P = 2A\hat{I}$ in the same segment;

$\therefore$ $AI \cdot PQ$;

$\therefore$ $AI \cdot PB = PQ \cdot IZ = 2R \cdot r$;

$$R^2 - OI^2 = 2R \cdot r$$.

Q.E.D.

Corollary.

By the same method, it may be proved that

$$OI^2 = R^2 + 2R \cdot r.$$
18. Prove the Corollary of Theorem 29.
You will prove that \( \frac{O}{P} = \frac{Q}{A} \) if \( P, O, A = P \) and \( O = A \).

19. A variable triangle circumscribes a fixed circle if its circumradius is given, find the locus of its circumcentre.


21. Prove that the ratio of the squares of the tangents from \( I \) and \( I \) to the circumcircle equals the ratio of the radii \( r_i \) and \( r \).

22. The perpendicular at \( I \) to \( I \) meets the incircle at \( K, K' \); prove that the perimeter of the triangle \( OKK' \) equals the diameter of the circumcircle.

23. A circle is drawn to touch \( OI \) at \( I \) and to touch the circumcircle; prove that it is equal to the incircle.

24. If the circumcircle meets the incircle at \( P \), and if the tangents at \( P \) to the two circles are at right angles, prove that the radius of the excircle equals the diameter of the incircle.

25. Given the base and vertical angle of a triangle, prove that the incircle is largest when the triangle is isosceles.

26. A remarkable property of the nine-point circle is due to Feuerbach.

Feuerbach's Theorem. The nine-point circle of a triangle touches the incircle and each of the excircles.

The following sequence of riders indicates one method of proof. Another method will be found on page 149.

27. Let \( AI \) meet \( BC \) at \( U \); from \( U \) draw \( UK \) to touch the incircle at \( K \). Let \( AK \) meet the nine-point circle at \( R \). Draw \( CM \) perpendicular to \( AI \) and let it meet \( AI, AB \) at \( M, L \); with the usual notation.

28. Prove \( AM = BL = \frac{1}{2}(a + b) \); \( AX = AH \); \( MD \parallel AC \); \( AD = AM \); \( AK = \frac{1}{2}(a + b) \); \( K = \frac{1}{2}(a + b) \).

29. Prove \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

30. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

31. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

32. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

33. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

34. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

35. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

36. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

37. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

38. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

39. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

40. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

41. Prove that \( OAH = \frac{1}{2}(a + b) \); \( OA = \frac{1}{2}(a + b) \); \( OB = \frac{1}{2}(a + b) \); \( OC = \frac{1}{2}(a + b) \).

42. A triangle of given shape has one vertex fixed and the opposite side passes through a fixed point, find the locus of the orthocentre.

43. Points \( P, Q, R \) are taken on \( BC, CA, AB \) so that \( BP = \frac{1}{2}(a + b) \); \( CQ = \frac{1}{2}(a + b) \); \( AR = \frac{1}{2}(a + b) \); \( RR = \frac{1}{2}(a + b) \).

\[ \text{prove that the area of the triangle} \ PQR = \frac{1}{2}(a + b) \Delta, \text{where} \ \Delta = \text{area of the triangle} \ ABC. \]
44. \( Q \) and \( R \) are the points of trisection of the arcs \( BE, CF \) of the nine-point circle, nearest to \( B \) and \( C \); prove that \( QR \) is the side of an equilateral triangle inscribed in the nine-point circle.

45. Squares \( AEFB, AGCH \) are described externally to the triangle \( ABC \), which is right-angled at \( C \); a point \( K \) is taken on \( AC \) (produced if necessary) so that \( AK = BC \); prove that \( A \) is the centroid of the triangle \( HEK \).

46. Points \( P, Q, R \) are taken on \( BC, CA, AB \) so that
\[
\frac{EP}{FC} = \frac{CQ}{QA} = \frac{AR}{RB} = 4.
\]
Calculate the ratio of the areas of the triangles \( PQR, ABC \).

47. A tangent \( LM \), parallel to \( BC \), is drawn to the circle escribed to \( BC \); if \( AB \) touches the excircle at \( Z_0 \), and if \( LM \) meets \( AB \), \( AC \) at \( P, Q \), prove that
\[
\frac{PQ}{BC} = \frac{AZ_0}{Z_0C}.
\]

48. The base of a triangle passes through a fixed point, and the other sides are bisected at right angles by two fixed lines \( OP, OQ \); prove that the locus of the vertex is a circle through \( O \).

49. Construct a triangle \( ABC \), given \( BC, D \) and the length of a side of the inscribed square which has one of its sides on \( BC \).

50. Construct a triangle given the base and vertical angle and one median. [Two cases.]

51. \( ABCD \) is a regular tetrahedron: \( AE \) is drawn perpendicular to the face \( BCD \); prove that \( 3AE = 2AB \).

52. Four equilateral triangles are arranged so that their vertices coincide and their bases form a square. Prove that the opposite edges of the solid angle at the vertex are at right angles.

53. If \( ABCD \) is a tetrahedron, prove that
\[
\sin BDC \sin DAC \sin ABC = \sin BDC \sin DAC \sin ABC = 1.
\]

54. If \( ABCD \) is a tetrahedron, prove that the square of the area of each face is proportional to the product of the sines of the angles of that face and the sines of the angles at the opposite vertex.

55. The sides of the triangle \( ABC \) are \( 6, 8, 10 \); and a point \( D \) is taken such that \( DA = DB = DC = 9 \); prove that the volume of the tetrahedron \( ABCD \) is 18.

56. If a cube and an octahedron have a common circumscribing sphere, prove that their surfaces are in the same ratio as their volumes.

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48. ON POINTS CONNECTED WITH A TRIANGLE

It is not proposed to include in this book any formal account of the modern geometry of the triangle which has been developed in great detail in recent years. The following sequence of riders (57-56), however, include a few of the more elementary theorems, and are so arranged that no difficulty will be found in working through them. Those which do not affect the logical connection of the set, regarded as a whole, are marked with an asterisk.

**Definition.**

If a straight line \( PQ \) cuts two fixed straight lines \( AB, AC \) at \( P, Q \) so that \( APQ = ACB \), then \( PQ \) is called an antiparallel to \( BC \) w.r.t. \( AB, AC \).

57. With the notation of this definition, prove that \( PQCB \) are concyclic.

58. Prove that the sides of the pedal triangle are antiparallels to the corresponding sides of the original triangle \( ABC \) w.r.t. the remaining sides of \( ABC \).

59. If two lines are antiparallel to \( BC \) w.r.t. \( AB, AC \), prove that they are parallel to each other.

60. Prove that the antiparallels to \( BC \) w.r.t. \( AB, AC \) are parallel to the tangent at \( A \) to the circumscribed circle of \( ABC \).

**Definition.** The line \( AK \) which bisects all the antiparallels to \( BC \) w.r.t. \( AB, AC \) is called the symmedian line of the triangle.

A triangle has therefore three symmedians, one through each vertex.

61. If the tangents at \( A, B, C \) to the circumcircle of the triangle \( ABC \) form the triangle \( A_1B_1C_1 \), where \( A_1, B_1, C_1 \) are respectively opposite to \( A, B, C \); prove that \( AA_1, BB_1, CC_1 \) are symmedians and that they are concurrent.

[Through \( A_1 \) draw an antiparallel \( QA_1R \) to \( BC \) to meet \( AB, AC \) at \( Q, R \); use Ex. 60 and prove \( QA_1 = RA_1 = CA_1 = AA_1 \).]

**Definition.** The point at which the symmedians intersect is called the symmedian point of the triangle and will be denoted by \( K \).

62. Prove that \( GAB = KAC \); \( GBC = KB \); \( GCA = KCB \).

[Reflect the figure about \( AI \) so that \( AB \) and \( AC \) interchange their positions, and note that antiparallels to \( BC \) become lines parallel to \( BC \) so that \( AC \) becomes \( AG \).]

63. \( P, P' \) are two points inside the triangle \( ABC \) such that \( BAP = CAP \); perpendiculars \( PN, PN' \); \( PM, P'M \) are drawn to \( AB, AC \) respectively. Prove that \( PN, PN' = PM, P'M \).

[Use similar triangles \( PAM, P'M \); \( PAN, P'M \).]

D, D
64. Conversely if, with the same construction as in Ex. 63, \( PN, PM' = PM, PM' \), prove that \( \angle BAP = \angle CAP \).

65.* Prove, with the same figure as Ex. 63, that \( MN \) is perpendicular to \( AP \).

Definition. \( AP, AP' \) are called isogonal lines w.r.t. \( AB, AC \) if \( \angle BAP = \angle CAP \); \( AP, AP' \) being between \( AB \) and \( AC \).

66. If \( AP, AP' \) are isogonal lines w.r.t. \( AB, AC \); and if \( BP, BP' \) are isogonal lines w.r.t. \( BC, BA \); then \( CP, CP' \) are isogonal lines w.r.t. \( CA, CB \). [Use Ex. 63, 64.]

Definition. Points \( P, P' \), situated as in Ex. 66 are called isogonal conjugates.

67.* Prove that \( O \) and \( H \) are isogonal conjugates.

68.* If \( P \) lies on the circumcircle, prove that its isogonal conjugate is at infinity.

69.* Prove that the incentre and excentres are isogonal self-conjugates (i.e. are isogonal to themselves).

70. Prove that \( G \) and \( K \) are isogonal conjugates.

71. Prove that the perpendiculars from \( K \) to \( BC, CA, AB \) are respectively proportional to those sides. [Use Ex. 65, 72.]

72. Prove that the sum of the squares on the perpendiculars from a point \( P \) to the sides of a triangle is least when \( P \) is at \( K \).

[If \( a, b, c \) are the lengths of the perpendiculars, use the identity

\[
(a^2 + b^2 + c^2 - a^2 - b^2 + c^2) - (a^2 + b^2 + c^2 - a^2 - b^2 + c^2) = (a^2 + b^2 + c^2 - a^2 - b^2 + c^2),
\]

and use Ex. 71.]

73. Prove that it is always possible to find one and only one point \( \Omega \) such that \( \Omega AB = \Omega BC = \Omega CA \).

[Prove that \( \Omega \) must lie on a circle, touching \( AB \) at \( A \) and passing through \( C \), and also on a circle touching \( AC \) at \( C \) and passing through \( B \).]

In the same way there exists one and only one point \( \Omega' \) such that \( \Omega' AC = \Omega' CB = \Omega' BA \).

Definition. The points \( \Omega, \Omega' \) obtained in Ex. 73 are called the Brocard points of the triangle \( ABC \).

74.* Prove that \( \Omega \) and \( \Omega' \) are isogonal conjugates.

75.* If \( AM \) is a chord parallel to \( BC \) of the circle which touches \( AB \) at \( A \) and passes through \( C \), prove that \( BM \) meets the circle at \( \Omega \).

76. Prove that \( \angle BAC = 180^\circ - ABC \). What is the corresponding value for \( BUC \)?

CHAPTER V.

CONCURRENCE AND COLLINEARITY.

In Coordinate Geometry it is of fundamental importance that account should be taken of the direction in which a line is measured. Thus lines measured along the \( x \)-axis, \( OX \), are taken as positive when measured towards the right hand from the point \( O \), and are taken as negative when measured towards the left. This is the usual convention; but it would be equally correct to regard lengths measured to the left as positive and to the right as negative.

Many geometrical theorems can be enunciated in a more general form, if the same connection between sign and direction is observed.

If then \( A, B \) are any two points, we say that

\[ AB = -BA \]

or

\[ AB + BA = 0. \]

This is equivalent to stating that if a point travels from \( A \) to \( B \) and then from \( B \) to \( A \), its final distance from its starting point is zero.

Definition.

Two segments \( AB, PQ \) of the same or parallel lines are said to have the same sense or opposite senses [or are sometimes called like or unlike] according as the directions \( A \rightarrow B \) and \( P \rightarrow Q \) are the same or opposite.

\[ \begin{align*}
A & \quad B & \quad P \\
A & \quad P & \quad B
\end{align*} \]

1. If \( ABP \) are three points in any order on a straight line, then

\[ AB = AP + PB. \]

This amounts to stating that if a point travels from \( A \) to \( P \) and then from \( P \) to \( B \), its final distance from \( A \) is equal to \( AB \).
2. If $A_P P_2 \ldots P_n B$ are points in any order on a straight line, then
   
   (1) $AB = AP_1 + P_1 P_2 + \ldots + P_n B$;
   
   (2) $AP_1 + P_1 P_2 + \ldots + P_n B + BA = 0$.

3. If $ABO$ are points in any order on a straight line, then
   
   $AB = OB = OA$.

4. Draw any finite straight line $AB$ and mark roughly on it a point $P$ such that $\frac{AP}{PB}$ equals
   
   (1) $+1$; (2) $-\frac{1}{2}$; (3) $-2$; (4) $-1$.

5. Describe the changes in the value of the ratio $\frac{AP}{PB}$ as $P$ moves on the unlimited line $AB$: and prove that this ratio can never have the same value for two different finite positions of $P$.

It follows from Ex. 5 that if $x$ is any real number (positive or negative), excluding $x = -1$, there is only one position of $P$ on the unlimited line $AB$ for which $\frac{AP}{PB} = x$. And as the distance of $P$ from $A$ or $B$ increases, the ratio $\frac{AP}{PB}$ approaches the value $-1$; and can be made to differ from $-1$ by a quantity less than any assigned quantity, however small, by increasing sufficiently the distance of $P$ from $A$.

If $x = -1$, there is no possible finite position for $P$. To preserve continuity, we shall therefore suppose that there is only one position of $P$ on the line $AB$ for which $\frac{AP}{PB} = -1$, and we shall call this point, the point at infinity on the line $AB$, and shall denote it by $\infty$. It should be understood that a point of this sort is purely ideal, in other words it is merely a convenient way of expressing in geometrical form an arithmetical idea. A more complete discussion will be found in the Second Part.

A number of relations exist which connect the lengths of various segments of a straight line. For example, if $A$, $B$, $C$, $D$ are any four collinear points, then $AB$, $CD = BC$, $AD + CA$, $BD = \infty$. These relations are best proved by taking an origin $O$ on the line and expressing the length of each segment in terms of the distances of its extremities from $O$.

Thus $AB = OB = CA$, etc.

v.) CONCURRENCE AND COLLINEARITY

6. Prove that if $ABCD$ are four collinear points in any order, then $AB$, $CD = BC$, $AD + CA$, $BD = \infty$.

7. $C$ is the mid-point of $AB$; $P$ is any other point on the line $AB$; prove that
   
   (1) $PC = \frac{1}{2}(PA + PB)$,
   
   (2) $AP \cdot BP + CP = CP$,
   
   (3) $AP^2 + BP^2 = 2AC^2 + 2CP^2$.

8. $AB$ is divided at $C$ so that $AB$, $BC = AC$; prove that
   
   $AP - AC = AB$, $AC$.

9. $AB$ is bisected at $C$ and produced to $D$; prove that
   
   $AC$, $AD = CB$, $BD = 2OB$.

Is the result still true if $D$ lies between $A$ and $B$?

10. $A$, $B$, $C$, $D$ are four collinear points; $P$, $Q$ are the mid-points of $AB$, $CD$; prove that $PQ = \frac{1}{2}(AD + BC)$.

11. $P$ is any point on the line $ABC$; prove that
   
   $PA^2 + BC + PP + CA + PC + AB + BC + CA = \infty$.

12. Prove the result of Ex. 11 when $P$ is any point in the plane.

[Draw $PO$ perpendicular to $ABC$.]

13. If $G$ is a point on the line $A_1 A_2 A_3 A_4$ such that
   
   $GA_1 + GA_4 + GA_2 = 0$, and if $P$ is any other point on the line, prove that
   
   $PA_1 + PA_2 + PA_3 + PA_4 = 0$.

Generalise this theorem for $n$ collinear points.

14. $G$ is a point on the line $A_1 A_2 A_3 A_4$ such that
   
   $GA_1 + GA_2 + GA_3 = 0$, if $P$ is any other point on the line, prove that
   
   $PA_1^2 + PA_2^2 + PA_3^2 + PA_4^2 = (GA_1^2 + GA_2^2 + GA_3^2 + GA_4^2) = 4PG^2$.

Generalise this theorem for $n$ collinear points.

15. Prove the result of Ex. 14 when $P$ is any point in the plane.

Theorem 31 was first discovered by Menelaus who died at the end of the first century A.D. It was, however, forgotten and then rediscovered afresh by Ceva (1650-1734) and by Carnot (1753-1823). The latter made it the basis of his researches on transversals. Ceva, to whom Theorem 30 is due, was an Italian by birth and an hydraulic engineer by profession. His treatise on geometry includes a number of practical applications. It is interesting to note that he was one of the earliest scientists to write on economics.
THEOREM 30. [CEVA'S THEOREM.]

Three concurrent straight lines are drawn through the vertices $A$, $B$, $C$ of a triangle to meet the opposite sides at $D$, $E$, $F$ respectively; then
\[ \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1. \]

Let the three lines concur at $O$.
If $O$ lies inside the triangle $ABC$ (Fig. 34) each of the ratios
\[ \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} \]
is positive; and if $O$ lies outside $ABC$ (Fig. 35) one ratio is positive and the other two are negative;
\[ \therefore \text{in either case the product of the three ratios is positive.} \]

By Theorem 6,
\[ \frac{BD}{DC} = \frac{\triangle BOC}{\triangle COA} = \frac{\triangle BOC}{\triangle COA} \cdot \frac{\triangle BOA}{\triangle BOA} \cdot \frac{\triangle BOA}{\triangle BOA} \cdot \frac{\triangle BOA}{\triangle BOA} = 1; \]
\[ \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1. \]

Corollary.
If $D$, $E$, $F$ are points on $BC$, $CA$, $AB$ such that
\[ \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = +1, \]
then $AD$, $BE$, $CF$ are concurrent.

[This may be easily proved by a 'reductio ad absurdum' method.]

16. Prove the Corollary of Theorem 30.
17. If in Fig. 34, $BD=\frac{1}{2}DC$, $CE=\frac{1}{2}CA$, calculate

\begin{align*}
(1) & \frac{AF}{FB}; \\
(2) & \triangle AOB : \triangle BOC : \triangle COA; \\
(3) & \frac{AO}{AD}.
\end{align*}
30. If in Fig. 34 P, Q, R are the mid-points of EF, FD, DE, prove that AP, BQ, CR are concurrent. [Use Theorem 8.]

31. If in Fig. 34 F, Q, R are points on EF, FD, DE such that DP, EQ, FR are concurrent, prove that AP, BQ, CR are concurrent. [Use Theorem 8.]

32. A triangle ABC is inscribed in a triangle XYZ and is circumscribed about a triangle PQR; the order of the letters round the triangles is ARBPCQA and XCYAZYB. If AP, BQ, CR are concurrent, and if AX, BY, CZ are concurrent, prove that PX, QY, RZ are concurrent. [Use Theorem 8.]

33. The sides AB, CD of the quadrilateral ABCD are parallel; CA, DB meet at E; CB, AD meet at H; CB, AD meet EFG, a parallel to AB, at G and F respectively; prove that AG, BF, EH are concurrent.

34. AP, BQ, CR are three concurrent lines meeting the opposite sides of the triangle ABC at P, Q, R; F, G, H, A', B', C' are the mid-points of AP, BQ, CR, BC, CA, AB; prove that A'F, B'G, C'H are concurrent.

Definition.

Any line cutting a system of straight lines is called a transversal of those lines.

**Theorem 31.** [Menelaus' Theorem.]

If a transversal meets the sides BC, CA, AB of a triangle at D, E, F respectively, then \(\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1\).

The transversal cuts either one side (Fig. 36) or three sides (Fig. 37) of the triangle externally; and therefore either one or three of the ratios \(\frac{BD}{DC}, \frac{CE}{EA}, \frac{AF}{FB}\) are negative, and therefore their product is negative in each case.

**Corollary.**

If D, E, F are points on the sides BC, CA, AB of a triangle such that \(\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1\), then D, E, F are collinear.

This may be easily proved by a 'reductio ad absurdum' method.

35. Prove the Corollary of Theorem 31.

36. If in Fig. 36 BD=3CD and CA=3CE, calculate \(\frac{BF}{FA}\).

37. If in Fig. 36 B', C' are the mid-points of the sides AC, AB of the triangle ABC. Q is the mid-point of B'C'; BQ meets AC at R. Calculate \(\frac{AR}{RC'}\).

38. If in Fig. 36 BE, CF are the internal bisectors of the angles ABC, ACR, prove that AD is the external bisector of BAC.

39. Prove that the points at which the external bisectors of the angles of a triangle meet the opposite sides are collinear.

40. H is a point inside the triangle ABC, prove that the external bisectors of the angles AHB, BHC, CHA meet AB, BC, CA respectively at three collinear points.
41. Points \( P, Q, R \) are taken on the sides \( BC, CA, AB \) of a triangle such that \( \frac{BP}{CD} = \frac{CQ}{AR} \); \( QR \) and \( CB \) produced meet at \( D \), prove that \( \frac{BD}{CD} = \frac{CQ}{AR} = \frac{QR}{CB} \).

42. A circle centre \( A \) touches externally each of two given circles centres \( B, C \) at \( D, E \); \( DE \) meets \( BC \) at \( F \). Another circle centre \( A' \) touches the same two circles at \( D', E' \); prove that \( D'E' \) meets \( BC \) at \( F \), and that the two original circles are homothetic with \( F \).

43. \( G \) is the centroid of the triangle \( ABC \); \( AG \) is produced to \( P \) so that \( GP = AG \). Parallels through \( P \) to \( CA, AB, BC \) meet \( CA, AB, BC \) at \( L, M, N \) respectively; prove that \( L, M, N \) are collinear.

44. The tangents to the circumcircle of a triangle \( ABC \) at the vertices \( A, B, C \) meet the opposite sides at \( L, M, N \). Prove that \( LMN \) are collinear.

[Show that \( \frac{BL}{BA} = \frac{CL}{CA} \) etc.]

45. If in Fig. 34, \( EF \) meets \( BC \) at \( K \), prove that \( \frac{BD}{BE} = \frac{DK}{EK} \).

46. If in Fig. 35, \( EF \) meets \( BC \) at \( K \), prove that \( \frac{BD}{BE} = \frac{DK}{EK} \).

47. If in Fig. 34, \( EF \) meets \( AD \) at \( H \) and \( BC \) at \( K \), prove that \( \frac{EH}{HE} = \frac{FK}{KF} \).

48. If in Fig. 34, \( EF \) meets \( BC \) at \( L \), \( FD \) meets \( CA \) at \( M \), \( ME \) meets \( AB \) at \( N \); prove that \( L, M, N \) are collinear.

49. Four points \( a, b, c, d \) are taken on the sides \( AB, BC, CD, DA \) of a quadrilateral \( ABCD \) such that \( \frac{AB}{CD} = \frac{CB}{DA} \); prove that \( ab, cd \) intersect at \( AC \).

50. If a straight line cuts the sides \( AB, BC, CD, DA \) of a quadrilateral at \( P, Q, R, S \), prove that \( \frac{AP}{BQ} \cdot \frac{BQ}{CR} \cdot \frac{CR}{DS} \cdot \frac{DS}{AP} = 1 \).

Generalise this theorem for an \( n \)-sided polygon.

51. Points \( P, P', Q, Q' \) are taken on the edges \( AB, CD, AD, BC \) of a tetrahedron \( ABCD \); if the lines \( PP' \) and \( QQ' \) intersect, prove that \( \frac{BP}{QD} \cdot \frac{AQ'}{DC'} \cdot \frac{CP}{QB} = \cdot \frac{CQ}{QB} \).

52. Given four lines of lengths \( a, b, c, d \), construct the ratio \( \frac{a}{b} = \frac{c}{d} \).

53. The incircle of the triangle \( ABC \) touches \( BC, CA, AB \) at \( X, Y, Z \); \( YZ \) is produced to meet \( BC \) at \( K \), prove that \( \frac{BX}{BK} \cdot \frac{AX}{CK} \).

54. \( A', B', C' \) are the mid-points of \( BC, CA, AB \); a transversal cuts \( BC, CA, AB \) at \( K, L, M \); \( AK, BL, CM \) meet \( BC, CA, AB \) at \( K', L', M' \); prove that \( K', L', M' \) are collinear.

55. \( ABCD \) is a quadrilateral; \( M, N, P \) are any points on \( BC, AD, AB \); \( PM \) meets \( BD \) at \( R \); \( KM \) meets \( DC \) at \( S \); \( PM \) meets \( AC \) at \( Q \); prove that \( Q, S, M \) are collinear.

56. A straight line meets the sides \( BC, CA, AB \) of a triangle at \( P, Q, R \); \( A', B', C' \) are the mid-points of these sides; \( AP, BQ, CR \) meet \( BC', CA', AB' \) at \( P', Q', R' \); prove that \( P', Q', R' \) are collinear.

57. A circle cuts each of the sides of a quadrilateral \( ABCD \) at two internal points so that the order of the letters round the quadrilateral is \( AAAA, BBPP, CCCC, DDDD, DDA \). If \( D_A, D_B, D_C, B \) are concurrent, prove the concurrency of the following sets:

\((1) D_A A, D_B, D_B C, D_C, D_B D; (2) D_c A, D_B, D_B C, D_B B; (3) D_A A, D_B C, D_B B, D_B D.\)

58. Three planes \( AOB, BOC, COA \) intersect at a point \( O \). Prove that the plane through the bisectors of the angles \( AOB, BOC \) cuts the plane \( COA \) in a line which bisects the angle \( COA \) externally.

59. If one of the transverse common tangents of two circles is perpendicular to one of the direct common tangents, prove that the eight points of contact lie on two straight lines.

60. [Desargues's Theorem on Perspective Triangles.]

\( ABC, A'B'C' \) are two triangles such that \( AA, BB', CC' \) meet at \( O \); prove that, if \( BC, B'C' \) meet at \( L \); \( CA, CA' \) at \( M \); \( AB, A'B' \) at \( N \), then \( LMN \) are collinear.

[Prove \( \frac{AN}{BL} \cdot \frac{BL}{CM} \cdot \frac{CM}{AN} = -1 \) by considering the transversals \( LFC, MCA, NA'B' \) applied to the triangles \( OBC, OCA, OAB \).]

61. [Pascal's Theorem.]

\( ABCDE \) is any hexagon inscribed in a circle; \( AB, DE \) meet at \( L \); \( BC, EF \) at \( M \); \( CD, FA \) at \( N \); prove that \( LMN \) are collinear.

[Produce \( AB, CD, EF \) to form a triangle \( XYZ \) and prove that \( LMN \) is a transversal of \( XYZ \) by considering \( BC, DE, FA \) as transversals of \( XYZ \).]
THEOREM 32.

Let the perpendiculars at $P$, $Q$, $R$ meet at $S$.

Join $SA$, $SB$, $SC$.

\[ BP^2 - FC^2 = (BP^2 + PS^2) - (PC^2 + PS^2) = BS^2 - SC^2. \]

Similarly, \[ CQ^2 - QA^2 = CS^2 - SA^2 \]

and \[ AR^2 - RB^2 = AS^2 - SB^2. \]

\[ \therefore \text{by addition } BP^2 - FC^2 + CQ^2 - QA^2 + AR^2 - RB^2 = 0; \]

\[ \therefore BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2. \]

Q.E.D.

Corollary.

If $P$, $Q$, $R$ are points on the sides $BC$, $CA$, $AB$ of a triangle such that \[ BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2, \]
then the perpendiculars at $P$, $Q$, $R$ to the sides are concurrent.

This may be easily proved by a 'reductio ad absurdum' method.

It is the converse theorem (in the Corollary) which is the more useful in rider work.

62. Prove the Corollary of Theorem 32.

63. Is Theorem 32 still true if $S$ lies outside the triangle $ABC$?

64. Use Theorem 32 Corollary to show that the altitudes of a triangle are concurrent.

65. If $I_1, I_2, I_3$ are the centres of the circles escribed to $BC$, $CA$, $AB$, prove that the perpendiculars from $I_1, I_2, I_3$ to these sides are concurrent.
CHAPTER VI.

VECTOR GEOMETRY.

The theory of vectors has been constructed entirely during the last century. It is usual to refer to Argand (1786-1829) as the first to attempt to represent graphically imaginary numbers. But this priority is questioned by some authorities. A great development was made by Hamilton (1805-1865) by his famous discovery of the properties of quaternions, published in 1852.

In the following pages, as an introduction to vector addition, the analogous properties which hold for the projections of a set of lines on an arbitrary line are first worked out. The advantage of this method lies in the fact that the law of addition, which is taken as a definition for vectors, is capable of simple proof [Theorem 33] in this special case, and its consequences are therefore more easily understood.

Definition.

If perpendiculars \( AA', BB' \) are drawn from the extremities \( A, B \) of a given line \( AB \) to a line \( Ox \), \( A'B' \) is called the projection of \( AB \) on \( Ox \) and will be denoted by \( \overrightarrow{AB} \). The projection is said to be with reference to \( Ox \). If no special line \( Ox \) is mentioned, it is implied that an arbitrary line may be selected for \( Ox \).

THEOREM 33.

If \( A, B, C \) are any three points, then \( AB + BC = AC \).

The proof is left to the reader.

1. If \( A_1A_2A_3...A_n \) is any polygon, the sum of the projections of the sides taken in order on any straight line is zero.

\[
[\sum A_1A_2A_3A_4A_5A_6A_7 = 0.]
\]

THEOREM 34.

If \( D \) is the mid-point of the base \( BC \) of a triangle \( ABC \), then \( \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD} \).

\[
\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} \quad [\text{Th. 33.}]
\]

\[
\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}.
\]

\[
\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{DC}.
\]

But \( \overrightarrow{DB} + \overrightarrow{DC} = 0 \), for their senses are opposite and their lengths are equal.

\[
\therefore \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}. \quad \text{Q.E.D.}
\]

THEOREM 35.

If \( D \) is a point on the base \( BC \) of a triangle \( ABC \) such that \( \lambda \cdot BD = \mu \cdot DC \); then \( \lambda \cdot \overrightarrow{AB} + \mu \cdot \overrightarrow{AC} = (\lambda + \mu) \overrightarrow{AD} \).

This may be proved in the same way as Theorem 34.

2. Prove Theorem 35.

3. If \( D \) is the mid-point of the base \( BC \) of a triangle \( ABC \), prove that \( \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{DC} \).

4. If \( D \) is a point on the base \( BC \) of a triangle \( ABC \), such that \( \lambda \cdot BD = \mu \cdot DC \), prove that

\[
\lambda \cdot \overrightarrow{AB} + \mu \cdot \overrightarrow{AC} = (\lambda + \mu) \overrightarrow{AD} + \lambda \cdot \overrightarrow{BD} + \mu \cdot \overrightarrow{DC}.
\]

5. If \( ABCD \) is any quadrilateral, prove that

\[
\overrightarrow{AB}, \overrightarrow{CD} + \overrightarrow{BC}, \overrightarrow{AD} + \overrightarrow{CA}, \overrightarrow{BD} = 0.
\]

6. If \( ABCD \) is any quadrilateral, prove that

\[
\overrightarrow{AD}, \overrightarrow{BC} + \overrightarrow{BD}, \overrightarrow{AC} + \overrightarrow{CD}, \overrightarrow{AB} + \overrightarrow{BC}, \overrightarrow{CA}, \overrightarrow{AB} = 0.
\]
THEOREM 36.

If two lines $AP, AQ$ are such that their projections on each of two lines at right angles are equal, prove that $P$ coincides with $Q$.

Let $Ox, Oy$ be the two given perpendicular lines.

Now $AP = AQ + PQ$.

But $AP = AQ$ referred to $Ox$ and $Oy$;

$\therefore \quad PQ = 0$ referred to $Ox$ and $Oy$;

$\therefore \quad PQ$ is perpendicular both to $Ox$ and to $Oy$, or else $Q$ coincides with $P$;

$\therefore \quad Q$ must coincide with $P$.

Hence it follows that if $AP = AQ$ referred to an arbitrary line, $P$ must coincide with $Q$.

THEOREM 37.

If $AB = CD$ referred to each of two perpendicular lines, then $AB$ is equal and parallel to $CD$.

[If possible draw $AB'$ equal and parallel to $CD$ and prove by Theorem 36 that $B'$ coincides with $B$.]

Hence it follows that if $AB = CD$ referred to an arbitrary line, then $AB$ and $CD$ are equal and parallel.

7. Prove Theorem 37.

8. If $AB$ is equal and parallel to $CD$, prove that $AB = CD$.

9. If $AB$ is parallel to $CD$ and if $AB = CD$ referred to a given line or, not parallel to $AB$, prove that $AB = CD$.

THEOREM 38.

$A'$ is the mid-point of the base $BC$ of the triangle $ABC$; $G$ is a point on $AA'$ such that $AG = \frac{1}{2}GA$. $O$ is a fixed point on an arbitrary line $OX$; then $OG = \frac{1}{2}(OA + OB + OC)$.

Since $BA' = AC$, $\overline{OB} + \overline{OC} = 2\overline{OA}$. [Th. 34]

Since $2AG = GA$, $2\overline{OA} + \overline{OA} = (2 + 1) \overline{OG}$; [Th. 35]

$\therefore \quad \overline{OG} = \frac{1}{3}(\overline{OA} + \overline{OB} + \overline{OC})$.

Q.E.D.

If $CA$ is bisected at $B'$ and $G_1$ is taken on $B'B$ so that $B'G_1 = \frac{1}{2}G'B$, in exactly the same way we may prove

$\overline{OG}_1 = \frac{1}{3}(\overline{OA} + \overline{OB} + \overline{OC})$;

$\therefore \quad \overline{OG} = \overline{OG}_1$ referred to an arbitrary line;

$\therefore \quad G_1$ coincides with $G$.[Th. 36.

And in the same way, if $AB$ is bisected at $C'$ and $G_2$ is taken on $C'C$ so that $C'G_2 = \frac{1}{2}G'C$, it follows that $G_2$ coincides with $G$.

Hence this method has proved that the medians of a triangle are concurrent, and divide each other at a point of trisection.

Definition.

1. The point $G$ determined by the equation

$\overline{OG} = \frac{1}{n}(\overline{OA} + \overline{OB} + \overline{OC})$

is called the mean centre of the three points $A, B, C$. It coincides with the centroid or centre of gravity of a triangular plate $ABC$.

2. If $A_1A_2 \ldots A_n$ are $n$ points, and if $G$ is a point determined by the equation $\overline{OG} = \frac{1}{n}(\overline{OA_1} + \overline{OA_2} + \ldots + \overline{OA_n})$, then $G$ is called the mean centre of the $n$ points $A_1, A_2 \ldots A_n$. It coincides with $D$.
the centre of gravity of \( n \) equal particles placed at the \( n \) points.

It is necessary to show that the position of \( G \) does not alter if the position of \( O \) is changed.

10. Prove that if \( G \) is defined by 
\[
\overrightarrow{OG} = \frac{1}{n}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \ldots + \overrightarrow{OA}_n),
\]
the position of \( G \) is independent of the position of \( O \).

[If \( O \) is any other point, and if \( \overrightarrow{OG'} = \frac{1}{n}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \ldots + \overrightarrow{OA}_n) \), show that \( G' \) coincides with \( G \).]

11. Prove that the centroid of the triangle \( ABC \) coincides with the centroid of the triangle formed by joining the mid-points \( A'B'C' \) of its sides. [Express \( \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) \) in terms of \( \overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC} \).]

12. Prove that the mean centre of the quadrilateral \( ABCD \) is at the mid-point of the line joining the mid-points of \( AB \) and \( CD \). Hence, prove that the lines joining the mid-points of opposite sides of a quadrilateral and joining the mid-points of the diagonals are concurrent and bisect each other.

13. If \( G \) is the mean centre of \( ABC \), prove that \( \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 0 \).

14. If \( A_1, B_1, C_1, D_1 \) are the centroids of the triangles \( BCD, CDA, DAB, ABC \), prove that \( AA_1, BB_1, CC_1, DD_1 \) are concurrent (at \( G \) say) and that \( AG = 3GD_1 \), etc.

15. If \( A_A, A_B, A_C, A_D \) are the mid-points of consecutive sides of a hexagon, prove that the triangles \( AA_A, AA_B, AA_C \) have the same centroid.

16. Points \( P, Q, R \) are taken on the sides \( BC, CA, AB \) of a triangle \( ABC \) so that \( \frac{BP}{PC} = \frac{CQ}{QA} = \frac{AR}{RB} \); prove that the centroids of the triangles \( ABC, PQR \) are coincident.

17. If \( A_1, B_1, C_1, D_1, E_1 \) are the mean centres of the quadrilaterals \( BCDE, CDEA, DEAB, EABC, ABCD \), prove that the sides of the pentagon \( A_1B_1C_1D_1E_1 \) are parallel to and equal to one quarter of the corresponding sides of the pentagon \( ABCDE \).

18. The sides \( AB, AC \) of a triangle are divided at \( P, Q \) so that \( \frac{AP}{PC} = \frac{AQ}{QC} \); the parallelograms \( APRC, ABSQ \) are drawn; prove that \( PS \) and \( QR \) are parallel to each other and to the median through \( A \) of the triangle \( ABC \).

19. (1) \( O \) is the centre of a circle of unit radius; \( A_1A_2A_3 \) are any three points on its circumference; another circle of unit radius is described through \( A_1 \) and \( A_2 \); if \( O_1 \) is its centre, prove that 
\[
\overrightarrow{OG} - \frac{1}{2}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \ldots + \overrightarrow{OA}_n)
\]
where \( O \) is any fixed point, then \( G \) is called the mean centre or the centroid of the system of points \( A_1, A_2, \ldots, A_n \).

[If further, two other circles of unit radius are described through \( A_1, A_2, A_3, A_4, A_5 \) and if their centres are \( O_2 \) and \( O_3 \), prove that the points \( O_2O_3 \) lie on another circle of unit radius (centre \( \beta \) say).]

[If \( O \) is any other point, and if \( \overrightarrow{OG'} = \frac{1}{n}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \ldots + \overrightarrow{OA}_n) \), show that \( G' \) coincides with \( G \).]

(2) If four points \( A_1A_2A_3A_4 \) lie on a circle of unit radius, and if from every set of three points a circle is constructed as in (1), then the centres \( \beta, \gamma, \delta, \epsilon \) of these four circles lie on a circle of unit radius, centre \( \gamma \). [\( \gamma \) is given by \( \overrightarrow{OG} = \overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \overrightarrow{OA}_3 + \overrightarrow{OA}_4 \).]

(3) If five points \( A_1A_2A_3A_4A_5 \) lie on a circle of unit radius, and if from every four points a circle of unit radius is constructed as in (2), then the centres \( \gamma, \kappa, \lambda, \mu, \nu \) of these circles lie on a circle of unit radius, centre \( \delta \) and so on indefinitely.

**SOLID GEOMETRY.**

This method of projection may be readily extended to three-dimensional geometry. Instead of taking two perpendicular lines for reference, it is now necessary to take three mutually perpendicular lines as axes. If \( AB \) is a line in space, \( AB \) will be taken to denote the projection of \( AB \) on a specified axis. Where no axis is mentioned, it is implied that an arbitrary line can be selected as the axis of reference.

20. If \( A, B, C, D \) are any four points in space, prove that \( AB + AC + CD = AD \).

21. Prove that \( AB + AC = 2AD \) where \( D \) is the mid-point of \( BC \).

**THEOREM 39.**

(1) If \( AB = AC \) referred to an arbitrary line, then \( B \) coincides with \( C \).

