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PREFACE

This text-book is divided into three parts: I. The Right-Angled Triangle; II. The General Triangle and Mensuration; III. The General Angle and Compound Angles. This corresponds to the Matriculation and School Certificate stage. A further volume will deal with Higher Certificate and Scholarship work.

The authors believe that the principles of Trigonometry are most easily grasped if the numerical work is at first of a simple nature; the less time that is required for purely arithmetical computation, the more time is available for illustrating the extensive applications of the subject. The material of Part I. has been so arranged that it may be taken very early in the school course; it assumes nothing more than a knowledge of decimals, simple ratios, and the ideas of drawing to scale. The numerical work is so simple that the use of logarithms is not required. Part II. can best be taken concurrently with Mensuration in Arithmetic and the Properties of Areas in Geometry.

Diagrams have been used to illustrate examples to a much greater extent than is customary; it has thus been possible to introduce an abundance and variety of examples, which make the subject-matter interesting, without burdening the pupil with tedious and complicated verbal descriptions.

The early chapters include supplementary exercises containing harder applications of the elementary principles; these
should, in general, be reserved for a second reading, but may be utilised to keep the quicker pupils in a class profitably occupied while the others are working through the straightforward applications.

The educational value of Trigonometry lies largely in its manifold practical applications and in problems which test insight rather than technique. But progress in later work is impossible without a considerable amount of skill in manipulation, so that a substantial number of examples have been inserted for drill purposes. These are straightforward, devoid of trimming, and are designed solely for the purpose of securing methodical arrangement of the work and facility in handling trigonometrical expressions.

The book-work and illustrative examples throughout the book have been set out in the way in which the pupil would normally be expected to write them out in an examination; but notes have frequently been appended to the work in order to suggest points which may usefully be emphasised in teaching.

Methods of solving the general triangle by division into right-angled triangles have been omitted; the authors consider that it is a wrong policy to teach a method which will shortly be superseded, and with the modern emphasis on the use of formulae in Algebra there should be little risk of pupils applying the sine and cosine formulae without understanding them. Further, every text-book on Arithmetic or Algebra now includes a chapter on Logarithms, which is usually taken comparatively early in the school course. It has therefore seemed unnecessary to add a similar chapter to this volume; a brief chapter only has been included to explain the methods of using the logarithm-tables of the trigonometrical ratios.

On the other hand, the treatment of mensuration is fairly complete, partly because many of its practical applications involve the use of Trigonometry and partly because it is valuable in showing how many of the formulae of Mensuration are simplified by the use of radian measure.

Geometrical proofs of the necessary half-angle formulae have been added at the end of Chapter IX., so that, if desired, these formulae can be used, instead of the cosine formulae, for the solution of triangles.

The treatment of the General Angle has been based on the idea of coordinates. This is undoubtedly advantageous now that graphs are included early in the school course, so that the pupil is familiar with the sign conventions employed. This treatment leads to a very simple proof of the addition theorem, valid for angles of any magnitude; the authors are indebted to Professor R. S. Heath for his kind permission to include this proof, which was first given in his text-book on Elementary Trigonometry.

As some teachers may prefer to keep to the more usual elementary method of proving this theorem, it has been included as an alternative, but in the opinion of the authors Professor Heath's method is much to be preferred, both for its intrinsic interest and the ease with which it demonstrates the truth of the theorem for angles of any magnitude. The proof by methods of projection was considered too difficult for inclusion at this stage.


C. V. D.
R. M. W.

October, 1926.
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TABLES

ANSWERS
FORMULAE

A. Ratios of a single angle.

\[
\begin{align*}
\tan \theta &= \frac{\sin \theta}{\cos \theta}, & \cot \theta &= \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} \\
\sec \theta &= \frac{1}{\cos \theta}, & \csc \theta &= \frac{1}{\sin \theta} \\
\sin^2 \theta + \cos^2 \theta &= 1. \\
\sec^2 \theta &= 1 + \tan^2 \theta; & \csc^2 \theta &= 1 + \cot^2 \theta. \\
\sin 45^\circ &= \cos 45^\circ = \frac{1}{\sqrt{2}}; & \tan 45^\circ &= 1. \\
\sin 30^\circ = \cos 60^\circ &= \frac{1}{2}; & \cos 30^\circ = \sin 60^\circ &= \frac{\sqrt{3}}{2}. \\
\tan 30^\circ &= \frac{1}{\sqrt{3}}; & \tan 60^\circ &= \sqrt{3}. \\
\sin 0^\circ = \cos 90^\circ &= 0; & \cos 0^\circ = \sin 90^\circ &= 1. \\
\tan 0^\circ = \cot 90^\circ &= 0; & \tan 90^\circ \text{ and } \cot 0^\circ \text{ are } \infty. \\
\sin 15^\circ = \cos 75^\circ &= \frac{\sqrt{3} - 1}{2\sqrt{2}}; & \cos 15^\circ = \sin 75^\circ &= \frac{\sqrt{3} + 1}{2\sqrt{2}}. \\
\sin 18^\circ = \cos 72^\circ &= \frac{\sqrt{5} - 1}{4}; & \sin 54^\circ = \cos 36^\circ &= \frac{\sqrt{5} + 1}{4}. \\
\text{If } \theta \text{ is small,} \\
\sin \theta \approx \theta; & \cos \theta \approx 1 - \frac{\theta^2}{2}; & \tan \theta \approx \theta. \\
\end{align*}
\]
FORMULÆ

B. Compound angles.

\[
\begin{align*}
\sin (90^\circ - \theta) &= \cos \theta; & 
\cos (90^\circ - \theta) &= \sin \theta; & 
\text{Page} & & 24 \\
\tan (90^\circ - \theta) &= \cot \theta. & & 40 \\
\sin (180^\circ - \theta) &= \sin \theta; & 
\cos (180^\circ - \theta) &= -\cos \theta. & 
\text{Page} & & 102, 103 \\
\sin (-\theta) &= -\sin \theta; & 
\cos (-\theta) &= \cos \theta. & & 198 \\
\text{For ratios of } 90^\circ + \theta, 270^\circ \pm \theta, 180^\circ + \theta, 360^\circ - \theta, \text{ see pp. 199-203} \\
\sin (A + B) &= \sin A \cos B + \cos A \sin B. & 
\sin (A - B) &= \sin A \cos B - \cos A \sin B. & 
\text{Page} & & 209-211 \\
\cos (A + B) &= \cos A \cos B - \sin A \sin B. & 
\cos (A - B) &= \cos A \cos B + \sin A \sin B. & \\
\tan (A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B}. & 
\tan (A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}. & & 218 \\
\text{Page} & & 218 \\
\tan (A + B + C) &= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}. & & 220 \\
\tan^{-1} x + \tan^{-1} y &= \tan^{-1} \left( \frac{x + y}{1 - xy} \right). & & 213 \\
\text{Page} & & 218 \\
2 \sin A \cos B &= \sin (A + B) + \sin (A - B). & 
2 \cos A \sin B &= \sin (A + B) - \sin (A - B). & 
\text{Page} & & 227 \\
2 \cos A \cos B &= \cos (A + B) + \cos (A - B). & 
2 \sin A \sin B &= \cos (A + B) - \cos (A - B). & \\
\sin x + \sin y &= 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}. & 
\sin x - \sin y &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}. & & 230 \\
\cos x + \cos y &= 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}. & 
\cos x - \cos y &= -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}. &
\sin^2 A - \sin^2 B &= \sin (A + B) \cdot \sin (A - B). & & 223 \\
\end{align*}
\]