(2) If \( AB = CD \) referred to an arbitrary line, then \( AB \) is equal and parallel to \( CD \).

[The methods of proof employed in Theorems 36, 37 apply here.]


**Definition.**

If \( A_1A_2 \ldots A_n \) are \( n \) points in space, and if \( G \) is a point determined by 
\[
\overrightarrow{OG} = \frac{1}{n}(\overrightarrow{OA}_1 + \overrightarrow{OA}_2 + \ldots + \overrightarrow{OA}_n)
\]
where \( O \) is any fixed point, then \( G \) is called the mean centre or the centroid of the system of points \( A_1, A_2, \ldots, A_n \).
Prove that the position of $G$, as just defined, is independent of the position of $O$. 

Prove that the lines joining the mid-points of opposite edges of a tetrahedron are concurrent and bisect each other.

[Prove that the mid-point of each line is at the mean centre of the tetrahedron.]

$G_4$ is the centroid of the base $BCD$ of the tetrahedron $ABCD$; $G$ is a point on $AG_4$ such that $GG_4 = \frac{1}{2}GA$; prove that $G$ is the mean centre of the tetrahedron.

$G_1, G_2, G_3, G_4$ are the centroids of the faces of the tetrahedron $ABCD$ opposite to $A, B, C, D$, respectively; prove that the lines $AG_1, BG_2, CG_3, DG_4$ are concurrent.

With the notation of Ex. 26, prove that the tetrahedra $G_1G_2G_3G_4$, $ABCD$ have the same mean centre.

With the notation of Ex. 26, prove that $G_1G_2$ is parallel to and equal to one-third of $AB$. 

$P, Q, R, S$ are the mid-points of the edges $AB, BC, CD, DA$ of a tetrahedron $ABCD$, prove that the tetrahedra $PQRS, ABCD$ have the same mean centre.

With the notation of Ex. 26, find the ratio in which $G_1G_2$ divides the line joining the mid-points of $AB$ and $CD$. 

**VECTORS.**

We have seen that it is possible to represent positive and negative numbers by points on the $x$-axis referred to a definite point $O$ as origin or zero-point. It is desirable to extend this method of representation.

Certain quantities, such as forces or velocities, etc., are not fully defined by their numerical magnitude; it is necessary in such cases to take account of their directions. These quantities can be represented by a straight line drawn in a definite direction, and of a definite length.

**Definition.**

Any quantity which can be completely represented by a straight line of appropriate length, drawn in an appropriate direction, is called a vector.

It follows from this definition that equal and parallel lines represent the same vector.

Thus a velocity of 20 feet per second in a direction N. 30° E. is a vector, and may be represented by any one of a series of equal and parallel straight lines.

In elementary arithmetic we are concerned with the addition of numbers in which no question of direction—except that of sense in a straight line—occurs. In the case of vectors, however, the idea of addition involves an entirely new conception. For it is necessary to require what is meant by adding together two quantities which involve, not merely magnitude, but also direction. For example, what is meant by adding a velocity of 10 feet per second due North to a velocity of 8 feet per second due East?

The same kind of problem arose when the reader passed from the conception of positive integral indices to fractional or negative indices. The Law of Multiplication, $x^m \times x^n = x^{m+n}$, where $m, n$ are positive integers, no longer admits of proof, just because the symbol $x^0$ has no longer any meaning attaching to it. It is therefore necessary to enlarge the original definition by stating that $x^m \times x^n$ is to be taken as $x^{m+n}$ for all real values of $m$ and $n$, and using this definition to interpret the meaning of $x^m$ when $m$ is not a positive integer. The present problem must be approached in just the same way.

If $A, B, C$ are any three points on the $x$-axis, so that $OA, OB, OC$ represent any three real numbers (positive or negative), we know that $AB + BC = AC$. If, however, $A, B, C$ are not collinear so that $AB$ and $BC$ represent two quantities, which involve different directions, we are compelled to enlarge our definition of what is meant by addition under these new circumstances. The original definition is therefore extended as follows:

**Definition.**

The result of adding two vectors $AB$ and $BC$ together is defined to be the vector $AC$.

This is a definition of what is meant by addition. Notice that it does not violate the theorem that the sum of the lengths of two sides of a triangle is greater than the length of the third; for it does not state that the length of $AB +$ the length of $BC$ $=$ the length of $AC$; but that the result of adding (combining) two quantities, whose lengths and directions are represented by $AB$ and $BC$, is taken to be a quantity whose length and direction is represented by $AC$.

To avoid confusion between a vector $AB$ and the length $AB$,
some difference of notation is desirable. The vector $\overrightarrow{AB}$ will therefore be denoted by $\overrightarrow{AB}$.

The Law of Addition of Vectors is therefore,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$ 

It is now necessary to show that vector quantities obey the fundamental Laws of Algebra, as regards addition.

The meaning of multiplication has yet to be considered.

**Theorem 40.**

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}.$$ 

Complete the parallelogram $ABCP$, By the law of addition $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, and

$$\overrightarrow{AP} + \overrightarrow{FC} = \overrightarrow{AC}.$$ 

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AP} + \overrightarrow{FC}.$$ 

But $\overrightarrow{AP} = \overrightarrow{PC}$ and $\overrightarrow{AP} = \overrightarrow{BC}$, since equal and parallel lines represent the same vector;

$$\therefore \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{BC} + \overrightarrow{AB}.$$  

Q.E.D.

**Corollary.**

If $\overrightarrow{AB}, \overrightarrow{AP}$ are two vectors, their sum is the vector represented by the diagonal through $A$ of the completed parallelogram $PABC$.

**Theorem 41.**

$$(\overrightarrow{AB} + \overrightarrow{AC}) + \overrightarrow{AD} = \overrightarrow{AB} + (\overrightarrow{AC} + \overrightarrow{AD}).$$

The proof is left to the reader.

[Complete parallelograms $BACP, DAPQ$ so that $\overrightarrow{AQ}$ represents the left hand side; then complete parallelogram $DACK$ and show that $\overrightarrow{AQ} = \overrightarrow{AB} + \overrightarrow{AK}$.

31. Prove Theorem 41.

VI] VECTOR GEOMETRY

The process of subtraction can be deduced at once from that of addition.

For $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = 0$;

$$\overrightarrow{AB} = -\overrightarrow{BA};$$

$$\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$ 

Starting then with this vector law of addition, it will be found instructive for the reader to work out the theorems and examples (omitting Ex. 2, 6) which have been given under the form of projections of segments of straight lines on an arbitrary line. These can be regarded afresh as vector theorems, for they depended simply on the initial fact $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Thus for example, Theorem 34 may be rewritten as follows: If $D$ is the mid-point of the line joining two points $B, C$, and if $O$ is the origin, the vector represented by $\overrightarrow{OD}$ equals half the sum of the vectors represented by $\overrightarrow{OB}, \overrightarrow{OC}$.

The fundamental identity of these two methods may be seen by noting that if two lines have equal projections referred to an arbitrary line, then they represent the same vector quantity for they are equal and parallel. [Th. 37]

Theorem 35 may be restated as follows: If $D$ is a point on $BC$ such that $\lambda, BD = \mu, DC$, then

$\lambda, \overrightarrow{BD} = \mu, \overrightarrow{DC} = (\lambda + \mu) \overrightarrow{OL}.$

Now, from elementary statics, if $\lambda, BD = \mu, DC$ it follows that $D$ is the centre of gravity of masses $\lambda$ at $B, \mu$ at $C$. Hence we have the theorem: $m_1 \overrightarrow{OA}_1 + m_2 \overrightarrow{OA}_2 + \ldots + m_n \overrightarrow{OA}_n = (m_1 + m_2 + \ldots + m_n) \overrightarrow{OG}$, where $G$ is the centre of gravity of masses $m_1, m_2, \ldots, m_n$ at $A_1, A_2, \ldots, A_n$.

This result can be readily extended to give the generalised vector theorem:

$$m_1 \overrightarrow{OA}_1 + m_2 \overrightarrow{OA}_2 + \ldots + m_n \overrightarrow{OA}_n = (m_1 + m_2 + \ldots + m_n) \overrightarrow{OG},$$

where $G$ is the centre of gravity of $n$ equal masses at $A_1, A_2, \ldots, A_n$; this point has been called the mean centre of the system of points $A_1, A_2, \ldots, A_n$.

The reader should now work through exercises 10-19.

The extension of vector addition to three dimensional geometry presents no new difficulty, and therefore Exercises 20-30 may be also taken.
VECTOR MULTIPLICATION.

When it is said that a point \( P \) on the axis \( Ox \) represents a positive or negative number by reference to its distance from \( O \), use is being made of the following idea.

If \( OA \) represents a unit of length on \( Ox \), and if \( OP \) represents \( k \) units of length (\( k \) being positive or negative), the ratio \( \frac{OP}{OA} \) is by definition equal to \( k \). In speaking of a length it is necessary to know the unit (inches, cms., etc.) in terms of which the length is given. But in the case of a ratio \( \frac{OP}{OA} \), the value does not depend at all on the particular unit chosen, and is in fact a pure number. When considering the meaning of multiplication, it is necessary to work with pure numbers. It is as impossible to multiply 1 inch by 1 inch as it is to multiply 1 chair by 1 chair (unless some new definition is made as to the meaning of multiplication under such circumstances).

For simplicity, however, it is usual to suppose that the product \( OP \cdot QQ \) is an abbreviation of \( \frac{OP}{OA} \cdot \frac{QQ}{OA} \) where \( OA \) represents the unit of length.

Given a unit of length \( OA \) on the \( x \)-axis, and given any real number \( k \), it is possible to find a point \( P \) on \( Ox \) such that \( \frac{OP}{OA} = k \); and for any point \( P \) on \( OA \), \( \frac{OP}{OA} \) is a real number.

Definition.

If \( P \) is a point outside \( Ox \), and if \( OA \) is a unit of length on the \( x \)-axis, the ratio \( \frac{OP}{OA} \) is called a complex number.

If \( OP = r \) and \( AOP = \theta \) (radians), \( \frac{OP}{OA} \) will also be denoted by \((r, \theta)\).

It follows that every positive real number can be denoted by \((r, 0)\) and every negative real number by \((r, \pi)\), where \( r \) represents any real positive number.

We have now to consider what is meant by multiplying together two complex numbers \((r_1, \theta_1)\), \((r_2, \theta_2)\).

The fundamental definition of multiplication cannot apply here, for as yet \((r_1, \theta_1)\) has received no interpretation.

Now in the case of real numbers the law of multiplication may be stated in the form

\[ (r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2) \]

This may be easily verified.

E.g.,

\[ -a \times (-b) = (a, \pi) \times (b, \pi) = (ab, \pi + \pi) = (ab, 2\pi) = -ab \]

32. Prove that \( -a \times (-b) = +ab \) and \(+a \times (-b) = -ab \) are particular cases of the law enunciated above.

In the case of complex numbers we shall therefore define multiplication as the operation of this Law, viz.,

\[ (r_1, \theta_1) \times (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2) \]

and then use this Law to discover what a complex number really is.

It is necessary first of all to prove that complex numbers, as now defined, obey the fundamental Laws of Algebra.

It has been already shown that they obey the Laws of Addition, viz.,

\[ (1) \quad (a_1 + a_2, \alpha_1 + \alpha_2) \]
\[ (2) \quad (a_1 + a_2, \alpha_1 + \alpha_2) = (a_1, \alpha_1) + (a_2, \alpha_2) \]

where \( a_1 \) is written for \((r_1, \theta_1)\), etc.

It remains to prove the following:

\[ (3) \quad (a_1 + a_2)^2 = a_1^2 + a_2^2 \]
\[ (4) \quad (a_1 + a_2)^2 = a_1^2 + a_2^2 \]
\[ (5) \quad a_1 a_2 = a_2 a_1 \]

These may be proved very easily by analysis.

For example, to prove (3):

Let \( a_1 = (r_1, \theta_1) \) and \( a_2 = (r_2, \theta_2) \).

Then

\[ a_1 \times a_2 = (r_1, \theta_1) \times (r_2, \theta_2) = (r_1, \theta_1 + \theta_2) = (r_1 r_2, \theta_1 + \theta_2) \]

for \( r_1, r_2 \) and \( \theta_1, \theta_2 \) are real numbers, and therefore satisfy the laws of algebra, so that \( r_1 r_2 = r_1 r_2 \) and \( \theta_1 + \theta_2 = \theta_1 + \theta_2 \).

In a similar manner, (4) and (5) may be established.

It is interesting however to regard them from a geometrical standpoint. For this purpose, it is necessary, first of all, to give a geometrical construction for \( a_1 \times a_2 \).
Let \( a_1 = (r_1, \theta_1) = \frac{OP_n}{OA} \), \( a_2 = (r_2, \theta_2) = \frac{OP_2}{OA} \) where \( OA \) represents a unit of length on the \( x \)-axis.

On \( OP_1 \) construct a triangle \( OP_1R \) directly similar to \( OAP \).

Then \( \frac{OR}{OP_2} = \frac{OP_1}{OA} \) by similar triangles;

\[
\frac{OR}{OP_2} = \frac{OP_1}{OA} = \frac{OP_1}{OP_2} \cdot \frac{OP_2}{OA} = \frac{r_1}{r_2};
\]

and \( A\hat{R} = A\hat{P_1} + P_1\hat{R} = A\hat{O}P_1 + A\hat{O}P_2 = \theta_1 + \theta_2; \)

\[
\frac{OR}{OA} = \left( \frac{OR}{OP_2} \right) A\hat{R} = (r_1 \theta_1 + r_2 \theta_2) = (r_1, \theta_1) \times (r_2, \theta_2) = a_1 \times a_2;
\]

\[
\text{this is the required construction.}
\]

If in Fig. 44 a triangle \( OP_1\hat{R} \) is drawn directly similar to \( OAP \),

then the same reasoning will show that \( \frac{OR}{OA} = a_1 \times a_1 \).

In order therefore to prove that \( a_1 \times a_2 = a_1 \times a_1 \), it is necessary to show that \( OR, OR' \) coincide; this is a simple geometrical construction.

33. If in Fig. 44 the triangles \( AO\hat{P}_2, PO\hat{R} \) are directly similar, prove that the triangles \( AO\hat{P}_2, PO\hat{R} \) are directly similar.

34. By making use of the same construction and method, prove geometrically that \( (x, \theta) = (x, \theta) \).

35. Make use of this idea to prove the remaining Law of Algebra, viz., \( a_1(a_1 + a_2) = a_1a_2 + a_1a_2 \).

36. Draw roughly figures to represent

\begin{align*}
(1) & \left( 1, \frac{\pi}{4} \right) \\
(2) & \left( 1, \frac{\pi}{4} \right) \\
(3) & \left( 1, \frac{\pi}{2} \right) \\
(4) & \left( 1, \frac{\pi}{3} \right) \times \left( 1, \frac{2\pi}{3} \right) \\
(5) & \left[ \left( 1, \frac{\pi}{3} \right) \right]^{\dagger} \text{ and } \left[ \left( 1, \frac{2\pi}{3} \right) \right]^{\dagger}.
\end{align*}

**INTERPRETATION OF** \( \frac{OP}{OA} = (x, \theta) \).

By the law of multiplication \( \left( 1, \frac{\pi}{2} \right) \times \left( 1, \frac{\pi}{2} \right) = (1, \pi) \);

\( \left( 1, \frac{\pi}{2} \right) \) is a quantity such that it, when multiplied by itself,

becomes \( -1 \), and may therefore be called the square root of \( -1 \),

and may be denoted by \( \sqrt{-1} \) or \( i \).

37. Prove that \( \left( 1, \frac{\pi}{2} \right) \) is equal to \( y.i \).

\[ \text{From} \ P \text{ draw} \ P\hat{V} \text{ perpendicular to} \ Ox. \]

Then \( \frac{OP}{OA} = \frac{ON + NP}{OA} = \frac{ON}{OA} + \frac{NP}{OA} \); \( \therefore \) if the lengths of \( ON, NP \) are \( x \) units and \( y \) units,

\( \frac{ON}{OA} = (x, 0) = x \),

\( \frac{NP}{OA} = (y, \frac{\pi}{2}) = y.i \);

\[ \therefore \frac{OP}{OA} = x + yi. \]
Again, if \( \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = (r, \theta) \), \( \overrightarrow{OP} \) contains \( r \) units of length;
\[
\therefore \quad \overrightarrow{ON} = r \cos \theta, \quad \overrightarrow{NP} = r \sin \theta;
\]
\[
\therefore \quad (r, \theta) - \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = (r \cos \theta, \varnothing) + \left( r \sin \theta \frac{\pi}{2} \right)
\]
\[
x \cos \theta = r \sin \theta + \tau i
\]
\[
x = r (\cos \theta + i \sin \theta).
\]

This then is the important interpretation of the new kind of number (defined above as complex) which is rendered necessary by the Law of Multiplication for vector quantities.

Certain purely geometrical theorems can be established by vector methods. We will prove the theorem of Pythagoras as an example.

Let \( OAP \) be a right angled triangle, to prove that
\[
\frac{\overrightarrow{ON}}{\overrightarrow{OA}} = \frac{\overrightarrow{NP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}}.
\]

To prove this, we consider the following:
\[
\overrightarrow{OP} = (r, \theta); \quad \overrightarrow{OP} = (r, -\theta);
\]
\[
\overrightarrow{OP} \cdot \overrightarrow{OP} = (r, \theta) - (r, -\theta) = \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}}.
\]

But
\[
\frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{\overrightarrow{OP}}{\overrightarrow{OA}}.
\]

Q.E.D.

**DEMOIVRE'S THEOREM.**

If \( n \) is any positive integer, then
\[
\cos n \theta + i \sin n \theta = (\cos \theta + i \sin \theta)^n.
\]

\[
(1, \theta)^n = (1, \theta) \times (1, \theta) \times \ldots \text{ to } n \text{ factors} = (1, n \theta) \text{ by the Law of Multiplication},
\]
\[
\cos n \theta + i \sin n \theta = (\cos \theta + i \sin \theta)^n.
\]

This theorem may be extended to the case in which \( n \) is any real number. But that would be beyond the scope of this volume.

38. From the identity \((1, \theta) \times (1, \varnothing) = (1, \theta + \varnothing)\) deduce the expansions of \( \sin (\theta + \varnothing) \) and \( \cos (\theta + \varnothing)\).

39. From the identity \((1, \theta)^n = (1, n \theta)\) deduce the expansions of \( \sin \theta \) and \( \cos \theta \).

40. Expand \( \sin \alpha \) in terms of \( \sin \alpha \), \( \cos \alpha \), where \( n \) is a positive integer.

41. Verify geometrically that \((1, \theta) + (1, -\theta) = 2 \cos \theta\).

42. Verify geometrically that \((1, \theta) + \left(1, \frac{2\pi}{3}\right) + \left(1, \frac{4\pi}{3}\right) = 0\), and that in general \((1, \theta) + \left(1, \frac{2\pi}{n}\right) + \left(1, \frac{4\pi}{n}\right) + \ldots \left(1, \frac{2\pi}{n} - \frac{2\pi}{n}\right) = 0\), where \( n \) is a positive integer.

43. Verify geometrically that \((1, \varnothing) + (1, \theta) = \left(1 - \cos \varnothing \frac{\theta}{2}\right)\).

44. Verify geometrically that \((1, \varnothing) + (1, \theta) = 2 \cos \theta\).

45. Prove by the multiplication Law that \((1, \frac{\pi}{2})\) is one of the cube roots of \((1, \theta)\); what are the other two cube roots?

46. Find the three cube roots of \((1, \pi)\); represent them graphically; and compare them with the roots of the equation \(x^3 + 1 = 0\) \((x + 1) (x^2 + x + 1) = 0\).

47. Find the three cube roots of \(i\) and of \(\frac{1 - i}{\sqrt{3}}\).

48. Write \(\left(1, \frac{\pi}{3}\right)\) in the form \(x + iy\); find its square roots by algebraic methods, and compare them with \(\left(\sqrt[4]{6}, \frac{\pi}{6}\right)\), \(\sqrt[4]{6}, \frac{7\pi}{6}\).
49. \( AD \) is a median of the triangle \( ABC \); interpret trigonometrically the relation \( AD = \frac{1}{2}(AB + AC) \).

50. Prove that the vector equation \( BC = BA + AC \) is equivalent to the two trigonometric formulae:
\[
\begin{align*}
\alpha = b \cos C + c \cos B \quad & \quad \frac{b}{\sin B} = \frac{c}{\sin C},
\end{align*}
\]

51. If \( ABCD \) is a parallelogram, prove that \( CA \cdot BD = AB^2 - BC^2 \).
Express this result trigonometrically, taking the \( x \)-axis along \( AB \).

52. If \( D \) is the mid-point of the base \( BC \) of the triangle \( ABC \),
prove that \( AB^2 + AC^2 = 2 AD^2 + 2 DB^2 \).
Express this result trigonometrically, taking the \( x \)-axis along \( BC \).

53. Ptolemy's Theorem. If \( ABCD \) is a plane quadrilateral, prove that \( AC \cdot BD = AB \cdot CD + BC \cdot AD \).
By expressing this result trigonometrically, prove that, if \( A, B, C, D \) are concyclic, then
\[
AC \cdot BD = AB \cdot CD + BC \cdot AD;
\]
but that otherwise \( AC \cdot BD < AB \cdot CD + BC \cdot AD \).

54. \( \lambda, \mu, \nu \) are given real numbers such that \( \lambda + \mu + \nu = 0 \); if \( A, B, C \)
are points such that \( \lambda \cdot OA + \mu \cdot OB + \nu \cdot OC = 0 \), prove that \( A, B, C \) are collinear.

55. Use Ex. 54 to establish Menelaus' Theorem, viz. if \( D, E, F \) are
points on the sides \( BC, CA, AB \) of a triangle such that
\[
\begin{align*}
DE = \lambda, \quad CE = \mu, \quad AF = \nu
\end{align*}
\]
and if \( \lambda \mu \nu = -1 \), then \( D, E, F \) are collinear.

CHAPTER VII.

THE MEAN CENTRE.

This chapter is concerned with properties which lie on the borderline separating Geometry and Statics. It frequently happens that a statical method affords a simple solution of a geometrical theorem (e.g. Ex. 25); while on the other hand certain types of statical questions are best treated geometrically (e.g. Ex. 34). An excellent account of the value of statical methods in geometrical problems is given by Mr. R. F. Davis in Milne's Companion to Weekly Problem Papers.

THEOREM 42.

\( C \) is a point on the straight line \( AB \) such that \( \lambda \cdot AC = \mu \cdot CB \);
if \( D, E, F \) are the feet of the perpendiculars from \( A, B, C \) to any straight line \( Ox \), then \((\lambda + \mu)CF = \lambda \cdot AD + \mu \cdot BE\).

\[ \text{Fig. 47.} \]

Draw \( Oy \) perpendicular to \( Ox \).
Apply Theorem 35 to the triangle \( OAB \).
\[
\therefore \lambda \cdot OA + \mu \cdot OB = (\lambda + \mu)OC,
\]
where \( OA, OB, OC \) denote the projections of \( OA, OB, OC \) on \( Oy \).
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But $\triangle D = DA, \triangle FC = \triangle BE$; 
$\therefore \lambda, D \delta + \mu, E B = (\lambda + \mu) F C$,
or $(\lambda + \mu) CE = \lambda, AD + \mu, BE$.
Q.E.D.

It should be noticed that $C$ is the centre of two parallel forces
$\lambda, \mu$ at $A, B$ respectively: this theorem is therefore equivalent
to the statical theorem that the sum of the moments of two parallel forces
$\lambda, \mu$ about an axis is equal to the moment of their resultant
$\lambda + \mu$ about that axis.

Definitions.

(1) $(x_1 y_1), (x_2 y_2), \ldots (x_n y_n)$ are the coordinates of $n$ coplanar
points $A_1 A_2 \ldots A_n$ referred to two rectangular axes $Ox, Oy$. If $G$
is a point whose coordinates are

$$\frac{1}{n}(x_1 + x_2 + \ldots + x_n), \frac{1}{n}(y_1 + y_2 + \ldots + y_n),$$

then $G$ is called the mean centre of the system of points.

It is clear that this definition is identical with the definition on
page 62.

(2) If $G$ is a point whose coordinates are

$$\frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}, \frac{m_1 y_1 + m_2 y_2 + \ldots + m_n y_n}{m_1 + m_2 + \ldots + m_n},$$

then $G$ is called the centre of the masses $m_1, m_2, \ldots m_n$ at the
points $A_1 A_2 \ldots A_n$.

The identification of $G$ with the statical centre of gravity of the
system of masses may be based on the work on page 74. Or again, it may be deduced from the statical theorem relating to moments about an axis.

Another method, which does not, however, differ materially from the
first, will be adopted here.

THEOREM 43 (1).

$A_1 A_2 A_3 \ldots A_n$ are $n$ coplanar points. $G_1$ is taken so as to divide
$A_1 A_2$ in the ratio $1:1$; $G_2$ divides $A_2 A_3$ in the ratio $1:2$; $G_3$
divides $A_3 A_4$ in the ratio $1:3$; and so on; $G_{n-1}$ divides $A_{n-1} A_n$ in
the ratio $1:n-1$. Then $G_{n-1}$ is the mean centre of the
points $A_1 A_2 \ldots A_n$.

THEOREM 43 (2).

$A_1 A_2 A_3 \ldots A_n$ are placed at the $n$ coplanar points
$A_1 A_2 \ldots A_n$; $G_1$ divides $A_1 A_2$ in the ratio $m_1 : m_1$; $G_2$ divides
$G_1 A_2$ in the ratio $m_2 : (m_1 + m_2)$; $G_3$ divides $G_2 A_3$ in the ratio
$m_3 : (m_2 + m_3 + m_4)$; and so on; $G_{n-1}$ divides $G_{n-2} A_n$ in
the ratio $m_n : (m_{n-1} + m_n)$; then $G_{n-1}$ is the centroid of $m_1, m_2, \ldots m_n$
at $A_1, A_2, \ldots A_n$.

[Use Theorem 42 and the method of Theorem 43 (1).]

The proof is left to the reader.

1. Prove Theorem 43 (2).

D.O.
THEOREM 44.

If \( G \) is the mean centre of \( n \) collinear points \( A_1, A_2, \ldots, A_n \), then
\[ GA_1 + GA_2 + \ldots + GA_n = 0. \]
[Take the origin at \( G \) and the \( y \)-axis along \( GA_1 \) and use the definition of the mean centre.]

2. Prove Theorem 44.

3. If \( G \) is the mean centre of masses \( m_1, m_2, \ldots, m_n \), at the \( n \) collinear points \( A_1, A_2, \ldots, A_n \), prove that
\[ m_1GA_1 + m_2GA_2 + \ldots + m_nGA_n = 0. \]

It is convenient to refer at this point to the following statical theorem.

If a system of forces, acting at a point \( O \), are represented completely by \( m_1OA_1, m_2OA_2, m_3OA_3, \ldots, \), then they are equivalent to a single force represented completely by
\[ (m_1 + m_2 + \ldots + m_n)OG, \]
where \( G \) is the centroid of masses \( m_1, m_2, \ldots, m_n \), at \( A_1, A_2, \ldots, A_n \).

This theorem is a particular case of the general vector theorem enunciated on page 71, since a force is a special case of a vector quantity. The proof suggested for the vector theorem may be applied equally easily in this case.

THEOREM 45.

If \( O \) is the mean centre of masses \( m_1, m_2, m_3 \) at \( A, B, C \) respectively, then
\[ \frac{m_1}{\triangle OBC} = \frac{m_2}{\triangle OCA} = \frac{m_3}{\triangle OAB}. \]

Draw \( OP, AD \) perpendicular to \( BC \).

By definition, regarding \( BC \) as the \( x \)-axis,
\[ OP = \frac{m_1AD + m_2AO + m_3AO}{m_1 + m_2 + m_3} = \frac{m_1AD}{m_1 + m_2 + m_3}, \]

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THE MEAN CENTRE

\[ \frac{m_1}{OP} = \frac{m_1 + m_2 + m_3}{AD}; \]
\[ \frac{1}{BC} = \frac{m_1 + m_2 + m_3}{4 AD \cdot BC}; \]
\[ \frac{m_1}{\triangle OBC} = \frac{m_1 + m_2 + m_3}{\triangle ABC}. \]

= similarly \( \frac{m_2}{\triangle OCA} = \frac{m_3}{\triangle OAB} \)

Q.E.D.

In the following examples, \( PQ \) denotes the force represented in magnitude and position by the line \( PQ \).

4. (1) \( A_1, A_2, \ldots, A_n \) are \( n \) collinear points; \( G \) is their mean centre.

If \( P \) is any other point on the line \( A_1A_n \), prove that \( \frac{A_1G \cdot P}{m} = h \cdot GP \).

(2) Extend part (1) to apply to the centroid \( G \) of masses \( m_1, m_2, \ldots, m_n \), at \( A_1, A_2, \ldots, A_n \).

5. \( G \) is the mean centre of \( n \) collinear points \( A_1, A_2, \ldots, A_n \); \( H \) is the mean centre of \( n \) points \( B_1, B_2, \ldots, B_n \) on the line \( A_1A_n \); prove that
\[ \frac{A_1B_1 \cdot A_2B_2 \cdot \ldots \cdot A_nB_n}{m} = \frac{m}{GH}. \]

6. \( G \) is the mean centre of \( n \) collinear points \( A_1, A_2, \ldots, A_n \); \( H \) is the mean centre of \( n \) points \( B_1, B_2, \ldots, B_n \) on \( A_1A_n \); prove that
\[ A_1B_1 + A_2B_2 + \ldots + A_nB_n = n \cdot GH. \]

7. Determine the masses to be placed at \( A, B, C \) in order that their centroid may be (1) the incentre, (2) the orthocentre, (3) the circumcentre, (4) the excentre opposite \( A \).

8. Determine the masses to be placed at the excentres so that their centroid may be at the incentre of the triangle.

9. \( P, Q, R, S \) are the mid-points of consecutive sides of a quadrilateral \( ABCD \); \( X, V \) are the mid-points of the diagonals \( AC, BD \); prove that the centroid of equal masses placed at \( A, B, C, D \) is at the mid-point of each of the lines \( PR, QS, XY \).

10. \( A, B, C \) are fixed points; \( P \) is a variable point on a fixed line. Find the locus of the centre of four equal and parallel forces at \( A, B, C, P \).

11. \( A', B', C' \) are the mid-points of \( BC, CA, AB \); \( P \) is any other point, prove that the forces \( PA', PB', PC', AP, BP, CP \) are in equilibrium.

12. If \( ABCD \) is a parallelogram, and if \( P \) is any other point, prove that the forces \( PA, PB, PC, DP \) are in equilibrium.
13. O, N are the circumcentre, nine-point centre of the triangle \(ABC\), prove that the forces \(NA, NR, NC\) can be balanced by a system of equal forces along \(OA, OB, OC\).

14. H, O are the orthocentre, circumcentre of the triangle \(ABC\), prove that the forces \(HA, HB, HC\) are balanced by the forces \(2OA, 2OB, 2OC\).

15. Forces \(P, Q, R, S\) act along the sides \(AB, BC, CD, DA\) of a quadrilateral \(ABCD\) and are in equilibrium, prove that \[
\frac{P \cdot R}{AB \cdot CD} = \frac{Q \cdot S}{BC \cdot DA}
\]

16. \(ABCD\) is a tetrahedron, \(P\) is a variable point such that the resultant of the forces \(FA, FB, FC, PD\) is of constant magnitude, find the locus of \(P\).

17. \(P\) is a variable point on the side \(CD\) of the quadrilateral \(ABCD\); \(PG\) is the resultant of the forces \(AP, BP, PC, PD\); find the locus of \(G\).

18. \(A', B', C'\) are the mid-points of \(BC, CA, AB\); prove that the forces \(AA', BB', CC'\) are in equilibrium.

19. Find a point \(P\) inside (i) the quadrilateral \(ABCD\), (ii) the tetrahedron \(ABCD\), such that the forces \(FA, FB, FC, PD\) are in equilibrium.

20. Prove that the straight lines joining the mid-points of opposite edges of a tetrahedron \(ABCD\) are concurrent and bisect each other. [Consider the resultant of the forces \(FA, FB, FC, PD, \) obtained by combining them in pairs in different ways.]

21. \(ABCD\) is a quadrilateral: if the forces \(l \cdot AB, m \cdot BC, n \cdot CD, r \cdot DA\) are in equilibrium, prove that \(lm = mn = nr = mr\).

22. Four straight lines in a plane intersect at the six points \(ABCD\); \(O\) is any other point; construct geometrically the resultant of the forces \(OA, OB, OC, OD, EO, FO\).

23. Forces act along the edges \(BA, BC, DA, DC\) of a tetrahedron \(ABCD\) and are inversely proportional to those edges; prove that if the forces give a single resultant, then \(AD = BC = AB = CD\).

24. \(A', C'\) are the mid-points of the sides \(BC, AB\) of an equilateral triangle \(ABC\); prove that the magnitude of the resultant of the forces \(AA', AC'\) equals the magnitude of the resultant of the forces \(AC, AB\). If \(H, K\) are the mid-points of \(CA, CC\), show that this is equivalent to the theorem that \(AHK\) is an isosceles triangle.

25. \(ABCD\) is a quadrilateral; \(AD\) meets \(BC\) at \(E\); \(AB\) meets \(DC\) at \(F\); prove that the resultant of the forces \(AB, AD, CA, CD\) passes through the mid-points of \(AC, BD, EF\); hence it follows that the mid-points of these lines are collinear.

26. \(A'B'C'\) are the mid-points of the sides \(BC, CA, AB\) of a triangle; prove that the centroid of three uniform rods \(BC, CA, AB\) is at the centre of the triangle \(A'B'C'\).

27. \(ABCD\) is a tetrahedron; \(P\) is a point on \(BC\) and \(Q\) is a point on \(AD\); if \(\frac{BP}{PC} = \frac{AQ}{QD} = m\), resolve the force \(PQ\) into components along \(BA, CA, BD, CD\). Is this resolution unique?

28. Three forces \(P, Q, R\) act along the edges \(DA, DB, DC\) of a tetrahedron \(ABCD\); their resultant meets the face \(ABC\) at \(X\), find, in terms of the edges of the tetrahedron, the lengths of the perpendiculars from \(X\) to \(BC, CA, AB\).

29. \(A_1A_2\ldots A_n\) and \(B_1B_2\ldots B_m\) are two polygons of \(n\) and \(m\) sides; construct the resultant of the forces \(A_iB_i\), where \(r, s\) may have any value from \(1\) to \(n\) and \(1\) to \(m\) respectively.

30. If in Ex. 29 \(n = m\), construct the resultant of the forces \(A_1B_1, A_2B_2, \ldots, A_nB_n\).

31. Find a point \(P\) in the plane of the triangle \(ABC\) such that the following sets of forces are in equilibrium:

   (1) \(a \cdot FA, b \cdot FB, c \cdot FC\).

   (2) \(sin 2A \cdot FA, sin 2B \cdot FB, sin 2C \cdot FC\).

   (3) \(sec A \cdot FA, sec B \cdot FB, sec C \cdot FC\).

   (4) \(a \cdot FA, b \cdot FB, c \cdot FC\).

32. \(AD, BE, CF\) are the altitudes of the triangle \(ABC\); find a point \(P\) such that the forces \(sin 2A \cdot PD, sin 2B \cdot PE, sin 2C \cdot PF\) are in equilibrium.

33. With the usual notation, if the forces \(\lambda, \mu, \nu, \omega\) are in equilibrium, find the ratios \(\lambda : \mu : \nu : \omega\).

34. Three forces \(P, Q, R\) act along the edges \(AB, AC, AD\) of a tetrahedron, if their resultant is parallel to the face \(BCD\), prove that \(P + Q + R = 0\).

35. \(A_1A_2\ldots A_n\) is a plane polygon and \(H\) is any point outside the plane: forces \(P_1, P_2, \ldots, P_n\) act along \(HA_1, HA_2, \ldots, HA_n\); if their resultant is parallel to the plane, prove that \(P_1 + P_2 + \ldots + P_n = 0\).
36. $ABCD$ is a tetrahedron; $G_1, G_2, G_3, G_4$ are the centroids of the faces opposite to $A, B, C, D$; $P$ is any other point; prove that the forces $\frac{1}{r_1}PA; \frac{1}{r_2}PB; \frac{1}{r_3}PC; \frac{1}{r_4}PD$ are equivalent; hence show that $AG_1, BG_2, CG_3, DG_4$ are concurrent (at $G$, say); and that $G = \frac{1}{4}G_1G_2G_3G_4$.

37. $P$ is any point; with the usual notation, prove that the resultant of the forces $\frac{1}{r_1}PA, \frac{1}{r_2}PB, \frac{1}{r_3}PC, \frac{1}{r_4}PD$ passes through the incentre of the triangle $ABC$.

38. $ABCD$ is a tetrahedron; $I_1, I_2, I_3, I_4$ are the incentres of the faces opposite to $A, B, C, D$; if $AI_1$ and $BI_2$ intersect, prove that

(i) $AD, BC = AC, BD$.
(ii) $CI_3$ and $DI_4$ also intersect, and if $AD, BC = AC, BD = AB, CD$, prove that $A I_1, B I_2, C I_3, D I_4$ are concurrent.

39. $A_1A_2A_3A_4$ is a tetrahedron; $B_1$ is a point on the face opposite to $A_1$ such that $A_1A_2, A_1A_3, A_1A_4 = A_1A_1, A_1A_2, A_1A_3, A_1A_4$; and similar points $B_2, B_3, B_4$ are taken on the faces opposite to $A_2, A_3, A_4$, prove that $A_1B_1, A_2B_2, A_3B_3, A_4B_4$ are concurrent.

40. Tangents at the angular points of a triangle $ABC$ to its circumcircle so as to form a triangle $PQR$. If $AP, BQ, CR$ intersect at $K$, prove that the forces $\frac{1}{r_1}PA, \frac{1}{r_2}QB, \frac{1}{r_3}RC$ are in equilibrium,

- $A_1, A_2, A_3, A_4$ are five points in space; $G_1$ is the centroid of the tetrahedron $A_1A_2A_3A_4$ and similarly for $G_2, G_3, G_4$. State what properties connect the lines $A_1G_1, A_2G_2, A_3G_3, A_4G_4$, and prove them.

41. If $P, Q, R$ are points on $BC, CA, AB$ such that $\frac{BP}{FC} = \frac{CQ}{QA} = \frac{AR}{KB}$ prove that the triangles $ABC, PQR$ have the same centroid.

42. Three forces $P, Q, R$ act along the sides $BC, CA, AB$ of a triangle $ABC$; if their resultant is along $GJ$, prove that $P:Q:R = \frac{a(b-c)}{b(c-a)} : \frac{b(c-a)}{c(a-b)} : \frac{c(a-b)}{a(b-c)}$.

43. Use the system of unit forces along $BA, BC, CA$ to prove that the internal bisectors of the angles of the triangle $ABC$ are concurrent.

44. $H$ is the orthocentre of the triangle $ABC$; $P, Q, R$ are the midpoints of $AH, BH, CH$; $A', B', C'$ are the midpoints of $BC, CA, AB$. By placing equal masses at $A, B, C, H$, prove that the lines $A'P, B'Q, C'R$ are concurrent (at $N$, say), and that $G$ is the centroid of the triangle $ABC$, prove that $G$ lies on $GH$ and that $3GN = NH$. Is this theorem true if $H$ is any point in the plane $ABC$?

45. Theorem 46. [Apollonius' Theorem.]

(a) If $J$ is the mid-point of the base $BC$ of a triangle $ABC$, then $AB^2 + AC^2 = 2JA^2 + JBA$.  
(b) If $J$ is a point on the base $BC$ of a triangle $ABC$ such that $mAB + nAC = (m+n)JA^2 + mAB^2 + nAC^2$.

Draw $AD$ perpendicular to $BC$.

$\begin{align*}
AB^2 &= AD^2 + DB^2 = AD^2 + (BA + AD)^2 \\
&= AD^2 + BA^2 + AD + BA + AD^2 \\
\text{and} \\
AC^2 &= AD^2 + DC^2 = AD^2 + (AC - AD)^2 \\
&= AD^2 + BA^2 - AD + AD^2 \\
\therefore AC^2 &= AD^2 + BA^2 - 2BA \
AB = AD^2 + 2BA^2 + 2AD^2 \\
\therefore AB^2 + AC^2 &= 2AB^2 + 2AC^2 = 2AA^2 + 2AB^2 \\
Q.E.D.
\end{align*}$

46. Calculate the lengths of the medians of a triangle whose sides are 7 cms., 8 cms., 9 cms.

47. Determine the lengths of the sides of a triangle, if the medians are of lengths $x, y, z$. 

\[ \text{Q.E.D.} \]
48. \( AA' \) is a median of the triangle \( ABC \) and is a mean proportional between \( AB, AC \); prove that \( a = \sqrt{3} (b + c) \).

49. Prove that the sum of the squares of the sides of a triangle is equal to four times the sum of the squares of the medians.

50. If \( ABCD \) is a parallelogram, prove that
\[
2(AB^2 + BC^2) = AC^2 + BD^2.
\]

51. \( P, Q \) are the mid-points of the diagonals \( AC, BD \) of the quadrilateral \( ABCD \), prove that \( AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2 \).

52. If \( a, b, c, d \) are the lengths of the sides of a quadrilateral, and if \( x, y \) are the lengths of the diagonals, calculate the line joining the mid-points of a pair of opposite sides.

53. If \( AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 \), prove that \( ABCD \) is a parallelogram.

54. If the sum of the squares on each pair of opposite edges of a tetrahedron is the same, prove that the lines joining the mid-points of opposite edges are equal.

55. \( ABCD \) is a rectangle; \( P \) is any point in space, prove that \( PA^2 + PC^2 = PB^2 + PD^2 \).

56. The diagonals \( AC, BD \) of a cyclic quadrilateral \( ABCD \) meet at \( E \); if \( BE = EP \), prove that \( 2AC^2 = AB^2 + BC^2 + CD^2 + DA^2 \).

57. Two straight rods are joined together at their mid-points; prove that the sum of the squares of the four joining their extremities is constant.

58. \( ABC \) is an equilateral triangle; \( AB \) is produced to \( D \) so that \( BD = AB \); \( P \) is any point such that \( DP = DC \); prove that \( AP^2 + AP^2 = 2BP^2 \).

59. In any tetrahedron, prove that the sum of the squares on any pair of opposite edges together with twice the square on the line joining their mid-points is the same, whatever pair of opposite edges is taken.

60. \( B \) is a fixed point, \( A \) is the centre of a fixed circle, prove that the locus of the mid-points of chords which subtend a right angle at \( B \) is a circle whose centre is the mid-point of \( AB \).

61. \( B \) is a fixed point; \( A \) is the centre of a fixed circle; \( PQ \) is a variable chord parallel to \( AB \), prove that \( BP^2 + PQ^2 \) is constant.

62. \( AP, BQ, CR \) are the tangents from three of the vertices of the rectangle \( ABCD \) to a given circle; if \( AP^2 + CR^2 = BQ^2 \), prove that \( D \) lies on the circle.

63. \( ABC, PQR \) are two triangles having \( AB, BC \) equal to \( PQ, QR \) respectively; if the angles \( ABC, PQR \) are supplementary, prove that \( AC^2 + PR^2 = 2AB^2 + 2BC^2 \).

64. Squares are described externally on the sides of a triangle and their adjacent corners are joined; prove that the sum of the squares on the joining lines is equal to three times the sum of the squares on the sides of the triangle.

65. In the triangle \( ABC \), \( AB = 8 \), \( AC = 6 \), \( BC = 9 \); \( D \) is a point on \( BC \) such that \( \frac{BD}{DC} = \frac{2}{3} \). Calculate \( AD \).

66. \( D, E \) are points on the sides \( BC, CA \) of the triangle \( ABC \), such that \( \frac{BD}{DC} = \frac{CE}{EA} = \frac{1}{2} \); if \( AD = BE \), prove that \( 2BC^2 = 3AB^2 + 3AC^2 \).

67. \( D \) is a point on the base \( BC \) of the triangle \( ABC \) such that \( \frac{BD}{DC} = \frac{\lambda}{\mu} \); \( AD \) is produced to meet the circumcircle at \( E \), prove that \( \mu \cdot AB^2 + \lambda \cdot AC^2 = (\mu + \lambda) \cdot AD \cdot AE \).

State what this theorem becomes when (i) \( AD \) is a median; (ii) \( AD \) bisects \( BAC \).

68. \( ABC \) is an equilateral triangle, \( AB \) is produced to \( D \) so that \( BD = BA \); \( P \) is any point such that \( DP = DC \); prove that
\[
\lambda \cdot AP^2 + \lambda \cdot DP^2 = (\lambda + 1) \cdot BD^2.
\]

69. If \( AD \) bisects the angle \( BAC \) and meets \( BC \) at \( D \), prove that
\[
AD^2 = b \left( 1 - \left( \frac{a}{b+c} \right)^2 \right) \quad \text{[Use Apollonius' (1) to get]} \quad PA^2 + PB^2
\]

70. \( A, B \) are two fixed points. \( P \) is a variable point such that \( m \cdot AP^2 + n \cdot BP^2 \) is constant, where \( m, n \) are constants, prove that the locus of \( P \) is a circle. Does the circle ever degenerate into a straight line?

71. (1) If \( G \) is the centroid of the triangle \( ABC \) and if \( P \) is any other point, prove that \( PA^2 + PB^2 + PC^2 - 3PG^2 = \frac{1}{2} (AB^2 + BC^2 + CA^2) \).

[Use Apollonius' (1) to get \( PA^2 + PB^2 \) and then use Apollonius (2).]

(2) By taking \( P \) at \( G \), prove that
\[
PA^2 + PB^2 + PC^2 = (GA^2 + GB^2 + GC^2) = 3PG^2.
\]

72. (1) \( A_1A_2A_3A_4 \) are four fixed points; \( G_2 \) is the mid-point of \( A_1A_2 \); \( G \) is a point on \( A_1A_2 \) such that \( 2G_2G = A_1A_2 \); \( G \) is a point on \( A_1A_2 \) such that \( 3G_2G = GA_2 \). If \( P \) is a variable point, prove that
\[
PA_1^2 + PA_2^2 + PA_3^2 + PA_4^2 = 4PG^2 + \text{constant}.
\]

(2) Hence prove that \( \sum_{i=1}^{4} (PA_i^2 - GA_i^2) = 4PG^2 \).

[Use the method of Ex. 71 (2).]

(3) State the analogous theorem for \( n \) fixed points.

73. \( ABC \) is a given triangle; find a point \( P \) for which \( PA^2 + PB^2 + PC^2 \) has its least value.
74. \(ABC\) is a given triangle; \(P\) is a variable point such that \(PA^2 + PB^2 + PC^2\) is constant; prove that the locus of \(P\) is a circle.

75. \(ABC\) is a given triangle; \(P\) is a variable point such that \(PA^2 + 3PB^2 + 3PC^2\) is constant; prove that the locus of \(P\) is a circle, and construct its centre.

**THEOREM 47.**

\(G\) is the mean centre of the \(n\) points \(A_1, A_2, \ldots, A_n\); if \(P\) is any other point, then \(\sum PA_i^2 - \sum GA_i^2 = n \cdot PG^2\).

\(\text{Fig. 31.}\)

Join \(PG\): let \(a_1, a_2, a_3, \ldots\) be the feet of the perpendiculars from \(A_1, A_2, A_3, \ldots\) to \(PG\).

Then, regarding \(G\) as the origin and \(PG\) as the \(x\)-axis,

\[\frac{1}{n}(Ga_1 + Ga_2 + \ldots + Ga_n) = x\text{-coordinate of } G = 0;\]

\[\sum Ga_i = 0.\]  \[\text{[Th. 44]}\]

\[PA_i^2 - GA_i^2 = PA_i^2 - (Ga_i^2 + a_iA_i^2) = (PG + Ga_i)^2 - Ga_i^2 = PG^2 + 2PG \cdot Ga_i;\]

\[\sum PA_i^2 - \sum GA_i^2 = n \cdot PG^2 + 2PG \cdot \sum Ga_i = n \cdot PG^2.\]  \[\text{Q.E.D.}\]

**THEOREM 48.**

If \(G\) is the centroid of masses \(m_1, m_2, \ldots, m_n\) at the \(n\) points \(A_1, A_2, \ldots, A_n\), respectively, and if \(P\) is any other point, then

\[\sum m_i \cdot PA_i^2 = \sum m_i \cdot GA_i^2 = (m_1 + m_2 + \ldots + m_n)PG^2.\]

This may be proved in exactly the same way as Theorem 47. Notice that Apollonius' Theorem, parts (1) and (2), are special cases of Theorems 47, 48.
91. \(A_1A_2...A_n, \, B_1B_2...B_n\) are two regular polygons inscribed in two circles, centres \(C, \, D\) respectively; prove that 
\[\sum A_iB_i^2 = \eta^2(A_iC_i + B_iD_i + CD)\]

92. Extend Ex. 91 to the case where the regular polygons have \(m\) and \(n\) sides.

93. \(A_1A_2...A_n\) is a regular polygon inscribed in a circle, centre \(O\). From any point \(P\), perpendiculars \(PN_1, \, PN_2, \ldots, \, PN_n\) are drawn to the radii \(OA_1, \, OA_2, \ldots, \, OA_n\). Prove that the mean centre of the points \(N_1, \, N_2, \ldots, \, N_n\) is the mid-point of \(OP\). [Draw the circle on \(OP\) as diameter.]

94. With the figure of Ex. 93, prove that \(\sum PN_1^2 = \sum ON_1^2\).

95. \(ABC\) is a given triangle, find the position of \(P\) for which
   (1) \(BC.PA + CA.PB + AB.PC\),
   (2) \(\lambda. PA + \mu. PB + \nu. PC\),
   is a minimum.

96. \(G\) is the centroid of masses \(l, m, n\) at \(A, \, B, \, C\); \(GT\) is the tangent from \(G\) to the circumcircle of the triangle \(ABC\). Prove that \(l, m, PA^2 + m, PB^2 + n, PC^2 = (l + m + n)(PG^2 - GT^2)\). [It is supposed that a negative mass may exist.]

97. If \(G\) is the centroid of masses \(m_1, m_2, m_3\) at \(A_1, \, A_2, \, A_3\) and if \(P\) is a point such that \(m_1, PA_1^2 + m_2, PA_2^2 + m_3, PA_3^2 = 0\), prove that the locus of \(P\) is a circle cutting the circle \(A_1A_2A_3\) at right angles. [\(\text{ie, the tangents at a point of intersection are at right angles.}\)]

98. \(P\) is a variable point on the nine-point circle of the triangle \(ABC\), prove that \(PA^2 \sin A \cos B - CB \sin B \cos C - AB \cos A = PA^2 \sin C \cos B - CB \sin B \cos A - PA^2 \sin C \cos A\) is constant.

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CHAPTER VIII.

HARMONIC RANGES AND PENCILS.

The fundamental ideas contained in this chapter date back to an early period. The word 'harmonic' was in use at the beginning of the fourth century B.C. But the properties known at that time were probably mainly of a simple metrical character. Theorem 51, which may be regarded as one of the foundation stones of modern descriptive geometry, was probably known to Euclid. Its extension to a pencil formed by any four concurrent lines was incorporated by Pappus in his great work, the Σωσική, published about the beginning of the fourth century A.D. It is most remarkable that this treatise, which approximates so closely to modern developments, received scarcely any attention before the 17th century.

Definition.

A straight line \(AB\) is divided internally at \(C\) and externally at \(D\) in the same ratio \([\frac{AC}{CB} = \frac{-AD}{DB}]\); then \(AB\) is said to be divided harmonically at \(C\) and \(D\); and the points \(ACBD\) are said to form a harmonic range.

\(C\) and \(D\) are called harmonic conjugates with respect to \(A\) and \(B\).

The range \(ACBD\) is denoted by \([ACBD]\) or \([AB; \, CD]\).

The reason for the name harmonic will be found in Ex. 5, which is due to Pythagoras.
THEOREM 49.

(1) If $C$, $D$ divide $AB$ harmonically, then $A$, $B$ divide $CD$ harmonically.

(2) If $O$ is the mid-point of $AB$, then $OB^2 = OC \cdot OD$.

\[ \begin{align*}
A & \quad O \quad C \quad B \quad D \\
\text{Fig. 59}
\end{align*} \]

(1) Since $\{ACBD\}$ is harmonic,

\[ \begin{align*}
AC &= AE \\
CB &= BE \quad \text{or} \quad AC \cdot BD = AD \cdot CB; \\
\dot{\ldots} \quad CB &= AC = CA \quad \text{or} \quad AD \cdot BD = AB \cdot CD.
\end{align*} \]

\[ \therefore CD \text{ is divided internally at } B \text{ and externally at } A \text{ in the same ratio.} \quad \text{Q.E.D.} \]

(2) Since $AC = AD$, then $AC + CB = AD + BD$

But $AC + CB = AB = 2OB,$

$AC - CB = (AO + OC) - (OB - OC) = 2OC,$

$AD + BD = (AO + OD) + (OD - OB) = 2OD,$

$AD - BD = AB = 2OB;$

$\therefore OB = OD;$

$\therefore OC = BD;$

$\therefore OB^2 = OC \cdot OD.$ \quad \text{Q.E.D.}

Corollary.

If $O$ is the mid-point of $AB$ and if $C$, $D$ are points on $AB$ such that $OB^2 = OC \cdot OD$, then $\{ACBD\}$ is harmonic.

This may be proved by taking each step in (2) in the reverse order.

1. Prove Theorem 49 (2) by putting $OC = c$, $OB = b$, $OD = d$ and substituting in the relation $AC = \frac{AD}{BD}$.

2. Prove the Corollary of Theorem 49 by the method of Ex. 1.

3. If $\{ACBD\}$ is harmonic, prove that $AC$, $AB$, $AD$ are in harmonical progression. [Put $AC = c$, $AB = b$, $AD = d$.]
THEOREM 49.

(1) If \( C, D \) divide \( AB \) harmonically, then \( A, B \) divide \( CD \) harmonically.

(2) If \( O \) is the mid-point of \( AB \), then \( OB = OC \cdot OD \).

\[ \text{Fig. 52.} \]

(1) Since \( \{ACBD\} \) is harmonic,
\[ \frac{AC}{AD} = \frac{CB}{BD} \quad \text{or} \quad AC \cdot BD = AD \cdot CB; \]
\[ \frac{CB}{AC} = \frac{CA}{AD} \]
\[ \frac{BD}{AD} = \frac{AD}{CB}. \]
\[ \therefore CD \text{ is divided internally at } B \text{ and externally at } A \text{ in the same ratio.} \]

Q.E.D.

(2) Since
\[ \frac{AC}{AD} = \frac{AC + CB}{AD + BD}; \]
But
\[ \frac{AC + CB}{AD + BD} = \frac{AB + 2OB}{AB + 2OD} = 1, \]
\[ \frac{AC}{CB} = \frac{AC + OB - 2OB}{AD + BD - 4OB} = \frac{AB - 2OB}{AB - 4OD}; \]
\[ \frac{CB}{AC} = \frac{2OB}{2OD} = \frac{OC}{OD}; \]
\[ \therefore OB = OC \cdot OD. \]

Q.E.D.

Corollary.

If \( O \) is the mid-point of \( AB \) and if \( C, D \) are points on \( AB \) such that \( OB = OC \cdot OD \), then \( \{ACBD\} \) is harmonic.

This may be proved by taking each step in (2) in the reverse order.

1. Prove Theorem 49 (2) by putting \( OC = c, OB = b, OD = d \) and substituting in the relation \( \frac{AC}{AD} = \frac{CB}{BD} \).

2. Prove the Corollary of Theorem 49 by the method of Ex. 1.

3. If \( \{ACBD\} \) is harmonic, prove that \( AO, AB, AD \) are in harmonic progression. [Put \( AO = a, AB = b, AD = d \).]
THEOREM 50.

\{ACBD\} is harmonic; \(O\) is any point outside \(AB\). If through \(B\) a line \(PBQ\) is drawn parallel to \(OA\), meeting \(OC\) at \(P\) and \(OD\) at \(Q\), then \(PB = BQ\).

Since \(AO\) is parallel to \(PB\), \(\frac{AC}{CB} = \frac{AO}{PB}\)

Since \(AO\) is parallel to \(BQ\), \(\frac{AD}{BD} = \frac{AO}{BQ}\)

But \(\frac{AC}{CB} = \frac{AD}{BD}\) given;

\(\frac{AO}{PB} = \frac{AO}{BQ}\);

\(\therefore PB = BQ\).

Q.E.D.

Corollary.

If \(OA, OC, OB, OD\) are four lines such that a parallel to \(OA\) through a point \(B\) in \(OB\) is divided into equal segments \((PB, BQ)\) by \(OC, OB, OD\); then every line through \(B\) is cut harmonically by \(OA, OC, OB, OD\).

This is proved by reversing each step in the above proof.

It is important to notice that if \(C\) is the mid-point of \(AB\), then \(\{ACBD\}\) is harmonic where \(\infty\) denotes a point on \(AB\) infinitely distant from \(A\).

15. Prove the Corollary of Theorem 50.
On account of the importance of this theorem, two alternative methods of proof are given below.

**THEOREM 51 (SECOND PROOF).**

Let $A'C'B'D'$ be the transversal.

- $AC = \Delta AOC \cdot \frac{1}{2}OA \cdot OC \cdot \sin AOC$
- $CB = \Delta AOB \cdot \frac{1}{2}OB \cdot CO \cdot \sin COB$
- $AD = OA \cdot \sin AOC$
- $DB = OB \cdot \sin COB$

Similarly,

- $AC \cdot AD = \sin AOC \cdot \sin BOD$
- $CB \cdot BD = \sin COB \cdot \sin AOD'$

which depends only on the sines of the angles at the vertex;

- $\frac{AC}{CB} \cdot \frac{AD}{BD}$ is equal to the same expression.

[Note that $\sin A'OC = \sin(180^\circ - AOC) = \sin AOC$;]

- $AC \cdot AD = AC \cdot AD$
- $CB \cdot BD = CB \cdot BD$

But

$AC \cdot AD = CB \cdot BD$ given;

- $\frac{AC}{CB} \cdot \frac{AD}{BD}$
- $\frac{AC}{CB} = \frac{AD}{DB}$

:: $(A'C'B'D')$ is harmonic.

Q.E.D.

Let $A'C'B'D'$ be the transversal.

Now

- $AC \cdot OA = OB$ [Th. 8]
- $AD \cdot OA = OB$ [Th. 8]

and

- $AC \cdot AC = DB$ [Th. 8]
- $CB \cdot CB = DB$ [Th. 8]

But

- $AC \cdot AD = \frac{AC}{CB} \cdot \frac{AD}{DB}$
- $\frac{AC}{CB} = \frac{AD}{DB}$

:: $(A'C'B'D')$ is harmonic.

Q.E.D.

The second proof of Theorem 51 puts in evidence the important fact that the harmonic property of a pencil depends on a definite relation between the angles at the vertex; namely, that (with the notation of Fig. 55) $\frac{\sin AOC \cdot \sin BOD}{\sin COB \cdot \sin AOD'} = 1$.

17. Use this angular relation to verify that the two bisectors of an angle are harmonic conjugates w.r.t. the arms of the angle.

18. Use this angular relation to prove Theorem 53.
THEOREM 52.

$O(ACBD)$ is harmonic; if $AO, BO$ are at right angles, then they are the two bisectors of the angle $COD$.

Draw any line parallel to $OA$ meeting $OB, OC, OD$ at $F, G, H$.

Then

$$GF = FH$$

also

$OFG - OEH$ right angles, since $OFP = 90^\circ$.

$OF$ is common;

the triangles $OFG, OEH$ are congruent;

$GOF = FOH$;

$OB$ bisects $COD$.

But $OA$ is perpendicular to $OB$;

$OA$ bisects $COD$ externally. Q.E.D.

Corollary.

If $\{ACBD\}$ is a harmonic range: and if $P$ is any point on the circle on $CD$ as diameter, then $PA/PB$ is constant and equal to $AC/BC$.

This is the converse of Theorem 9. [Apollonius’ Circle]

19. Prove the Corollary of Theorem 52.

20. In the triangle $ABC$, $AI, AI_1$ meet $BC$ at $P, Q$; prove that $BPCQ$ is harmonic.

21. In the triangle $ABC$, prove that (1) $D(BFAE)$; (2) $A(OH; HM)$ are harmonic pencils.

22. $A, B$ are two given points, $CD$ is a given straight line, prove that in general there is one and only one position of $P$ on $CD$ such that $APP$ is bisected by $CD$. What is the exceptional case?

23. The lines joining a point on a circle centre $O$, to the extremities of a chord $PQ$ meet the diameter perpendicular to $PQ$ at $A, B$. Prove that $OA, OB = OP$.

24. A circle is described touching a straight line $AB$ at $C$; if its diameter is the harmonic mean between $AC, CB$, prove that it touches the circle on $AB$ as diameter.

25. $OA, OC, OB$ are three concurrent lines, construct a line $OD$ such that $OACBD$ is harmonic.

26. The incircle of the triangle $ABC$ touches $BC, CA, AB$ at $X, Y, Z$; $YZ$ produced meets $BC$ at $T$; prove that $TXCT$ is harmonic. [Use Menelaus.]

27. $ABCD$ is a parallelogram; $AE$ is drawn parallel to $BD$, prove that $AD = EC$; $BD$ is harmonic.

28. $A', B', C'$ are the mid-points of $BC, CA, AB$, prove that $A'A, A'C'$ are harmonic conjugates with $AA', AC'$.

29. $AD$ is an altitude of the triangle $ABC$; $A'$ is the mid-point of $BC$; $A'P, A'O$ are drawn parallel to $AB, AC$ to meet $AD$ at $P, Q$; prove that $A'POQ$ is harmonic.

30. With the usual notation for the triangle $ABC$, if $DF$ meets $BE$ at $K$, prove that $[BKHE]$ is harmonic; also, if $EF$ meets $BC$ at $M$, prove that $[BDFM]$ is harmonic.

31. Through a fixed point $K$ draw a straight line to meet $BC, CA, AB$ at $PQR$ so that $[KPQR]$ is harmonic.

32. The tangent at a point $C$ of a circle meets $AB$ of the circle at $T$, prove that the other tangent from $T$ to the circle is divided harmonically by $CA, CB, CT$ and its point of contact.

33. $A'BC$ are the mid-points of $BC, CA, AB$; $CC'$ meets $AA', AB$ at $L, M$; prove that $CL, MC = CC, LM$.

34. $O$ is the circumcentre of the triangle $ABC$; $AK, AL$ are two lines between $AB, AC$ such that $BAK = LAC$ and meet the perpendicular from $O$ to $BC$ at $K, L$; prove that $OK, OL = OA$.

35. $ABCD$ is a quadrilateral; $AB, CD$ meet at $P$; $AD, BC$ meet at $G$; $AG, BD$ meet at $E$; $FE$ meets $BC, AD$ at $Q, P$; $GE$ meets $AB, CD$ at $H, K$; $FG$ meets $AC, BD$ at $K, S$. Prove that $[DKCF]$ is harmonic. [Apply Menelaus and Ceva to the triangle $DSC$]

Hence prove that every other set of four collinear points in the figure forms a harmonic range.

36. If $ax^2 + bx + c = 0, a^2x^2 + bx + d = 0$, give two pairs of points on the $x$-axis which are harmonic conjugates, prove that $ab + a'b = 2hk$. 

viii. HARMONIC RANGES AND PENCILS
37. If a variable straight line is cut harmonically by two circles, centres \(A, B\), conjugate points being on the same circle, prove that the locus of the perpendiculars let fall on it from \(A\) is a circle whose centre is at the mid-point of \(AB\).

38. ABCD PQ are six points on a circle; if \(P \parallel A E C D\) is harmonic, prove that \(Q \parallel A E C D\) is harmonic.

Definition. If four points on a circle are such that a harmonic pencil is formed by joining them to one other point on the circle, then by Ex. 38 the same is true for any other point on all the circle. The four points are then said to form a harmonic system of points on the circle.

If four tangents to a circle are cut harmonically by another tangent to the circle, the four tangents are said to form a harmonic system of tangents to the circle. It follows from Ex. 40 that four such tangents are cut harmonically by every other tangent to the circle.

39. What does Ex. 38 become when \(Q\) is very close to \(A\)?

40. (1) Two tangents \(TA, TB\) to a circle, centre \(C\), are met by another tangent at \(P, Q\); prove that \(PQ\) is equal or supplementary to \(\overline{A C B}\).

(2) Hence show that if four tangents meet \(TA, TB, PA, PB, PA, PB\) and \(TA, TB\) at \(Q_1, Q_2, Q_3, Q_4\), and if \(P, P, P, P\) is harmonic, then \(Q_1, Q_2, Q_3, Q_4\) is harmonic.

(3) Hence prove that if four tangents to a circle cut another tangent harmonically, then they cut every other tangent harmonically.

41. \(PQ, AB\) are two parallel chords of a circle; \(C\) is the mid-point of \(AB\); \(PC\) meets the circle at \(R\); \(QR\) is the tangent at \(Q\); prove that \(Q \parallel A R B T\) is harmonic. [Use Ex. 38.]

42. \(AB\) is a diameter of a circle; \(AP\) is any chord; the tangents at \(A, P\) meet at \(O\); \(PQ\) is the foot of the perpendicular from \(P\) to \(AB\); prove that \(BO\) bisects \(PR\).

[Prove that \(OP, OB, OA, OA\) cut \(AB\) harmonically.]

43. \(T\) is a point on the base \(BC\) produced of the triangle \(ABC\); \(N\) is the harmonic conjugate of \(T\) with \(BC\); \(TP\) is a tangent from \(T\) to the circumcircle of the triangle \(ABC\); \(PN\) meets \(AB, AC\) at \(L, M\); and the circle at \(Q\); prove that \(\{P, L, Q, M\}\) is harmonic.

44. The tangents at the extremities of a chord \(AD\) of a circle meet at \(P\); \(PCB\) is any line through \(P\) cutting the circle at \(C, B\) and \(AD\) at \(L, M\); prove that \(\{P, B, C, A\} \sim \{P, B, C, A\}\) and \(\{P, B, C, A\}\) is harmonic.

This proves the important theorem that any straight line drawn from a point \(P\) outside the circle is cut harmonically by the circle and the chord of contact of the tangents from \(P\).

Another method is suggested in Ex. 45.

45. Prove Ex. 44 (3).

[Let \(E\) be the mid-point of \(AD\) and let \(CD\) meet the circle at \(E\); produce \(BE\) to meet the circle at \(C\). Join \(C\); notice that the figure is symmetrical about \(EP\) and that \(E\) is harmonic.]

46. If \(A, A'; B, B'; C, C; D, D'\) are harmonic conjugates w.r.t. the same two points \(H, K\); prove that \(\frac{AB}{CD} \sim \frac{A'B'}{C'D'}\).

47. \(ABC\) is a straight line; \(AT\) is the tangent from \(A\) to the circle on \(BC\) as diameter; \(O\) is its centre; \(N\) is the foot of the perpendicular from \(T\) to \(BC\); prove that \(AO, AT, AN\) represent the Arithmetic, Geometric, and Harmonic means between \(AB, AC\); and prove that these three quantities are in geometrical progression.

**CROSS RATIO.**

If a transversal meets four fixed lines, which pass through a point \(O\), at \(A, B, C, D\), it follows from either of the last two methods of proof of Theorem 51 that the value of the ratio \(\frac{AC \cdot BD}{AD \cdot BC}\) is equal to \(\frac{\sin AOC \cdot \sin BOD}{\sin AOD \cdot \sin BOC}\) and is independent of the position of the transversal. For brevity this ratio is represented by \(\{ACBD\}\). In the particular case when \(O\) is harmonic, this ratio is by definition equal to \(-1\). Consequently the fact that \(O\) is harmonic is sometimes denoted by \(\{ACBD\} = -1\).

**Definitions.**

(1) Any set of collinear points are said to form a range.

(2) Any set of concurrent lines are said to form a pencil; the lines are called rays of the pencil, and the point of concurrency is called the vertex of the pencil.

(3) If a transversal meets a pencil \(O\{ACBD\}\) at \(A, B, C, D\), the ratio \(\frac{AC \cdot BD}{AD \cdot BC}\) or \(\frac{\sin AOC \cdot \sin BOD}{\sin AOD \cdot \sin BOC}\) is called the cross ratio of the pencil \(O\{ACBD\}\) or the cross ratio of the range \(\{ACBD\}\).

(4) If two ranges or pencils are of equal cross ratio, they are said to be equal-cross. To remember the ratio \(\{ACBD\}\), place the four points in order on a circle; the numerator is obtained by
starting at A and reading forwards and the denominator by starting at A and reading backwards.

48. Prove that the value of the cross ratio \( \{ACBD\} \) is unaltered if any two letters are interchanged, the other two letters are also interchanged.

\[ \{ACBD\} = \{CABD\} \]

49. If \( \{ACBD\} = l \), prove that (1) \( \{ABCD\} = \frac{1}{l-1} \); (2) \( \{BADC\} = \frac{1}{1-l} \); (3) \( \{ACDB\} = \frac{l}{l-1} \); (4) \( \{ADBC\} = \frac{1}{1-l} \); (5) \( \{DACB\} = \frac{l}{1-l} \).

The relation in Ex. 6 in chapter V. will be of use.

50. Prove that there are 24 cross ratios of four collinear points and that they fall into sets of four, each of which is equal to one of the six ratios in Ex. 49.

51. Prove that the six different cross ratios of four collinear points may be written as \( l, m, n, \frac{1}{l}, \frac{1}{m}, \frac{1}{n} \) where \( l + \frac{1}{m} = m + \frac{1}{n} = n + 1 = -1. \)

52. Prove that \( AB \cdot CB = ABC \) where \( in \) represents a point on \( AC \) infinitely distant from \( A \).

53. A variable line meets four fixed planes which have a common line of intersection at \( PQRS \), prove that \( |PQRS| \) is constant.

54. \( \{ABCD\}, \{ABC\} \) are two equicross ranges on different lines \( AD, AB \), prove that \( BB, CC, DD \) are concurrent.

55. \( O \{ABCD\}, O \{ABCD\} \) are two equicross pencils, if \( O \) lies on \( OA \), prove that \( B, C, D \) are collinear.

56. A pencil \( O \{ABCD\} \) is cut by a transversal at \( A, B, C, D \); through \( B \) is drawn a line \( FB \) parallel to \( OD \) to meet \( OA, OC \) at \( F, G \); prove that \( \{ABCD\} = \frac{FB}{GB} \) [Use Ex. 52.]

---

**Theorem 53.**

If \( \{ACBD\} = \{ADBC\} \), then the range \( \{ACBD\} \) is harmonic.

\[ \frac{AC}{BD} = \frac{AD}{BC} \]

\[ \frac{AC}{CB} = \frac{AD}{DB} \]

Here it is necessary to take the negative sign; for if \( \frac{AC}{CB} + \frac{AD}{DB} \) the points \( C \) and \( D \) coincide.

(The proof is left to the reader; see Ex. 57.)

\[ \frac{AC}{CB} = \frac{AD}{DB} \]

\[ \frac{AC}{CB} = \frac{AD}{DB} \]

\[ \{ACBD\} \) is harmonic.

**Corollary.**

If \( O \{ACBD\} = O \{ADBC\} \), the pencil \( O \{ACBD\} \) is harmonic.

57. If \( ABCD \) are collinear, and if \( \frac{AC}{CB} = \frac{AD}{DB} \), then \( C, D \) coincide.

There are three methods

(a) \( \frac{AC}{CB} = \frac{AD}{DB} \) prove that \( AB = AD \)

(b) \( \frac{AC}{CB} = \frac{AD}{DB} \), and if \( AC < AD \), then \( CB < DB \).

(c) Draw any other line \( AX \) and divide it at \( P \) so that \( \frac{AC}{CB} = \frac{AD}{DB} \) and show that \( PC, PD \) are both parallel to \( BX \).

---

**Theorem 54.**

If \( \{ACBD\} \) and \( \{ACBD\} \) are two harmonic ranges, and if \( AA', CC', BB' \) are concurrent (at \( O \), say), then \( DD \) also passes through \( O \).

Join \( OD \) and produce it to meet \( AB \) at \( E \).

Since \( O \{ACBD\} \) is harmonic, \( \{ACBE\} \) is harmonic;

\[ \frac{AC}{CB} = \frac{AD}{DB} \]

*Theorems similar to Th. 54. Th. 55 hold for equicross ranges and pencil and are of frequent use in solid-work. They are therefore stated below, but their proofs are as similar to those of Th. 54, 55 that they are left as exercises for the reader. Figures should be drawn with the pencils in several different positions, e.g. with \( O \) on different sides of \( AB \), or with \( AD \) and \( A'D \) intersecting each other, etc., so that familiarity may be obtained in recognising the presence of ranges related to each other in this way.*
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But \[ \frac{AC}{CB} = \frac{AD}{BD} \]
\[ \therefore \frac{AE}{BE} = \frac{AD}{BD} \]
\[ \therefore E \text{ coincides with } D; \quad \text{[see Ex. 57]} \]
\[ \therefore OD \text{ passes through } D. \quad \text{Q.E.D.} \]

Corollary.
If \{ACBD\}, \{ACBD'\} are harmonic ranges on two lines \(AD, AD'\), then \(CC', BB', DD'\) are concurrent.

This is a most important theorem.

68. Prove the Corollary of Theorem 54.

THEOREM 55.

If two harmonic pencils \(O\{ACBD\}, O\{ACBD'\}\) are such that \(A, C, B\) are collinear, then \(D\) also lies on \(ACB\).

\[ \text{Produce } ACB \text{ to meet } OD, OD' \text{ at } E, E'. \]

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viii.

Since \(O\{ACBD\}\) is harmonic, \(O\{ABCD\}\) is harmonic;
\[ \begin{align*}
\therefore \quad & \frac{AC}{CB} = \frac{AE}{BE} \\
\therefore \quad & \frac{AE}{BE} = \frac{AD}{BD}
\end{align*} \]

\[ \therefore \quad E \text{ coincides with } D; \quad \text{[see Ex. 57]} \]
\[ \therefore \quad OD \text{ passes through } D. \quad \text{Q.E.D.} \]

Corollary.
If two harmonic pencils \(O\{ACBD\}, O\{ACBD'\}\) are such that \(O\) lies on one of the rays of the first pencil, e.g. \(OA\), then \(C, B, D\) are collinear.

This is a most important theorem.

59. Prove the Corollary of Theorem 55.

THEOREM 56.

\(ABCD\), \(AB'C'D'\) are two equicross ranges on different lines \(AD, AD'\), then \(BB', CC', DD'\) are concurrent.

THEOREM 57.

\(ABCD\), \(A'B'C'D'\) are two equicross ranges so placed that \(AA', BB', CC'\) intersect at a point \(O\), then \(DD'\) passes through \(O\).

THEOREM 58 (1).

\(O\{ABCD\}, O\{ABCD\}\) are two equicross pencils; if \(O\) lies on \(OA\), then \(B, C, D\) are collinear.

THEOREM 58 (2).

\(O\{ABCD\}, O\{ABCD\}\) are two equicross pencils so placed that \(ABC\) are collinear, then \(D\) also lies on \(ABC\).

It is supposed in the above that \(OA\) are corresponding rays of the two pencils, i.e. \(O\{ABCD\} = O\{ABCD\}\).
60. If \( \{APQR\} \), \( \{AP'QR'\} \) are two harmonic ranges, prove that \( PR, PR', OQ \) are concurrent.

61. In the triangle \( ABC \), the internal and external bisectors of \( \triangle ABC \) meet \( BC \) at \( L, M \); also the internal and external bisectors of \( \triangle BCA \) meet \( AB \) at \( P, Q \); prove that \( PL, AC, QM \) are concurrent; and that \( PM \) and \( QL \) intersect on \( AC \).

62. If \( ABCD \) form a harmonic system of points on a circle, prove that the points of intersection of the following pairs of straight lines are collinear: \( AC, BD \); \( AB, CD \); the tangents at \( A, D \); the tangents at \( B, C \).

63. Through a fixed point \( A \) is drawn a variable line meeting two fixed lines \( OC, OD \) at \( P_1, P_2 \); a point \( Q \) is taken on \( AP_1P_2 \) such that \( \frac{AQ}{AP_1} = \frac{1}{4} \frac{AP_1}{AP_2} \); prove that the locus of \( Q \) is a straight line.

64. Through a fixed point \( A \) is drawn a variable line meeting \( N \) fixed lines at \( P_1, P_2, ..., P_n \); on this line is taken a point \( Q \) such that
\[
\frac{AQ}{OP_1} = \frac{AQ}{OP_2} = \cdots = \frac{AQ}{OP_n}
\]
prove that the locus of \( Q \) is a straight line.

65. With the usual notation for the triangle \( ABC \), if \( DE, EF \) meet \( AB, BC \) at \( F, D \); prove that \( FD, FU \) intersect on \( AC \). [Use Ex. 21 (1)].

66. With the usual notation for the triangle \( ABC \), if \( A_0, B_0, C_0 \) meet \( BC, DE, DF \) at \( P, Q, R \); prove that \( PQR \) are collinear.

67. If the internal and external bisectors of \( \triangle ABC \) meet \( BC \) at \( L, M \), and if \( C' \) is the mid-point of \( AB \), apply Theorem 54 Corollary to the harmonic ranges \( \{BLCM\}, \{BC'CA\} \) to invent a property of concurrency of certain lines, where \( \omega \) denotes a point on \( AB \) at an infinite distance from \( A \).

68. With the usual notation for the triangle \( ABC \), \( CD \) meets \( AC \) at \( K \); \( EF \) meets \( BC \) at \( E \); prove that \( KL \) is parallel to \( AB \).

69. With the notation of Ex. 68, if \( CP \) meets \( AC \) at \( Q \); prove that \( DQ \) is parallel to \( AB \).

70. \( T \) is a point on the diameter, \( BA \) produced, of a circle. \( TP \) is the tangent from \( T \) to the circle; \( N \) is the foot of the perpendicular from \( P \) to \( AB \); the line joining \( B \) to the mid-point of \( TP \) cuts \( PN \) at \( Q \); prove that \( AQ \) is parallel to \( TP \).

* The following exercises involve cross ratios.

71. Prove Theorem 56.

72. Prove Theorem 57.

73. Prove Theorem 58 (1).

74. Prove Theorem 58 (2).

75. If \( \{APQR\} = \{AP'QR'\} \) prove that \( PR, PR', OQ \) are concurrent.

76. \( PQR \) is a variable triangle; if each side passes through a fixed point, the three fixed points being collinear, and if \( P, Q \) move on fixed lines, prove that the locus of \( R \) is a fixed line.