FORMULÆ

C. Double and half angles, etc.

\[
\begin{align*}
\cos 2A &= \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A. & & 218 \\
\sin 2A &= 2 \sin A \cos A; & 
\tan 2A &= \frac{2 \tan A}{1 - \tan^2 A}. & & 219 \\
1 + \cos \theta &= 2 \cos^2 \frac{\theta}{2}; & 
1 - \cos \theta &= 2 \sin^2 \frac{\theta}{2}. & & 219 \\
\cos \frac{\theta}{2} &= \pm \sqrt{\frac{1 + \cos \theta}{2}}; & 
\sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}}. & & 219 \\
\text{If } \tan \frac{\theta}{2} = t, \quad \sin \theta &= \frac{2t}{1 + t^2}; \quad \cos \theta &= \frac{1 - t^2}{1 + t^2}. & & 220 \\
\sin 3\theta &= 3 \sin \theta - 4 \sin^3 \theta; & 
\cos 3\theta &= 4 \cos^3 \theta - 3 \cos \theta. & & 221 \\
\end{align*}
\]

D. Triangle formulæ.

\[
\begin{align*}
\frac{b - c}{\sin A} &= \frac{c - a}{\sin B} = \frac{a - b}{\sin C}; & & 118, 184 \\
\frac{a}{\sin A} &= \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. & & 124 \\
c^2 &= b^2 + c^2 - 2bc \cos A; & 
\cos A &= \frac{b^2 + c^2 - a^2}{2bc}. & & 124 \\
a &= b \cos C + c \cos B. & & 109 \\
\Delta &= \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(s-c)}. & & 108, 178, 246 \\
\sin A &= \sqrt{\frac{(s-b)(s-c)}{bc}}; & 
\cos A &= \sqrt{\frac{(s-a)}{bc}}. & & 244 \\
\tan A &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}. & & 245 \\
\tan \frac{B-C}{2} &= \frac{b-c}{b+c}; & 
\tan \frac{B+C}{2} &= \frac{b-c}{b+c} \cot \frac{A}{2}. & & 247 \\
r &= \Delta = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. & & 184, 277 \\
r_1 &= \frac{\Delta}{s-a} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. & & 185, 277 \\
\end{align*}
\]
E. Mensuration. [For meaning of letters, see page-reference.]

\[ \pi \text{ radians} = 180^\circ; \quad \theta^\circ = \frac{180\pi}{\pi} \text{ degrees}. \]

\[ \pi \approx \frac{22}{7} \approx 3.1416; \quad \log \pi \approx 0.4681. \]

Area of trapezium = \( \frac{1}{2}(x+y) \cdot h \).

Area of parallelogram = \( xy \sin \theta \).

Volume of pyramid = \( \frac{1}{3} \text{ base} \times \text{height} \).

Volume of frustum of pyramid = \( \frac{h}{3} [(a^2 + ab + b^2)] \).

Circumference of circle = \( 2\pi r \).

Area of circle = \( \pi r^2 \).

Length of arc \( (x', \theta') = \frac{\pi x}{180} = r \theta \).

Area of sector \( (x', \theta') = \frac{\pi r^2}{360} = \frac{1}{2}r^2\theta \).

Area of curved surface of cylinder = \( 2\pi rh \).

Volume of cylinder = \( \pi r^2 h \).

Area of curved surface of cone = \( \pi rl \).

Volume of cone = \( \frac{1}{3} \pi r^2 h \).

Area of curved surface of frustum of cone = \( \pi l(a+b) \).

Volume of frustum of cone = \( \frac{r^2 h}{3} (a^3 + ab + b^2) \).

Area of surface of sphere = \( 4\pi r^2 \).

Volume of sphere = \( \frac{4}{3}\pi r^3 \).

Dip = \( \sqrt{\left(\frac{2h}{r}\right)} \) radians.

Distance of horizon = \( \sqrt{2hr} = \sqrt{\left(\frac{2h}{r}\right)} \) miles.

---

CHAPTER I.

THE TANGENT OF AN ANGLE.

Historical Note. Numerical Trigonometry was originally devised to meet the needs of the astronomer. The first ideas may be traced as far back as the time of Ahmes, about 1700 B.C., but the earliest systematic treatment is attributed to an astronomer, Hipparchus (150 B.C.), who not only constructed the equivalent of a Table of natural sines but also investigated right-angled spherical triangles. Progress was slow owing to the absence of any suitable notation. Some advance was made by the Arabic School of Bagdad between 800 A.D. and 1400 A.D., the purely trigonommetrical treatise being written by a Persian in the thirteenth century. Knowledge of what the Arabs had done gradually reached Europe through Spain; and by the sixteenth century English mathematicians had obtained a general acquaintance with the methods of plane and spherical numerical Trigonometry, while about this time the six ratios received their standard names. The construction of Tables had naturally attracted the attention of mathematicians and astronomers from early times. The most famous of these is the Opus Palatinum, compiled by Rheticus and a number of assistants and published in 1596; it gives all six ratios at 10° intervals to ten decimal places. In the seventeenth century progress became rapid, elementary algebra was assuming its modern form, and this invention of a simple symbolic notation transformed Trigonometry into an analytical subject. Newton's expansions for \( \sin x \) and \( \cos x \), date from 1666; De Moivre's theorem probably from 1707, De Lagny's expansion for \( \tan x \) from 1710, and Lambert's hyperbolic functions from 1760.

The practical application of Trigonometry to the problems of surveying was an afterthought; one of the earliest books...
dealing with this aspect is the *Practica Geometrias* of Leonardo of Pisa (1220 A.D.). The reader has already learnt in his elementary geometry how to apply the method of scale-drawing to problems in surveying, the data for which are obtained by using a chain to measure lengths and a theodolite to measure angles both in a vertical and in a horizontal plane. Thales (600 B.C.) had made use of the same idea, the principle of similar figures, to find the height of the pyramids by measuring the lengths of their shadows. By the aid of Trigonometry such problems may now be solved—and to a higher degree of accuracy—by calculation, but the theory is based on the same principles.

**Angles.** The existing method of measuring angles is modern. In early days astronomers took a circle of some convenient fixed radius and divided the circumference into a number of equal arcs, and worked with these arcs where we now work with angles, that is to say, they measured the length of an arc where we measure the angle standing at the centre on that arc, and they measured the half-chord cutting off an arc where we measure the sine of half the angle at the centre standing on that arc. Whereas we divide four right angles into 360 degrees, the Greeks in the time of Ptolemy (85-165 A.D.), divided the circumference into 360 equal arcs, each arc being called a degree and regarded as the unit measure; they then called \( \frac{1}{60} \) of a degree a first part (Latin, *pars minuta prima*, hence our name “minute”), and \( \frac{1}{60} \) of a degree a second part (Latin, *pars minuta secunda*, hence our name “second”). This is called the *sexagesimal measure of angles*:

- 1 degree = 60 minutes (60’) ; 1 minute = 60 seconds (60”).