[Take three positions of the triangle, and prove that the three positions of \( R \) are collinear, using Th. 55 (1).]

77. \( PQR \) is a variable triangle; \( PR, QR, PQ \) pass through three fixed points \( A, B, H \); \( P, Q \) move on two fixed lines \( CD, CE \); if \( A, B, C \) are collinear, prove that the locus of \( R \) is a straight line. [Use the method of Ex. 76.]

78. \( PQRS \) is a quadrilateral; \( PQ, PS \) are given in position, \( QR, QS \) each pass through one of three fixed collinear points, prove that the locus of \( R \) is a straight line.

79. \( PQR \) is a variable triangle with its vertices on three fixed concurrent lines; \( PQ, QR, PR \) pass through fixed points; prove that \( PR \) also passes through a fixed point. [Take three positions of \( PQR \) and prove that the three positions of \( PR \) are concurrent.]
CHAPTER IX.
THE QUADRILATERAL AND QUADRANGLE.

This chapter is intended to be an introduction to the Principle of Duality, which may be most simply illustrated by the relationship of the quadrilateral to the quadrangle. It is essentially a modern idea, to which attention was first drawn in a definite form by Poncelet (1788–1867). His famous treatise on projective geometry was written while he was held a prisoner by the Russians who captured him during the retreat from Moscow. To some extent, however, he was anticipated by Brianchon, another French soldier, who published in 1806 the theorem with which his name is associated and which was obtained by a method of the same kind. Theorem 60 was in substance included by Pappus in his Syntagmata.

Definitions.
(1) If \( A \) and \( B \) are two points, the unlimited line joining \( A \) and \( B \) is called the join of \( A \) and \( B \).
(2) If \( a \) and \( b \) are two lines, the point of intersection of \( a \) and \( b \) is called the meet of \( a \) and \( b \).

In the first instance it is usual to regard a curve as consisting of a series of points each of which is so situated as to obey some characteristic law. For example, a circle may be considered as formed by a series of points each of which is at a constant distance from a fixed point. In order to pass from the series of points to the continuous curve, it is necessary to imagine that a number of lines are drawn connecting consecutive points of the series; by taking a large enough number of these points, it is possible to make each connecting line as short as we please.

and the limiting result of this process is said to constitute the curve.

But we may equally well consider a circle as generated by a series of straight lines each of which is at a constant distance from a fixed point. On each line is cut off a segment by the two lines which come, one before and the other after it, in the series. By taking a large enough number of these lines, it is possible to make each segment as short as we please; and in the limit, these segments are said to constitute the curve.

In the first case, the curve is defined as the locus of a series of points; and in the second case as the envelope of a series of lines. Thus the part played by a moving (tracing) point in the first case corresponds to that played by a moving (tangent) line in the second case.

In the first case the point is regarded as the fundamental conception, and the straight line as the join of two points; while in the second, the straight line is regarded as the fundamental conception, and the point is regarded as the meet of two straight lines.

If then we have any geometrical system built up on one of these conceptions, it is possible to build up the corresponding geometrical system which is based on the other.

In what follows, it is convenient to take single capital letters to represent points and single small letters to represent lines.

Correspondence will be represented in parallel columns.

\[ AB \] is the join of two points \( A, B \); \( [ABCD] \) denotes the range formed by four collinear points \( A, B, C, D \).
\[ PA \] (or \( \{ABCD\} \)) denotes the pencil, whose vertex is \( P \) and whose rays pass through the four points \( A, B, C, D \).

\[ ab \] is the meet of the two lines \( a, b \); \( \{abcd\} \) denotes the pencil formed by four concurrent lines \( a, b, c, d \).

\[ \cdot \] \( \{abcd\} \) denotes the range intercepted on the line \( a \) (the base of the range) by the four lines \( a, b, c, d \).

Hitherto a triangle has been regarded indiscriminately as a figure formed by the joins of three points or the meets of three lines; it is convenient to distinguish by different names these two standpoints.

A figure formed by the joins of three points is called a triangle. A figure formed by the meets of three lines is called a trilateral.

This correspondence is exhibited very plainly in theorems which can be stated without reference to the measurement of lines or
angles. It will be found that all such theorems occur in pairs, so that the knowledge of one includes the knowledge of the other. This correspondence is usually spoken of as The Principle of Duality.

The following theorems will illustrate the application of this principle:

the joins of \( n \) points, no three of which are collinear, taken in pairs, form \( \frac{1}{2}n(n-1) \) lines.

\( A, B, C; A', B', C' \) are the vertices of two triangles; if the joins of \( A, A' \); \( B, B' \); \( C, C' \) are concurrent, then the meets of \( BC, B'C; CA, C'A; AB, A'B' \) are collinear.

The sides \( BC, CA, AB \) of a variable triangle \( ABC \) pass through three fixed collinear points \( P, Q, R \) respectively; if \( A, B \) lie on two fixed lines, then the locus of \( C \) is a straight line.

\([A, C \text{ lies on a fixed straight line}].\)

If \( \{ABCD\}, \{A'B'C'D'\} \) are two harmonic ranges on different bases \( x, y \); then the joins of \( BB'; CC'; DD' \) are concurrent.

A curve which is met by any straight line at two and only two points is called a curve of the second degree.

The vertices \( b, c, a, ab \) of a variable trilateral \( abc \) lie on three fixed concurrent lines \( p, q, r \) respectively; if \( a, c \) pass through two fixed points, then the envelope of \( a \) is a point.

\([A, c \text{ passes through a fixed point}].\)

If \( \{abcd\}, \{ab'c'd'\} \) are two harmonic pencils with different vertices \( X, Y \); then the meets of \( b'b'; c'c'; d'd' \) are collinear.

A curve to which from any point two and only two tangents can be drawn is called a curve of the second degree.

Write down without proof the theorems which correspond, in virtue of the principle of duality, to the following: and draw figures for the original and the dual theorems exhibiting this correspondence. [Compare Figs. 61, 62.]

1. Chapter VIII, Ex. 79.

2. \( A_1, A_2, \ldots, A_n \) are \( n \) points in a plane which move on fixed concurrent lines; if \( A_1A_2, A_2A_3, \ldots, A_{n-1}A_n \) pass through fixed points, then every other join of a pair of the \( n \) points passes through a fixed point.

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3. From a fixed point \( D \) two variable lines \( DP, DR \) are drawn to cut two fixed lines \( AB, AC \) at \( P, Q \) and \( R, S \); then the locus of the meet of \( PS, QR \) is a straight line.

4. \( \{ABCD\}, \{A'B'C'D'\} \) are two harmonic ranges on different bases \( x, y \); if the joins of \( A, A' \); \( B, B' \); \( C, C' \) concur at \( Q \), then \( BB' \) passes through \( O \).

5. If \( \{ABCD\}, \{A'B'C'D'\} \) are two harmonic ranges on different bases \( x, y \), then the join of \( BB', CC' \) is concurrent.

6. Two lines \( FDA, FCB \) are met by two other lines \( GBA, GCD \); \( BD \) meets \( AC \) at \( E \); then \( F(AB; BG) \) is a harmonic pencil.

7. One and only one curve of the second degree can be drawn to pass through five given points.

8. \( A_1A_2A_3A_4A_5 \) are six points on a curve of the second class. Then the points of \( A_1A_2, A_1A_3; A_2A_3, A_2A_4; A_3A_4, A_3A_5 \) are collinear.

9. A variable curve of the second degree passes through four fixed points \( A, B, C, D \); if two fixed lines \( AX, BY \) cut the curve at \( P, Q \); then \( PQ \) passes through a fixed point.

10. The sides of a variable triangle pass through three non-collinear fixed points; if two of the vertices lie on fixed lines, then the locus of the third vertex is a curve of the second degree.

We shall now proceed to work out in detail the dual properties of the figure generated by four straight lines (the quadrilateral) and the figure generated by four points (the quadrangle).
A complete quadrangle is formed by four points \( A, B, C, D \), called its vertices, no three of which are collinear; their joins form six straight lines, grouped in pairs, \( AB, CD; BC, AD; CA, BD \), called its sides: the sides in each pair being called opposite sides.

The meets of opposite sides form three points \( G, F, E \) called diagonal points, and are the vertices of a triangle called the diagonal point triangle. If \( A, B, C, D \) lie on a circle, the circle is called the circumscribing circle of the quadrangle.

\* Now we have
\[
\{AECP\} = F\{AEP\} = G\{DEBQ\} = (CEAP).
\]

\( \therefore \) by Theorem 53
\( \{AECP\} \) is a harmonic range.

A complete quadrilateral is formed by four lines \( a, b, c, d \), called its sides, no three of which are concurrent; their meets form six points, grouped in pairs, \( ab, cd; bc, ad; ca, bd \), called its vertices: the vertices in each pair being called opposite vertices.

The joins of opposite vertices form three lines \( eg, f, c \) called diagonal lines, and are the sides of a triangle called the diagonal line triangle. If \( a, b, c, d \) envelope a circle, the circle is called the inscribed circle of the quadrilateral.

\* Now we have
\[
\{aep\} = f\{aep\} = g\{deby\} = \{aep\}.
\]

\( \therefore \) by Theorem 53, Corollary,
\( \{aep\} \) is a harmonic pencil.
The harmonic property of the quadrangle and quadrilateral may be established in a simpler manner.

**THEOREM 59.**

$ABCD$ is a quadrilateral: $AD$, $BC$ meet at $F$; $AB$, $CD$ meet at $G$; $AC$, $BD$ meet at $E$; $FE$ meets $AB$ at $K$; then $[AKBG]$ is harmonic.

![Diagram](image)

From the triangle $FAB$, since $FK$, $AC$, $BD$ concur,

\[
AK \cdot BC \cdot PD = KB \cdot CF \cdot DA = 1, \quad \text{by Ceva.}
\]

Since $DCG$ is a transversal of the same triangle,

\[
AG \cdot BC \cdot FD = GB \cdot CF \cdot DA = 1, \quad \text{by Menelaus;}
\]

\[
AK \cdot BC \cdot FD = AG \cdot BC \cdot FD,
\]

\[
KB \cdot CF \cdot DA = GB \cdot CF \cdot DA
\]

\[
\therefore AK = AG,
\]

\[
KB = GB,
\]

\[
\therefore [AKBG] \text{ is harmonic.}
\]

**THEOREM 60.**

If in the figure of the complete quadrangle (Fig. 64), $FE$ is produced to meet $DC$, $AE$ at $H$, $K$ and $GE$ is produced to meet $AD$, $CB$ at $L$, $M$, then every set of four collinear points forms a harmonic range and every set of four concurrent lines forms a harmonic pencil.

But it intercepts on the transversals $FA$, $FK$, $FB$ the ranges $[ALDF]$, $[KEHF]$, $[BMCF]$ which are therefore harmonic.

Also $E[BMCF]$ intercepts on $QF$ the range $[QGPF]$ which is therefore harmonic.

Hence each of the nine lines in the figure is cut harmonically.

Therefore each of the sets (three in number through $E$, $F$, $G$) of four concurrent lines form harmonic pencils, for each has a transversal which is cut harmonically.

Q.E.D.

11. State the dual theorem of Theorem 60, draw the corresponding figure and work through the proof.

12. If in Fig. 63 $AC$ meets $FG$ at $P$, use the method of Theorem 59 to prove that $[AECB]$ is harmonic.

13. Given three collinear points $A$, $B$, $C$, by using a ruler only, show how to find a point $D$ such that $[ABCD]$ is harmonic.

14. Given three concurrent lines $PA$, $PB$, $PC$, by using a ruler only, show how to construct a line $PD$ such that $P[ABCD]$ is harmonic.

15. $K$ is the given mid-point of the line joining two given points $A$, $B$; $O$ is a given point outside $AB$; draw through $O$ a line parallel to $AB$, using a ruler only.
The mid-points of the three diagonals of a complete quadrilateral are collinear.

\[ \text{Fig. 65.} \]

\[ \text{This method of proof was given by Mr. C. E. Hillyer in the Educational Times.} \]
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But the system is also equivalent to
\[ \frac{AF}{FD} = \frac{AG}{GB} = \frac{CG}{GD} = \frac{CF}{FB}, \]
or to
\[ \frac{AP}{PD} = \frac{AG}{GD} = \frac{FB}{GB} = \frac{CG}{CG} = \frac{CF}{CF}. \]
or to
\[ \frac{zAN}{zND} = \frac{zNB}{zCN}; \]
\[ \therefore \text{the resultant passes through } N; \]
\[ \therefore L, M, N \text{ are collinear.} \quad \text{Q.E.D.} \]

This method fails if the forces are in equilibrium, in which case \( LM \) is zero so that the diagonals bisect each other; the quadrilateral is then a parallelogram and \( FG \) is at infinity.

If a circle can be inscribed in the quadrilateral its centre must lie on \( AMN \).

For suppose the circle touches \( AB, BC, CD, DA \) at \( P, Q, R, S \); the force-system is then equivalent to \( AP, AS; FB, QB; CR, CK; KD, SD \); the resultant of each pair bisects the corresponding chord of contact and therefore passes through the centre of the circle; therefore the final resultant passes through the centre of the circle.

The reader who is acquainted with the corresponding property of the conic will see that this method yields an immediate proof of the theorem that "The locus of the centres of conics touching four straight lines is a straight line."

Statistical methods should be used in Ex. 23-28.

23. Through a point \( F \) inside a rectangle \( ABCD \) lines are drawn parallel to the sides to meet \( AP \) in \( G \), \( BC \) in \( E \), \( CD \) in \( K \), \( DA \) in \( H \).
Prove that \( RH, CK, DG \) are concurrent.

24. \( ABCD \) is a quadrilateral. If \( AR = CD \), prove that the line joining the mid-points of \( BC \) and \( AD \) is equally inclined to the other two sides.

25. On the sides of a quadrilateral four points are taken such that each forms with the extremities of the opposite side a triangle whose area is half that of the quadrilateral. Prove that these points lie on the line joining the mid-points of the diagonals.

26. A parallelogram \( ABCD \) is divided into four parallelograms by two lines \( PK, KQ \), meeting \( BC, AB \) at \( P, Q \) and \( DA, CR \) at \( R, S \).
Prove that the triangle \( AKC \) is equal to half the difference of the parallelograms \( KB, DK \).

27. \( ABCD \) is a quadrilateral in which \( AD \) and \( BC \) are parallel; find the magnitude and direction of the resultant of the forces \( AB, BC, CD, DA, BD, CA \).

28. Prove that the three pairs of biseectors of the opposite angles of a complete quadrilateral intersect at three collinear points.

29. \( O \) is any point inside the triangle \( ABC \); \( AO, BO, CO \) meet the opposite sides at \( D, E, F \); \( P, Q, R \) are harmonic conjugates of \( D, E, F \) w.r.t. \( B, C \); \( C, A \); \( A, B \); prove that the mid-points of \( AP, BQ, CR \) are collinear.

30. If two of the opposite angles of a complete quadrilateral are right angles, prove that the third diagonal is perpendicular to one of the internal diagonals.

31. \( ABCD \) is a quadrilateral, \( UV \) is the projection of \( BD \) on \( AC \); prove that \( AB^2 + CD^2 - BC^2 - AD^2 = AC \cdot UV \).

32. With the notation of Fig. 86, if \( ABCD \) is inscribed in a circle, prove that \( FG^2 = FR^2 + GS^2 \) where \( FR, GS \) are the tangents from \( F, G \) to the circle.

33. \( ABCD \) is a quadrilateral having \( AB \) parallel to \( CD \); \( AC, BD \) meet at \( P \); \( AD, BC \) meet at \( Q \); prove that \( PQ \) bisects \( AB \) and \( CD \).

34. The base \( BC \) of a triangle is given, and the vertex \( A \) moves on a given straight line. A given line cuts \( AB, AC \) at \( H, K \); find the locus of the meet of \( CH, BK \).

35. If in Fig. 63,
(i) \( FC, FB \) join \( G \), \( GD = FG^2 \) or (ii) \( FA, FD \) join \( A \), \( GB = FG^2 \), prove that \( ABCD \) is a cyclic quadrilateral.
CHAPTER X.

ORTHOGONAL CIRCLES.

Definitions.

(i) If two curves intersect at a point A, and the tangents at A to the curves make an angle $\alpha$ with each other, the two curves are said to cut each other at A at an angle $\alpha$.

In particular, if $\alpha$ is a right angle, the curves are said to cut each other orthogonally or at right angles; and the curves are called orthogonal curves.

(ii) If $CA$ is a radius of a circle, centre C, and if $P$, $Q$ are two points on $CA$ such that $CP \cdot CQ = CP_{1}P_{1}$, then $P$, $Q$ are called inverse points with respect to the circle.

1. If two circles intersect at A and B, prove that the angles at which they cut each other at A and B are equal.

2. If $P$, $Q$ are inverse points w.r.t. a circle, and if $PQ$ meets the circle at $A$, $B$, prove that $\{AB, PQ\}$ is harmonic.

3. Two circles, centres $A$, $B$, intersect orthogonally at C; prove that $CA$, $CB$ are the tangents at C to the two circles.

4. Two circles, centres $A$, $B$, radii $a$, $b$, cut each other orthogonally; prove that $AB = \sqrt{a^{2} + b^{2}}$.

5. If $a$, $b$ are the radii of two circles, centres $A$, $B$, and if $AB = \sqrt{a^{2} + b^{2}}$, prove that the circles cut orthogonally.

6. If $P$ is a point on a circle, centre $C$; $AB$ is a diameter; prove that the circles $APC$, $BPC$ intersect orthogonally.

7. If $ PQ $ is a common point of two circles; $ A $ is a centre of one; if $ AC $ touches the other, prove that the circles are orthogonal.

8. If $ PQ $ is a common tangent to two circles, $ P $, $ Q $ being the points of contact; prove that the circle on $ PQ $ as diameter is orthogonal to each of them.

9. $\{ACBD\}$ is a harmonic range; prove that the circles on $AB$, $CD$ as diameters are orthogonal.

10. Two circles, centres $A$, $B$, intersect at $C$; prove that the angle at which they cut each other is supplementary to the angle $ACB$.

11. A circle $\Sigma$ cuts a circle $C_{1}$ at $P$, $Q$ and a circle $C_{2}$ at $R$, $S$; $PQ$ meets $RS$ at $H$; prove that it is possible to draw a circle with $H$ as centre to cut $C_{1}$, $C_{2}$ orthogonally, $C_{1}$, $C_{2}$ being non-intersecting circles.

12. If a straight line cuts a circle orthogonally, prove that it passes through the centre of the circle.

13. $\{ACBD\}$ is a harmonic range; prove that any circle through $C$, $D$ cuts the circle on $AB$ as diameter orthogonally.

14. Two circles, centres $A$, $B$, intersect at $C$, $D$; if $ACBD$ is a cyclic quadrilateral, prove that the circles are orthogonal.

15. If $P$, $Q$ are two points on the major arcs $CD$ of two circles which intersect at $C$, $D$; if $\alpha$ is the angle $CQD = 90^\circ$, prove that the circles cut orthogonally. [Use Ex. 14.]

What is the test if $P$, $Q$ lie on the minor arcs $CD$?

16. Prove the converse of Ex. 13.

17. From a point $P$ outside a circle, centre $C$, tangents are drawn to the circle; if the chord of contact meets $CP$ at $Q$, prove that $P$, $Q$ are inverse points w.r.t. the circle.

18. If $a$, $b$, $c$ are the given lengths of the sides of the triangle $ABC$; circles, centres $A$, $B$, $C$, radii $x$, $y$, $z$, are described so that each is orthogonal to the other two. Calculate $x$, $y$, $z$.

19. If $P$, $Q$ are inverse points w.r.t. a circle $\Sigma$, prove that any circle through $P$, $Q$ is orthogonal to $\Sigma$.

20. If two circles are orthogonal, prove that any diameter of one is cut harmonically by the other circle.

THEOREM 62.

If two circles, centres $A$ and $B$, intersect orthogonally at $C$, then $\{\}$ $CA$, $CB$ are tangents to the circles.

(i) $CA^{2} + CB^{2} = AB^{2}$.

(ii) Draw the tangents $CP$, $CQ$ to the two circles.

Then $QC = CP$ given; $\therefore CQ$ is perpendicular to the tangent $CP$ to the circle centre $A$; $\therefore CQ$ passes through $A$ and similarly $CP$ passes through $B$; $\therefore AQ$, $BP$ are tangents to the two circles. Q.E.D.
Corollary.

Conversely, if two circles, centres $A$, $B$, intersect at $C$, and if either $A\hat{C}B = 90^\circ$ or $AB^2 = AC^2 + CB^2$ or if $AC$ is a tangent to the circle, centre $B$, then the two circles intersect orthogonally.

This Corollary is most important.

21. Prove the Corollary of Theorem 62.

22. If two circles are orthogonal, prove that the rectangle contained by their common chord and the line joining their centres is equal to twice the rectangle contained by their radii.

23. If two orthogonal circles are equal, calculate the ratio of the area common to both of them to the area of either.

24. The straight line joining the centres $O$, $O'$ of two orthogonal circles, meets the first in $A$, $B$ and the second in $C$, $D$; prove that

$$DA \cdot DB = DC \cdot DD.$$

25. Prove that, with the usual notation for the triangle $ABC$, the three circles whose centres are $A$, $B$, $C$ and the squares of whose radii are $AH$, $AD$, $BH$, $BE$, $CH$, $CF$ cut orthogonally.

26. A line $QPR$ is drawn through the point of intersection $P$ of two circles, and meets them again at $Q$, $R$; with $Q$, $R$ as centres two circles are described orthogonal to the given circles, prove that they intersect on the circle on $QR$ as diameter.

27. Prove that the locus of the centres of circles which pass through a given point and cut a given circle orthogonally is a straight line.

28. Prove that the circle, whose diameter is the third diagonal of a quadrilateral inscribed in a circle, is orthogonal to that circle.

[Use Ex. 32 in Chapter IX.]

THEOREM 63.

Two circles intersect at $C$, $D$. If the sum of the angles in the segments of the two circles on opposite sides of $CD$ is $90^\circ$ or $270^\circ$, then the circles cut orthogonally.

$A$, $B$ are the centres of the circles; $P$, $Q$ are points on segments of the circles on opposite sides of $CD$.

In the case of the figure $C\hat{P}D + C\hat{Q}D = 90^\circ$.

Now

$$C\hat{A}D = 2C\hat{P}D \text{ and } C\hat{B}D = 2C\hat{Q}D;$$

$$C\hat{A}D + C\hat{B}D = 2(C\hat{P}D + C\hat{Q}D) = 180^\circ;$$

$$A\hat{C}B + A\hat{D}B = 180^\circ,$$

since the angles of a quadrilateral add up to four right angles.

But

$$A\hat{C}B = A\hat{D}B,$$

$$A\hat{C}B = 90^\circ,$$

then the circles cut orthogonally.

Q.E.D.

If $P$, $Q$ had been taken on the minor arcs $CD$, we should have the case $C\hat{P}D + C\hat{Q}D = 270^\circ$.

29. Give the proof of Theorem 63 for the case $C\hat{P}D + C\hat{Q}D = 270^\circ$.

30. Use Theorem 63 to prove Ex. 6.

31. $H$ is the orthocentre of the triangle $ABC$, prove that the circles on $AH$ and $BC$ as diameters are orthogonal.

32. $AB$ is the diameter of a given circle; $C$ is a point off the circle; $AC$, $BC$ meet the circle again at $P$, $Q$; prove that the circle $CPQ$ is orthogonal to the given circle.
33. If $TP$, $TQ$ are the tangents from a point $T$ to a circle; $R$ is any point such that $TR=TP$, $RP$, $RQ$ meet the circle again at $A$, $B$, prove that $AB$ is a diameter.

34. A point $P$ is taken on the base $BC$ produced of a triangle $ABC$ so that $APB=ABC$; $AD$ is the internal bisector of the angle $BAC$ and meets $BC$ at $D$, prove that the circle, centre $P$, radius $PD$, is orthogonal to the circumcircle of the triangle $ABC$.

35. With the usual notation, prove that the circle $IBC$ is orthogonal to the circle on $I_3 D_3$ as diameter.

36. The diagonals of a quadrilateral $ABCD$ meet at $O$, if the circumcircles of the triangles $AOB$, $COD$ are orthogonal to each of the circles $AOE$, $BOC$, prove that the quadrilateral must be a rectangle.

**THEOREM 64.**

1. If $\{ACBD\}$ is harmonic, then the circle on $AB$ as diameter is orthogonal to any circle through $C$ and $D$.

2. If $C$, $D$ are inverse points w.r.t. a circle, any circle through $C$, $D$ is orthogonal to that circle.

![Diagram](image)

Corollary 1.

If two circles cut orthogonally, then any diameter of one of them is cut harmonically by the other. Let $\Sigma$ and $S$ be orthogonal circles (see Fig. 69). $\Sigma$ meets $S$ at $C$, $D$. $O$ is the centre of $\Sigma$; $S$ cuts $S$ at $H$. Since the circles are orthogonal, $OH$ is a tangent;

$$OC, OD = OH^2 = OA^2 \text{ radii;}$$

$\therefore \{ACBD\}$ is harmonic. Q.E.D.

Corollary 2.

If a circle $S$ is orthogonal to a circle $\Sigma$ and cuts a diameter of $\Sigma$ at $C$, $D$; then $C$, $D$ are inverse points w.r.t. $\Sigma$. For from Corollary 1, $OC, OD = OB^2$;

$\therefore \{ACBD\}$ is harmonic. Q.E.D.

87. A fixed diameter $AB$ of a circle $S$ meets a given orthogonal circle $\Sigma$ at $C$, $D$; $P$ is a variable point on $S$, prove that $\frac{PC}{PD}$ is constant.

38. Prove that the circumcircle of the triangle formed by the three diagonals of a complete quadrilateral is orthogonal to the three circles whose diameters are the three diagonals.

39. Two circles $S_1$, $S_2$ intersect at $P$, $Q$; another circle $\Sigma$ cuts $S_1$ and $S_2$ orthogonally; prove that $P$, $Q$ are inverse points w.r.t. $\Sigma$.

40. Show how to describe a circle to pass through two given points and to be orthogonal to a given circle.

41. Two circles, centres $A$, $B$, cut orthogonally at $Q$. A line $PQ$ cuts the circles again at $P$ and $R$; if $L$ is the mid-point of $PR$, prove that $\angle ALB = 90^\circ$.

[Use $\angle LQ, LR = \angle LQ, LP$.]

42. Two circles intersect orthogonally at $P$; the line joining their centres meets the first circle at $A$, $B$; prove that $PA$, $PB$ cut the second circle at the ends of a diameter perpendicular to $AB$.

43. If $A$, $B$, $C$, $D$ are any four points, prove that the circles $ABC$, $ABD$ cut at the same angle as the circles $ACD$, $BCD$. 
CHAPTER XI.

INVERSION.

The method of Inversion, described in this chapter, is employed to
deduce a new theorem or set of theorems from a known property.
It appears to be due in the first instance to Quetelet: but it was
discovered independently a few years later (1843) by two Dublin
University lecturers, Stubbs and Ingram. Since then it has been
elaborated by Cayley, Charles and Roberts. [See Salmon's Higher
Plane Curves.] And Lord Kelvin has applied it to the solution of
certain electrostatical problems.

Many of the properties of a coaxal system of circles can be
obtained by using the process of inversion. A discussion of this
application is given in Chapter XIII. Since the inverse curve of a
conic is in general a curve of degree 3 or 4, inversion may be used
to discover properties of higher plane curves from known properties
of the conic. The number of useful properties however, which can
be thus obtained, is not very considerable; since it will often be
found that the new theorem is of so complicated a character, in
enunciation, as to be hardly worth possessing.

Definition.

If on the line joining a given point $O$ to a point $P$, a point $P'$ is
taken such that \( OP \cdot OP' = k^2 \) where $k$ is a given constant, then $P'$
is called the inverse point of $P$ with respect to the centre or origin
of inversion $O$; and $k$ is called the radius of inversion.

The circle, centre $O$, radius $k$, is called the circle of inversion.

It is clear that this definition agrees with the definition of inverse
points given in Chapter X.

1. If $P$ and $Q$ are inverse points of $P$ with respect to an origin $O$, prove
that $PQOP'$ are concyclic.
2. Draw roughly the systems of points given in Figs. 70, 71, 72, and then construct the approximate positions of the inverse points w.r.t. $O$ for any convenient value of $\kappa$.

![Diagram 1](image1)

Suppose now that a point $P$ moves along any continuous curve, then it is clear that the point $P'$, connected with $P$ as in the definition, will trace out another continuous curve which is called the inverse curve of the locus of $P$ w.r.t. an origin $O$ and a radius of inversion $\kappa$. Unless special mention is made of $\kappa$, it is implied that any constant quantity may be chosen for $\kappa$. (See Ex. 8.)

3. Draw roughly the curves on the opposite page and then construct the approximate positions of the inverse curves w.r.t. $O$ for any convenient value of $\kappa$. In Figs. 73, 75, each arm of the curve is supposed to extend to infinity.

4. If a figure is inverted, which points of the figure will remain unaltered in position?

![Diagram 2](image2)

5. A straight line is inverted w.r.t. a point on itself, what is the inverse figure?

6. A given circle is inverted w.r.t. its centre, prove that the inverse figure is a concentric circle: what does the inverse figure become as the given circle shrinks into a point?

![Diagram 3](image3)

7. The curve $S'$ is the inverse of the curve $S$ w.r.t. origin $O$. $OA$ is a tangent to $S$, prove that it touches $S'$.

8. If $S'$ and $S''$ are inverse curves of the curve $S$ w.r.t. the same origin $O$ but different radii $\kappa$, $\kappa''$ of inversion, prove that $S'$ and $S''$ are homothetic curves and that $O$ is their centre of similitude.

It is due to this property that the particular value of $\kappa$ to be chosen is "invariably" immaterial.
9. OT is the tangent from a point O to a circle S; prove that with origin O and radius of inversion OT, S will coincide with its inverse.

10. If two curves touch each other, prove that their inverse curves w.r.t. any point O also touch each other.

11. If P is the inverse of O, and if PP' meets the circle of inversion at C, D; prove that \( \angle PCD \) is harmonic.

12. Regarding a straight line as the limiting form of a circle when the centre is at infinity, if \( P, P' \) are inverse points w.r.t. the straight line, prove that it bisects PP' at right angles. [Use Ex. 11.]

13. \( P', Q', O' \) are the inverse points of \( P, Q, O \) w.r.t. a point \( O \); prove that \( P'Q'R' = PQR \), if \( O \) lies on \( PR \). [Use Ex. 1.]

14. If two curves cut at \( P \), and if their inverse curves cut at \( P' \), prove that the angle of intersection at \( P \) is equal to the angle of intersection at \( P' \). [Regard a tangent as the limiting position of a line joining two adjacent points on the curve.]

16. Points \( P, Q \) are taken on the diameter \( AB \) of a circle so that \( \angle APB = \angle BQC \) is harmonic; prove that \( P, Q \) are inverse points w.r.t. the circle.

17. \( J \) is the incentre of the triangle \( ABC \); \( I, C' \) are the inverse points of \( B, \) centre \( I \), radius \( IA \); prove that \( J \) is the orthocentre of the triangle \( ABC \).

18. \( OA \) is a diameter of a circle; \( P \) is a variable point on \( AB \); \( A', P' \) are the inverse points of \( A \) and \( P \) w.r.t. \( O \); prove that \( \angle OA'P' = 90^\circ \); (2) the locus of \( P' \) is a straight line. This proves that the inverse of a circle w.r.t. a point on it is a straight line.

19. From a point \( O \) a perpendicular \( OB \) is drawn to a given straight line \( BC \); \( P \) is a variable point on \( BC \); \( B', P' \) are the inverse points of \( B \) and \( P \) w.r.t. \( O \); prove that \( \angle OBP' = 90^\circ \); (2) the locus of \( P' \) is a circle through \( O \). What therefore is the inverse of a straight line w.r.t. a point outside the line?

Before giving geometrical proofs for the inverse curves of given curves, we shall apply analytical methods for this purpose. By this means it is also possible to include some mention of less familiar curves.

To prove that the inverse of a straight line which does not pass through the origin is a circle through the origin.

Let \( O \) be the origin.

Draw \( OA \) perpendicular to the given line.

Let \( P \) be any point on the given line.

Take \( OX \) along \( OA \). Let \( OX = a \).

Produce \( OP \) to \( P' \) so that \( OX \cdot OP' = b^2 \).

Let \( (r_1, \theta_1) \) be coordinates of \( P \).

Then \( r_1 \cos \theta_1 = a. \)

If the coordinates of \( P' \) are \( (r_1', \theta_1') \),

\[ r_1 = r_1; \quad r_1' = \frac{b^2}{a} \cos \theta_1; \quad r_1' = \frac{b^2}{a} \cos \theta_1; \]

the equation of the locus of \( P' \) is \( r = \frac{b^2}{a} \cos \theta \), or \( x^2 + y^2 = \frac{b^2}{a} \).

In Cartesian's, its equation is \( x^2 + y^2 = \frac{b^2}{a} \).

which is a circle through \( O \) with its centre on \( OA \). Q.E.D.

19. Prove that the inverse of a circle through the origin is a straight line.

Take the equation of the circle to be \( r = a \cos \theta \).

20. Prove that the inverse of a circle which does not pass through the origin is another circle.

[The equation of any circle can be written \( x^2 + y^2 + 2gx + 2fy + c = 0 \), which in polar becomes \( r^2 + 2r(g \cos \theta + f \sin \theta) + c = 0 \).]

21. Prove that the inverse of a parabola w.r.t. its focus is a cardioid.

Draw a rough figure of the cardioid.

[The equation of a parabola referred to its focus is \( \frac{r}{r_0} = 1 + \cos \theta \), and the equation of a cardioid is of the form \( r = a(1 + \cos \theta) \). Fig. 75 represents a parabola with \( O \) as focus.]
22. Prove that the inverse of any conic w.r.t. its focus is a Limaçon.
Draw a rough figure of a Limaçon.

[The equation of any conic referred to its focus is \( r^2 = a + 2bc \cos \theta \), the equation of a Limaçon is of the form \( r = a + b \cos \theta \). Fig. 74 represents an ellipse with \( O \) as focus.]

23. Find the equation of the inverse of a parabola w.r.t. its vertex and indicate roughly the shape of the curve.

[The Cartesian equation of a parabola with the vertex as origin is \( y^2 = 4ax \). The inverse curve is called the Cissoid of Diocles.

24. Prove that the inverse of a rectangular hyperbola w.r.t. its centre is a Lemniscate whose equation is of the form \( r^2 = a^2 \cos 2\phi \). It is represented in Fig. 76. The Cartesian equation of a rectangular hyperbola with the centre as origin is \( x^2 - y^2 = a^2 \).

25. Prove that the inverse of the curve whose equation is \( f(x, y) = 0 \) has an equation of the form \( f'(x', y') = 0 \).

26. Prove that the inverse of the curve whose equation is \( f(x, y) = 0 \) has an equation of the form \( f'(x', y') = 0 \).

27. Prove that the degree of the inverse of a curve of degree \( n \) cannot be greater than \( 2n \).

THEOREM 85.

1. If a straight line is inverted w.r.t. a point \( O \) on it, the inverse is the same straight line.

2. If however \( O \) is not on the line, the inverse is a circle through \( O \).

3. If a circle is inverted w.r.t. a point \( O \) on the circle, the inverse is a straight line.

---

(1) If \( Q \) is any point on the given line \( OA \), the inverse of \( Q \) also lies on \( OA \) by definition; so that every point on \( OA \) inverts into a point on \( OA \).

(2) \( \hat{P} \) drawn \( OA \) perpendicular to the given line.

Take any point \( P \) on the given line.

Let \( A \), \( P' \) be the inverses of \( A \), \( P \) w.r.t. \( O \).

Since \( OA \cdot OP = OP' \cdot OA' \), \( \triangle OP'A \) is a cyclic quadrilateral;

\[ \angle OP'A = OAP = 90^\circ \]

\( P' \) lies on the circle on \( OA' \) as diameter;

the inverse of the straight line is a circle through \( O \), such that its centre lies on the perpendicular from \( O \) to the line.

Q.E.D.

(3) Let \( OA \) be the diameter through \( O \) of the given circle.

Take any point \( P \) on its circumference.

Let \( A \), \( P' \) be the inverses of \( A \), \( P \) w.r.t. \( O \).

Then since \( A \), \( A \), \( P \), \( P' \) are concyclic, \( \angle OP'A = OAP = 90^\circ \), since \( OA' \) is a diameter;

\( P \) lies on a line through \( A \) perpendicular to \( OA \);

the locus of \( P \) is a straight line;

the inverse of a circle through \( O \) w.r.t. \( O \) is a straight line.

Q.E.D.

28. Prove that a system of parallel lines invert into a system of circles touching each other at a common point.

29. A parallelogram is inverted w.r.t. any point, what is the inverse figure?

30. A system of circles have two common points \( A \), \( B \); when inverted w.r.t. \( A \), what is the inverse figure?

31. What is the inverse of the theorem: a straight line can be drawn to pass through any two points?

32. Invert w.r.t. \( O \) the theorem: from any point outside a circle two lines can be drawn to touch a given circle \( O \).

[Use Ex. 19.]

33. Invert w.r.t. \( O \) the theorem: \( ABC \) is a triangle, \( DE \) is drawn parallel to the base \( BC \) and meets \( AB \), \( AC \) at \( D \), \( E \), then \( AD = AE \).

34. Invert w.r.t. \( O \) the theorem: \( O \) is a fixed point; \( P \) is a variable point on a fixed line; \( OP \) is produced to \( Q \) so that \( \triangle OPQ \) is constant, then the locus of \( Q \) is a straight line parallel to the fixed line.
THEOREM 66.

If a circle is inverted w.r.t. a point \( O \) which does not lie on the circle, then the inverse is another circle.

![Diagram of circle inversion](image)

Let \( P \) be any point on the given circle \( \Sigma \), and let \( OP \) meet \( \Sigma \) again at \( Q \).

Let \( P' \) be the inverse of \( P \).

Then \( OP' = h^2 \), where \( h \) = radius of inversion,
also \( OP \cdot OQ = t^2 \), where \( t \) = tangent from \( O \) to \( \Sigma \);
\[
\frac{OP'}{OP} = \frac{r^2}{h^2} = \text{constant}.
\]

\( P' \) traces out \( \Sigma \) again.

Q.E.D.

[Th. 3.]

It is important to notice that the centre \( D \) of \( \Sigma \) is not the inverse of the centre \( C \) of \( \Sigma \). (See Ex. 56.)

Notice also that \( P' \) describes \( \Sigma \) in a sense opposite to the sense in which \( P \) describes \( \Sigma \), if \( C \) lies outside \( \Sigma \).

What difference is there in the proof when \( O \) lies inside \( \Sigma \)?

35. Prove Theorem 66 when \( O \) lies inside \( \Sigma \).

36. With the notation of Theorem 66 prove that \( OC \cdot OP = OQ^2 \).

And hence show that \( C, D \) cannot be inverse points.

37. Prove that a pair of common tangents to a circle and its inverse meet at the centre of inversion.

38. If \( O \) lies outside the given circle, radius \( a \), prove with the notation of Theorem 66 that the radius of the inverse circle is \( \frac{a^2}{b^2} \).

39. Two circles intersect at \( A, B \); \( O \) is any point on \( AB \); prove that it is possible to invert each circle into itself with \( O \) as centre of inversion.

40. A circle \( \Sigma \) passes through the centre of a circle \( \Sigma \); the external common tangents to \( \Sigma \) and \( \Sigma \) touch \( \Sigma \) at \( P, Q \) and meet at \( O \). Prove that \( PQ \) touches \( \Sigma \).

[Invert w.r.t. \( O \) so that each circle inverts into the other, and consider the original figure and the new figure taken together.]

41. Two non-intersecting circles are inverted w.r.t. any point, what do their four common tangents invert into?

42. Two circles intersect at \( O \); if the system formed by the two circles and their common tangents is inverted w.r.t. \( O \), what is the inverse system composed of?

43. \( P, Q, R \) are three points on a circle, centre \( A \); \( PR \), \( PR \) are produced to \( H, K \) so that \( PQ \cdot PH = PR \cdot PK \); prove that \( FA \) is perpendicular to \( HK \).

44. \( A, B, C \) are three collinear points. Two variable equal circles \( PAB, PBC \) meet at \( P \); prove that the locus of \( P \) is a straight line.

[Invert w.r.t. \( B \).]

45. Prove that the inverse of a plane w.r.t. a point \( O \) outside it is a sphere passing through \( O \).

46. Prove that the inverse of a sphere w.r.t. a point on it is a plane.

47. Prove that the inverse of a sphere w.r.t. a point, not lying on it, is another sphere.

48. Prove that the inverse of a circle w.r.t. a point \( O \) not in its plane is another circle. [Regard the circle as the section of a sphere through \( O \) made by a plane.]

49. If \( C \) is the mid-point of \( AB \) and if \( A, C, B \) are inverted w.r.t. a point \( O \) on \( AB \) into \( A', C', B' \), prove that \( OAC' \cdot OAB' \) is harmonic.

50. \( PQ, RS \) are two common tangents to two circles \( PAR, QAS \), prove that the circles \( PAG, RAS \) touch each other.

51. If \( (ABCD) \) is a harmonic range, and if \( A', C', B', D' \) are their inverses w.r.t. a point \( O \) not on \( AB \), prove that \( A', C', B', D' \) form a harmonic system of points on a circle.

52. \( T \) is a variable point on a circle \( \Sigma \); \( PQ \) is the chord of contact of the tangents from \( T \) to another given circle, prove that the mid-point of \( PQ \) is in general a circle. What is the exceptional case?

53. \( A, B, C \) are three collinear points. \( O \) is any other point, prove that the circumcentres of the triangles \( OBC, OCA, OAB \) are concyclic with \( O \).

54. In the triangle \( ABC \), the point \( A \) is fixed and the angle \( BAC \) is constant; if the area of the triangle \( ABC \) is also constant and if \( B \) describes a straight line or a circle, prove that the locus of \( C \) is in general a circle. Under what conditions is the locus, however, a straight line?
55. Three circles \(BOC\), \(COA\), \(AOB\) are such that the centres of \(BOC\), \(COA\) lie on \(OA\), \(OB\) respectively; prove that the centre of the circle \(AOB\) lies on \(OC\), and that the three centres lie on the circumcircle of the triangle \(ABC\).

**THEOREM 67.**

The angle at which two curves cut is equal to the angle at which their inverse curves cut at the corresponding point of intersection.

\[ O \] is the centre of inversion.
\[ P, P' \] are the corresponding points of intersection of the two curves.

Let \( Q, R \) be points near \( P \) on the original curves, such that \( O, Q, R \) are collinear, and let \( Q', R' \) be their inverses.

Since \( OQ, OQ' = OP, OP' = OR, OR' \)
\[ PQQ'P' \] and \( PRR'P' \) are cyclic quadrilaterals;
\[ \
\begin{align*}
\therefore QP &= OQ'P' \quad \text{and} \quad OR = OR'P' \\
\therefore QR &= OR'P' - OPQ = OR'P' - OQ'P' \\
&= QR'.
\end{align*} \]

But in the limit when \( Q, R \) become very close to \( P \), \( Q', R' \) become the tangents at \( P \), and \( P', R' \) become the tangents at \( P' \) to the two curves, which therefore proves what was required.

Q.E.D.

Hence if two circles cut orthogonally at \( A \), it follows that when inverted w.r.t. \( A \), they become two perpendicular straight lines.

If they are inverted w.r.t. a point \( B \) on one circle but not on the other, they become a circle and a straight line which is a diameter of the circle.

The following example will illustrate the use of inversion in solving problems. It is convenient to use dashes to denote corresponding points in the inverse figure.

**EXAMPLE 1.**

Three circles cut one another orthogonally; their common chords are \(AB, CD, EF\); to prove that the circles \(ACE, ADF\) touch at \(A\).

Invert w.r.t. \( A \) and denote by dashes the corresponding points in the inverse figure.

Let \( S_1, S_2, S_3 \) be the three circles \(DECF, FBFA, DBCA\).

\( S_1 \) and \( S_3 \) become straight lines \( S_2 \) orthogonal to the circle \( S_3 \), and are therefore diameters of \( S_1 \), so that \( B' \) is the centre of \( S_2 \).

Hence \( DBCF, FBE \) are diameters of \( S_2 \).

Now the circles \( ACE, ADF \) become the straight lines \( C'E', D'F' \).

But it is obvious that \( C'E', DT' \) are parallel;

\( \therefore \) the circles \( ACE, ADF \) touch at \( A \).

Q.E.D.

Fig. 83 represents the inverse of Fig. 82. It has been drawn out separately to avoid confusion with the original figure. But in reality Figs. 82, 83 are so related that lines joining corresponding points pass through \( A \).

56. \( OABCH \) are five points such that the circles \(OAB, OCH\) and also the circles \(OBC, OAH\) are orthogonal, prove that the circles \(OCA, OBH\) are also orthogonal.

57. \{\( ABCD\)\} is a harmonic range; \( O \) is any point on \( AD\); \( A, B, C, D \) are the inverse points of \( A, B, C, D \) w.r.t. \( O \). By using the fact that the circles on \( AB, CD \) as diameters are orthogonal, prove that \{\( ABCD\)\} is harmonic.

58. Four circles, centres \( C_1C_2C_3 \), pass through two common points \( A, B \). If \( A(C_1C_2C_3) \) is harmonic, prove that any line through \( A \) is cut harmonically by the circles.
59. Prove that if a straight line is cut harmonically by two orthogonal circles it must be a diameter of one of them. [Invert w.r.t. a point of intersection of the line with the circles.]

60. Two circles are inverted w.r.t. any point, what does the line joining their centres invert into?

61. \( P, Q \) are inverse points of \( P, Q \) w.r.t. \( O \), prove that
\[
OP \cdot PQ = OP' \cdot PQ = OP \cdot PQ = OP \cdot PQ.
\]

62. Two lines are drawn from a fixed point to meet a fixed line and to contain a constant angle, prove that the circumcircle of the triangle so formed touches a fixed circle.

63. A variable circle touches a fixed circle and cuts another fixed circle orthogonally, prove that it also touches another fixed circle.

64. \( A, B \) are fixed points on a fixed circle \( \Sigma \); \( P \) is a variable point on \( \Sigma \); \( O \) is any other fixed point. Prove that the circles \( OAP, OB \) cut at a constant angle.

65. Circles are described to touch two given intersecting circles, prove that they cut orthogonally one of the two circles which pass through the points of intersection of the given circles and bisect the angles between them.

66. Invert w.r.t. \( O \) the theorem: \( O \) is a fixed point on the fixed circle on \( AB \) as diameter, \( P \) is a variable point, then the angle \( APB \) is a right angle.

67. Invert w.r.t. \( O \) the theorem: \( OP, OQ \) are variable chords drawn through a fixed point \( O \) on a fixed circle and are equally inclined to another fixed chord \( OA \), then \( PQ \) is fixed in direction.

68. If four points \( A, B, C, D \) are inverted w.r.t. any point, prove that \( A'BC' + A'B'C' \) is unaltered in value.

69. If the circumcircles of the triangles \( ABC, ABD \) are orthogonal, prove that the circles \( CAD, CBD \) are also orthogonal.

70. Invert w.r.t. any point the theorem: two variable circles touch one another and each of two fixed circles, then their point of contact lies either on one of the fixed circles or on one of the two lines bisecting the angle between the fixed lines.

71. Two circles \( S_1, S_2 \) intersect at \( A, B \); two other circles \( S_3, S_4 \) are drawn to touch both \( S_1 \) and \( S_2 \) externally; prove that a circle can be drawn through \( A \) and \( B \) to cut \( S_3 \) and \( S_4 \) orthogonally.

72. Invert w.r.t. \( A \) the theorem: if in the quadrilateral \( ABCD \), \( BADC = 180^\circ \), then the points \( A, B, C, D \) lie on a circle.

73. Prove that circles which cut one given circle orthogonally and another given circle at a constant angle, also cut a third fixed circle at the same constant angle.

74. \( A, B \) are two fixed points on a circle; \( PQ \) is a variable diameter; \( AP \) meets \( BQ \) at \( R \); prove that the locus of \( R \) is a circle through \( A, B \) orthogonal to the given circle.

75. From a given point \( O \), straight lines \( OA, OB, OC \) are drawn to cut another straight line at \( A, B, C \); a circle \( OBD \) is drawn orthogonal to the circle \( OAC \) and meets \( AC \) again at \( D \); prove that the angles \( AOB, COD \) differ by a right angle.

76. Two circles intersect orthogonally at \( P \); \( O \) is any point on any circle touching the former circles at \( Q, R \); prove that the circles \( GPQ, OPR \) intersect at an angle of \( 45^\circ \).

77. Explain the fallacy in the following argument:
\( AB \) and \( CD \) are any two straight lines: \( O \) is a point, lying on neither. \( AB \) and \( CD \) invert w.r.t. \( O \) into two circles intersecting at \( G \). But \( O \) is the inverse of a point at infinity, therefore the original lines \( AB \) and \( CD \) intersect at a point at infinity, and are therefore parallel.

78. Invert w.r.t. any point the following theorem: it is possible to draw an infinity of concurrent straight lines cutting a given circle orthogonally.

**THEOREM 69.**

\[
P' \text{ and } Q' \text{ are the inverse points of } P, Q \text{ w.r.t. } O; \quad \frac{P'O}{PQ} = \frac{Q'O}{PQ} = \frac{O'Q}{OP} = \frac{O'P}{OP} \quad \text{where } h \text{ is the radius of inversion.}
\]

**Fig. 14.**

Draw \( OF, OG \) perpendicular to \( PQ, PQ' \).

Since \( P, Q, Q', P' \) are concyclic, \( P'Q' = O'O'P' \) and \( O'P' = O'Q' \).
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EXAMPLE 2.

[Ptolemy's theorem and its extension.]

In a non-cyclic quadrilateral, the sum of the rectangles contained by two pairs of opposite sides is greater than the rectangle contained by the diagonals.

If, however, the quadrilateral is cyclic, these two expressions are equal.

\[ \begin{align*}
\text{Let } OABC \text{ be the quadrilateral;} \\
A', B', C' \text{ are the inverse points of } A, B, C.
\end{align*} \]

(1) Since \( OABC \) is non-cyclic, \( A'B'C' \) are not collinear.

Now \( \frac{A'B'}{A'O} = \frac{A'B'}{OB} \), where \( R \) = radius of inversion; \[ \text{[Th. 68]} \]

\[ \frac{AB}{A'O} = \frac{x_B}{k} \]

Similarly \[ \frac{BC}{OB} \cdot \frac{AC}{OC} = \frac{A'C'}{k^2} \]

\[ \text{Q.E.D.} \]

The reader is reminded that a vector proof of this theorem is suggested in Ex. 53 in Chapter VI.

79. Invert w.r.t. \( O \) the following theorem:
A variable line is drawn through a fixed point \( O \) to meet two fixed lines at \( P, Q \); on \( OP \) is taken a point \( R \) such that \( \frac{1}{OR} = \frac{1}{OP} + \frac{1}{OQ} \), then the locus of \( R \) is a straight line.

80. Invert w.r.t. \( O \) the following theorem:
If \( PAB, QAB \) are two chords of a circle, then \( PA \cdot AQ = RA \cdot AS \), where \( O \) is any point on the circle.

81. Two circles intersect at \( O, P \); their tangents at \( O \) meet the circles again at \( A, B \); if the circle \( AOB \) cuts \( OP \) produced at \( Q \), prove that \( OQ = 2OP \).

82. \( A, B, C \) are three collinear points, \( O \) is a point such that \( AOB = \alpha \), prove that \( \frac{1}{OB} = \frac{1}{OA} + \frac{1}{OC} \).

83. Three straight lines \( OP, OQ, OR \) are of lengths \( x, y, z \); \( \widehat{OR} = \alpha, \widehat{RP} = \beta, \widehat{PO} = \gamma \); if \( P, Q, R \) are collinear, prove that \( \sin \alpha \cdot \sin \beta \cdot \sin \gamma = \alpha \).

Hence prove by inversion that \( PQR \) are concyclic if \( x \sin \alpha + y \sin \beta + z \sin \gamma = 0 \).

[This theorem was published in the Mathematical Gazette.]

84. \( O \) is a point on the minor arc \( BC \) of the circumcircle of the triangle \( AABC \); \( OP, OQ, OR \) are the perpendiculars from \( O \) to \( BC, CA, AB \); prove that \( \frac{BC}{OB} \cdot \frac{BA}{AC} = \frac{OP}{OR} \).

85. State and prove a theorem similar to Ex. 84 which holds for any polygon inscribed in a circle.

86. A line \( AB \) of constant length moves with its extremities on two fixed lines \( ox, oy \); \( A', B' \) are inverse points of \( A, B \) w.r.t. \( O \). Prove that \( A'B' \) touches a fixed circle, centre \( O \).
87. O is a point outside a given circle, centre A; OPQ is any line cutting the given circle at P, Q and meeting the circle on OA as diameter at R, then \[ OR = \frac{1}{2} (OP + OQ). \]
By inverting w.r.t. O so that the given circle inverts into itself, prove that any line through O is cut harmonically by the given circle, by the chord of contact of the tangents from O to the circle and by O.

88. \( OA_1A_2...A_n \) are a system of points on a circle, prove that
\[ \frac{A_1A_2}{OA_1} + \frac{A_2A_3}{OA_2} + \frac{A_3A_4}{OA_3} + ... + \frac{A_{n-1}A_n}{OA_{n-1}} + \frac{A_nA_1}{OA_n} = 1. \]

89. If \( A, B, C, D \) are any four collinear points, prove that the ratio \( \frac{AE}{ED} \) is unaltered by inversion w.r.t. any point.

90. Invert w.r.t. any point the theorem that the line joining the centres of two intersecting circles bisects their common chord.

91. Three circles have a common point O; \( OA, OB, OC \) are the common chords and meet the remaining circles at \( D, E, F \), prove that
\[ \frac{BD}{CE} \cdot \frac{AE}{DF} = 1. \]

92. Three circles \( OAB, OBC, OCA \) pass through a common point O; a line through O meets \( OAB, OBC, OCA \) again at \( F, D, E \), prove that
\[ \frac{BD}{CE} \cdot \frac{AE}{DF} = 1. \]

Peaucellier's Cell.

This is a system of linkages for describing mechanically a curve inverse to a given curve. It consists of four equal rods, of length \( a \), jointed together so as to form the rhombus \( ABCD \).

Fig. 86.

Two other rods, each of length \( b (b > a) \), are jointed together at one end \( O \), and the other extremities are jointed to the rhombus at \( B, D \). The point \( O \) is kept fixed. If \( \alpha \) is made to describe any curve, then a pencil placed at \( C \) will trace out the inverse curve.

Fig. 86, \( FQRS \) is a rhombus formed of four equal rods.
\( OQ, OS \) are two other equal rods, and the point \( O \) is fixed.
\( OQ \) is less than \( PQ \).
\( A \) is a fixed point and \( AP \) is a rod of length equal to \( AO \).
Then \( P \) is compelled to describe a circle, centre \( A \), which therefore passes through \( O \). It can then be proved that \( K \) describes a straight line perpendicular to \( OA \).

It is necessary to prove (1) \( K, O, P \) are collinear.
(2) \( OK, OP = \beta = \) constant.

\( P, K \) therefore describe inverse curves.

D.C.
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It should be noted that \( \rho^2 \) is negative, so that the radius of inversion is imaginary.
A model can be made without much difficulty and will be found to work well.

95. With the notation of Fig. 87, prove that \( R, O, P \) are collinear and that \( RO, OP \) is constant.

96. In Fig. 87, \( PQ = a, \ OG = b, \ AP = c. \)

(1) Prove that the length of the perpendicular from \( O \) to the straight line traced by \( K \) is \( \frac{a^2 - b^2}{2c}. \)

(2) Prove that the length of line which can be traced out by \( K \) is \( 2(a+b)\sqrt{1-\frac{(a-b)^2}{2c^2}}. \)

(3) Prove that the length of line traced out by \( R \) can, for different values of \( h, c, \) never exceed \( 4\rho. \)

(4) Describe what happens in the limiting case when \( a = b. \) What is the general effect of the radius of inversion becoming zero? [This question will be made much easier if a model is used.]

**THEOREM 69.**

If a circle and two inverse points are inverted w.r.t. any point, then the inverse system is a circle and two inverse points.

\[ S \]
\[ \overline{S} \]

Let \( P, Q \) be two inverse points w.r.t. circle \( S, \) centre \( A. \)

Any point \( O \) is taken as centre of inversion.

Draw a circle through \( O, P, Q \) and call it \( S. \)

Call the line \( APQ, \) \( L. \)

When inverted w.r.t. \( O \) (see Fig. 89), \( S \) becomes a circle \( S' \) and \( S \) becomes a straight line \( S' \) orthogonal to \( S \) (and therefore a diameter of \( S \)), because \( S' \) passes through two inverse points of \( S \) and is therefore orthogonal to \( S. \) [Th. 64]

Also the diameter \( APQ \) of \( S \) becomes a circle through \( O \) orthogonal to \( S'. \)

... the inverse points \( P', Q' \) of \( P, Q \) are the intersections of a diameter of \( S' \) with a circle orthogonal to \( S, \) and are therefore inverse points of \( S. \) [Th. 64, Cor. 2.]

Q.E.D.

**Corollary.**

If \( C_1 \) and \( C_2 \) are inverse curves w.r.t. a circle \( S, \) and if the system is inverted w.r.t. any point so that \( C_1', C_2', S' \) become \( C_1', C_2', S'; \) then \( C_1' \) and \( C_2' \) are inverse curves w.r.t. the circle \( S'. \)

The method of Theorem 69 leads to an important special case.

The point \( O \) lies on \( S, \) it will be found that \( S \) inverts into the line bisecting \( PQ \) at right angles. In other words, \( Q \) is the reflection of \( P \) in the line \( S. \)

Hence if \( C_1 \) and \( C_2 \) are inverse curves w.r.t. a circle \( S, \) and if the system is inverted w.r.t. a point \( O \) on \( S, \) \( C_1 \) and \( C_2 \) invert into equal curves, being reflections of one another in the straight line \( S. \) [Compare Ex. 12.]

97. Two intersecting circles \( C_1 \) and \( C_2 \) are orthogonal to a circle \( S, \) prove that the points of intersection of \( C_1 \) and \( C_2 \) are inverse points w.r.t. \( S. \)

98. \( P \) is the inverse of \( O \) w.r.t. a circle \( S, \) if the system is inverted w.r.t. \( O, \) prove that \( P' \) is the centre of \( S'. \)

99. Two given circles intersect at \( O. \) By inverting w.r.t. \( O, \) prove that the inverse of \( O \) w.r.t. any circle touching the given circles lies on one or other of two fixed orthogonal circles.

100. \( A', B' \) are inverse points of \( A, B \) w.r.t. a given circle \( S; \) \( O \) is a point on \( S, \) prove that the circles \( OAB, O'A'B' \) intersect at a second point on \( S. \)

101. \( S \) is a given point, \( S \) and \( S' \) are two given circles; \( P_1 \) is the inverse of \( S \) w.r.t. \( S; \) \( S_1 \) is the inverse of \( S_1 \) w.r.t. \( S; \) \( S_2 \) is the inverse of \( S_2 \) w.r.t. \( S; \) \( Q_1 \) is the inverse of \( Q_1 \) w.r.t. \( S; \) prove that \( P_1 \) and \( Q_1 \) coincide if and only if \( S \) and \( S' \) are orthogonal. This may be stated as follows: the operations of inversion w.r.t. two circles are commutative if and only if the circles are orthogonal.
102. \( Q \) and \( Q' \) are inverse points w.r.t. a circle \( C_1 \); \( R, R' \) are the inverse points of \( Q, Q' \) w.r.t. a circle \( C_2 \) orthogonal to \( C_1 \); prove that \( R, R' \) are inverse points w.r.t. \( C_2 \).

103. Three circles \( C_1, C_2, C_3 \) all pass through a point \( A \), and cut each other at angles of 120°. \( P \) is any other point; \( P_1, P_2, P_3 \) are the inverses of \( P \) w.r.t. \( C_1, C_2, C_3 \); prove that the circles \( AP, P_1A, AP_2, AP_3 \) also cut each other at angles of 120°.

104. Generalise Ex. 103 so as to obtain a theorem for \( n \) circles.

105. \( A, B, C, D \) are any four points: three circles are drawn each passing through \( D \), and such that \( A, B; B, C; C, A \) are respectively inverse points w.r.t. them, prove that the circles have a second common point.

106. \( C_4 \) is the inverse of the circle \( C_1 \) w.r.t. a circle \( \Sigma \); \( O \) is any point on \( \Sigma \); of the four circles which can be drawn through \( O \) to touch \( C_1 \) and \( C_4 \) prove that one pair touch each other at \( O \), and that the other pair intersect on \( \Sigma \).

107. \( C_5 \) is the inverse of a circle \( C_1 \) w.r.t. a circle \( \Sigma \); another circle touches \( C_1 \) and \( C_5 \) at \( P, Q \), if both the contacts are internal or both external, prove that \( P, Q \) are inverse points w.r.t. \( \Sigma \).

108. \( C_1, C_2, C_3 \) are pairs of inverse circles w.r.t. a circle \( \Sigma \); \( O \) is any point on \( \Sigma \); prove that the inverses of \( O \) w.r.t. \( C_1, C_2, C_3 \) are concyclic.

**Theorem 70.**

1. \( O \) is the point of intersection of the exterior common tangents of two circles \( C_1, C_2 \); with \( O \) as centre of inversion, it is possible to invert either circle into the other.

2. If \( H \) is any point on the circle of inversion and if the system is inverted w.r.t. \( H \), the circles \( C_1, C_2 \) invert into equal circles.

**Example 3.** [Feuerbach's Theorem.]

The nine-point circle of a triangle touches the incircle and each excircle.

The usual notation for the triangle \( ABC \) is observed. \( AH_A \) meets \( BC \) at \( L \); the remaining common tangent \( C \) to the incircle and excircle is the reflection of \( CLB \) in \( AH_A \), so that \( ACL = A\overline{CE} \).
Now \( CI, CI \) are the bisectors of the angle \( ACB \);
\[
\therefore [AILI] \text{ is harmonic;}
\]
\[
\therefore \{DXLX\} \text{ is harmonic, by parallels.}
\]
But \( A' \) is the mid-point of \( XX_1 \) \( \because A'LAD = A'X^2 = A'X_1^2 \);
\[
\therefore \text{the inverse of the nine-point circle w.r.t. } A' \text{ and radius of inversion } A'X \text{ is a straight line through } L \text{ since the nine-point circle passes through } A', D.
\]

![Diagram](image)

**Fig. 91.**

Again (see Fig. 91), \( A'F = A'E \); for \( A' \) is centre of circle \( BFEC \);
\[
\therefore \text{the diameter through } A' \text{ of the nine-point circle is perpendicular to } FE \text{ (a chord of the nine-point circle).}
\]
But \( \overline{A'E} = \overline{A'F} = A'C \); \( \because \) the diameter through \( A' \) of the nine-point circle is perpendicular to \( FE \) (a chord of the nine-point circle).
\[
\therefore C'F = \overline{A'C} = \overline{A'C} \text{ (a chord of the nine-point circle).}
\]
\[
\therefore C'F = \overline{A'C} \text{ (a chord of the nine-point circle).}
\]
\[
\therefore \text{the inverse of the nine-point circle w.r.t. } A' \text{ and radius of inversion } A'X \text{ is a straight line through } L \text{ since the nine-point circle passes through } A', D.
\]

But the incircle and excircle each invert into themselves in this case;
\[
\therefore \text{since } C' \text{ touches the incircle and excircle, by considering the inverse figure, it follows that the nine-point circle touches the incircle and excircle.}
\]

Q.E.D.

109. Show how to find a centre of inversion such that three given circles can be inverted into themselves.

110. The perpendicular bisector of \( BC \) meets the sides \( AB, AC \) of the triangle \( ABC \) at \( PQ \); prove that \( P, Q \) are inverse points w.r.t. the circumcircle of the triangle \( ABC \).

111. \( P' \) is the inverse of \( P \) w.r.t. a circle; \( QP' \) is any chord; prove that \( P'Q' \) bisects the angle \( QP' \).

112. Invert w.r.t. any point the theorem that the three altitudes of a triangle are concurrent.

113. If the circles inverse to the given circles \( ACD, BCD \) w.r.t. a point \( P \) are equal, prove that the circle \( PGD \) bisects the angle of intersection of the given circles.

114. Deduce from Ex. 113 a method for inverting three circles which cut each other into three equal circles.

115. Use Theorem 70 (2) to invert three given circles into three equal circles.

116. The perpendicular bisectors of the sides \( AB, AC \) of a triangle \( ABC \) meet \( AG, AH \) at \( O, R \).\( \) Circles are drawn with centres \( O, D \), radius \( QA, RA \); prove that they intersect on the diameter through \( A \) of the circumcircle of the triangle \( ABC \).

117. Four circles \( A, B, C, D \) all touch a fifth circle; also \( A \) touches \( B \) at the point \( P \); \( B \) touches \( C \) at \( O \); \( C \) touches \( D \) at \( R \). If the circle \( PQR \) cuts \( A, B, C, D \) orthogonally, prove that a second circle can be drawn to touch \( A, B, C, D \).

118. Invert w.r.t. \( O \) the theorem: \( AB \) is a diameter of a circle, centre \( O \); \( C \) is any other point; \( AG, BC \) meet the circle again at \( P, Q \); then the circle \( CPQ \) is orthogonal to the given circle.

119. \( OABC, OABC \) are two straight lines; if \( B, C \) are the mid-points of \( AG, AC \), prove that the circles \( OAA', OBB, OOC' \) have a common chord.

120. A variable circle touches a fixed circle and is such that the tangent to it from a fixed point is of constant length; prove that it also touches another fixed circle.

121. [This requires some knowledge of cross ratios. Th. 56–58] Two circles intersect at \( O, O' \); a variable line \( OPQ \) cuts the circles at \( P, Q \) and is divided at \( V \) so that \( PV:VQ \) is constant; prove that the locus of \( V \) is a circle through \( O \) and \( O' \).

122. \( G \) is the centroid of the triangle \( ABC \); \( D \) is the mid-point of \( BC \). The circles \( CDG, BDG \) meet at \( F \); the circles \( CDG, BDA \) meet at \( E \); prove that \( BC \) touches the circle \( DEF \); that \( DE \) touches the circle \( BDF \); and that \( DF \) touches the circle \( CDG \).
123. \(ABCD\) are any four points. Prove that in an infinite number of ways, two circles can be drawn making an assigned angle with each other and such that \(A, B\) are inverse points w.r.t. one circle, and \(C, D\) w.r.t. the other.

124. Invert w.r.t. \(A\) the theorem: \(PQ\) is a variable chord of a fixed circle, centre \(O\), such that \(OA\) bisects externally the angle \(PAG\), where \(A\) is a fixed point; then the circle \(PAQ\) passes through another fixed point.

125. Invert w.r.t. \(P\) the theorem: \(A, B, C\) are three collinear points; \(P\) is any other point, \(AF, BF, CF\) are drawn perpendicular to \(PA, PB, PC\), then \(PDEF\) are concyclic.

126. A circle touches two other circles at \(A, B\), and meets their common chord at \(P, Q\); prove that \(AP, BP, AQ, BQ\) meet two of the circles at the points of contact of their common tangents.

127. A point \(P\) is inverted successively w.r.t. four circles each of which is orthogonal to the others; prove that it returns finally to its original position.

128. \(OA', OB'\) are two straight lines; \(AB', AB\) intersect on the bisector of the angle between them, prove that

\[
\frac{1}{OA} + \frac{1}{OB} = \frac{1}{OA'} + \frac{1}{OB'}. \]

129. \(BC, CA, AB\) are arcs of circles orthogonal to the circle \(ABC\). Through \(A, B, C\) three circles \(AD, BE, CF\) are drawn orthogonal to the circle \(ABC\) and to \(BC, CA, AB\) respectively. Prove that the circles \(AD, BE, CF\) intersect at two common points.

130. Two perpendicular straight lines \(MX, AD\) meet at \(X\); \(AX = XD\). Points \(P, Q, R, S\) are taken on \(AD\) such that \(BX = XC\). If \(O\) is any point in \(AD\), and if four circles are drawn through \(O, M\) orthogonal to \(ABM, ACM, BDM, CMD\) respectively, and cutting them at \(P, Q, R, S\), prove that \(FQRS\) are concyclic.

131. The extremities of a line \(PQ\) of constant length move on two fixed lines \(OX, OY\); \(PK, QK\) are drawn perpendicular to \(OP, OQ\) to meet \(K\), prove that the focus of \(K\) is a circle.

132. From a fixed point \(O\), a tangent \(OM\) is drawn to a variable circle through two fixed points \(G, D\); if \(OM\) touches another circle through \(G, D\) at \(N\) and if \(P\) is the harmonic conjugate of \(O\) w.r.t. \(M, N\), prove that the focus of \(P\) is a circle.

133. Two circles \(S_1, S_2\) touch each other at \(O\). \(S_1, S_2\) are two other circles also touching each other at \(O\). \(S_1\) meets \(S_2\) at \(D\) and \(S_2\)

134. A system of circles is drawn to cut a given circle orthogonally at two points of intersection, and to pass through a given point not in the plane of the circle. Prove that they intersect in another common point.

Hence show how a circle and a point not in its plane can be inverted into a circle and its centre.

135. Prove that the inverse of a conic w.r.t. any point outside its plane is the section of a cone on a sphere through the vertex of the cone.

136. **Casey's Theorem.** \(J\) is the length of a common tangent of two circles, radii \(a, b\); \(J\) is the length of the corresponding common tangent of their inverses w.r.t. any point, and \(a', b'\) the lengths of their radii; prove that

\[
\frac{a^3}{a - b} = \frac{b^3}{a + b}. \]

The following riders (137-167) depend on the analytical results obtained in Ex. 21-24. Where a theorem is to be proved, an elementary knowledge of geometrical conics is required.

These results may be enumerated as follows:

1. The inverse of a parabola w.r.t. its focus \(S\) is a cardioid with its pole at \(S\).

2. The inverse of a cardioid w.r.t. its pole \(S\) is a parabola with its focus at \(S\).

3. The inverse of a conic w.r.t. a focus \(S\) is a Limaçon with its pole at \(S\).

4. The inverse of a Limaçon w.r.t. a focus \(S\) is a conic having \(S\) as one focus.

5. The inverse of a parabola w.r.t. its vertex \(A\) is a Conicoid with its axis at \(A\); and conversely.

6. The inverse of a rectangular hyperbola w.r.t. its centre \(C\) is a Lemniscate with its double point at \(C\); and conversely.

137. Invert the following theorem w.r.t. \(O\): if the tangents at the points \(P, Q, R\) on a cardioid, whose pole is \(O\), are parallel, then of the quantities \(\sqrt{OP}, \sqrt{OQ}, \sqrt{OR}\), the sum of two is equal to the third.

138. Invert w.r.t. \(S\): the circumcircle of the triangle formed by three tangents to a parabola passes through the focus \(S\) of the parabola.

139. Invert w.r.t. \(S\): \(PSP'\) is a variable chord of a parabola, focus \(S\); then \(\frac{1}{SP} - \frac{1}{SP'}\) is constant.

140. Invert w.r.t. \(S\): \(PSP'\) is a variable chord of a cardioid, focus \(S\); \(R\) is a point on PP' such that \(\{PP', SR\}\) is harmonic, then the locus of \(R\) is a straight line.
144. A circle is drawn through the pole $O$ of a cardioid to touch the curve at $P$ and meet the axis at $T$; prove that $OP = OT$.

145. A variable circle is drawn through the pole $O$ of a cardioid to touch the curve; prove that the locus of the extremity of the diameter through $O$ is a circle.

146. If in Ex. 144, $P$ is the point of contact of the variable circle with the curve, prove that its diameter varies as $\sqrt{OP}$.

147. A circle is drawn through the pole of a cardioid to touch the curve at a variable point $P$; prove that the locus of the extremity of the diameter through $P$ is another circle.

148. Invert w.r.t. $S$: two tangents $TP$, $TQ$ are drawn from a point $T$ to a parabola, focus $S$, then

$$\frac{1}{ST^2} + \frac{1}{SO^2} = \frac{1}{SP^2} + \frac{1}{OQ^2}.\sqrt{QP}.\sqrt{TP}.$$

149. A circle is drawn through the pole $O$ of a cardioid to cut the curve orthogonally at $P$; if it cuts the axis at $Q$, prove that $OG = OP$.

150. $PQQ'$ is a variable chord of a Limaçon whose pole is $O$; two circles are drawn through $O$ to touch the curve at $P$, $Q$, respectively; if they meet again at $R$, prove that the locus of $R$ is a circle.

151. A variable circle is drawn through the pole $O$ of a Limaçon to touch the curve; prove that the locus of the extremity of the diameter through $O$ is a circle.

152. Invert w.r.t. a focus the locus of the intersection of perpendicular tangents to a conic is a circle.

153. Obtain a theorem similar to Ex. 149, for a Limaçon.

154. Invert w.r.t. a focus; $P$, $Q$, $O$, $Q'$ are two variable focal chords of a conic perpendicular to each other, then

$$\frac{1}{PS} + \frac{1}{SP} + \frac{1}{QS} + \frac{1}{SQ} = \text{constant}.$$

155. Invert w.r.t. a focus; a circle of constant radius passes through the centre $S$ of a conic and cuts it at $A$, $B$, $C$, $D$; then

$$\frac{1}{SA} + \frac{1}{SB} + \frac{1}{SC} + \frac{1}{SD} = \text{constant}.$$

156. Invert w.r.t. a focus; $PQ$ is a variable chord of a conic subtending a constant angle at the focus, then the envelope of $PQ$ is a conic with the same focus.

157. Invert w.r.t. $O$: $P$ is a variable point on a fixed circle; $O$ is a fixed point; a line $PR$ is drawn so that $\hat{OPR}$ is constant, then $PR$ envelopes a conic, focus $O$.

158. Invert w.r.t. $O$: the tangents at the ends of a chord of a cardioid, through the pole $O$, are at right angles.

159. Invert w.r.t. $O$: the normals at the ends of a chord through the pole of a Limaçon intersect on a fixed circle.

160. Invert w.r.t. $O$: two cardioids have the same pole $O$, and their axes lie in opposite directions along the same line, then they cut orthogonally.

161. Invert w.r.t. $C$: $PP'$ is a diameter of a rectangular hyperbola, centre $C$; a circle is drawn through $P$, $P'$ and meets the hyperbola at $Q$, $Q'$; then $QQ'$ is a diameter of the circle.

162. Invert w.r.t. $C$: two rectangular hyperbolas have the same centre $C$; also the axes of one are the asymptotes of the other, then they cut orthogonally.

163. Three circles touch a cardioid and pass through its pole, prove that their other three points of intersection are collinear.

164. If two circles touch a cardioid and pass through its pole, prove that their common chord will bisect the angle between the two lines joining the pole to the points of contact of the circles with the cardioid.

165. The cusp of a cissoid is at $A$, and $AX$ is the tangent to the cusp: a circle is drawn through $A$ to touch the cissoid at $P$ and meets $AX$ at $Q$; a line through $P$ perpendicular to $AP$ meets $AX$ at $R$: prove that $QA = AR$.

166. Invert w.r.t. $A$: the locus of the point of intersection of perpendicular tangents to a parabola, vertex $A$, is a straight line.

167. [This needs the differential Calculus.] If the curve $f(r, \theta) = 0$ is transformed by the substitution $r = r^2$, $\theta = \pi$, a generalised form of inversion, prove that the angle at which two curves cut is equal to the angle at which the inverse curves cut at the corresponding point.
CHAPTER XII.

POLES AND POLARS.

When considering the Principle of Duality in Chapter IX., nothing was said as to the way in which the correspondence alluded to may be determined. It is essential that this correspondence should be unique. In other words, to any one point \( A \) in Fig. 1 there must correspond one and only one line \( a \) in Fig. 2, and vice versa. In the following pages some indication will be found of the manner in which this relationship may be effected. It is, however, impossible to realise the generality of the process before the analogous theorems for the conic are enunciated.

The theorems of this chapter cover a wide period of time. The fundamental property, proved in Theorem 73, is given by Apollonius for the case in which the pole lies outside the circle, and he shows further that the same property holds good for the conic. Pappus, in the \( \Sigma \beta \gamma \zeta \alpha \gamma \zeta \), completes the theorem by proving it for any position of the pole. The general theory is developed in detail by Desargues, for the conic as well as for the circle, in his treatise on conic sections, published in 1639. Owing to the analytical discoveries of Descartes during the same period, his work received little attention, and was lost until Chasles discovered a copy in 1845. The harmonic properties on pages 168-9 were also in substance contained in this treatise. Dr. Taylor has pointed out that Newton (1647-1727) was also acquainted with the general theory, but whether through Desargues or from independent research, it is impossible to say.

Definition.

\( P \) is a point on the diameter \( AB \) (or \( AB \) produced) of a circle \( \Sigma \), centre \( O \); \( Q \) is the harmonic conjugate of \( P \) w.r.t. \( A, B \); \( QR \) is drawn parallel to the tangent at \( A \) to the circle; then \( QR \) is called the polar of \( P \) w.r.t. \( \Sigma \), and \( P \) is called the pole of \( QR \) w.r.t. \( \Sigma \).

THEOREM 71.

If the line joining a point \( P \) to the centre \( O \) of a circle \( \Sigma \) meets the polar of \( P \) w.r.t. \( \Sigma \) at \( Q \), then the polar of \( P \) is perpendicular to \( OP \) and \( OQ^2 = (\text{radius})^2 \).

The proof is very simple and is left to the reader.

The form of the definition given above has been so chosen that it will also apply to the case of a conic.

1. Prove Theorem 71.

2. If \( P \) and \( P' \) are inverse points w.r.t. a circle \( \Sigma \), prove that the polar of \( P' \) w.r.t. \( \Sigma \) passes through \( P \).

3. If the point \( P \) is outside the circle \( \Sigma \), prove that the chord of contact of the tangents from \( P \) to \( \Sigma \) is the polar of \( P \) w.r.t. \( \Sigma \).

4. What is the polar of a point situated on the circle \( \Sigma \)?

5. What is the polar of a point situated in a given direction at an infinite distance from the centre of \( \Sigma \)?

6. What is the polar of the centre of \( \Sigma \)?

7. What is the pole of a diameter of \( \Sigma \)?

8. \( R \) is a point on the polar \( QR \) of a point \( P \) w.r.t. a circle \( \Sigma \), centre \( O \); \( PH \) is drawn perpendicular to \( OR \) to meet it at \( H \); prove that \( H \) is the inverse point of \( R \) and that \( HP \) is the polar of \( R \). Hence prove that if \( R \) lies on the polar of \( P \), then \( P \) lies on the polar of \( R \).