The reader is reminded of the following definitions:

1. Two angles are said to be *complementary* if their sum is 90°.
2. Two angles are said to be *supplementary* if their sum is 180°.
3. **Bearings.** There are two principal methods of indicating the direction of any point \( P \) from a given point or origin \( O \) in the same horizontal plane.

(c) **The surveyor’s method.** The direction of a horizontal line \( OP \) is given in terms of the cardinal directions N., E., S., W.; thus, if the direction \( OP \) is given as N. 53° E., a man standing at \( O \) facing due North and then turning through 53° towards the East is now facing \( P \). Similarly, if the direction \( OQ \) is S. 19° E., or, in other words, if the “bearing” of \( Q \) from \( O \) is S. 19° E., a man standing at \( O \) facing due South and then turning through 19° towards the East is now facing \( Q \).

**Note.** In using this method bearings should always be reckoned from the North or from the South, not from the East or West. Thus \( OP \) should be described as N. 53° E., not E. 37° N.; similarly one should say S. 78° W., not W. 12° S. This practice is adopted to avoid the possibility of mis-reading.

(b) **The soldier’s method.** In the army all bearings are given from the geographical or the true North. The “true bearing” of a line is the angle the line makes with the true
TRIGONOMETRY

North, the angle being measured in a clock-wise direction, i.e. from the North through East and South.

Thus the bearing of A from O in Fig. 2 (i) would be given as a bearing of 150° by this method, and not as S. 30° E.; and the bearing of B from O in Fig. 2 (ii) would be given as a bearing of 240°, and not as S. 60° W.

Note. Complications are introduced in practice from the fact that “True” North, “Magnetic” North, and “Grid” North differ; for details reference should be made to books on practical surveying and military manuals on map-reading.

EXERCISE 1. a.

1. How many degrees are there (i) in 3 right angles, (ii) in half a right angle, (iii) in § right angle?
2. What is (i) the complement of 20°, (ii) the supplement of 25°, (iii) the supplement of 54° 20', (iv) the complement of 72° 50', (v) the supplement of 137° 25'?
3. What is the third angle of a triangle if two of the angles are (i) 90°, 40°; (ii) 90°, 20° 30'; (iii) 90°, 63° 59'; (iv) 27°, 66°; (v) 105° 15', 30° 55'?
4. How many degrees are there between (i) N.E. and S.E.; (ii) S. 10° W. and S. 45° E.; (iii) S. 10° W. and N. 50° W.; (iv) a bearing of 100° and a bearing of 210°; (v) a bearing of 20° and a bearing of 530°?
5. Give the following bearings in the army form: (i) N. 70° E.; (ii) S. 10° E.; (iii) S. 20° W.; (iv) N. 50° W.
6. Give the following true bearings in the surveyor’s form: (i) 50°; (ii) 210°; (iii) 308°; (iv) 110°.
7. What is the bearing of O from A if the bearing of A from O is (i) N. 10° E.; (ii) S. 14° E.; (iii) 15°; (iv) 310°?
8. The angle of elevation of Q from P is observed to be 18° 45'. What is the observation of P from Q?
9. The angle of depression of a boat from the top of a cliff is observed to be 15° 27'. What is the elevation of the top of the cliff as seen by a man in the boat?

(Written.)

10. Express in degrees, minutes, seconds, correct to the nearest second, (i) 28° 37' 2"; (ii) § right angle; (iii) 10,000 seconds; (iv) 232' 4" minutes.
11. Express as a decimal of a degree correct to 3 places of decimals, (i) $25^\circ 30' 25''$; (ii) $10^\circ 17' 20''$.

12. Through what angle does the earth turn in one minute of time?

13. Through what angle does the hour hand of a clock turn in one minute of time?

14. Cape Town has latitude $33^\circ 40' S$, and longitude $18^\circ 30' E$. Cologne has latitude $50^\circ 55' N$, and longitude $7^\circ E$. What is their difference of latitude and longitude?

15. From A the bearing of B is N. $80^\circ$ E. and the bearing of C is N. $50^\circ$ E.; also B and C are equidistant from A. What is the bearing of C from B?

**Similar triangles.** The subject of Trigonometry depends in the first instance upon the fact that two equiangular triangles have their corresponding sides proportional. If a number of triangles are drawn with the same set of angles they will all have the same shape. The following experiment can be performed by a class of pupils:

"Draw a triangle ABC having $\angle A=40^\circ$, $\angle B=90^\circ$, $\angle C=50^\circ$. Measure AB, BC. Work out the value of the ratio $\frac{BC}{AB}$.”

The triangles thus drawn will differ in size, but they should all have the same shape and, subject to errors of experiment,

the values obtained for the ratio $\frac{BC}{AB}$ should all be the same, viz., about 0.84.

**Example I.** A pole 10' high casts a shadow $6\frac{1}{2}'$ long; at the same time the shadow of a church tower is 52' long. What is the height of the tower?

$\triangle ABC, XYZ$ represent the triangles formed by the pole and tower and their shadows (see Fig. 5). Since the sun's rays strike the earth at the same angle, $\angle O = \angle Z$. Also $\angle Z = \angle Y = 90^\circ$. $\therefore \triangle A\triangle B$, the $\triangle s$ are equiangular and similar.

If $XY$, the height of the tower, is $h$ ft.,

\[
\begin{align*}
 h & = \frac{10 \times 52}{6\frac{1}{2}} \\
 & = 80 \\
\end{align*}
\]

\[\therefore\] the height of the tower is 80 ft.

**Exercise I. a.**

1. Draw two triangles $ABC, PQR$, having $\angle A = \angle P = 25^\circ$, $\angle B = \angle Q = 90^\circ$, $AB = 4$ in., $PQ = 3$ in. Measure $BC$ and $QR$ and find the values of the ratios $\frac{BC}{AB}, \frac{QR}{PQ}$.

2. Draw two triangles $ABC, XYZ$ having $\angle A = \angle X = 33^\circ$, $\angle B = \angle Y = 90^\circ$, $AB = 10$ cm., $XY = 3$ in. Measure $BC$ in cm. and $YZ$ in inches, and find the values of the ratios $\frac{BC}{AB}, \frac{YZ}{XY}$.

3. If, in Fig. 6, $BC$ is drawn 2 cm. long, it is found that $AC$ is 1.20 cm. long.

\[\text{What is PR if} (i) \, QR = 5 \text{ cm.}; (ii) \, QR = 5 \text{ in.} \]
\[\text{What is QR if} (i) \, PR = 3 \text{ cm.}; (ii) \, PR = 3 \text{ in.} \]
\[\frac{AC}{QR} \text{ and PR?} \]
TRIGONOMETRY

4. ABC, PQR are two triangles, right-angled at C and R, and such that the angles at B, Q are each 58°. If BC is 5 cm., it is found by measurement that CA is 8 cm., and

What is PR if (i) QR = 7 cm.? (ii) QR = 8 in.?
What is QR if (i) PR = 6 cm.? (ii) PR = 1 ft.?
What are the values of AC and GB?

5. When the shadow of a vertical stick 3 ft. high is 3 ft. 9 in. long, the shadow of a tower is 90 ft. long. What is the height of the tower?

6. A halfpenny (diameter 1 inch) placed at a distance of 3 yds. from the eye will just obscure the disc of the sun or moon. Taking the distance of the sun as 93 million miles, find its diameter. Taking the diameter of the moon as 2160 miles, find its distance.

7. How far in front of a pinhole camera must a man 6 ft. high stand in order that a full-length photograph may be taken on a film 3 in. high and 24 in. from the pin-hole?

8. Two scale-drawings are made of a rectangular court 100 yd. long, 50 yd. wide, one on a scale of 10 yd. to the cm., the other on a scale of 20 yd. to the inch. What are the dimensions of the drawings? Are they the same shape?

9. A path 1 yd. wide runs all round a rectangular lawn 20 yd. long, 15 yd. wide. Is the rectangle formed by the outer edge of the path the same shape as the lawn?

10. The radius of the base of a cone is 8° and its height is 15°. What is the radius of a section parallel to the base and 6° from it?

The tangent of an angle. Example 1. on p. 7 and the examples in Exercise 1. b. are illustrations of the fact that if two triangles are of the same shape the ratio of a pair of sides in one triangle is equal to the ratio of the pair of corresponding sides in the other.

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For instance, the \( \triangle \text{APM}, \text{BQN} \) in Fig. 7 are the same shape, and the ratio \( \frac{\text{MP}}{\text{AM}} \) is equal to the ratio \( \frac{\text{NQ}}{\text{BN}} \).

The reader can draw two triangles of this shape, making, for instance, in one \( \text{AM} = 10 \) cm., and in the other \( \text{BN} = 2 \). He should find \( \text{MP} = 8 \) cm. and \( \text{NQ} = 1.62 \) cm., and obtain the results

\[
\frac{\text{MP}}{\text{AM}} = \frac{8}{10} = 0.81, \quad \frac{\text{NQ}}{\text{BN}} = \frac{1.62}{2} = 0.81.
\]

The value of this ratio depends only on the fact that in both these triangles one angle is 39° and one angle is 90°.

The ratio \( \frac{\text{MP}}{\text{AM}} \) in Fig. 7 is called the tangent of the angle \( \angle \text{PAM} \), and is written \( \tan \angle \text{PAM} \) or \( \tan 39° \), since \( \angle \text{PAM} = 39° \).

The general statement may now be made:

If a perpendicular is drawn from any point in either arm of an angle to the other arm, the tangent of the angle \( \angle \text{PAM} \) is defined as the ratio of the side opposite the angle \( \angle \) to the side adjacent to the angle \( \angle \), i.e., \( \frac{\text{MP}}{\text{AM}} \).

The approximate value of the tangent of an angle may be found by measurement; for instance, if Fig. 7 is drawn accurately with \( \text{AM} = 10 \) cm., it will be found that

\[
\tan 39° = \frac{\text{MP}}{\text{AM}} = \frac{8.1}{10} = 0.81.
\]

But the tangents of angles have been calculated once for all by mathematicians, and have been entered in books of Tables from which they can be obtained when required. A book of seven-figure Tables will give \( \tan 39° = 0.8097840 \). Seven-figure Tables are required by astronomers, but for most practical purposes four-figure Tables are sufficiently accurate; they will give

\[
\tan 39° = 0.8096.
\]

Use of tangent Tables. A book of four-figure Tables (see p. 10) gives the tangents of angles from 0° to 90° at intervals of 6
minutes, and by means of the difference-columns at the side it is possible to find the values for intervals of 1 minute.

Extract from Table of Natural Tangents.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
</tr>
<tr>
<td>56</td>
<td>13</td>
<td>91</td>
<td>13</td>
<td>19</td>
<td>25</td>
<td>31</td>
</tr>
</tbody>
</table>

This extract shows that \(\tan 50^\circ = 1.1918\) (to 4 places).

Similarly, \(\tan 50^\circ 12'' = 1.2002\), etc.

To find \(\tan 50^\circ 42''\), we say

\[\tan 50^\circ 42'' = 1.2218\]

Difference for 2' = 0.0014;

\[\therefore \tan 50^\circ 44'' = 1.2232\]

Note. The difference columns can only give average differences, and the fourth decimal place is not therefore reliable. The results, as in any work with four-figure logarithms, are only approximate. Four figures should always be retained throughout the working of an example, but the final result should be given correct to three significant figures as a general rule.

Note. The "tangent" of an angle was first used by Abul-Wefa (940-998 A.D.); he also formulated the relations

\[\tan \theta = \sin \theta / \cos \theta; \quad \sec^2 \theta = 1 + \tan^2 \theta; \quad \csc^2 \theta = 1 + \cot^2 \theta; \quad \sin 2\theta = 2 \sin \theta \cos \theta.\]

The relation \(\sin^2 \theta + \cos^2 \theta = 1\) was recognised by Ptolemy.

Example II. Find, by drawing, approximate values of \(\tan 12^\circ, \tan 24^\circ, \tan 36^\circ, \tan 48^\circ\).

(Note. The reader should draw his own figure and take 1 decimetre as his unit. Figure 8 represents part of a circle of radius 1 inch.)

Draw the tangent at \(A\) to the circle and draw lines \(AP, AQ, OR, OS\) cutting the tangent at \(P, Q, R, S\), and such that

\[\angle AOP = 12^\circ, \quad \angle AOQ = 24^\circ, \quad \angle AOR = 36^\circ, \quad \angle AOS = 48^\circ.\]

The angle at \(A\) is \(90^\circ\);

\[\therefore \quad \tan 12^\circ = \frac{AP}{OA}.\]

By measurement \(AP = 0.21\) in.; also \(OA = 1\) in.;

\[\therefore \quad \tan 12^\circ = \frac{0.21}{1} \approx 0.21\] approx.

Similarly by measurement,

\(AQ = 0.45\) in., \(AR = 0.73\) in., \(AS = 1.11\) in.

\[\therefore \quad \tan 24^\circ = \frac{0.45}{1} = 0.45\] approx.,

\[\tan 36^\circ = \frac{0.73}{1} = 0.73\] approx.,

\[\tan 48^\circ = \frac{1.11}{1} = 1.11\] approx.

If then the radius is unity, the length (or rather the number of units in the length) cut off on the tangent, as drawn in Fig. 8, represents the tangent of the corresponding angle. This is the reason for the name chosen for this particular ratio. But it is important to notice that the tangent of an angle is a ratio, i.e. a pure number independent of any unit of length used in finding or applying it. Further, the tangent of an angle is not
directly proportional to the size of the angle. Thus \( \tan 24^\circ \) is more than twice \( \tan 12^\circ \), and \( \tan 48^\circ \) is more than twice \( \tan 24^\circ \); and the nearer the angle approaches 90° the larger the value of its tangent becomes; by taking an angle sufficiently near 90° we can make the value of its tangent as large as we please.