9. \( R \) is any point on the polar of \( P \) w.r.t. a circle \( \Sigma \); prove that the circle on \( PR \) as diameter is orthogonal to \( \Sigma \). [Note that it passes through \( O \); Fig. 93.]

10. Any line through a point \( P \) meets the polar of \( P \) at \( R \), and the circle at \( H, K \); prove that \( PH \; PK \) is harmonic. [Use Ex. 9.]

11. \( PR \) is a diameter of a circle \( \Sigma \), orthogonal to another circle \( \Sigma \); \( O \) is the centre of \( \Sigma \). If \( OP \) meets \( \Sigma \) at \( Q \), prove that \( QR \) is the polar of \( P \) w.r.t. \( \Sigma \).

Before giving the geometrical treatment of polars properties, it is instructive to obtain some of the fundamental theorems by analysis.
I. To find the equation of the polar of a point \( P (x_1, y_1) \) w.r.t. the circle \( x^2 + y^2 = a^2 \),

![Diagram of a circle and a point P](image)

The equation of \( OP \) is \( \frac{x}{x_1} = \frac{y}{y_1} \).

\[ \therefore \text{any line perpendicular to } OP \text{ is } xx' + yy' = k. \]

The length of the perpendicular from \( O \) to this line is \( \frac{k}{\sqrt{x'^2 + y'^2}} \).

But if this line is the polar of \( P \), the length of the perpendicular is \( \frac{a^2}{\sqrt{x'^2 + y'^2}} \). \[ \text{[Th. 71]} \]

\[ \therefore k = a^2; \]

\[ \therefore \text{the polar of } P (x', y') \text{ is } xx' + yy' = a^2. \quad \text{Q.E.D.} \]

II. If the polar of \( P \) passes through \( R \), then the polar of \( R \) passes through \( P \). See Fig. 94.

Let the coordinates of \( P, R \) be \( (x_1, y_1) \) and \( (x_2, y_2) \).

The equation of the circle is \( x^2 + y^2 = a^2 \).

The polar of \( P \) is \( xx_1 + yy_1 = a^2 \).

Since this passes through \( R \), \( x_2x_1 + y_2y_1 = a^2 \). \[ \therefore \text{the polar of } R \text{ is } xx + yy = a^2. \]

But the polar of \( R \) is \( xx_1 + yy_1 = a^2 \).

\[ \therefore \text{by (i) the point } P \text{ lies on the polar of } R. \quad \text{Q.E.D.} \]

**Definition.**

If two points are so situated that the polar of either passes through the other w.r.t. the same circle, then the points are called conjugate points w.r.t. the circle.

It follows from II. that the points \( (x_1, y_1) \) and \( (x_2, y_2) \) are conjugate points w.r.t. the circle \( x^2 + y^2 = a^2 \) if \( xx_1 + yy_1 = a^2 \).

And the converse is also true.

### Theorem 72.

If the polar of \( P \) passes through \( R \), then the polar of \( R \) passes through \( P \).

\( O \) is the centre of the circle.

\( OP \) meets the polar of \( P \) at \( Q \).
Corollary 3.

(i) If any number of points are collinear, then their polars are concurrent.

(ii) If any number of lines are concurrent, then their polars are collinear.

For if in 3 (i) \( P \) is a variable point on a fixed line \( L \) whose pole is \( L \), then the polar of \( P \) passes through \( L \), which is a fixed point. And if in 3 (ii) \( P \) is a variable line through a fixed point \( L \), whose polar is \( L \), then the pole of \( P \) lies on \( L \), which is a fixed line.

Corollary 3 is frequently useful, for if it is required to prove three or more particular lines concurrent, it may be easier to prove that their polars are collinear, which would be sufficient; or vice versa.

Definitions.

(i) If two points are such that the polar of either w.r.t. a circle \( \Sigma \) passes through the other, then the points are called conjugate points w.r.t. \( \Sigma \).

(ii) If two lines are such that the pole of either w.r.t. a circle \( \Sigma \) lies on the other, then the lines are called conjugate lines w.r.t. \( \Sigma \).

12. If \( P, R \) are conjugate points w.r.t. a circle \( \Sigma \), prove that the circle on \( PR \) as diameter is orthogonal to \( \Sigma \).

13. A variable chord \( PQ \) of a fixed circle passes through a fixed point \( H \), prove that the tangents at \( P, Q \) intersect on a fixed line, viz. the polar of \( H \).

14. \( T \) is a variable point on a fixed line; \( TP, TQ \) are the tangents from \( T \) to a fixed circle, prove that \( PQ \) passes through a fixed point.

15. The tangent at \( A \) to the circumcircle of the triangle \( ABC \) meets \( BC \) at \( T \) and is produced to \( U \) so that \( AT = TU \); prove that \( A \) and \( U \) are conjugate points w.r.t. any circle through \( B, C \).

16. \( P, R \) are conjugate points w.r.t. a circle, centre \( O \), prove that the pole of \( PR \) is the orthocentre of the triangle \( OPR \).

17. \( P \) is a point inside a circle, centre \( O \); \( OP \) meets the polar of \( P \) at \( \xi \); any line through \( P \) meets the circle at \( H, K \) and the polar of \( P \) at \( \xi \); prove that

(i) \( OH \) touches the circle \( HPQ \);

(ii) \( OP = HP \) and \( OK = KP \);

(iii) \( \{O, HPK, OP, K \} \) is harmonic.

[This proves that any line through a point \( P \) is cut harmonically by \( P \), the polar of \( P \), and the circle.]
18. If \( P \) is a point outside a circle, and if \( PHRK \) is any line meeting the circle at \( H, K \) and the polar of \( P \) at \( R \), prove that \( \{PR; HK\} \) is harmonic. [Adapt the method of Ex. 17.]

19. Prove that the angle between two lines is equal to the angle which their poles subtend at the centre of the circle.

20. \( AB \) is the diameter of a circle \( C_1 \); if the polar of \( A \) w.r.t. a circle \( C_2 \) passes through \( B \), prove that \( C_1 \) and \( C_2 \) are orthogonal.

21. The incircle of the triangle \( ABC \) touches \( BC, CA, AB \) at \( X, Y, Z \). Prove that the diameter through \( X \) of the incircle meets \( YZ \) on the line joining \( A \) to the mid-point \( A' \) of \( BC \). [Prove that \( AA' \) is conjugate to the line through \( A \) parallel to \( BC \) by Ex. 17 (3), and use Theorem 72, Cor. 3.]

22. \( P_1, P_2 \) are two points on the circles \( S_1, S_2 \) which have a common centre \( O \). \( N_1, N_2 \) are the feet of the perpendiculars from \( P_1, P_2 \) to their polars w.r.t. \( S_1, S_2 \) respectively. Prove that \( P_1N_1 = O P_1 \), \( P_2N_2 = O P_2 \).

23. \( PQ \) is a diameter of a circle \( C_1 \); \( C_2 \) is a circle orthogonal to \( C_1 \); prove that the polar of \( P \) w.r.t. \( C_2 \) passes through \( Q \).

24. \( I \) is the incentre of the triangle \( ABC \). The tangent at any point \( K \) on the incircle is met by perpendiculars through \( I \) to \( IA, IB, IC \) at \( P, Q, R \). Construct the positions of the polars of \( AP, BQ, CR \) w.r.t. the incircle, and hence prove that these three lines are concurrent.

25. Prove that the locus of a point whose polars w.r.t. three circles is concurrent is a circle orthogonal to each of the given circles.

26. \( ABC \) is a triangle right-angled at \( B \); \( BB' \) is a median. Any circle is drawn to touch \( BB' \) at \( B \); prove that \( A, C \) are conjugate points w.r.t. it.

27. \( TP, TQ \) are the tangents at the extremities of a chord \( PQ \) of a circle; the tangent at any other point \( R \) on the circle meets \( PQ \) at \( S \); prove that \( TR \) is the polar of \( S \).

28. \( PA, QB \) are the tangents from two points \( P, Q \) to a circle, centre \( C \); if \( PQ = PA + QB \), prove that \( Q \) lies on the polar of \( P \). [Draw \( QN \) perpendicular to \( CP \).]

29. \( ABC \) is a given triangle. \( O \) is the centre of a variable circle through \( B, C \); find the locus of the point of intersection of \( AO \) with the polar of \( A \) w.r.t. the circle. [Invert w.r.t. \( B \).]

30. A variable circle passes through a fixed point \( C \), and has two fixed points \( H, K \) as conjugate points, prove that the locus of its centre is a straight line perpendicular to the line joining \( C \) to the mid-point of \( HK \). [Use Ex. 12.]

31. \( P, Q \) are two points subtending a right angle at a point \( C \) on a circle; if the tangents at \( C \) bisect \( PQ \), prove that \( P, Q \) are conjugate points.

32. Prove that the polar of a fixed point w.r.t. a variable circle which touches a fixed line at a fixed point passes through a fixed point.

33. \( P, R \) are conjugate points w.r.t. a fixed circle centre \( O \); if \( PQR \) is constant, and if \( P \) moves on a straight line, prove that the locus of \( R \) is a circle.

34. Assuming Pascal's theorem that, if a hexagon is inscribed in a circle, then the meets of opposite sides are collinear, deduce Brianchon's theorem that, if a hexagon is circumscribed about a circle, the joins of opposite vertices are concurrent. [Use Th. 72, Cor. 3.]

35. Assuming the theorem that, if a circle touches the sides \( BC, CA, AB \) of a triangle at \( X, Y, Z \) then \( AX, BY, CZ \) are concurrent, deduce by using Th. 72, Cor. 3, a theorem for the circumcircle of a triangle. [The incircle of \( ABC \) is the circumcircle of \( XYZ \).]

36. \( PQR \) is a triangle; \( P', Q', R' \) are the poles of \( QR, RP, PQ \) w.r.t. a circle; prove that \( P, Q, R \) are the poles of \( QR', RP', PQ' \).

Definitions.

(i) If two triangles are such that the vertices of either are the poles of the sides of the other w.r.t. a circle, then the triangles are called conjugate triangles w.r.t. the circle.

(ii) If a triangle is conjugate to itself (i.e. each vertex is the pole of the opposite side w.r.t. a circle), the triangle is called a self-conjugate triangle w.r.t. the circle.

37. \( P, Q \) are any two conjugate points, \( R \) is the pole of \( PQ \); prove that \( PQR \) is a self-conjugate triangle.

38. If \( PQR \) is a self-conjugate triangle w.r.t. a circle, prove that the centre of the circle is the orthocentre of the triangle.

39. Given a triangle, prove that there exists one, and only one circle (real or imaginary) w.r.t. which the triangle is self-conjugate, and that the circle is real if, and only if, the triangle is obtuse angled.

40. \( P \) is any point on the base \( BC \) of an obtuse-angled triangle \( ABC \); prove that the circle on \( AP \) as diameter is orthogonal to the circle w.r.t. which the triangle \( ABC \) is self-conjugate.

41. \( PQR, P'Q'R' \) are two conjugate triangles of a circle; assuming the theorem that \( PP', QQ', RR' \) are concurrent, deduce another theorem by using Th. 72, Cor. 2, 3.
THEOREM 73.

If $P$ is the pole of a line $QR$ with respect to a circle $\Sigma$, then any line through $P$ is cut harmonically by $P$, $QR$, and $\Sigma$.

Let $O$ be the centre of $\Sigma$; and let $OP$ meet the polar of $P$ at $Q$.

$PR$ is any line through $P$ cutting $\Sigma$ at $H, K$.

On $PR$ as diameter, describe a circle $\Sigma'$, and let it cut $\Sigma$ at $C, D$.

Since $PQ \perp QR = 90^\circ$, $\Sigma'$ passes through $Q$.

Bisect $PR$ at $O$, so that $O$ is the centre of $\Sigma'$.

Now $OP \cdot OQ = OC^2$, since $P$ is the pole of $QR$; 

$\therefore \Sigma'$ is orthogonal to $\Sigma$; 

$\therefore OH \cdot OK = OD^2 = OP^2$; 

$\therefore \{OP, HK\}$ is harmonic. Q.E.D.

Corollary.

If $P, R$ are conjugate points with respect to a circle, and if $PR$ meets the circle at $H, K$, then $\{PR, HK\}$ is harmonic.

Another proof of this fundamental theorem is given on page 187.

42. Prove the Corollary of Theorem 73.

43. Deduce from the Corollary of Theorem 73 a new theorem by using Th. 72, Cor. 3.

THEOREM 74.

If $\rho$ and $\tau$ are two conjugate lines with respect to a circle, and meet each other at a point $T$ outside the circle, then $\rho, \tau$ are harmonic conjugates with respect to the tangents $TX, TY$ from $T$ to the circle.

Let $P$ be the pole of $\rho$; and let $X, Y$ be the points of contact of $TX, TY$.

Then it is given that $P$ lies on $\tau$.

Now the polar of $P$ passes through $T$; 

$\therefore$ the polar of $T$ (i.e. $XY$) passes through $P$; 

$\therefore P$ lies on $XY$.

Then if $PQY$ meets $\rho$ at $Q$,

$\{PQ; XY\}$ is a harmonic range; [Th. 73]

$\therefore T\{PQ; XY\}$ is a harmonic pencil; 

$\therefore \rho, \tau$ are harmonic conjugates with respect to $TX, TY$. Q.E.D.

44. A variable chord $PQ$ of a fixed circle passes through a fixed point $H$; $K$ is a point on $PQ$ such that $\{PQ; HK\}$ is harmonic; find the locus of $K$.

THEOREM 75.

If four points form a harmonic range, then their polars with respect to any circle form a harmonic pencil.

Let $\{ABCD\}$ be the harmonic range.

Let $P$ be the pole of $AD$, $O$ the centre of the circle, radius $a$.

Let $OP$ meet $AD$ at $Q$.

Draw $PA, PB, PC, PD$ perpendicular to $OA, OB, OC, OD$. 
Now the polar of $P$ passes through $A$.
\[ \therefore \text{ the polar of } A \text{ passes through } P. \]

But $PA'$ is perpendicular to $OA$.
\[ \therefore PA' \text{ is the polar of } A. \]

Similarly, $PB$, $PC$, $PD$ are the polars of $B$, $C$, $D$.

Now the rays of the pencil $PABCD$ are respectively perpendicular to the rays of the pencil $OAABCD$.
\[ \therefore \text{ each angle at the vertex of the one pencil is equal to supplementary to the corresponding angle at the vertex of the other.} \]

But $OABCD$ is harmonic.
\[ \therefore PABCD \text{ is harmonic.} \quad \text{QED.} \]

45. Draw a figure in which $Q$ lies between $A$ and $D$, and point out how this affects the proof.

46. The tangents at the extremities of a chord $PQ$ of a circle meet at $T$, prove that any other tangent to the circle is cut harmonically by $TP$, $TQ$; $PQ$ and its point of contact.

47. $ABCD$ are four points on a circle; $AB$, $CD$ meet at $G$; $AD$, $CB$ meet at $E$; $AC$, $BD$ meet at $F$; $EG$ meets $AD$, $CB$ at $P$, $Q$; prove that the polar of $P$ passes through $F$ and $Q$. Hence prove that $EFG$ is a self-conjugate triangle w.r.t. the circle.

48. With the notation of Ex. 47, prove that (1) the tangents at $C$, $D$ meet on $EF$; (2) the tangents at $B$, $D$ meet on $FG$; (3) the tangents at $A$, $B$, $C$, $D$ form a complete quadrilateral whose diagonal lines pass through the diagonal points of the complete quadrangle formed by the points $A$, $B$, $C$, $D$.

49. $AB$, $CD$ are chords of a circle, intersecting at $O$; the tangents at $A$, $D$ meet at $P$; the tangents at $B$, $C$ meet at $Q$. Prove that $P$, $O$, $Q$ are collinear.

50. A line $PAB$ meets a circle at $A$, $B$ and the polar of $P$ at $Q$; if $O$ is the mid-point of $AB$, prove that $PQ$, $PQ=PA \cdot PB$.

51. A line $PAB$ meets a circle at $A$, $B$ and the polar of $P$ at $Q$; $O$ is the mid-point of $AB$; if the polar of $P$ meets the circle at $X$, $Y$, prove that $PX$ touches the circle $OKX$.

52. $P$ is the pole of the chord $AB$ of a circle; $O$ is the mid-point of $AB$, any line through $P$ meets the circle at $C$, $D$; prove that $AB$ bisects the angle $COD$.

53. Through the mid-point $M$ of a chord $AB$ of a circle, two straight lines $MP$, $MQ$ are drawn to the circumference so that $AB$ is bisected internally and externally by the angle $PMQ$; prove that the tangents at $P$, $Q$ and the chord $AB$ are concurrent.

54. $TP$, $TQ$ are tangents to a circle; $THK$ is a line cutting the circle at $H$, $K$; $PV$ is a chord of the circle parallel to $TH$, prove that $QV$ bisects $HK$.

55. $C$ is the mid-point of a chord $AB$ of a circle; $PQ$ is another chord through $C$; if the tangents at $A$, $B$ meet at $H$, prove that $PHC=QHC$.

56. Prove that the harmonic mean of the perpendiculars from a given point $P$ inside a circle to the tangents drawn from any point on the polar of $P$ is constant.

57. $T$ is a variable point on a fixed line; $TP$, $TQ$ are the tangents to a fixed circle; $PN$, $QM$ are the perpendiculars to the fixed line, prove that $PQ=NQ$ is constant.

58. If a given quadrilateral is circumscribed to a given circle, and if a variable tangent to the circle is drawn, prove that the rectangles contained by the perpendiculars on it from each pair of opposite vertices are in a constant ratio. [Use Salmon's Theorem.]

59. $PQ$, $PR$ are two chords of a circle. The perpendicular bisector of $PQ$ meets $PR$ at $H$, prove that the line joining $H$ to the pole of $QR$ is parallel to $PQ$.

60. If two chords $AB$, $CD$ of a circle are conjugate lines, prove that $ABCD$ form a harmonic system of points on the circle.

61. [Involves cross ratios.] Prove that the cross ratio of four collinear points is equal to the cross ratio of the pencil formed by their polars.

62. [Involves cross ratios.] Two chords $AB$, $CD$ of a circle meet at $P$; $AC$ and $BD$ meet the minimum chord through $P$ in $X$ and $Y$; prove that $PX=PY$.

63. [Involves cross ratios.] $PQ$ is a chord of a circle bisecting the chord $AB$ at $C$; the tangents at $P$, $Q$ meet $AB$ at $H$, $K$; prove that $CH=CK$. 
THEOREM 76.

To prove that the diagonal point triangle $EFG$ is a self-conjugate triangle w.r.t. the circle.

$\{ABC\}$ is a harmonic range:
- the polar of $E$ passes through $P$.

$\{DEFG\}$ is a harmonic range:
- the polar of $E$ passes through $Q$;
- the polar of $E$ is $PF$ or $FG$.

Again $\{DHCG\}$ and $\{AKBG\}$ are harmonic ranges:
- the polar of $G$ passes through $H$ and $A$;
- the polar of $G$ is $HK$ or $EF$.

Similarly, the polar of $F$ is $GE$;
- the triangle $EFG$ is self-conjugate.

Q.E.D.

THEOREM 78.

With the notation of Theorem 76.

To prove that the tangents at $A, B, C, D$ form a quadrilateral having for its diagonal line triangle, the diagonal point triangle of the quadrangle $ABCD$.

$BC$ contains the pole $F$ of $EG$;
- $EG$ contains the pole of $BC$;
- the tangents at $B, C$ meet on $EG$.

Similarly, the tangents at $A, D$ meet on $EG$;
- and the tangents at $G, D$ meet on $EF$;
- and the tangents at $A, B$ meet on $EF$.

Again, $BD$ contains the pole $F$ of $FG$;
- $FG$ contains the pole of $BD$;
- the tangents at $B, D$ meet on $FG$.

Similarly, the tangents at $A, C$ meet on $FG$;
- $EF, FG, GE$ are diagonal lines of the quadrilateral formed by the tangents at $A, B, C, D$.

Q.E.D.

[The reader should draw his own figure and draw in the tangents at $A, B, C, D$.]
64. \( PQ \) is a variable diameter of a fixed circle; \( A \) is any fixed point. Find the locus of the orthocentre of the triangle \( APQ \).

65. Two tangents to a fixed circle are given; two others are drawn so as to form with the two fixed tangents a quadrilateral having a pair of opposite sides along the fixed tangents. Prove that the locus of the meet of the internal diagonals of the quadrilateral is a straight line.

66. If a quadrilateral circumscribes a circle, prove that each pair of lines joining the points of contact intersect on some one of the diagonals of the quadrilateral.

67. \( P \) is a variable point on the circumcircle of the triangle \( ABC \); \( AP, BP, CP \) meet \( BC, CA, AB \) at \( F, G, H \); prove that each side of the triangle \( FGH \) passes through a fixed point.

68. \( PQ \) is a variable diameter of a fixed circle. \( A \) is any fixed point. \( AP, AQ \) meet the circle again at \( R, S \); prove that \( RS \) passes through a fixed point.

69. Using a ruler only, construct the polar of a given point w.r.t. a given circle.

70. Using a ruler only, construct the pole of a given line w.r.t. a given circle.

71. \( ABCD \) is a cyclic quadrilateral inscribed in a circle, centre \( O \); the diagonals \( AC, BD \) meet at \( G \); \( TN, TN' \) are the perpendiculars from \( T, T' \); the poles of \( AD, BC \), to \( GO \); prove that \( TN = TN' \).

72. If in Fig. 99, \( PEQ \) is a chord through \( E \) parallel to \( FG \), prove that \( OEP \) is a right angle, where \( O \) is the centre of the circle.

73. \( PAQ \) is a chord of a circle; \( A \) is a fixed point; \( B \) is any other fixed point. \( BP, BQ \) meet the circle again at \( R, S \); prove that \( RS \) passes through a fixed point.

74. Enunciate the dual theorem corresponding to Ex. 73.

75. The tangents at two points \( A, B \) on a circle meet at right angles at \( T \). The tangent at any other point \( P \) on the circle cuts \( AB, AT \) at \( Q, R \); prove that \( TR \) bisects the angle \( PTQ \).

76. With the notation of Fig. 99, prove that the circle on \( AC \) as diameter is orthogonal to the circle \( PEG \).

77. Two sides of a triangle self-conjugate to a given circle cut the circle at four points forming the vertices of a quadrangle \( ABCD \), prove that the bisectors of the angles \( ABC, ADC \) meet on \( AC \).

78. Any two chords \( AOB, COD \) of a fixed circle meet at a given point \( O \); find the locus of the meet of \( AC, BD \).

79. Enunciate the dual theorem corresponding to Ex. 78.
CHAPTER XIII.

GOAXAL CIRCLES.

Definition.
The locus of a point \( P \) which moves so that the tangents from \( P \) to two circles are of equal length is called the **radical axis** of the two circles.

If \( A \) is the centre of a circle, radius \( a \), the length of the tangent from \( P \) to the circle is equal to \( \sqrt{AP^2 - a^2} \). If \( P \) lies inside the circle, the length is therefore imaginary. To secure continuity in the locus of \( P \) it is however desirable to include such positions of \( P \) if there are any in the locus, defined above as the radical axis. The expression \( AP^2 - a^2 \) is called the **power** of \( P \) w.r.t. the circle. Accordingly the definition, given above, may be restated as follows:

**Definition.**
The locus of a point \( P \) which moves so that its powers w.r.t. two circles are equal is called the **radical axis** of the two circles.

1. Through a point \( P \) two lines \( HP, LP \) are drawn to meet two circles, the first at \( H, K \) the second at \( L, M \); if \( HP = LP, PK = PM \), prove that \( P \) lies on their radical axis.

2. If two circles intersect at \( A, B \), prove that the straight line \( AB \) is the **radical axis** of the two circles.

3. \( AB, CD \) are chords of two circles \( S, \Sigma \); if \( ABCD \) is a cyclic quadrilateral, prove that the needs of \( AB, CD \) lies on the radical axis of \( S, \Sigma \).

4. If two circles touch at \( A \), prove that their radical axis is their **common tangent** at \( A \).

5. \( S_1, S_2, S_3 \) are three circles; if the radical axis of \( S_1 \) and \( S_2 \) meets the radical axis of \( S_3 \) and \( S_1 \) at \( H \), prove that \( H \) lies on the radical axis of \( S_1 \) and \( S_3 \).

6. \( P \) is a point on the radical axis of two circles, centres \( A, B \), radii \( a, b \); \( PN \) is the perpendicular from \( P \) to \( AB \); prove that \( AN^2 - BN^2 = a^2 - b^2 \). Hence prove that \( N \) is a fixed point and that the radical axis consists of a straight line perpendicular to \( AB \).

7. \( \Sigma \) is a given circle; \( S_1, S_2, S_3, S_4 \), etc. are a system of circles such that the radical axis of any pair is the same as that of any other pair, prove that the radical axis of \( \Sigma \) and \( S_1 \) passes through a fixed point.

8. \( H, K \) are the mid-points of the external common tangents to two circles, prove that \( HK \) is their radical axis. [Use Ex. 6.]

9. \( P, \varnothing, R, S \) are the mid-points of the common tangents of four non-intersecting circles, prove that \( PR, \varnothing R, S \) are collinear. [Use Ex. 6.]

10. Three circles are such that the radical axis of any pair is the same as that of any other pair. If \( P \varnothing \) is a real common tangent to two of them, and if it meets the third circle at \( R, S \), prove that \( \varnothing R \) is harmonic.

11. If a circle is orthogonal to each of two circles, prove that its centre lies on their radical axis.

12. A system of circles is such that the radical axis of any pair is the same as that of any other pair. If one circle is drawn orthogonal to two of the circles, prove that it is orthogonal to every circle of the system.

13. Each of a given system of circles is orthogonal to two fixed circles, prove that the radical axis of any two circles of the system is a fixed line.

Many of the properties of coaxal circles admit of a simple analytical treatment. For this purpose the following abbreviations will be useful:

\[
S_i = x^2 + y^2 + 2x_i x + 2y_i y + c_i,
\]

and similarly for \( S_1, S_2 \), etc.

The equation of any circle \( S_i = 0 \) can be written in the form

\[
(x + x_i)^2 + (y + y_i)^2 = r_i^2 + f_i^2 - c_i.
\]

This shows that its centre is \((-x_i, -y_i)\) and its radius

\[
= \sqrt{r_i^2 + f_i^2 - c_i}.
\]

1. The length of the tangent from \((x, y)\) to \( S_i = 0 \) is \( \sqrt{S_i(x, y)} \).

\( TP \) is the tangent from \( T \) to the circle, centre \( A \), whose equation is \( S_i = 0 \).

Now, \( TP^2 = TA^2 - AP^2 \).
But \( T A^2 \) equals the square of the line joining \((\xi, \eta)\) to \((-\delta_1, -f_1)\):
\[
T A^2 = (\xi + \delta_1)^2 + (\eta + f_1)^2;
\]
\[
T P^2 = (\xi + \delta_1)^2 + (\eta + f_1)^2 - (\delta_1^2 + f_1^2 - \gamma_1)
= \xi^2 + \eta^2 + 2\xi\delta_1 + 2\eta f_1 + \gamma_1
= S_1(\xi, \eta);
\]
\[
T P = \sqrt{S_1(\xi, \eta)}.
\]
Q.E.D.

II. The radical axis of two circles is a straight line perpendicular to the line joining their centres, and passes through the points of intersection (real or imaginary) of the two circles.

Let \( S_1 = 0 \) and \( S_2 = 0 \) be the two circles.

Let \((\xi, \eta)\) be a point on their radical axis:
\[
\sqrt{S_1(\xi, \eta)} = \sqrt{S_2(\xi, \eta)} \text{ by definition;}
\]
squaring,
\[
\xi^2 + \eta^2 + 2\xi\delta_1 + 2\eta f_1 + \gamma_1 = \xi^2 + \eta^2 + 2\xi\delta_2 + 2\eta f_2 + \gamma_2 \quad (1)
\]
or
\[
2\xi(\delta_1 - \delta_2) + 2\eta(f_1 - f_2) + \gamma_1 - \gamma_2 = 0; \quad \text{..............(2)}
\]
the locus of \((\xi, \eta)\) is a straight line.
Q.E.D.

Now the centres of the circles are \((-\delta_1, -f_1)\) and \((-\delta_2, -f_2)\):
\[
\therefore \text{the line joining the centres is } \frac{x + \delta_1}{\delta_1 - \delta_2} = \frac{y + f_1}{f_1 - f_2},\text{ which is clearly perpendicular to the radical axis given by}
\]
\[
x(\delta_1 - \delta_2) + y(f_1 - f_2) + \frac{1}{2}(\gamma_1 - \gamma_2) = 0 \text{ from (2).} \quad \text{Q.E.D.}
\]

Further, \((1)\) shows that any point whose coordinates satisfy \( S_1 = 0 \) and \( S_2 = 0 \) must satisfy the equation of the radical axis.
\[
\therefore \text{the radical axis passes through the common points of } S_1 = 0 \text{ and } S_2 = 0. \quad \text{Q.E.D.}
\]

**Definition.**
If a system of circles is such that the radical axis of any pair is the same as that of any other pair, the circles are said to form a coaxal system.

III. The three radical axes of three circles taken in pairs are concurrent.

Let \( S_1 = 0 \), \( S_2 = 0 \), \( S_3 = 0 \) be the three circles.

Let the radical axis of \( S_1 = 0 \) and \( S_2 = 0 \) meet the radical axis of \( S_1 = 0 \) and \( S_3 = 0 \) at \((\xi, \eta)\):
\[
\therefore S_1(\xi, \eta) = S_2(\xi, \eta) \quad \text{and} \quad S_1(\xi, \eta) = S_3(\xi, \eta) \quad \therefore S_1(\xi, \eta) = S_2(\xi, \eta).
\]
\((\xi, \eta)\) also lies on the radical axis of \( S_1 = 0 \) and \( S_3 = 0 \). Q.E.D.

14. How is this proof affected if two of the radical axes are parallel?

IV. A point \( P \) moves so that the lengths of the tangents from \( P \) to two given circles are in a constant ratio \( \lambda \); then the locus of \( P \) is a circle coaxal with the given circles.

Let \( S_1 = 0 \) and \( S_2 = 0 \) be the given circles.

Let \((\xi, \eta)\) be the coordinates of \( P \) in any one position.

Then \[
S_1(\xi, \eta) - \lambda^2 S_2(\xi, \eta) = 0 \quad \therefore \text{the locus of } P \text{ is } S_1 - \lambda^2 S_2 = 0,
\]
which is a circle, for the coefficients of \( x^2 \) and \( y^2 \) are equal and the coefficient of \( xy \) is zero.

Further, it is coaxal with \( S_1 = 0 \) and \( S_2 = 0 \), for the coordinates of any point which satisfy \( S_1 = 0 \) and \( S_2 = 0 \) must also satisfy \( S_1 - \lambda^2 S_2 = 0 \); so that it passes through the points of intersection of \( S_1 = 0 \) and \( S_2 = 0 \) and is therefore coaxal with them by II.
Q.E.D.

15. Prove independently, by using the method of I., that the tangents from any point \((\xi, \eta)\) on the line \( S_1 - S_2 = 0 \) to the circles \( S_1 = 0 \), \( S_2 = 0 \), \( S_1 - \lambda^2 S_2 = 0 \) are equal.

V. \( P \) is a variable point on a given circle coaxal with two given circles \( S_1 = 0 \), \( S_2 = 0 \); then the ratio of the lengths of the tangents from \( P \) to these two circles is constant.

The proof is left to the reader.

16. Prove \( V \), using the fact that the equation of any circle coaxal with \( S_1 = 0 \) and \( S_2 = 0 \) is of the form \( S_1 - k S_2 = 0 \), since it must pass through their points of intersection.
VI. To find the general equation of a coaxal system of circles.

Take the line of centres as \(x\)-axis, and the radical axis as the \(y\)-axis. [All the centres must be collinear, because the line joining every pair is perpendicular to a definite line, viz. the radical axis.]

The equation of any circle whose centre is on the \(x\)-axis is

\[x^2 + y^2 + 2g_xx + c = 0.\]

Let \(x^2 + y^2 + 2g_xx + c = 0\) represent any other circle of the coaxal system;

\[\therefore \text{the radical axis is } 2(g_y - g_x)x + c_1 - c_2 = 0.\]

But the radical axis is \(x = 0\);

\[\therefore c_1 - c_2 = 0 \text{ or } c_1 = c_2;\]

\[\therefore \text{every circle of the coaxal system has an equation of the form } x^2 + y^2 + 2g_xx + c = 0, \text{ where } c \text{ is the same for every circle};\]

which is the required result.

The equation can be written \((x + g_x)^2 + y^2 = g_y^2 - c\).

Its radius is therefore zero if \(g_y^2 - c = 0 \text{ or } g_y = \pm \sqrt{c}.\)

In this case, the circle reduces to a point circle

\[(x + \sqrt{c})^2 + y^2 = 0.\]

These two points (or point circles), \((+\sqrt{c}, 0)\) and \((-\sqrt{c}, 0)\), belong to the coaxal system, and are called the limiting points of the system.

17. Prove that the limiting points are real; if the radical axis does not meet any circle of the system in real points.

18. Prove that the limiting points are inverse points w.r.t. any circle of the system.

19. Prove that any circle through the points \((+\sqrt{c}, 0); (-\sqrt{c}, 0)\) is orthogonal to any circle of the form \(x^2 + y^2 + 2g_xx + c = 0\).

20. Prove that the distance of any point \(P\) on the radical axis from either limiting point is equal to the length of the tangent from \(P\) to any circle of the coaxal system.

VII. The polars of a given point \(P\), w.r.t. the circles of a coaxal system, all pass through the same point \(Q\); and \(PQ\) is bisected by the radical axis.

Let \((x, y)\) be the coordinates of \(P\), and let \(x^2 + y^2 + 2g_xx + c = 0\) be any circle of the system.

The polar of \((x, y)\) is \(x\xi + y\eta + (x + \xi) + c = 0\), or

\[\begin{cases} x\xi + y\eta + c = 0, \\ x + \xi = 0. \end{cases}\]

The coordinates of \(Q\) are therefore \((-\xi, \frac{\xi^2 - c}{\eta})\);

\(\therefore\) the coordinates of the mid-point of \(PQ\) are \((\frac{c - \xi}{2\eta}, \frac{c - \xi}{\eta})\);

\(\therefore\) the mid-point of \(PQ\) lies on \(x = 0\), i.e., on the radical axis.

Q.E.D.

21. \(P\) is a point on the radical axis of two circles, prove by analysis that the polars of \(P\) w.r.t. the circles intersect on their radical axis.

THEOREM 10.

The radical axis of two circles is a straight line perpendicular to the line joining the centres of the circles.

\[\begin{array}{c}
A, B \text{ are the centres of the two circles.} \\
\text{Let } P \text{ be a point such that the tangents } PH, PK \text{ are equal.} \\
\text{Draw } PO \text{ perpendicular to } AB. \\
\text{Let } F = PH^2 = PK^2; \\
\therefore \text{the distance of } P \text{ from the radical axis is } AF^2 = AH^2 + BK^2; \\
\text{the distance of } P \text{ from the radical axis is } AP^2 + OF^2 = AF^2 = FH^2 = FK^2; \\
\text{the distance of } P \text{ from the radical axis is } OF^2 = AP^2 - BK^2; \\
\text{the distance of } P \text{ from the radical axis is } AO + OB; \\
\text{the distance of } P \text{ from the radical axis is } AO - OB; \\
\therefore \text{since } AO + OB \text{ and } AO - OB \text{ are each constant, it follows that } O \text{ is a fixed point}; \\
\therefore \text{the locus of } P \text{ is a line through } O \text{ perpendicular to } AB. \\
\text{Q.E.D.}
\end{array}\]
If the circles intersect at real points \( C, D \); then \( P \) lies on the common chord \( CD \). When \( P \) lies between \( C \) and \( D \), the tangents are imaginary, but the relation \( PA^2 - AP^2 = PB^2 - BK^2 \) still holds good. The segment \( CD \) is therefore also regarded as part of the radical axis. [See the second definition on page 172.]

22. Prove the following construction for the radical axis of two circles, centres \( A, B \).

Draw a circle cutting one circle at \( L, M \) and the other at \( X, Y \); produce \( LM, XY \) to meet at \( P \). Draw \( PO \) perpendicular to \( AB \); then \( PO \) is the radical axis.

23. Prove that the radical axis of the incircle of the triangle \( ABC \) and the circle escribed to \( BC \) bisects \( BC \).

24. \( A, B, C, D \) are four circles; the radical axis of \( A \) and \( B \) is perpendicular to that of \( C \) and \( D \); the radical axis of \( A \) and \( C \) is perpendicular to that of \( B \) and \( D \); prove that the radical axis of \( A \) and \( D \) is perpendicular to that of \( B \) and \( C \).

25. \( A, B, C \) are the centres of three circles; if the three radical axes of these circles, taken in pairs, are such that one is equally inclined to the other two, prove that \( ABC \) is an isosceles triangle.

**THEOREM 61.**

\( S_1, S_2, S_3 \) are three circles; prove that the radical axes of \( S_1 \) and \( S_2 \); \( S_2 \) and \( S_3 \); \( S_1 \) and \( S_3 \) are concurrent.

Let \( L_{12}, L_{23}, L_{31} \) be the radical axes of \( S_1, S_2 \); \( S_2, S_3 \); \( S_1, S_3 \).

Either \( L_{12}, L_{23}, L_{31} \) are all parallel, or else at least two (say \( L_{12} \) and \( L_{23} \)) intersect; call their point of intersection \( O \).

Since \( O \) lies on \( L_{12} \), the tangents from \( O \) to \( S_1 \) and \( S_2 \) are equal.

Since \( O \) lies on \( L_{23} \), the tangents from \( O \) to \( S_2 \) and \( S_3 \) are equal.

\( \therefore \) the tangents from \( O \) to \( S_1 \) and \( S_3 \) are equal;

\( \therefore \) \( O \) lies on \( L_{13} \).

Q.E.D.

Note that the point \( O \) may be at infinity. What verbal changes should be made if \( O \) lies inside a circle?

**Definition.**

(1) The point of intersection of the three radical axes of three circles taken in pairs is called the radical centre of the three circles.

(2) A system of circles is said to be coaxal if the radical axis of any pair is the same as that of any other pair. Hence if a system of circles pass through two fixed points \( A, B \), they form a coaxal system, having \( AB \) as their radical axis.

26. Three circles are such that each touches the other two; if \( A, B, C \) are the points of contact, prove that the tangents at \( A, B, C \) are concurrent.

27. A variable circle \( \Sigma \) cuts two fixed circles \( S_1, S_2 \); prove that the common chords of \( S_1, \Sigma \) and \( S_2, \Sigma \) meet on a fixed line.

28. Three circles are such that each intersects the other two at real points, prove that their common chords are concurrent.

29. \( PP', QQ', RR' \) are pairs of points on the sides \( BC, CA, AB \) of the triangle \( ABC \). If \( PP'QQ' \); \( RR'QQ' \); 

30. Straight lines \( AP, BQ, CR \) are drawn from the vertices of the triangle \( ABC \) to the opposite sides; prove that the radical centre of the circles whose diameters are \( AP, BQ, CR \) is the orthocentre of the triangle \( ABC \).

31. A variable circle is drawn through a fixed point to touch a given circle. Find the locus of the point of intersection of the tangent at the point of contact of the two circles with the tangent at the fixed point.