**Example III.** Given a triangle ABC such that \( \angle AOB = 90^\circ \), \( \angle ABC = 56^\circ \), \( AC = 6^\prime \), calculate BC.

Since \( \angle ABC = 56^\circ \), \( \angle BAC = 90^\circ - 56^\circ = 34^\circ \).

Then \[ BC = AC \cdot \frac{BC}{AC} = 6 \tan 34^\circ \]
\[ = 6 \times 0.6745 = 4.047 = 4.05 \text{ inches (to 3 figures).} \]

Note. We might also argue as follows:

\[ \frac{AC}{BC} = \tan 56^\circ ; \quad \frac{BC}{AC} = \tan 56^\circ ; \]
\[ \therefore BC = \frac{AC \cdot tan 56^\circ}{6} = 4\text{.}047 = 4\text{.}05 \text{ inches (to 3 figures).} \]

The first method is obviously simpler. The reader should note that \( \tan 56^\circ = \frac{1}{\tan 34^\circ} \) and that in general the tangent of any angle is equal to the reciprocal of the tangent of the complementary angle.

**Notation.** If ABC is a triangle, it is usual to denote the lengths of the sides BC, CA, AB by \( a \), \( b \), \( c \) respectively, and the magnitudes of the opposite angles by A, B, C respectively.

Thus in the above example \( b = 6^\prime \), \( B = 56^\circ \), \( C = 90^\circ \).

**Exercise I.**

[All results involving calculation should be given correct to three significant figures, unless otherwise stated (see note on p. 16).]

1. Find by drawing the values of \( \tan 20^\circ \), \( \tan 40^\circ \), \( \tan 45^\circ \), \( \tan 50^\circ \), \( \tan 60^\circ \), \( \tan 75^\circ \). (It saves time to use squared paper.) Then find their values from the Tables.

2. Find by drawing (preferably using squared paper) the angles whose tangents are \( 4 \), \( 0 \)₉, \( 16 \), \( 23 \). Then find these angles from the Tables.

3. Use the Tables to write down the values of \( \tan 62^\circ \), \( \tan 32^\circ 24^\prime \), \( \tan 13^\circ 45^\prime \), \( \tan 63^\circ 30^\prime \).
4. Use the Tables to write down the angles whose tangents are 0·4245, 2·9042, 0·2754, 3·0061, 28·64, 0·2636, 1·2016, 0·8922.

5. Find the marked angles in Fig. 11, (i), (ii), (iii), (iv), given that the triangles are right-angled.

6. Find the marked angles in Fig. 12, (i), (ii), (iii).

7. Find the lengths of the marked sides in Fig. 13, (i), (ii), (iii), (iv), given that the triangles are right-angled.

8. Find the lengths of the marked sides in Fig. 14, (i), (ii), (iii).

9. ABC is an equilateral triangle of side 2 inches; AD is perpendicular to BC (Fig. 15). Use Pythagoras to prove that AD = √3 inches. Then calculate tan 60° and tan 30°, and compare with the Tables.

10. Find from a suitable figure the value of tan 45°.

11. ABC is an isosceles triangle with AB = AC; AD is drawn perpendicular to BC.
   (i) If B = 37°, a = 6 cm., calculate AD.
   (ii) If B = 42°, AD = 5 cm., calculate BC.
   (iii) If A = 62°, a = 4 in., calculate AD.

12. From a point on the ground, 100 yards away from a tower, the angle of elevation of the top of the tower is 33° 30′. Find the height of the tower.

13. From the top of a cliff 250 feet high the angle of depression of a boat is 17°. Find the distance of the boat from the cliff.

14. The shadow of a vertical pole 12 ft. high is 17 ft. 4 in. long. What is the altitude of the sun?

15. The vertical angle of a cone is 102°, and the diameter of its base is 5 inches. What is its height?

16. A ladder leaning against a vertical wall makes an angle of 21° with the wall; the foot of the ladder is 5 ft. from the wall. How high up the wall does the ladder reach?

17. A man starts from O and walks 2 miles East and then 3 mile South. What is his bearing from O? What is his new bearing from O when he walks another half mile South?

18. A chest of drawers, 3 ft. high, stands in an attic with a roof sloping down to the floor. If the chest can only just reach 3 ft. from the edge of the room, find the slope of the roof.

19. What is the angle of elevation of the top of a spire 240 ft. high from a point on the ground 200 yd. from the foot of it?

20. The pole of a bell tent is 8 ft. high, and the diameter of the base of the tent is 14 ft. What angle does the slant side of the tent make with the ground?

21. Using tables, find A, B and A+B, if (i) tan A = 2, tan B = 3,
   (ii) tan A = 1, tan B = 3.

22. Fig. 16 represents the roof of a villa; find the height of the ridge B of the roof above the top of the walls.
22. The diagonals of a rhombus are 6 in., 4 in. long. What are the angles of the rhombus?

24. One angle of a rhombus is 144°; the shorter diagonal is 5 cm. long. Find the other diagonal.

25. The vertical angle of an isosceles triangle is 45° and the base is 6 in. long. Find the area of the triangle.

26. A cricket ball is rolled in a straight line down the pitch from immediately alongside one of the stumps at one end of the pitch. Find within what angle its direction of motion lies if it does not miss the wickets at the other end. Take the diameter of the ball as 3 inches and the extreme width of the stumps as 8 inches.

27. A chord of a circle is 6 cm. long and subtends an angle of 103° 30' at the centre. Find its distance from the centre.

28. The steps of a staircase are 10 inches deep and 6 inches high. What angle does the bannister rail make with the horizontal?

29. The greatest and least heights of a lean-to shed are 10 ft. and 7 ft. 3 in.; the floor is 12 ft. wide. Find the slope of the roof.

30. Fig. 17 represents the section of a railway cutting; the base BC is horizontal and 15 ft. wide; the tops A, D are each 18 ft. above BC. Find AD.

31. AP, AQ are tangents to a circle of radius 4 inches; \( \angle PAQ = 41^\circ \). Find AP.

32. A map shows a straight road crossing two contour levels 100 ft., 200 ft. at P, Q. The length of PQ is 1-2 inches, and the scale of the map is 4 inches to the mile. What average angle does the road make with the horizontal?

33. A circle is inscribed in an equilateral triangle of side 6 inches. What is its radius?

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34. AB = 3 cm., BC = 4 cm., AP bisects \( \angle BAC \). Find PG.

The following exercise may be reserved for a second reading.

Example 7. ABC is a triangle such that:

BC = 4 in., \( \angle ABC = 61^\circ \), \( \angle ACB = 68^\circ \).

A circle is drawn to touch AB produced, AC produced and BC. Calculate its radius.

Let K be the centre of the circle, so that KB, KC bisect \( \angle BAC \), \( \angle CFB \); let KO touch the circle at N; let KN = r inches.

\( \angle DBC = 180^\circ - 61^\circ = 119^\circ \)

\( \angle KEN = 59^\circ 30' \)

\( \therefore \angle BNK = 90^\circ - 59^\circ 30' = 30^\circ 30' \)

Similarly

\( \angle KCN = \frac{1}{2}(180^\circ - 68^\circ) = 56^\circ \)

\( \angle NKC = 90^\circ - 56^\circ = 34^\circ \)

\( BN = NK \times r \tan 30^\circ 30' \)

and \( NC = r \tan 34^\circ \)

\( \therefore r \tan 30^\circ 30' + r \tan 34^\circ = BN + NC = BC = 4 \)

\( r(0.5880 + 0.6745) = 4 \)

\( r = \frac{4}{1.2635} = 3.17 \) inches.
EXERCISE 1.

1. Two men, one due North and the other due South of a tower, measure the angles of elevation of the top of its spire as 28° and 37°; the height of the spire is 120 feet. How far apart are the men?

2. The shadow of a vertical pole is 10 ft. long when the sun’s elevation is 35°. What is the length of the shadow when the sun’s elevation is 23°?

3. The angle of elevation of the top of a tower from a point on the ground 120 yards from its foot is 21° 48’; what will it be from a point on the ground 20 yards nearer the tower?

4. The sides of a rectangle are 4, 5 inches long. What is the angle between the diagonals?

5. Find a value of \( \theta \) if \( \tan \theta = 3 \tan 30° \).

6. Simplify (i) \( \tan 20° \times \tan 70° \); (ii) \( \tan 34° - \frac{1}{\tan 56°} \).

7. From the top of a cliff 300 feet high the angles of depression of two boats in a vertical plane with the observer are 25°, 24°, 37°, 32°. Find the distance between the boats.

8. A man stands at a distance of 90 ft. from the foot of a tower and observes that the angles of elevation of the top and bottom of a flagstaff on it are 56° and 53° respectively. What is the length of the flagstaff?

9. A conical funnel, vertical angle 52°, rests inside a glass of height 7 inches and diameter 3 inches, internal measurements. Find the height of the apex of the funnel above the base of the glass.

10. In Fig. 23,

\[ \angle ABC = 90° = \angle BCD, \quad \angle ACB = 41° 27’, \quad \angle CBD = 32° 44’, \]

\( BC = 10 \text{ cm.} \) Calculate \( AB, CD \) and \( \angle AEB \).

11. The base of a tank is 2 ft. square, and contains water to a depth of 1 ft. It is tilted about one edge as shown, through 10°. What is the length of \( AP \)?

12. From halfway up a tower the angle of depression of a mark on the ground is 52°, 27’. What will it be from the top of the tower?

13. Two circles of radii 5, 3 cm. are drawn touching a line at points A, B, 7 cm. apart; the other external common tangent PQ cuts AB at T. Calculate \( \angle ATP \).

14. \( \angle BAC = 90° = \angle ADB; \quad BD = 12 \text{ in.}, \quad DC = 6 \text{ in.} \) Find \( \angle ABC \).

15. O is the centre and C is the apex of a thin hemispherical shell or bowl. When suspended from a point A of the rim the shell hangs so that the mid-point of OC is vertically below A. What angle will AC make with the vertical?

16. ABC is a triangle such that \( \angle ABC = 37° 15’, \angle ACB = 59° 40’ \). BC = 8 cm.; the perpendicular bisector of BC cuts BA, CA produced at P, Q. Find the length of PQ.
17. ABC is a triangle such that
   \[ \angle ABC = 40^\circ, \quad \angle ACB = 62^\circ. \]
   A circle is inscribed in the triangle, i.e., touches the three sides. Calculate its radius.

18. A, B are points on opposite sides of a street 32 ft wide, each at a height of 25 ft above the street. Lights are attached to points P, Q, R on a wire suspended from A and B as shown, and are arranged so as to be at equal horizontal intervals across the street. Find the heights of P, Q, R above the ground.

19. ABC is a triangle such that \( \angle ABC = 24^\circ, \quad \angle ACB = 110^\circ; \) O is the mid-point of BC and AD is drawn perpendicular to BC produced. Show that \( DO = \frac{1}{2}(DB + DC) \). Hence prove that \( \tan \angle OAD = \frac{1}{2}(\tan \angle BAD + \tan \angle CAD) \), and calculate \( \angle AOC \).

20. ABC is a triangle such that
   \[ \angle ABC = 56^\circ, \quad \angle ACB = 42^\circ, \quad BC = 5 \text{ inches}. \]
   Calculate the length of the perpendicular from A to BC.

---

**CHAPTER II.**

**THE SINE AND COSINE.**

The sine of an angle. We saw in Fig. 7, p. 8, that if the angles at A, B are equal, and if perpendiculars are drawn from points P, Q on either arm of the angle to the other arm, the two right-angled triangles so obtained are the same shape.

Consequently \( \frac{MP}{NQ} = \frac{AP}{BQ} \).

Therefore the value of the ratio \( \frac{MP}{AP} \) does not depend on the length of AP, but only on the size of the angle MAP.

We can test this approximately by measurement:

- \( MP = 2.65 \text{ cm}, \quad AP = 4.2 \text{ cm}, \quad \frac{MP}{AP} = \frac{2.65}{4.2} = 0.63, \text{ approx.} \)
- \( NQ = 0.79 \text{ in}, \quad BQ = 1.25 \text{ in}, \quad \frac{NQ}{BQ} = \frac{0.79}{1.25} = 0.63, \text{ approx.} \)

The ratio \( \frac{MP}{AP} \) in Fig. 7 is called the **sine of the angle** MAP.
and is written \( \sin \angle MAP \) or \( \sin 39^\circ \), since \( \angle MAP = 39^\circ \). We may state this as follows:

If a perpendicular is let fall from any point on either arm of an angle to the other arm,

the sine of the angle \( \angle MAP \) is \[ \frac{\text{side opposite angle}}{\text{hypotenuse}} \], i.e., \[ \frac{MP}{AP} \].

or, more shortly, for any angle \( \theta \), \[ \sin \theta = \frac{o}{h} \].

The cosine of an angle. We also know that \[ \frac{AM}{AP} \] does not depend on the length of \( AP \), only on the size of the angle \( MAP \).

The ratio \[ \frac{AM}{AP} \] in Fig. 7 is called the cosine of the angle \( MAP \), and is written \( \cos MAP \) or \( \cos 39^\circ \), since \( \angle MAP = 39^\circ \). We may state this as follows:

If a perpendicular is let fall from any point on either arm of an angle to the other arm,

the cosine of the angle \( \angle MAP \) is \[ \frac{\text{side adjacent to angle}}{\text{hypotenuse}} \], i.e., \[ \frac{AM}{AP} \].

or, more shortly, for any angle \( \theta \), \[ \cos \theta = \frac{a}{h} \].

**Summary of definitions.**

With the notation of Fig. 30 and Fig. 31, we have

\[ \sin \theta = \frac{o}{h}; \cos \theta = \frac{a}{h}; \tan \theta = \frac{o}{a} \]

From the definitions we see that

\[ \sin \theta = \frac{o}{h}; \cos \theta = \frac{a}{h}; \tan \theta = \frac{o}{a} \].