32. Two circles intersect at \( P \); \( Q \) and cut a third circle \( S \) orthogonally, prove that \( P, Q \) are inverse points with \( S \).

33. Prove that the orthocentre of a triangle is the radical centre of the three circles described on the sides of the triangle as diameters.

34. \( ABCD \) is a cyclic quadrilateral. Two circles \( PAB, PCD \) are drawn to touch at \( P \); prove that the locus of \( P \) is a circle.

35. Prove that the locus of the centre of a circle which bisects the circumferences of two given circles is a straight line.

36. \( AD \) is an altitude of the triangle \( ABC \); the circles \( BAD, CAD \) meet the bisector of the angle \( BAC \) at \( P, Q \); prove that the tangents at \( P, Q \) meet on \( AD \).

37. \( A'B'C' \) are the mid-points of the sides \( BC, CA, AB \) of a triangle; prove that the external bisector of the angle \( C'A'B' \) is the radical axis of the incircle of the triangle \( ABC \) and the circle escribed to \( BC \).

38. Show how to construct a circle orthogonal to each of three given circles.
THEOREM 82.

If a system of circles is coaxal, then the centres of the circles are collinear; also, if the line of centres meets the radical axis at O, and if \( a \) is the radius of a circle of the system whose centre is \( A \), then \( OA^2 - a^2 \) is a constant for the system.

Definition.
The two point-circles of a coaxal system are called the limiting points of the system.

With the notation of Theorem 82, two points \( L, L' \) are taken on the line of centres such that \( L'0 = OL = k \); then the radii of the two circles of the system, centres \( L, L' \), are given by \( OF^2 = (\text{radius})^2 = k^2 = OI^2 \), and therefore each radius is zero.

Hence there are two point-circles \( L \) and \( L' \) which belong to the coaxal system.

Corollary.
If the centres of a system of circles are collinear; and if \( A \) is the centre of one of the circles, radius \( a \); and if \( OA^2 - a^2 \) is constant, where \( O \) is a fixed point on the line of centres, then the circles form a coaxal system.

What modification must be made in this proof; if the circles intersect at real points?

39. Prove the Corollary of Theorem 82.
47. \( P \) is any point on the radical axis of a coaxial system; \( PH \) is the tangent from \( P \) to a circle of the system, prove that \( PH = PL \).

48. \( PQ \) is a common tangent to two circles and \( L \) is one of their limiting points, prove that \( PL = 90^\circ \).

49. Prove that if a circle is orthogonal to each of two circles, centres \( A \) and \( B \), it meets \( AB \) at the limiting points of the two circles.

50. If two systems of coaxial circles have one limiting point in common, prove that the radical axis of any pair of circles, one from each system, passes through a fixed point.

51. If two systems of coaxial circles have one common circle, prove the property of Ex. 50.

52. Show how to construct a circle of a given coaxal system to touch another given circle; is there more than one solution?

53. If three circles are coaxal, prove that the common tangent of two of them is divided harmonically by the third.

54. A system of coaxial circles meet at \( A, B \); \( AC \) is a fixed line drawn through \( A \) meeting one of the circles at \( P \); prove that the locus of the foot of the perpendicular from \( B \) to the tangent at \( P \) is a straight line passing through the foot of the perpendicular from \( B \) to \( AC \).

55. A variable circle passes through two fixed points \( A, B \) and cuts a fixed circle at \( P, Q \). Prove that \( \frac{AP}{AQ} \cdot \frac{BP}{BQ} \) is constant.

56. If two coaxal systems have one circle in common, prove that they also have one common orthogonal circle.

57. Prove that the radical axes of a fixed circle and each of a system of coaxial circles are concurrent.

58. Prove that the tangent from a limiting point to any circle of the coaxal system is bisected by the radical axis.

59. A variable circle passes through two fixed points \( A, B \) and cuts a fixed circle at \( P, Q \); prove that \( PQ \) passes through a fixed point.

60. \( AB, CD \) are two parallel chords of two circles \( S_1, S_2 \); \( AC \) and \( BD \) meet at \( H, K \); prove that the locus of the meet of \( CD, HK \) is a straight line.

61. \( S_1, S_2 \) are two circles each orthogonal to the two circles \( S_1, S_2 \); prove that the radical axis of \( S_1 \) and \( S_2 \) passes through the centres of \( S_1, S_2 \).

62. If two circles cut each of a system of circles orthogonally, prove that the system is coaxal.

63. Describe a circle to pass through two given points and to bisect the circumference of a given circle.

64. \( AD, BE, CF \) are the altitudes of the triangle \( ABC \); \( P \) is any other point, prove that the circles \( PDA, PEB, PGC \) are coaxal.

65. \( ABCD \) is a cyclic quadrilateral; \( S \) is any circle having \( A, B \) as limiting points; \( Z \) is any circle having \( C, D \) as limiting points. Prove that the radical axis of \( S \) and \( Z \) passes through a fixed point.

66. A system of coaxal circles passes through the points \( A, B \); another system of coaxal circles has \( A, B \) as limiting points; prove that any circle of one system is orthogonal to any circle of the other.

67. A system of circles is drawn so that each bisects the circumferences of two given circles, prove that the system is coaxal.

**THEOREM 83.**

The limiting points of a coaxal system are inverse points w.r.t. any circle of the system. Conversely, if two points are inverse points w.r.t. each of a system of circles, then the system is coaxal, and has these two points as limiting points.

(1) Let \( OP \) be the radical axis of the system, and \( L, L' \) the limiting points.

\( A \) is the centre of any one circle of the system; the line of centres meets that circle at \( C, D \) and the radical axis at \( G \).

Draw \( OF \) to touch the circle.

Then \( OF^2 = OF' = OF^2 \), since \( L, L' \) are point-circles of the system \( = OC, OD \); \( L, L' \) are inverse points w.r.t. the circle. Q.E.D.
(2) Let $L$, $L'$ be inverse points w.r.t. each circle of a system.
Then the centres of the circles are collinear, for each must lie on $LL'$.

Let $O$ be the mid-point of $LL'$ and let $LL'$ meet one of the
circles, centre $A$, radius $a$, at $C$, $D$.

Since $L$, $L'$ are inverse points $AD^2 = AC^2 = AL \cdot AL'$;

$\therefore \{CD, LL'\}$ is harmonic;

$\therefore OA^2 = OC, OD = OA^2 - a^2$, since $O$ is the mid-point of $LL'$;

$\therefore OA^2 - a^2$ is constant and $O$ is a fixed point;

$\therefore$ the system of circles is coaxal, and $L$, $L'$ are the limiting
points. [Th. 82, Cor.]
Q.E.D.

Another method may be used to prove Theorem 83 (2). This
is embodied in Ex. 68.

68. $L$, $L'$ are inverse points w.r.t. each of a system of circles; $P$ is
a point on the perpendicular bisector of $LL'$. Use Theorem 64 (3)
to prove that the tangents from $P$ to each of the circles of the system
are equal, and hence show that the system is coaxal and has $L$, $L'$ as
limiting points.

69. Show how to draw through a given point a circle orthogonal to
two given circles.

70. Draw a circle, coaxal with two given circles, to touch a given
straight line. Is there more than one solution? Consider the case
where the straight line passes through one or both limiting points.

71. $A$, $B$ are two fixed points; $P$ is a point such that $\frac{PA}{PB} = \lambda$; then
$P$ describes a circle $\Sigma$. Prove that, as $\lambda$ varies, the circles $\Sigma$ form a
coaxal system.

72. $L$ is a limiting point of two circles; the line joining their centres
meets one circle at $A$, $B$ and the other at $A'$, $B'$. If $\frac{SL}{LA} = \frac{PL}{AL'}$, prove
that the circles are equal.

73. A straight line $ABCD$ meets one circle at $A$, $B$ and another
circle at $C$, $D$; if $L$ is a limiting point, prove that $\angle ALD' + \angle BL'C = 180^\circ$.

74. A chord $PQ$ of one circle touches a second circle at $R$; if $L$ is a
limiting point of the two circles, prove that $LR$ bisects $PQ$.

75. If a circle cuts each of two circles orthogonally, prove that its
centre lies on the radical axis, and that it cuts every circle coaxal with
the given circles orthogonally. What is meant by the circle cutting
the point-circles orthogonally?

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**THEOREM 89.**

If a circle cuts each of two circles orthogonally, it passes through
their limiting points, and is orthogonal to every circle coaxal
with them.

---

[Diagram showing circles $S_1$, $S_2$, $S_3$ centered at $H$, $K$]

Let the circle $\Sigma$, centre $P$, cut the circles $S_1$, $S_2$ orthogonally
at $H$, $K$.

Then $PH = PK$ radii; but $PH$, $PK$ are tangents to $S_1$, $S_2$;

$\therefore P$ lies on the radical axis of $S_1$ and $S_2$.

Let $L$, $L'$ be the limiting points of $S_1$ and $S_2$; and regard them
as point-circles. Since $P$ lies on the radical axis, and since
$PL, PL'$ are tangents to the point-circles $L$, $L'$,

$\therefore PL = PL' = PH$; $\therefore L$, $L'$ lie on $\Sigma$. Q.E.D.

Let any other circle $S_3$ of the system cut $\Sigma$ at $E$.
Then $PE = PH$, radii.

But $P$ lies on the radical axis, therefore $PF$ is a tangent to $S_3$;

$\therefore \Sigma$ is orthogonal to $S_3$. Q.E.D.

76. Prove that the three pairs of limiting points of three circles
taken in pairs are concyclic.

77. $AB$ is a common tangent to two non-intersecting circles, prove
that the circle on $AB$ as diameter passes through their limiting points.

78. The points $A$, $B$, $C$, taken in pairs, are the limiting points of
three systems of coaxal circles, prove that the radical centre of any
three circles, selected one from each system, is a fixed point.

79. Prove that the four circles, whose diameters are the four common
tangents of two non-intersecting circles, are coaxal.

80. Prove that any circle through the limiting points of a coaxal
system is orthogonal to every circle of the system.
THEOREM 85.

The polars of a given point $P$ w.r.t. each circle of a coaxal system pass through a fixed point $R$; and $PR$ is bisected by the radical axis.

$P$ is the given point.
Let $L, L'$ be the limiting points of the coaxal system.
Describe a circle $\Sigma$ through $P, L, L'$.
Then $\Sigma$ is orthogonal to every circle of the coaxal system, for $L, L'$ are inverse points w.r.t. any circle of the system.
Let $PR$ be a diameter of $\Sigma$ so that $R$ is a fixed point.
Let $A$ be the centre of any circle $S$ of the system.
$AP$ meets $\Sigma$ at $Q$; join $QR$.
Since $S$ is orthogonal to $\Sigma$, $AP$ touches $\Sigma$;
$\therefore$ $AQ \cdot AP = AP^2$;
$\therefore$ $Q, P$ are inverse points w.r.t. $S$.
But $PQ = 2\angle AQP = 90^\circ$ since $PP$ is a diameter of $\Sigma$;
$\therefore$ $AQ$ is the polar of $P$ w.r.t. $S$.
$\therefore$ the polar of $P$ w.r.t. $S$ passes through a fixed point $R$. Q.E.D.

Further, since $\Sigma$ passes through $L, L'$, its centre lies on the perpendicular bisector of $LL'$, i.e. on the radical axis;
$\therefore$ $PR$ meets the radical axis at the centre of $\Sigma$;
$\therefore$ $PR$ is bisected by the radical axis. Q.E.D.

It is well to point out the importance of regarding the limiting points as point-circles of the coaxal system.
Readiness in recognising this fact is frequently of use in solving riders. The fundamental pole and polar property might have been obtained in this way, as the following example will show.

COAXAL CIRCLES

I. $P$ is a given point; a variable line through $P$ meets a given circle $\Sigma$, centre $O$, at $H, K$; $Q$ is a point on $HK$ such that $(HK, PQ)$ is harmonic; then the locus of $Q$ is a straight line perpendicular to $OP$.

$\therefore$ Bisect $PQ$ at $C$.
Since $(HK, PQ)$ is harmonic, $CH = CK = CP^2$;
$\therefore$ $C$ is a point on the radical axis of $\Sigma$ and the point-circle $P$;
$\therefore$ the locus of $C$ is a straight line perpendicular to $OP$.
But $P$ is a fixed point and $PC = CQ$;
$\therefore$ the locus of $Q$ is a straight line perpendicular to $OP$. Q.E.D.

Another example is given to emphasise the same idea.

II. $H, K$ are the mid-points of the tangents from a point $T$ to a circle $\Sigma$; $P$ is any point on $HT$; prove that the tangent $PQ$ from $P$ to $\Sigma$ is equal to $PT$.

Consider the coaxal system formed by $\Sigma$ and the point-circle $T$.
Since the tangents from $H$ and also from $K$ are equal, it follows that $HK$ is the radical axis;
$\therefore$ $PT = PQ$, since $P$ lies on the radical axis. Q.E.D.
81. A variable circle passes through a fixed point and cuts a given circle orthogonally; prove that it belongs to a fixed coaxal system.

[Use the idea of 11.]

82. A variable circle passes through a fixed point and cuts a given circle orthogonally; prove that it belongs to a fixed coaxal system.

[Use the idea of 11.]

83. From a limiting point $L$ a tangent $LP$ is drawn to a circle of the coaxal system and cuts another circle of the system at $HK$; prove that $\{LP, HK\}$ is harmonic.

84. $L$ is the polar of a fixed point $P$ w.r.t. any circle of a given coaxal system; by considering $P$ as a point-circle, prove that the line parallel to $L$ and midway between $L$ and $P$ passes through a fixed point, and hence prove Theorem 85.

85. $PQ$ is a variable chord of a fixed circle $S$; $PQ$ passes through a fixed point $A$; $\Sigma$ and $\Sigma'$ are the circles on $AP, AQ$ as diameters. Prove that the common chord of $S$ and $\Sigma$ meets the common chord of $S$ and $\Sigma'$ on a fixed straight line.

86. $OP, OP'$ are the tangents from $O$ to a circle; $T$ is any point on the line $KK'$ bisecting $OP, OP'$; if the polar of $T$ meets $KK'$ at $Q$, prove that $TOQ$ is a right angle.

87. $QR$ is the tangent at $Q$ to a circle; $P$ is any other point of $QR$ is joined, and $PR$ is drawn perpendicular to $PQ$. If the rectangle $PQRS$ is completed, prove that the polar of $P$ passes through $S$.

[Prove that the centre of the rectangle lies on the radical axis of the given circle and the point-circle at $P$.

88. From two fixed points $A, B$, the tangents $AB, AO, BI, BM$ are drawn to a variable circle; if $P, Q, L, M$ are the mid-points of these four lines, prove that the locus of the meet of $PQ$ and $LM$ is a straight line.

88. The polars of a variable point $P$ w.r.t. two fixed circles meet at $Q$; if $PQ$ subtends a right angle at a fixed point $A$, find the locus of $P$.

THEOREM 86.

1. The orthocentres of the four triangles formed by four straight lines are collinear.
2. The mid-points of the three diagonals of a complete quadrilateral are collinear.
3. The circles on the three diagonals of a complete quadrilateral are coaxal, and are cut orthogonally by the polar circles of the four triangles formed by the sides of the quadrilateral.

$FG$ is the third diagonal of the quadrilateral $ABCD$.

Let $H_1$ be the orthocentre of the triangle $ABF$, $AH_1, BH_1, FH_1$ meet the opposite sides $BF, EA, AB$ at $A_1, B_1, F_1$.

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COAXAL CIRCLES

Since $H_1$ is the orthocentre,

$H_1A, H_1B, H_1C = H_1F, H_1E, H_1D = r_1$ (say).

But $AA_1, BB_1, FF_1$ are pairs of points on the circles whose diameters are $AC, BD, FG$ respectively.

Fig. 109.

\[ \therefore \text{the tangents from } H_1 \text{ to these three circles are equal.} \]

\[ \text{In exactly the same way the tangents from } H_2, H_3, H_4, \text{ the orthocentres of the triangles } ADG, DCE, CBA, \text{ are equal;} \]

\[ \therefore \text{ these three circles are coaxal and } H_1, H_2, H_3, H_4 \text{ lie on their radical axis;} \]

\[ \therefore \text{ the points } H_1, H_2, H_3, H_4 \text{ are collinear.} \]

Q.E.D.

Also the centres of the circles, which are the mid-points of $AC, BD, FG$ are collinear, since the circles are coaxal.

Q.E.D.

Finally, the circle, centre $H_1$, radius $r_1$, is orthogonal to each of these circles. But this is the polar circle of the triangle $ABF$;

\[ \therefore \text{ the four polar circles are orthogonal to the circles on diameters } AC, BD, FG. \]

Q.E.D.

It should be observed that the polar circles are not all real, for the polar circle is real, only if the triangle is obtuse-angled. And the tangents from $H_1, H_2, H_3, H_4$ to the circles on $AC, BD, FG$ are not all real, but this fact does not affect the proof.

[See the second definition, page 172.]

The reader is reminded that the circumcircle of the diagonal line triangle is also orthogonal to these three circles. [See Ex. 89.]
89. If in Fig. 109 \(AC\) meets \(BD\) at \(P\) and \(FG\) at \(R\), use the fact that \(\{APCQ\}\) is harmonic to prove that the circumcircle of the diagonal line triangle of the complete quadrilateral is orthogonal to the circles on \(AG, BD, FG\) as diameters.

90. Find the locus of a point \(P\) which moves so that its polars w.r.t. three given circles are concurrent.

91. If each of a system of circles is cut orthogonally by two given circles, prove that the system is coaxal.

92. A variable circle touches a given line at a given point, prove that the polar of any other fixed point passes through a fixed point.

93. A line is drawn cutting two non-intersecting circles; find two points on this line such that each is the point of intersection of the polars of the other w.r.t. the two circles.

94. \(PQ\) is a common tangent to two circles; prove that \(P\) and \(Q\) are conjugate points w.r.t. any coaxal circle.

95. \(P\) is a point on the radical axis of two circles, prove that the polars of \(P\) w.r.t. the two circles intersect on the radical axis. [Do not use the result of Theorem 85.]

96. \(L, L'\) are the limiting points of a coaxal system. \(A\) is the centre of any circle of the system; \(AP\) is a diameter of any circle through \(A, L\); prove that the polar of \(P\) w.r.t. any circle of the system passes through a fixed point.

97. \(AB\) is a common tangent of two circles; prove that the polar of \(A\) w.r.t. any coaxal circle passes through \(B\).

98. If the polars of a point w.r.t. three circles, whose centres are collinear, are concurrent, prove that the three circles are coaxal.

99. If the polar of a given point w.r.t. a variable circle is fixed, prove that the polar of any other given point passes through another fixed point.

100. A variable circle is drawn through two fixed points. \(C\) is a third fixed point. Find the locus of the meet of the diameter through \(C\) with the polar of \(C\) w.r.t. the circle.

101. A variable circle has two pairs of fixed conjugate points; prove that it belongs to a fixed system of coaxal circles.

102. \(P\) is any point on a given circle \(S_1\); the tangent at \(P\) to \(S_1\) meets the polar of \(P\) w.r.t. a second circle \(S_2\) at \(Q\); prove that the circle on \(PQ\) as diameter belongs to a fixed coaxal system.

103. In a trapezium \(ABCD, AB\) and \(CD\) are parallel; \(AD\) meets \(BC\) at \(E\). Prove that the common chord of the circles on \(AC\) and \(BD\) as diameters passes through \(E\) and is perpendicular to \(AB\).

104. Two circles lying outside each other are met by a variable straight line at \(A, B\) and \(P, Q\) respectively; prove that there are two possible positions of a point \(H\) such that the bisectors of the angles \(ABH, PHQ\) are at right angles.

105. Prove that the circumcentre of the diagonal line triangle of a complete quadrilateral is collinear with the orthocentres of the triangles formed by the sides of the quadrilateral.

106. Prove that the line joining the orthocentres of the triangles formed by the four sides of a quadrilateral is perpendicular to the line joining the mid-points of the diagonals of the quadrilateral.

107. A system of concentric circles, centre \(A\), is inverted w.r.t. any point \(B\); prove that the inverse system is cut orthogonally by every circle through \(B\) and the inverse point \(A'\) of \(A\), and hence show that a system of concentric circles inverts into a system of coaxial circles.

108. If a system of coaxial circles has real points of intersection, prove that it can be inverted into a system of concurrent straight lines.

109. If a system of coaxal circles has real limiting points, prove that it can be inverted into concentric circles.

[Invert w.r.t. a limiting point. What does the other limiting point invert into?]

**THEOREM 87.**

\(PH, PK\) are the tangents from a point \(P\) to two circles, centres \(A, B\); \(PN\) is the perpendicular to their radical axis; prove that \(PK^2 - PH^2 = 2AB \cdot PN\).

![Diagram](image-url)
PLANE GEOMETRY

Now since $O$ lies on the radical axis, the tangents from $O$ are equal;

\[ OA^2 - AH^2 = OB^2 - BK^2, \]

or \[ BK^2 - AT^2 = OB^2 - OA^2; \]

\[ \therefore PK^2 - PH^2 = (PM^2 + MB^2) - (PM^2 + MA^2) - (OB^2 - OA^2) \]

\[ = MB^2 - MA^2 - (OB^2 - OA^2) \]

\[ = BA(BM - MA - BO + OA) \]

\[ = BA(BM - BO + OA - MA) \]

\[ = BA(2M = 2AB, MO) \]

\[ = 2AB, PN. \]

Q.E.D.

Notice that if $P$ is in the left of the radical axis, $PK^2 - PH^2$ is negative, but in this case $PN$ and $AB$ are in opposite senses, so that the relation holds good in every case.

Theorem 87 admits of a simple analytical proof.

In Fig. 110, take $OA, ON$ as $x$-axis and $y$-axis.

Let $(x, y)$ be the coordinates of $P$.

The equations of the two circles may therefore be written (page 176).

\[ x^2 + y^2 = 25_2 x + e = 0, \text{ and } x^2 + y^2 = 25_2 x + e = 0; \]

\[ \therefore PK^2 - PH^2 = (x^2 + y^2 - 25_2 x + e) - (x^2 + y^2 - 25_2 x + e) \]

\[ = 2(x^2 - e) \]

But $OA = 5_2; OB = 2_3; \therefore x_1 - x_2 - OA - OB = OA + BO = BA \]

and $\xi = NP;$

\[ \therefore PK^2 - PH^2 = 2NP, BA = 2PN, AB. \]

Q.E.D.

Corollary 1.

From a point $P$ on a circle, centre $A$, a tangent $PK$ is drawn to a circle, centre $B; PN$ is the perpendicular from $P$ to the radical axis of the two circles; then $PK^2 = 2AB, PN$.

[Take $P$ at $H$ in Theorem 87.]

Corollary 2.

From a variable point $P$ on a fixed circle, centre $A$, tangents $PH, PK$ are drawn to two other circles, centres $B, C$; if the three circles are coaxal, then $PH$ is constant.

The proof is left to the reader (see Ex. 119).

110. Prove Corollary 2 of Theorem 87 by using Corollary 1.

111. Prove Corollary 1 of Theorem 87 without using the result of Theorem 87.

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THEOREM 88.

A point $P$ moves so that the tangents $PH, PK$ from it to two fixed circles, centres $A, B$, are in a constant ratio, then the locus of $P$ is a circle coaxal with the given circles.

Take any one position of $P$ and draw through $P$ a circle coaxal with the given circles and let $C$ be its centre.

[See Fig. 110.] $PN$ is the perpendicular from $P$ to the radical axis.

By Theorem 87, Corollary 1.

\[ PH^2 = 2CA, PN; \]

and \[ PK^2 = 2CB, PN; \]

\[ \therefore \frac{CA}{PB^2} = \text{constant, by hypothesis}; \]

\[ \therefore C \text{ is a fixed point}; \]

\[ \therefore \text{every position of } P \text{ is on the same coaxal circle.} \]

Q.E.D.

112. Prove that the locus of a point, which moves so that the difference of the squares of the tangents from it to two given circles is constant, is a straight line.

Give an analytical in addition to a geometrical proof.

113. $L$ is a limiting point of a coaxal system. $P$ is any point on a given circle of the system, prove that $PL^2$ varies as the distance of $P$ from the radical axis.

114. The external common tangents of two non-intersecting circles meet at $O$ and the internal common tangents at $O$. From the circle on $O\xi$ as diameter is coaxal with the given circles. [Use Theorem 88, taking $P$ successively at $O, O$.]

115. Given three circles $S_1, S_2, S_3$ find a point $P$ such that the tangents from $P$ to $S_1$ and $S_2$ are in a given ratio, and such that the tangent from $P$ to $S_3$ is of a given length.

116. The square of the tangent from a variable point $P$ to a fixed circle is proportional to the sum of the squares of the tangents from $P$ to two other fixed circles; if the three circles are coaxal, prove that the locus of $P$ is a circle of the same coaxal system.

117. $S_1, S_2, S_3, S_4$ are four coaxal circles; the product of the tangents from a variable point $P$ to $S_1, S_4$ is proportional to the product of the tangents from $P$ to $S_3$ and $S_4$ prove that the locus of $P$ is in general one of two coaxal circles.

118. Generalise Ex. 117 so as to obtain a theorem for $2^n$ circles.
119. Find the locus of a point $P$ which moves so that the tangent from $P$ to a fixed circle is $(i)$ equal to the distance of $P$ from a fixed point, $(ii)$ in a constant ratio to the distance of $P$ from a fixed point.

120. Two circles touch each other internally at $O$, and a line $PQRS$ is drawn, meeting one circle at $P$, $Q$, and the other at $Q$, $R$. The tangent at $P$ cuts the tangents at $Q$, $R$ in $A$, $B$; the tangent at $S$ cuts the tangents at $Q$, $R$ in $C$, $D$ : prove that $ABCD$ lie on a circle touching the given circles at $O$.

121. The tangents at $A$, $B$, $C$ to the circumcircle of the triangle $ABC$ cut the opposite sides at $P$, $Q$, $R$; $U$, $V$, $W$ are the mid-points of $AP$, $BQ$, $CR$. Prove that $UVW$ is the radical axis of the circumcircle and the polar circle of the triangle $ABC$.

122. If each of two pairs of opposite vertices of a complete quadrilateral is conjugate w.r.t. a circle, prove that the third pair is also conjugate w.r.t. the circle; and that the circle is one of a coaxal system whose radical axis is the line of collinearity of the mid-points of the diagonals.

123. A straight line $PQ$ cuts two fixed circles so that the chords $PQ$, $QR$ are in constant ratio; the tangents at $P$ and $Q$ meet at $T$: prove that the locus of $T$ is a circle coaxal with the given circles.

124. A circle $\Sigma$ touches two given circles and cuts their radical axis at $Q$, $Q$: prove that the tangents at $Q$, $Q$ are parallel to a pair of common tangents of the given circles.

[Use Theorem 87.]

125. From a point $P$, three tangents $PA_1$, $PB_1$, $PC_1$ are drawn to three coaxal circles, centres $A$, $B$, $C$; prove that $PA_1^2 + PB_1^2 + PC_1^2 = AB$. 

128. A straight line cuts one circle at $A$, $B$ and a second circle at $C$, $D$. The tangent at $A$ meets the tangents at $C$, $D$ at $P$, $Q$, and the tangent at $B$ meets the tangents at $C$, $D$ at $K$, $S$. Prove that $PQRS$ lie on a circle coaxal with the given circles.

THEOREM 89.

A system of coaxal circles can be inverted either into a system of concurrent straight lines or into a system of concentric circles.

Case I. If the circles intersect at two real points $A$, $B$, invert w.r.t. $A$; then each circle becomes a straight line passing through the inverse of $B$, and therefore the inverse figure is a system of concurrent straight lines.

Q.E.D.

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Case II. If the circles do not intersect at real points, they have two real limiting points $L$, $M$ (page 181).

Draw any circle $\Sigma$ through $L$, $M$, and let $C$ be any circle of the coaxal system. Invert w.r.t. $L$.

Denote by dashes corresponding points and curves in the inverse figure.

Since the centre of $C$ lies on $LM$, the centre of $C'$ lies also on $LMM'$.

Since $\Sigma$ passes through the limiting points, it is orthogonal to $C$;

\[ \therefore \Sigma \text{ is a straight line orthogonal to } C' \]

\[ \therefore C' \text{ is a diameter of } C' \]

\[ \text{FIG. 81.} \]

The centre of $C'$ therefore lies on $\Sigma$ and $LMM'$.

But $M'$ is the point of intersection of $\Sigma$ with $LMM'$, each of which is a diameter of $C'$;

\[ \therefore M' \text{ is the centre of } C' \]

\[ \therefore \text{every circle of the system inverts into a circle having its centre at the inverse of the other limiting point.} \quad \text{Q.E.D.} \]

The proof of Case II may run also as follows:

Draw any two circles $\Sigma_1$, $\Sigma_2$ through $L$, $M$. $C$ is orthogonal to $\Sigma_1$, $\Sigma_2$. After inversion w.r.t. $L$, the circles $\Sigma_1$, $\Sigma_2$ become two straight lines through $M'$. Since $C'$ is orthogonal to each of these lines, its centre is at $M'$;

\[ \therefore \text{the centre of } C' \text{ is a fixed point.} \quad \text{Q.E.D.} \]
127. Prove that a coaxal system when inverted w.r.t. any point inverts into a coaxal system.

[Consider what the system of circles orthogonal to the given coaxal system invert into, when the points of intersection are imaginary.]

Analysis shows that every two circles have two points of intersection (real or imaginary). If then we take an imaginary point as centre of inversion, it is possible to invert a system of coaxal circles into a system of concurrent straight lines under all conditions. Since we may regard inversion as an analytical process (see pages 123-4) any results which may be obtained by this method will be true. Greater continuity is therefore obtained in the method of proof required to establish many geometrical theorems if this extended form of inversion is admitted.

128. When inverted by the method of Theorem 90, what will the radical axis become?

THEOREM 90.

If a circle $S$ when inverted w.r.t. a circle $\Sigma$ becomes a circle $S'$, then the three circles $S$, $\Sigma$, $S'$ are coaxal.

The points common to $S$ and $\Sigma$ remain unaltered by inversion, and therefore lie on $S'$; [See Ex. 129.]

$\therefore$ the three circles have two common points.

Corollary.

A system of concentric circles may be regarded as coaxal, having the line at infinity as their radical axis.

[I invert w.r.t. any point, and use the fact that the inverse of a coaxal system is a coaxal system.]

129. Prove Theorem 90 in the case in which $S$ and $\Sigma$ do not meet at real points without using imaginary points. [Consider what the system of circles orthogonal to $S$ and $\Sigma$ invert into.]

130. $X$ is a variable circle touching two fixed circles $A, B$; the two circles $P, Q$ are drawn, which are orthogonal to $A$, $B$, and touch $X$; prove that they cut at a constant angle.

131. $X$ is a variable circle touching two fixed circles $A, B$; $L$ is a limiting point of $A, B$; $P$ is the foot of the perpendicular from $L$ to its polar w.r.t. $X$; prove that the locus of $P$ is a circle coaxal with $A, B$.

132. Any circle is drawn through two fixed points. $C$ is a third fixed point, find the locus of the inverse of $C$ w.r.t. the variable circle.

133. $T$ is a point on the radical axis of two circles; the line of centres meets the first circle at $A, B$ and the second at $C, D$. If $TA, TB$ meet the first circle at $P, Q$, and if $TD, TG$ meet the second at $Q, T$; prove that $TQPP'$ are concyclic.

134. $S_1$ and $S_2$ are two circles touching externally two circles $\Sigma_1$ and $\Sigma_2$, prove that a circle can be drawn coaxal with $\Sigma_1$ and $\Sigma_2$ to cut $S_1, S_2$ orthogonally.

135. Two variable circles touch a fixed circle and cut each other at a constant angle, if one of their points of intersection is fixed, the locus of the other.

136. A variable circle touches each of two fixed circles, prove that it cuts any other fixed circle, coaxal with them, at a constant angle.

137. $L$ is the limiting point of two circles $S_1, S_2$; two circles $\Sigma_1$ and $\Sigma_2$ are drawn through $L$ to touch $S_1$ and $S_2$ respectively and to cut at a given angle, prove that the locus of their other point of intersection is a circle coaxal with $S_1$ and $S_2$.

138. (1) If a circle $C$, diameter $d$, is inverted into a circle $C'$, diameter $d'$ w.r.t. a point outside $C'$; and if $L$ is the tangent from $O$ to $C$, and if $d$ is the radius of inversion, prove that $\frac{d'}{d} = \frac{d'}{d}$

(2) Hence if $r$ is the radius of a variable circle, which touches two fixed circles, and if $L$ is one of their limiting points, and if $LP$ is the tangent from $L$ to the variable circle, prove that $\frac{LP}{r}$ is constant.

139. Prove that if a variable circle is drawn to touch two fixed circles, the ratio of the tangents drawn from the limiting points to the variable circle is constant.

140. $P$ is a point such that the inverse of two fixed circles is a two equal circles, prove that the locus of $P$ is a circle coaxal with the given circles. Hence prove that any three circles can be inverted into three equal circles.

141. If a variable circle $P$ cuts three fixed circles at the same angle, prove by inversion that $P$ belongs to a fixed coaxal system. [Use Ex. 125 (C).]

142. Prove that the circumcircle, the nine-point circle and the polar circle of a triangle are coaxal. [Use Theorem 90.]

143. The line joining the centres of two circles $S_1, S_2$ meets $S_1$ at $A, B$ and $S_2$ at $C, D$; any circle orthogonal to $S_1$ and $S_2$ meets $S_1$ at $P, Q$ and $S_2$ at $R, S$; prove that $AP, BQ, CS, DK$ are concurrent.

144. $A, B$ are two fixed points on a fixed circle; $O$ is any other fixed point. Two circles are described to touch each other at $O$ and to pass through $A$ and $B$ respectively; if they meet the fixed circle again at
145. A straight line $PQRS$ meets one circle at $P$, $Q$ and another at $R$, $S$; if $L$ is a limiting point, prove that the angles $PLS$, $QLR$ are equal or supplementary.

148. If three circles are such that two of them do not intersect at real points, show how inversion may be used to describe a circle to touch the three given circles.

147. A system of circles pass through two fixed points $A$, $B$; $T$ is any fixed point on $AB$; a variable line through $T$ meets any circle of the system at $P$, $Q$; $O$ is any other fixed point, prove that the circle $OPQ$ belongs to a fixed coaxial system.

148. Given two circles, one inside the other, prove that a point $H$ can be found such that extreme portions of any straight line cutting both circles subtend equal angles at $H$.

149. $PT$, $PT'$ are the tangents from two points $P$, $P'$ to a circle; if $PT + PT' = PP'$, prove that $PP'$ must touch the circle. [Use Theorem 84.]

150. The tangents at $A$, $B$, $C$ to the circumcircle of the triangle $ABC$ cut $BC$, $CA$, $AB$ at $Q$, $R$, $W$; prove that the circles on $AQ$, $BW$, $CW$ as diameters are coaxal; and that the line joining the circumcentre to the orthocentre of the triangle $ABC$ is their radical axis.

151. A system of circles have two common points $A$, $B$; any line through $A$ is drawn making the circles at $P$, $Q$, $R$, $S$, ... prove that the ratios $PO:QR:RS$, etc., are constant.

152. By inverting Ex. 151 w.r.t. $A$, prove that it is equivalent to a fundamental cross ratio theorem.

153. State and prove the converse of Ex. 151.

154. Two circles, $S_1$, $S_2$, centres $A$, $B$, touch internally a third circle, centre $C$, at $P$, $Q$. If $S_1$ and $S_2$ intersect at $H$, $K$, and if $K$ lies on $PQ$, prove that $CH$ is parallel to $AB$.

155. $a$, $b$, $c$ are the radii of three coaxial circles, centres $A$, $B$, $C$; prove that $ab + bc + ca + abc = 0$.

156. The tangents at $A$, $A'$ to a given circle cut a given non-intersecting circle at $P$, $Q$ and $P'$, $Q'$; $AA'$ cuts $PP'$ at $X$; $O$ is a limiting point of the two circles; prove that (1) $PO\parallel PP'$; (2) $OX$ bisects $PP'$.

157. $L$, $L'$ are the limiting points of two circles $C_1$, $C_2$ which both enclose $L$; from any point $P$ on the line through $L'$ perpendicular to

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**Coaxial Circles**

$LL'$, tangents $PHK$, $PFK$ are drawn to the inner circle $C_1$ and meet $C_2$ at $H$, $K$ and $F$, $G$; prove that $HG$ and $FK$ intersect at $L$.

158. A quadrilateral is inscribed in one circle and circumscribed to another circle; prove that the point of intersection of the diagonals is a limiting point of the two circles.

159. $AB$ is a common tangent to two circles $HAP$, $QBR$; $O$ is any fixed point on $AB$; any line $OHFQK$ is drawn through $O$ to meet one circle at $H$, $P$ and the other at $Q$, $F$; $AP$ meets $BQ$ at $T$; prove that the locus of $T$ is a coaxal circle.

160. The incircle of the triangle $ABC$ and the circle escribed to $BC$ touch $BC$ at $X$, $X'$, and $CA$ at $Y$, $Y'$; prove that the circles on $XX'$ and $YY'$ as diameters intersect on the bisector of the angle $BAC$ at the feet of the perpendiculars to it from $B$ and $C$.

161. Prove that the centroid of three uniform thin rods forming a triangle is the radical centre of the three ex-circles of the triangle.

162. $A$, $B$ are limiting points of a coaxal system; $C$, $D$ are the centres of two circles of the system enclosing $A$, $B$ respectively. One of the common tangents of these two circles touches the circle, centre $B$, at $E$, and cuts the line through $B$, perpendicular to $AB$, at $F$; prove that $E$ is parallel to $CF$.

163. $F$ is any point on a fixed circle $S_x$. The tangent at $F$ to $S_x$ meets a second fixed circle at $H$, $K$; $Q$ is the harmonic conjugate of $P$ w.r.t. $H$, $K$; find the locus of the mid-point of $PQ$.

164. $O$ is the meet of an exterior common tangent to two circles with their radical axis; $L$, $L'$ are their limiting points; prove that $OL$, $OL'$ are parallel to the internal common tangents of the two circles.

165. $P_1Q$ is a variable chord of a fixed circle of a given coaxal system; $L$ is a limiting point; $PH$, $QK$ are the perpendiculars from $P$, $Q$ to the radical axis; prove that $PH\cdot QK$ is constant.

166. $A$ is a fixed point; $PQ$ is a variable chord of a fixed circle $\Sigma$ such that $PAQ = 90^\circ$. Prove that the locus of the mid-point of $PQ$ is a circle coaxal with $\Sigma$ and the point circle $A$. [Use Theorem 83.]

167. With the notation of Ex. 166, if the perpendicular from $A$ to $PQ$ meets $PQ$ at $D$, prove that the locus of $D$ is a circle. [Lachlan.]

168. From a limiting point $L$ of two circles, tangents $LP$, $LQ$ are drawn to the circles; if $LQ$ meets the other circle at $H$, $K$; prove that $LP = LQ.HO\cdot QK$.

169. By inverting Ex. 166 w.r.t. $A$, deduce a new theorem.

170. $O$ is a fixed point; $POQ$, $ROS$ are two variable perpendicular chords of a fixed circle; the tangents at $PQRS$ form a quadrilateral $ABCD$; prove that $A$, $B$, $C$, $D$ lie on the same fixed circle. [Use Ex. 166.]
171. Two circles are drawn in different planes. Prove that there is in general one point on the line of intersection of the planes, from which the tangents to the circles are of equal length; and that if there be more than one such point, then the circles are sections of the same sphere.