**Note.** (i) The letters O.H.M.S. may be used to remember the fact that "Opposite over Hypotenuse Means Sine."

(ii) The reader must accustom himself to the different ways in which a figure can be turned round, cf. Fig. 30 and Fig. 31 above.

**Example I.** Find, by drawing and measurement, approximate values of 

- \( \sin 20^\circ, \cos 20^\circ \)
- \( \sin 40^\circ, \cos 40^\circ \)
- \( \sin 70^\circ, \cos 70^\circ \).

![Fig. 32](image)

Draw a circle of unit radius, centre O, diameter AOB.

(Note. The reader should draw his own figure, preferably on squared paper, and take 1 dm or 5 inches as his unit. Fig. 32 represents part of a circle of radius 1 inch. It is unnecessary to draw more than a quarter of the circle.)

Draw lines OP, OQ, OR cutting the circle at P, Q, R and such that \( \angle AOP = 20^\circ \), \( \angle AOP = 40^\circ \), \( \angle AOP = 70^\circ \); draw the perpendiculars PM, QN, RK to AB.

By definition,

\[ \sin 20^\circ = \frac{MP}{OP} = 0.34 \text{ in.; } \cos 20^\circ = \frac{MP}{OP} = 0.94, \text{ approx.} \]

Similarly, \( \cos 20^\circ = \frac{OM}{OP} = 0.94 \text{ in.; } \cos 20^\circ = \frac{OM}{OP} = 0.34, \text{ approx.} \)

And from the other necessary measurements we have

\[ \sin 40^\circ = \frac{NQ}{OQ} = 0.64; \cos 40^\circ = \frac{ON}{OQ} = 0.77. \]

\[ \sin 70^\circ = \frac{KR}{OK} = 0.94; \cos 70^\circ = \frac{KR}{OK} = 0.34. \]
Note. (i) The construction used in this Example shows that as the angle \( \theta^\circ \) increases from 0° to 90°, \( \sin \theta^\circ \) increases steadily from 0 to 1, since it is represented by \( \frac{MP}{\text{radius}} \), and \( \cos \theta^\circ \) decreases steadily from 1 to 0, since it is represented by \( \frac{OM}{\text{radius}} \).

(ii) \( \sin 40^\circ \) is less than twice \( \sin 20^\circ \); the value of the sine of an angle is not proportional to the angle.

(iii) The values obtained above show that \( \sin 20^\circ = \cos 70^\circ \), and \( \cos 20^\circ = \sin 70^\circ \). This follows from the fact that the triangles \( \triangle OMP, \triangle RKO \) are congruent; but it is easier to deduce it from Fig. 33.

By definition, with the notation of Fig. 33,

\[
\sin \theta^\circ = \frac{y}{z} = \cos (90^\circ - \theta^\circ),
\]

and

\[
\cos \theta^\circ = \frac{z}{z} = \sin (90^\circ - \theta^\circ).
\]

Hence the sine of any angle equals the cosine of its complement.

Use of Tables. The sine-table is used in exactly the same way as the tangent-table (see p. 10). But, in using the cosine-tables, the figures in the difference-column must be subtracted, for the cosine decreases as the angle increases.

Thus, to find \( \cos 53^\circ 20' \),

\[
\cos 53^\circ 18' = 0.6976
\]

Difference for 2' = 0.0005

\[
\therefore \cos 53^\circ 20' = 0.6971.
\]

Note. Since the sine of any angle equals the cosine of its complement, and vice-versa, the Table of Natural Cosines is obtained by writing the Table of Natural Sines backwards: some books of Tables do not therefore print the values of natural cosines separately.

The name sine, or rather its Latin equivalent \( \sinus \), was first used in the twelfth century, but was not adopted universally till the seventeenth century, when the abbreviation \( \sin \) was first employed (1634); the cosine came into use first in India, about 600 A.D., simply as the sine of the complementary angle, and it was a long time before it received any recognised name of its own; the term \( \cosinus \) was used by Gunter in 1620, and the abbreviation to \( \cos \) was made fifty years later.

Gradient. The statement that the gradient of a road is 1 in 12 is ambiguous.

It may either mean that the road rises 1 ft. vertically for each 12 ft. measured horizontally (Fig. 34 (i)), or that it rises 1 ft. vertically for each 12 ft. measured along the slope (Fig. 34 (ii)).

If \( \theta^\circ \) is the angle which the road makes with the horizontal,

in (i), \( \tan \theta^\circ = \frac{1}{12} = 0.0833 \); \( \therefore \theta = 4^\circ 46' \); 

in (ii), \( \sin \theta^\circ = \frac{1}{12} = 0.0833 \); \( \therefore \theta = 4^\circ 47' \).

This example shows that for small slopes the exact meaning is almost immaterial; but for large slopes the precise meaning must be specified. The former meaning (\( \tan \theta^\circ \)) is commonly attributed in work with graphs; the latter (\( \sin \theta^\circ \)) is adopted by surveyors and engineers.
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Example II. A ladder 10 ft. long leans against a wall and is inclined at 53° to the ground. How far from the wall is the foot of the ladder? How high up the wall does the ladder reach?

AB is the ladder; AB = 10 ft.

\[ \text{AC} = \text{AB} \times \frac{\text{AO}}{\text{AB}} = 10 \cos 53° \text{ ft.} \]

\[ = 10 \times 0.6068 = 6.07 \text{ ft.} \]

\[ \text{OB} = \text{AB} \times \frac{\text{OB}}{\text{AB}} = 10 \sin 53° \text{ ft.} \]

\[ = 10 \times 0.7986 = 7.99 \text{ ft.} \]

**EXERCISE II. a.**

1. Draw (on squared paper) a quadrant of a circle of convenient radius and use it to find the values of \( \sin 25° \), \( \cos 25° \), \( \sin 35° \), \( \cos 35° \), \( \sin 65° \), \( \cos 65° \). What are the values of \( \sin 90° \), \( \cos 90° \), \( \sin 0° \), \( \cos 0° \)?

2. Use Tables to write down the sines of the following: (i) 17°; (ii) 43°; (iii) 64°; (iv) 88°; (v) 23° 30'; (vi) 23° 36'; (vii) 23° 31'; (viii) 23° 35'; (ix) 35° 21'; (x) 34° 11'; (xi) 49° 2'; (xii) 35° 14'.

3. Use Tables to write down the cosines of the following: (i) 14°; (ii) 28°; (iii) 56°; (iv) 89°; (v) 66° 36'; (vi) 60° 42'; (vii) 60° 39'; (viii) 66° 41'; (ix) 62° 40'; (x) 58° 12'; (xi) 45° 23'; (xii) 63° 10'.

4. Show that the triangles in Fig. 36 are right-angled:

(i)  \[ \text{(i)} \]

(ii)  \[ \text{(ii)} \]

(iii)  \[ \text{(iii)} \]

(iv)  \[ \text{(iv)} \]

(iii) contains three such triangles. Write down the sine and cosine of each marked angle.