172. [Casey's method for proving Poncelet's theorem.] Let $A$, $A'$ be two points on a circle $C$, and the tangents from them to a circle $P$ meet at $G$, and cut $X$ again at $B$, $B'$; $O$, $N$ are the centres of $X$, $P$; $A, A'$ are the perpendiculars from $A$ and $A'$ to the radical axis of $X$, $P$; $P, P'$ are the perpendiculars from $B, B'$ to the radical axis; prove that

1. $AC + A'C = \sqrt{2} \cdot ON (\sqrt{P} + \sqrt{P'})$;
2. $AA' + BB' = (\sqrt{P} + \sqrt{P'}) (\sqrt{P} + \sqrt{P'})$.

173. [Poncelet's Theorem.] If a variable triangle is inscribed in a fixed circle of a coaxal system, and if two of its sides touch two other fixed circles of the system, prove that the third side touches another fixed circle of the system. [Take two positions of the triangle and use Ex. 172.]

174. Enunciate and prove a theorem similar to Ex. 173 for a polygon.

175. $PQ, P'Q'$ are two circles of a circle; $PP', QQ'$ meet at a point $O$ inside the circle; a circle $\mathcal{S}$ is drawn coaxal with the given circle, and the point circle $O$ is so as to touch $PQ, P'Q'$, prove that it will also touch $P'Q'$.

176. $LH$ is the tangent from a limiting point $L$ to any circle, centre $A$, of the coaxal system, prove that $L'H$ varies as $LA$. 

CHAPTER XIV.

CENTRES OF SIMILITUDE.

Definition.
Let $A, B$ be the centres of two circles, radii $a, b$; if two points $O, O'$ are taken on $AB$ such that $\frac{a}{b} = \frac{AO}{OB} = \frac{AO'}{OB'}$ then $O$ and $O'$ are called the external and internal centres of similitude of the two circles.

Further, the circle on $OO'$ as diameter is called the circle of similitude of the two circles. The reason for this name is contained in the property of Ex. 22.

From Theorem 4, page 5, the following results are evident:
1. The centres of similitude of two circles are points w.r.t. which the circles are homothetic.
2. The common tangents of two circles pass through one or other of the centres of similitude.
3. The lines joining the extremities of parallel diameters of two circles pass through one or other of the centres of similitude.
4. Any line through a centre of similitude meets the circles in two pairs of corresponding points, which lie at the extremities of parallel radii.

These results should however be worked through independently; they present little difficulty.

1. Work out independently the theorems 1-IV, given above.

Definition.
Two pairs of corresponding points of two circles are said to be similar, if their joints pass through the same centre of similitude.

2. Where is the internal centre of similitude of two circles which touch each other? And where is the external centre of similitude of two equal circles.
3. Where are the centres of similitude of a circle and a straight line?

4. Prove that the two centres of similitude are harmonic conjugates w.r.t. the centres of the two circles.

5. Prove that the centroid and orthocentre are the centres of similitude of the nine-point circle and the circumcircle of the triangle.

6. If \( I_1, I_2, I_3, I_4 \) are the incentre and excentres of a triangle, prove that \( I \) is a centre of similitude of the circumcircle and the circle \( I_1 I_2 I_3 I_4 \).

7. A straight line drawn through the external centre of similitude \( O \) of two circles, centres \( A, B \), meets the first circle at \( P, Q \), and the second at \( P', Q' \). Prove that the lines \( AP, AQ \) are parallel to the lines \( BP', BQ' \) and that \( OP \cdot OQ = OP' \cdot OQ' \), where \( P, Q \) are separated by both or neither of the points \( P, Q' \).

8. Prove that the six centres of similitude of three circles, centres \( A, B, C \), taken in pairs, lie in sets of three points, on four straight lines. [Apply Menelaus to the triangle \( ABC \).]

9. \( OXY, O'X'Y' \) are two of the common tangents to two circles, where \( O, O' \) are their centres of similitude. If \( a, b \) are the radii, prove that \( \frac{OX}{O'X} = \frac{OP}{OP'} \). Hence prove that the circle of similitude of the two circles is coaxal with them. [Use Theorem 88.]

10. Prove that the centres \( A, B \) of two circles, radii \( a, b \), are inverse points w.r.t. their circle of similitude.

If \( P \) is any point on the circle of similitude, prove that \( \frac{PA}{PB} = \frac{a}{b} \).

Hence prove that if two circles intersect at \( P, Q \), then their circle of similitude also passes through \( P, Q \).

11. \( Z_1, Z_2, Z_3 \) are the circles of similitude of the three circles \( Z_1, Z_2, Z_3 \); \( A, B, C \) are the centres and \( a, b, c \) the radii of \( Z_1, Z_2, Z_3 \). If \( Z_1, Z_2, Z_3 \) intersect at \( P, Q \), prove that \( \frac{PA}{PC} = \frac{PB}{PD} ; \) and that \( \frac{QA}{QO} = \frac{QB}{QC} ; \)

and hence prove that the circles \( Z_1, Z_2, Z_3 \) are coaxal. [Use Ex. 10.]

**THEOREM 91.**

\( P, Q \) are any two points on a circle; \( P', Q' \) are similar corresponding points on a second circle; then \( P, Q \) and \( P', Q' \) are parallel.

Further, if \( PP', QQ \) meet at \( O \), and if \( OP, OQ \) meet the first circle at \( X, Y \) and the second at \( X', Y' \), then

\[ OP \cdot OX' = OP' \cdot OX = OQ \cdot OY' = OQ' \cdot OY. \]

\[ A, B \text{ are the centres of the two circles.} \]

Since \( P, P' \) and \( Q, Q' \) are similar pairs of corresponding points, \( PP', QQ \) meet at a centre of similitude, \( O \).

(1) Now \[ \frac{OP}{OP'} = \frac{OA}{OQ} = \frac{OQ'}{OQ} \]

by parallels;

\[ \therefore \ QP \text{ is parallel to } PQ. \]

Q.E.D.

(2) \( AX \) is parallel to \( BX' ; \)

\[ \frac{OP}{OP'} = \frac{OA}{OQ} = \frac{OX}{OX'} \]

by parallels;

\[ \therefore \ PQX' \text{ are concyclic.} \]

Q.E.D.

(3) Also \( OXY = P'QY' \), since \( P'Q \) are coaxal with \( X'Y' \) by a method similar to (2).

Q.E.D.

12. Prove that \( PP' \) meets \( X'Y' \) on the radical axis of the two circles, with the notation of Theorem 91.

13. Prove that the meets of \( P, Q, X' \); \( Q, P', Y' \); \( X, P', Y' \); \( P', Y', X' \) are collinear, with the notation of Theorem 91.

14. Prove that \( \frac{X'Y'}{XY} = \text{ratio of radii with notation of Theorem 91.} \)

15. \( O \) is the external centre of similitude of two circles, \( O' \) is an external common tangent; \( X, Y \) are the extremities of two parallel radii drawn in the same sense; prove that \( QT; O' \) are parallel.

16. Eiculate and prove a theorem similar to Ex. 15 for the internal centre of similitude, \( O \).

17. Work through the proof of Theorem 91 for an internal centre of similitude \( O \). Is there any difference, if the circles cut at real points?
THEOREM 92.

The six centres of similitude of three circles, taken in pairs, lie in sets of three points on four straight lines forming a complete quadrilateral, whose diagonal line triangle is the triangle formed by joining the centres of the circles.

Also the three circles of similitude are coaxal.

\[ O_p, O'_p \] are the centres of similitude for the circles, centres \( A, B \); \( O_q, O'_q \); \( O_r, O'_r \) are the centres of similitude for the circles, centres \( B, C \); \( C, A \).

Let \( a, b, c \) be the radii of the circles, centres \( A, B, C \).

Then
\[
\frac{AO_o}{O_oB} = \frac{a}{b} \cdot \frac{BO'_o}{O'_oC} = \frac{c}{a} \cdot \frac{CO'_o}{O'_oA} = \frac{b}{c}.
\]

\[
\frac{AO'_o}{O'_oB} = \frac{b}{c} \cdot \frac{BO'_o}{O'_oC} = \frac{c}{a} \cdot \frac{CO'_o}{O'_oA} = \frac{a}{b}.
\]

\[
\therefore \text{by applying Menelaus to the triangle } ABC, \text{ it follows that } O_p, O'_p \text{ are collinear.}
\]

Similarly, \( O_p, O_q, O'_q \); \( O_q, O'_q, O_r \); \( O_r, O'_r, O_o \) are sets of collinear points.

Hence clearly \( ABC \) is the diagonal line triangle of the quadrilateral formed by the four lines
\( O_pO'_o, O'_oO'_q, O'_qO'_r, O'_rO'_o \).

Qu. E. D.

Again, the circles of similitude are the circles whose diameters are the three diagonals of this complete quadrilateral and are therefore coaxal. [Th. 86 (3)].

Corollary

If a circle \( S \) touches two circles \( S_1 \) and \( S_2 \) at \( P, Q \) then \( PQ \) passes through a centre of similitude of \( S_1 \) and \( S_2 \).

For \( P \) is a centre of similitude of \( S \) and \( S_1 \) and \( Q \) is a centre of similitude of \( S \) and \( S_2 \).

18. Work out in detail the proof (in Fig. 113) that \( O_p, O'_q, O'_r \) are collinear.

19. Prove independently the Corollary of Theorem 92.

20. Prove that the centres of the three circles of similitude of three circles, taken in pairs, are collinear.

21. Prove, with the notation of Theorem 92, that the circle \( ABC \) is orthogonal to each of the circles of similitude.

THEOREM 93.

The circle of similitude of two circles \( S_1 \) and \( S_2 \) is coaxal with \( S_1 \) and \( S_2 \).

Let \( OT, OT' \) be two common tangents through the centres of similitude \( O, O' \).

Let \( a, b \) be the radii of \( S_1 \) and \( S_2 \).

Then
\[
\frac{OT}{OT'} = \frac{a}{b} \cdot \frac{o'}{o'}.
\]

If then a circle is drawn through \( O \) coaxal with \( S_1 \) and \( S_2 \) it will also pass through \( O \).  [Th. 88]
But its centre must lie on \( AO'BO \) since it is coaxal with \( S_p, S_q \); therefore the circle through \( G \) coaxal with \( S_p, S_q \) is the circle on \( O'O \) as diameter.

Q.E.D.

If \( S_p \) and \( S_q \) intersect at real points, the proof suggested in Ex. 20 may be used; and if imaginary points are admitted, from analysis, this proof could be regarded as sufficient in every case.

22. \( P \) is any point on the circle of similitude of two circles \( S_p \) and \( S_q \); prove that \( S_p \) can be made homothetic to \( S_q \) by a rotation about \( P \).

23. Two circles touch at \( A \); two variable secants \( APQ, ABK \) are drawn at right angles and meet one circle at \( P, H \) and the other at \( Q, K \); prove that \( PK, QH \) intersect at a fixed point.

24. Two circles touch externally at \( S \); a tangent to one of them at \( P \) cuts the other at \( Q, R \); \( PP' \) is a diameter of the first circle, prove that \( SP'' \) bisects \( QR \).

25. Parallel chords of two circles are drawn through the points of contact of a common tangent, prove that the line joining their other extremities passes through a fixed point.

26. Prove that a line through a centre of similitude cuts off from the two circles chords whose lengths are in a constant ratio.

27. If two circles intersect, prove that the tangents at a point of intersection are equidistant from either centre of similitude.

28. Any circle through the centres of two circles is orthogonal to their circle of similitude.

29. Prove that two circles subtend equal angles at any point on their circle of similitude.

30. \( OPQ \) is the external common tangent to two circles which intersect at \( A, B \); if \( O \) is a centre of similitude, prove that \( O.P.OQ = O.A.P \).

31. The angles of a triangle \( ABC \) are given in magnitude. The point \( A \) is fixed; \( B \) moves on a fixed circle, find the locus of \( C \).

32. \( O \) is the external centre of similitude of two circles; one of the internal common tangents touches the circles at \( P, Q \); \( OP \) meets the other circle at \( L, M \), prove that \( LQ \) or \( MQ \) is a diameter of that circle.

33. \( H \) is the orthocentre and \( O \) the circumcentre of the triangle \( ABC \); \( A' \) is the mid-point of \( BC \); \( HA' \) meets the circumcircle at \( P \); prove that \( HA' = AP \) and that \( A, O, P \) are collinear.

34. A straight line \( ABCD \) cuts two circles at \( A, B \) and \( C, D \) respectively, cutting off similar segments on the same side of the line; prove that circles can be drawn to touch the given circles at \( A, D \) or at \( B, C \).

35. A straight line \( OPQQ' \) is drawn through the external centre of similitude of two circles to meet the first at \( P, Q \) and the second at \( Q', P' \); prove that \( PQ, P'Q' \) is constant.

36. Through a given point, draw a circle to touch two given circles.

37. Two spheres, radii \( a \) and \( b \) (\( a > b \)) are glued together at a point \( P \). They are placed so as to be both in contact with a horizontal table and are made to roll, without slipping, on the table. Prove that \( P \) describes a circle of radius \( \frac{a^2 - b^2}{2a} \).

38. Find when possible two real points \( P, Q \) at each of which three given circles subtend equal angles, and prove that the radical centre of the three circles lies on \( PQ \).

39. Two variable circles touch one another, and each of them touches two fixed circles; find the locus of the point of contact of the variable circles.

40. \( S_p, S_q \) are two circles touching externally each of two circles \( \Sigma_1, \Sigma_2 \); prove that a circle \( \Sigma \) can be described coaxal with \( \Sigma_1, \Sigma_2 \) and orthogonal to \( S_p, S_q \).

41. \( O \) is a centre of similitude of two circles, \( P \) is a variable point; \( H, K \) are the poles of \( OP \) w.r.t. the two circles; prove that \( HK \) passes through a fixed point.

42. \( O \) is the external centre of similitude of two circles; prove that the two circles can be inverted into each other w.r.t. \( O \) and that their radical axis inverts into the circle of similitude.

43. The incircle of the triangle \( ABC \) touches \( BC \) at \( X \), and the circle escribed to \( BC \) touches \( BC \) at \( X' \); if \( AX' \) meets the incircle at \( E, G \) prove that \( EX' \) or \( GX' \) is a diameter of the incircle.

44. Two circles intersect at \( C, D \); a third circle touches them at \( P, P' \); prove that \( CP : CP' = DP : DP' \).

45. Two circles \( APQ, ABK \) cut \( A' \); \( PH, QK \) are their common tangents; prove that the circles \( APH, AQK \) touch each other.

46. Three circles \( \Sigma_1, \Sigma_2, \Sigma_3 \) are touched externally by a circle \( \Sigma_4 \) and internally by a circle \( \Sigma_5 \); prove that the radical centre of \( \Sigma_1, \Sigma_2, \Sigma_3 \) is a centre of similitude of \( \Sigma_4 \) and \( \Sigma_5 \).

47. In the triangle \( ABC \), the circle escribed to \( BC \) touches \( BC \) at \( X' \); if \( F \) is the incentre and if \( A' \) is the mid-point of \( BC \), prove that \( FA' \) is parallel to \( AX \)
48. If two circles touch each of two other circles, so that four, or two or none of the contacts are external, prove that the radical axis of either pair passes through a centre of similarity of the other pair.

49. The internal bisector of the angle \( ABC \) meets \( BC \) at \( D \), prove that the circle on \( AD \) as diameter is coaxal with the incircle of the triangle \( ABC \) and the circle escribed to \( BC \).

50. \( P \) is any point on the circumcircle of the triangle \( ABC \); \( H \) is the orthocentre; prove that \( BP \) is bisected by the nine-point circle.

51. Given the base \( BC \) and the ratio of the sides \( \frac{AB}{AC} \) of the triangle \( ABC \), prove that the external centre of similarity of the circles escribed to \( AB \) and \( AC \) is a fixed point.

52. \( A, B, C \) are the centres of three circles, prove that the radical axis of the three circles of similarity passes through the circumcentre of the triangle \( ABC \).

53. \( O \) is the circumcentre of the triangle \( ABC \), prove that there are two points \( P, Q \) such that \( \frac{PA}{PB} = \frac{PC}{PA} \) and that \( I, P, Q \) are collinear, where \( I, P, Q \) are given constants.

54. Prove that the circle whose diameter is the third diagonal of a cyclic quadrilateral is the circle of similarity of the two circles whose diameters are the other diagonals.

55. The incircle of the triangle \( ABC \) touches \( BC \) at \( X \); \( A' \), \( K \) are mid-points of \( BC \), \( AX \); \( I \) is the incentre; prove that \( A', I, K \) are collinear. [Use Ex. 43.]

56. \( O \) is a point of intersection of two circles \( S_1 \) and \( S_2 \); \( OA, OB \) are their diameters through \( O \), prove that their circle of similarity is orthogonal to the circle \( OAK \).

MISCELLANEOUS EXAMPLES.

1. \( AP \) is a fixed diameter of a given circle, centre \( O \); \( P \) is a variable point on the circumference. \( BP \) is produced to \( Q \) so that \( BP = PQ \). Find the locus of the meet of \( AP, OQ \).

2. The sides of a variable triangle pass through three fixed points. If the triangle is similar to a fixed triangle, prove that the locus of its centroid is a circle.

3. Inscribe in a given triangle a rectangle similar to a given rectangle.

4. \( A \) is a fixed point on a fixed circle; \( AP \) is a variable chord. \( Q \) is a point on \( AP \) such that \( AP, AQ \) is constant, find the locus of \( Q \).

5. Construct a square \( ABCD \) such that \( AC = AB \) is equal to a given length.

6. Inscribe in a given triangle a parallelogram similar to a given parallelogram.

7. Construct a quadrilateral, similar to a given quadrilateral, and so that each of its sides passes through a given point.

8. In a given quadrilateral inscribe a quadrilateral, similar to another given quadrilateral.

9. \( ABC \) is a given triangle; \( O \) is any other point. The figure is rotated about \( C \), in a direction \( CA \rightarrow CB \), through an angle \( \angle ACB \); \( O, A' \) are the new positions of \( O, A \); \( O'A' \) meets \( OB \) at \( H \); and a line through \( B \) parallel to \( O'A' \) meets \( CH \) at \( K \), prove that \( \frac{CK}{CB} = \frac{CH}{CR} \). [Rouché.]

10. Construct a triangle \( ABC \) similar to a given triangle, given the point \( C \), and that \( A \), \( B \) lie on given lines \( OA, OB \). [Use Ex. 9.]

11. Construct a triangle \( ABC \) similar to a given triangle, and such that its vertices lie on three parallel straight lines.

12. \( B, B' \) are two points on two lines \( AB, AB' \). \( O \) is a point such that, by a rotation about \( O \), the line \( AB \) and the point \( B \) can be made homothetic to the line \( AB' \) and the point \( B' \); prove that the locus of \( O \) is the circle \( ABB' \).
13. \( AB, AB' \) are two given lines; \( B, B' \) being given points. Find points \( P, P' \) on \( AB, AB' \) such that \( \frac{BP}{BP'} \) is equal to a given ratio and such that \( PP' \) passes through a given point \( C \). \([\text{Rouché}]\)

14. Two sources of light of different intensities are situated at two given points. Find the locus of a point which is equally illuminated by each source.

15. If \( ABCD \) is a cyclic quadrilateral, prove that 
\[
\frac{AC}{AB} = \frac{AD + BC}{BC + AD - CD}
\]

16. If \( AB, AC \) of a triangle \( ABC \) is the mid-point of \( DE \); \( AQ \) meets \( BC \) at \( R \); prove that \( BR = AR, AE \).

17. \( C \) is a point on the diameter \( AB \) of a circle; \( P \) is any point on the circumference. A line through \( C \) perpendicular to \( AB \) cuts \( AP \), \( PB \) at \( K, H \) and the circle at \( L \), prove that \( CL = CH, CK \).

18. \( A, B, C \) are any three points, prove that the locus of a point (in space) which is equally illuminated by each of three sources of light of any intensities situated at \( A, B, C \) is a circle.

19. If \( ABC \) are taken on the sides \( BC, CA, AB \) of a triangle \( ABC \), prove that 
\[
\frac{CA}{AB} = \frac{BC}{AC} + \frac{BA}{AC} \sin\angle CBA - \sin\angle ABA.
\]

20. If \( A, B, C \) are any three points on the sides \( BC, CA, AB \) of a triangle, prove that 
\[
\frac{AC}{AB} = \frac{BC}{AC} + \frac{BA}{AC} \sin\angle CBA - \sin\angle ABA.
\]

21. \( ABC \) is a right-angled triangle at \( A \); \( PQMK \) is a square with its vertices \( P, Q \) on \( BC \) and \( K, H \) on \( AC, AB \) respectively; prove that \( PQ = BP, QC \).

22. In the triangles \( ABC, PQR \), if 
\[
ABC = PQR \quad \text{and} \quad ACB + P\hat{R}Q = 180^\circ,
\]
prove that 
\[
\frac{AC}{AC} = \frac{AB}{BP} = \frac{PQ}{\hat{P}Q}.
\]

23. With the usual notation of the triangle \( ABC \), prove that 
\[
\frac{AC}{AB} = \frac{AD}{BD} - \frac{AE}{EC}.
\]

24. \( PQ \) is a diameter of the circumscribed circle of the triangle \( ABC \). \( \rho, \gamma \) are the pedal lines of \( P, Q \) with the triangle. The perpendiculars from \( P, Q \) to \( \rho, \gamma \) respectively meet at \( R \); prove that \( R \) lies on the circumscribed circle, and that the pedal line of \( R \) is parallel to \( PQ \).

25. \( ABC \) is a triangle right-angled at \( B \); the incircle touches \( AC \) at \( J \), prove that \( AY, VC = AB, BC \).

26. With the usual notation of the triangle \( ABC \), if \( L, M \) are the feet of the perpendiculars from \( B, C \) to \( AC \); prove that \( A, D, L, M \) are concyclic.

27. With the usual notation of the triangle \( ABC \), if the triangles \( FBD, EDC \) are congruent, prove that they are congruent.

28. \( ABC \) is a cyclic quadrilateral; \( a, b, c, d \) are the orthocentres of the triangles \( BCD, CDA, DAB, ABC \); prove that \( An, Bm, Cn, Dn \) are concyclic.

29. \( J \) is the incentre of the triangle \( ABC \); the circle \( JBC \) cuts \( AB, AC \) again at \( M, N \); prove that \( MN \) touches the incircle.

30. A variable chord \( PQ \) of the circumscribed circle of the triangle \( ABC \) is parallel to \( BC \), prove that the locus of the meet of the pedal lines of \( P, Q \) with \( ABC \) is a straight line.

31. \( ABC \) is a triangle right-angled at \( A \); from any point \( P \) on \( AC \) a perpendicular is drawn to \( BC \) cutting \( AB \) at \( Q \). If \( BP \) cuts \( QC \) at \( R \), prove that the loci of \( R \) is a circle \( \Sigma \); and that the tangent at \( R \) to \( \Sigma \) bisects \( PQ \).

32. A chord \( PQ \) of the circumscribed circle of the triangle \( ABC \) is parallel to \( BC \), prove that the pedal line of \( P \) is perpendicular to \( AQ \).

33. With the notation of Ex. 26, prove that \( ab \) is a straight line to the circle \( ABCD \).

34. Two variable chords \( AB, CD \) of a circle intersect at \( O \); if \( AD \) is a diameter, prove that \( AO, AB + DO, DC \) is constant.

35. \( ABC \) is an equilateral triangle inscribed in a circle, centre \( O \). \( P \) is any point on the circumference, prove that the pedal line of \( P \) bisects \( OP \).

36. With the usual notation of the triangle \( ABC \), prove that 
\[
AB + BC + CA = 2AH + AD + 2BH, BE + 2CH, CF.
\]

37. \( AD \) is an altitude of the triangle \( ABC \), right-angled at \( A \). \( r_1, r_2 \) are the radii of the triangles \( ADC, ADB, ABC \); prove that 
\[
r_1^2 + r_2^2 = 2a^2 - 2bc.
\]

38. Enumerate and prove a theorem, similar to Ex. 37, connecting the radii of the excircles of the same three triangles.

39. \( \Delta \) is the area, and \( K \) is the circumradius of the triangle \( ABC \), prove that the perimeter of the pedal triangle is equal to \( \frac{2\Delta}{K} \).

40. \( K \) is the circumradius and \( \sigma \) is the perimeter of a triangle, prove that the area of the triangle formed by its excentres is equal to \( \sigma K \).

B.G. 0
41. Two circles, centres $A, B$, touch at $C$. $P$ is a point outside both circles. If $CP$ bisects $APB$, prove that the rectangle contained by the tangents from $P$ to the two circles is equal to $PC^2$. 

42. $ABC$ is a given triangle; $O$ is any other given point; on $OA, OB, OC$ are taken points $a, b, c$ such that $\frac{OA}{a}, \frac{OB}{b}, \frac{OC}{c} = k$; if $A', B', C'$ are the mid-points of $BC, CA, AB$, prove that the point $P$ determined by $OP = \frac{L}{1 + 2k}(OA + OB + OC)$ lies on $aA'$; hence or otherwise prove that $aA', bB', cC'$ are concurrent.

43. Two equilateral triangles $ABC, A'B'C'$ have a common orthocentre. $P, P'$ are any points on the circles $ABC, A'B'C'$ respectively; prove that $PA^2 + PB^2 + PC^2 = P'A^2 + P'B^2 + P'C^2$.

44. $ABCDEF$ is a hexagon; find a line $L$ such that the sum of the perpendiculars from $A, B, C$ to $L$ is in a constant ratio to the sum of the perpendiculars from $D, E, F$ to $L$, and such that $L$ passes through another given point.

45. $PQ$ is a variable chord of a fixed circle and subtends a right angle at a fixed point $O$; if the rectangle $POQR$ is completed, find the locus of $Q$.

46. $RS$ is a diameter of a circle; $K$ is any point on the circumference; the tangent at $K$ meets the tangents at $R, S$ in $P, Q$; if $PS$ cuts $QR$ at $L$, prove that $KL$ is perpendicular to $RS$.

47. If $P, Q, R$ are points on the sides $BC, CA, AB$ of a triangle, such that the circles on $AP, BQ, CR$ as diameters have a common point, prove that $P, Q, R$ are collinear.

48. $ABCD$ is a quadrilateral; if $AB$ is parallel to $CD$, prove that $AC + BD = AD + BC + 2AB, CD$.

49. A straight line meets the sides $BC, CA, AB$, produced of a triangle $ABC$ at $P, Q, R$; if $x, y, z$ are the lengths of the tangents from $P, Q, R$ to the circumcircle of $ABC$, prove that $\frac{AP}{BP} \cdot \frac{BQ}{CQ} \cdot \frac{CR}{AR} = 1$.

50. $P, Q$ are points on the sides $AB, AC$ of a triangle $ABC$ such that $\frac{PA}{PB} \cdot \frac{QA}{QC} = k$. $PQ$ is divided at $D$ so that $\frac{PD}{PD} = \frac{AP}{PB}$, prove that the area of the triangle $APQ$ is half that of $BDC$.

51. A straight line meets the sides $BC, CA, AB$ of a triangle at $P, Q, R$; three concurrent lines $AO, BO, CO$ meet $PR$ at $P, Q, R$; prove that $\frac{OP}{PR} \cdot \frac{OQ}{Q'P} \cdot \frac{OR}{RQ} = 1$.
66. Two chords $AB$, $CD$ of a circle are conjugate lines w.r.t. the circle: any line through $A$ cuts the circle at $P$ and $CD$ at $Q$; prove that $R(CD; PQ)$ is harmonic.

67. $ABCD$ is a quadrilateral circumscribing a circle, centre $O$; $A'$, $B'$, $C'$, $D'$ are the points of contact of $AB$, $BC$, $CD$, $DA$. If $A'P$, $B'C$ meet at $P$, prove that $PO$ is perpendicular to $AC$.

68. $FG$ is the third diagonal of a quadrangle $ABCD$, inscribed in a circle $\Sigma$; if $K$ is the mid-point of $FG$, prove that $FG$ is equal to the sum of the tangents from $K$ to $\Sigma$.

69. $O$ is a given point outside a circle. The line bisecting the tangents from $O$ to the circle cuts any chord $AB$ at $P$, prove that $P\overline{OB} = O\overline{AO}$.

70. $A$ is a fixed point: $O$ is the centre of a fixed circle; $PQ$ is a variable chord of the circle such that $OA$ bisects $P\overline{AQ}$; prove that $PQ$ passes through a fixed point. [Two Cases.]

71. $R$ is the pole of a variable chord $PQ$ of the incircle of the triangle $ABC$. If $R(AB, PC)$ is harmonic, prove that $Q$ is a fixed point.

72. The tangents from the vertices $A$, $B$, $C$ of a triangle to any circle meet the opposite sides at $P$, $P'$; $Q$, $Q'$; $R$, $R'$; if $P$, $Q$, $R$ are collinear, prove that $P'$, $Q'$, $R'$ are also collinear.

73. Two chords $AB$, $CD$ of a circle are conjugate lines w.r.t. the circle, prove that $AC$, $BD = AC +\frac{1}{2}AB + CD$.

74. With the notation of Ex. 73, if $P$ is a variable point on $AB$, and if $CP$, $DP$ cut the circle again at $Q$, $R$, prove that $QR$ passes through a fixed point.

75. $ABC$ is a self-conjugate triangle w.r.t. a circle; $P$ is any point on the circle; $BP$ meets the circle again at $Q$; $CQ$ meets the circle again at $R$; prove that $P$, $A$, $R$ are collinear.

76. With the notation of Ex. 75, prove that $BR$, $AQ$, $CP$ are concurrent.

77. $L$ is a limiting point of two circles $S_1$, $S_2$; $T$ is a point conjugate to $L$ w.r.t. $S_1$; the tangents $TPQ$, $TRS$ from $T$ to $S_1$ cut $S_2$ at $P$, $Q$ and $R$, $S$; prove that $QLS = PLR$.

78. $A$, $B$ are the centres of two circles; a variable line through $B$ cuts the circle, centre $A$ at $P$, $Q$: prove that the circle $APQ$ belongs to a fixed coaxial system.

79. A variable circle subtends given angles at two given points, prove that the locus of its centre is a circle $\Sigma$. If the given angles vary, prove that the circles $\Sigma$ generate a coaxial system.

80. Prove that the six radical axes of the incircle and excircles of a triangle, taken in pairs, form a triangle and its orthocentre.

81. If in Fig. 99, $L$, $M$, $N$ are the mid-points of $AC$, $BD$, $FG$, prove that $LM$, $FG$, $MN$, $AC$, $+\frac{1}{2}BL + BM, MN, NL = 0$.

82. $AD$, $BE$, $CF$ are the altitudes of the triangle $ABC$. $EF$, $BC$; $FD$, $CA$; $DE$, $AB$ intersect at $P$, $Q$, $R$; prove that $PQR$ is the radical axis of the circumcircle and nine-point circle of the triangle $ABC$.

83. A variable circle passes through two fixed points $A$, $B$, and cuts two fixed circles $AG$, $BC$ at $P$, $Q$. If $AQ$, $BP$ and $AB$, $PQ$ intersect at $X$, $Y$, prove that $XY$ passes through a fixed point.

84. "If a circle is inscribed in a quadrilateral $ABCD$, then its centre lies on the line joining the mid-points of the diagonals." What does this theorem become, when $AD$, $BC$ form one straight line.

85. $ABC$, $PQR$ are two self-conjugate triangles w.r.t. a circle $\Sigma$; prove that the centre of $\Sigma$ lies on the radical axes of the circles $ABC$, $PQR$.

86. $PQ$ is a variable chord of a given circle: $PH$, $QK$ are the tangents from $P$, $Q$ to a second given circle. If $PH^2 + QK^2$ is constant, find the locus of the mid-point of $PQ$.

87. Give an analytical solution of Ex. 86.

88. $ABCD$ is a quadrilateral: $AC$, $BD$ intersect at $E$. Lines, drawn through $E$ parallel to $AB$, $BC$, $CD$, $DA$, meet $CD$, $DA$, $AB$, $BC$ at $Q$, $R$, $S$; prove that $P$, $Q$, $R$, $S$ are collinear, and that $PS$ is parallel to the third diagonal of the quadrilateral.

89. Given three circles $S_0$, $S_1$, $S_2$ and three points $A_0$, $A_1$, $A_2$, construct a circle $\Sigma$ such that the radical axes of $\Sigma$, $S_1$; $\Sigma$, $S_2$; $\Sigma$, $S_0$ pass through $A_0$, $A_1$, $A_2$ respectively.

90. $A_0$, $A_1$, $A_2$ are points on the radical axes of the circles $S_0$, $S_1$; $S_1$, $S_2$; $S_2$, $S_0$. Prove that the meets of $A_1A_2$, $A_2A_3$, $A_3A_1$, $S_0$ are concyclic.

91. A point is inverted successively w.r.t. any number of circles of a coaxial system, prove that its final position could be obtained by a single inversion w.r.t. a circle of the system.

92. $ABCD$ is an isosceles trapezium, $AB$, $CD$ being parallel. In this figure $AD$, $BC$, $AC$, $BD$ represent four rods jointed together at their extremities. $O$, $P$, $P$ are points on $DA$, $DB$, $CA$ such that $DA = DB = CA$. The point $O$ is fixed. Prove that as the system of rods is allowed to move (1) $O$, $P$, $P$ are always collinear; (2) $AB$, $CD$ is constant; (3) if $P$ describes any curve, then $P$ describes an inverse curve.

This instrument was invented by Dr. Hart.
93. A sphere is inverted w.r.t. any point on its surface; determine the inverses of the parallels of longitude and latitude.

94. (1) $C$ is a fixed point on a fixed chord $AB$ of a circle; a variable chord $PQ$ is divided in a constant ratio at $R$: if $AP$, $BQ$, $CR$ concur at a point $O$, prove that the locus of $O$ is a circle $\Sigma$.

(2) If $C$ moves along the chord $AB$, prove that the circle $\Sigma$ generates a conical surface.

95. $a$, $b$, $c$, $d$ are four points on the sides $AB$, $BC$, $CD$, $DA$ of a quadrilateral inscribed in a circle $\Sigma$. $k_a$ denotes the power of $P$ w.r.t. a circle, centre $Q$, and the square of whose radius is $k_q$. Also $P_{Qa}$ stands for $((PQ)_a)$. Prove that

(1) $A_{abcd}$ coincides with $A$.

(2) If $Q$ is the point $k_{abcd}$, then $(AQ)_{abcd}$ is the same as $A_{abcd}$ and passes through $A_{abcd}$.

[Note that $(AQ)_{abcd}$ and $(AQ)$ necessarily make the same angle with $\Sigma$.]

(3) Hence show how to inscribe in a circle $\Sigma$ a quadrilateral whose sides pass through four given points $a$, $b$, $c$, $d$. [Reache.] $\Sigma$.

96. Show how to inscribe in a circle $\Sigma$ a polygon, each of whose sides passes through a given point.

97. $S$ is a variable circle of a given coaxal system. $\Sigma$ is any other fixed circle. $O$ is a fixed point; a circle $S'$ is drawn through $O$ coaxal with $S$ and $\Sigma$, prove that it belongs to a coaxal system.

98. Three given circles have two common points $A$, $B$; a variable line through $A$ cuts the circles at $P$, $Q$, $K$, prove that the tangents at $P$, $Q$, $R$ form a triangle whose circumcircle passes through the fixed point $K$.

99. Two chords $AB$, $CD$ of a circle are conjugate lines w.r.t. the circle; $P$ is any other point on the circumference. $GP$, $DP$ cut $AB$ at $H$, $K$; prove that $HD$, $KC$ intersect on the circle.

100. $S$ and $\Sigma$ are two given circles: the tangents at a variable point $P$ on $S$ meet the polar of $P$ w.r.t. $\Sigma$ at $Q$, prove that the circle on $PQ$ as diameter belongs to a fixed coaxal system.

101. Invert w.r.t. $P$ the following theorem: the tangents at the extremities of a chord $AB$ of a circle meet at $T$; $P$ is any point on the circle; $PL$, $PM$, $PN$ are the perpendiculars to $AB$, $AT$, $BT$, then $PD = PM/PN$.

102. A circle $\Sigma$ cuts each of three circles $S_1$, $S_2$, $S_3$ orthogonally. If $\Sigma$ cuts $S_1$ at $P$, $Q$, prove that there are two circles coaxal with $S_1$, $S_2$ which touch $S_3$ either at $P$ or $Q$.

103. $A$, $B$, $C$, $D$ form a harmonie system of points on a circle $\Sigma$. The figure is inverted w.r.t. any point; prove that the inverses of $A$, $B$, $C$, $D$ form a harmonie system of points on the inverse of $\Sigma$.

104. $O$ is a fixed point. $POQ$, $ROS$ are two variable chords of a fixed circle. Find the locus of the second point of intersection of the circles $PO$, $QS$.

105. $OA$, $OA'$ are two fixed lines. $C$ is a fixed point. Two lines through $C$ meet $OA$ at $P$, $Q$ and $OA'$ at $P'$, $Q'$; prove that the circles $CPQ$, $CP'Q'$ cut at the same angle as the lines $OA$, $OA'$.

106. $ABCD$ is a complete quadrangle inscribed in a circle; prove that each pair of diagonal points are the extremities of a diameter of the circle of similitude of two circles whose diameters are a pair of opposite sides.

107. Given the orthocentre and circumcentre of a triangle, find the locus of the mid-points of the sides.

108. Two circles touch at $O$: $PQ$ is a chord of one; $PH$, $QK$ are tangents to the other, prove that $\frac{PH}{PO} = \frac{QK}{QQ'}$.

109. If $PQ$ is a diameter of the circumcircle of the triangle $ABC$; prove that the pedal lines of $P$, $Q$ w.r.t. $ABC$ intersect at right angles on the nine-point circle.

110. $S_1$, $S_2$, $S_3$ are three circles. If the centres of $S_1$, $S_2$ lie on the radical axes of $S_2$, $S_3$ and $S_3$, $S_1$ respectively, prove that the centre of $S_3$ lies on the radical axis of $S_1$, $S_2$.

111. With the usual notation of the triangle $ABC$, calculate the values of the different cross ratios formed by the points $G$, $H$, $O$, $N$.

112. $ABC$, $A'B'C'$ are two sets of three collinear points. $AC$, $AC'$; $BC$, $BC'$; $AC$, $AC'$; $AB$, $AB'; AC$, $AC'$; $BC$, $BC'$ intersect at $O$, $H$, $O$, $Q$, $P$, $E$. Prove that $(1) \frac{LAHPQ}{(ABCQ)} = [KPQ]$.

(2) $P$, $Q$, $R$ are collinear.

113. $C$ is the mid-point of a chord $AB$ of a circle; two other chords $PQ$, $RS$ are drawn: $PS$, $QR$ cut $AB$ at $L$, $M$, prove that $LC = CM$.

114. $[ABCD]$, $[ABCD]$ are two equiangular ranges. $P$, $P'$ are any two points on $AP$; prove that the meets of $PB$, $PB'$; $PC$, $PC'$; $PD$, $PD'$ are collinear.

115. $ABC$ is a given triangle; $D$ is a fixed point on its circumcircle; a variable line through $D$ cuts $AB$, $BC$, $CA$ at $C'$, $A'$, $B'$ and the circumcircle at $P$; prove that $[A'B'C']$ is of constant cross ratio.
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