5. Using the data of Fig. 37, write the following as trigonometrical ratios, in more than one way if possible:

\[ \text{(i) } \frac{\text{AC}}{\text{BC}}; \text{ (ii) } \frac{\text{PQ}}{\text{PR}}; \text{ (iii) } \frac{\text{GF}}{\text{EF}}; \text{ (iv) } \frac{\text{XY}}{\text{YZ}}; \text{ (v) } \frac{\text{QR}}{\text{PR}}; \text{ (vi) } \frac{\text{AC}}{\text{AB}}; \text{ (vii) } \frac{\text{YZ}}{\text{XZ}}; \text{ (viii) } \frac{\text{EG}}{\text{GF}}; \text{ (ix) } \frac{\text{AB}}{\text{BC}}; \text{ (x) } \frac{\text{QR}}{\text{PR}}; \text{ (xi) } \frac{\text{EG}}{\text{EF}}; \text{ (xii) } \frac{\text{XY}}{\text{XZ}}. \]

6. In Fig. 38 the triangles are right-angled, and the given side is in each case the hypotenuse. Find the other sides.

7. In Fig. 37 solve the following:

\[ \text{(i) } \text{BC} = 3, \angle C = 22° 34', \text{ find } AC; \]

\[ \text{(ii) } \text{PR} = 2, \angle R = 21° 44', \text{ find } PQ; \]

\[ \text{(iii) } \text{EF} = 100, \angle E = 71° 8', \text{ find } FG; \]

\[ \text{(iv) } \text{XZ} = 4, \angle X = 35° 53', \text{ find } XY; \]

\[ \text{(v) } \text{QR} = 10, \angle P = 61° 30', \text{ find } PQ; \]

\[ \text{(vi) } \text{BC} = 5, \angle B = 70° 45', \text{ find } AB; \]

\[ \text{(vii) } \text{XY} = 3, \angle X = 31° 24', \text{ find } YZ; \]

\[ \text{(viii) } \text{EF} = 6, \angle E = 74° 22', \text{ find } EG. \]
8. Find the sine, cosine and tangent of each marked angle in Fig. 39; note that the triangles are not right-angled.

![Fig. 39](image)

9. A hill slopes upwards at an angle of 18° with the horizontal. What height does a man rise when he walks 100 yd. up the slope?

10. B is 2000 yd. N. 34° E. from A. How much is (i) East, (ii) North of A?

11. The string of a kite is 400 ft. long, and makes an angle of 62° with the horizontal. What is the height of the kite?

12. Find the area of the parallelogram in Fig. 40.

13. What are the values of \( \cos 32° \) and \( \sin 68° \)? Why are they equal?

14. Find a value of \( x \) if
   (i) \( \cos x = \sin 48° \);
   (ii) \( \sin x = \cos 47° 30' \);
   (iii) \( \cos x = \sin 21° 47' \);
   (iv) \( \sin x = \cos 15° 21' \).

15. Find from a Table of sines the value of \( \cos 14° 27' \).

16. The legs of a pair of dividers are each 12 cm. long, and are opened to an angle of 31°. Find the distance between their points.

17. Repeat No. 16, if the angle is 170°.

18. Fig. 41 represents a semicircle; find the length (i) of the chord, (ii) of the portion of the tangent it cuts off.

19. The vertical angle of a cone is 23° and the length of a slant edge is 2.5 in. What is the diameter of the base?

20. A telegraph pole AB, 18 ft. high, is stayed by a tie CD, 12 ft. long, making 67° with the horizontal. How far is the point of attachment C from A?

21. A regular pentagon is inscribed in a circle of radius 5 cm. What is the length of its side?

22. A ladder 20 ft. long leans against the side of a house. What distance must the foot of the ladder be pushed to increase the angle of slope of the ladder from 60° to 65°?

23. The diagonals of a rectangle are 12 cm. long and contain an angle of 17° 30'. Find its length and breadth.

24. Find the projection CD of AB on the ground line HK.

![Fig. 43](image)

25. A straight path is inclined at 4° to the horizontal. What is the distance along the path between the points where the 100 ft. and 200 ft. contours are crossed?

Construction of acute angles of given sine or cosine.

We have seen that the sine and cosine of an acute angle can have any value between 0 and 1. By using the tables we can find approximately the size of the angle if the sine or cosine is given.

Example III. Find \( x \), given that
   (i) \( \sin x = 0.86 \),
   (ii) \( \cos x = 0.74 \).

(i) From the tables, \( \sin 59° 15' = 0.8699 \).
   Difference for 1' = 0.0001;
   \( \therefore \sin 59° 19' = 0.8660 \).
(ii) From the tables, $\cos 42^\circ 18' = 0.7396$.
Difference for $2' = 0.0004$ ;
$\therefore \cos 42^\circ 16' = 0.7400$.

Note that the $2'$ is subtracted because the angle decreases if its cosine is increased.

**Definition.** The angle whose sine is $x$ is often written $\sin^{-1}(x)$ ; the angle whose cosine is $x$ is written $\cos^{-1}(x)$, and the angle whose tangent is $x$ is written $\tan^{-1}(x)$ ; or, more shortly $\sin^{-1}x, \cos^{-1}x$ and $\tan^{-1}x$.

Thus $\sin^{-1}0.86 = 58^\circ 19'$ and $\cos^{-1}0.74 \approx 42^\circ 16'$.

**Example IV.** Construct the angle whose sine is equal to 0.77.

![Diagram of circle and triangle](image)

Draw a circle of unit radius, centre O, diameter AOB. (Note. The reader should draw his own figure, preferably on squared paper, and take 1 dm. or 5 inches as his unit. Fig. 44 represents part of a circle of radius 1 inch. It is unnecessary to draw more than a quarter of the circle.)

Draw the radius OC at right angles to OA, and cut off ON equal to 0.77 units : the parallel through N to OA cuts the circle at P ; PM is drawn perpendicular to OA.

Then $\sin AOP = \frac{MP}{OP} = \frac{ON}{OP} = 0.77$.

By measurement we find $\angle AOP \approx 50^\circ 5'$.

From the Tables we see that $\angle AOP \approx 50^\circ 21'$.

**The Sine and Cosine**

Note. (i) To construct (say) $\cos^{-1}(0.65)$, cut off a length OM from OA, such that OM = 0.65 units, and draw MP perpendicular to OA, cutting the circle at P, then $\angle AOP$ is the required angle. (ii) The use of squared paper is advised, merely because it saves time.

**Exercise II. b.**

1. Find, by drawing and measurement,
   (i) $\sin^{-1}(0.4)$; (ii) $\sin^{-1}(0.7)$; (iii) $\sin^{-1}(0.8)$; (iv) $\sin^{-1}(0.92)$.
   [Use squared paper.]

2. Find, by drawing and measurement,
   (i) $\cos^{-1}(0.31)$; (ii) $\cos^{-1}(0.63)$; (iii) $\cos^{-1}(0.81)$; (iv) $\cos^{-1}(0.91)$.
   [Use squared paper.]

3. Use Tables to write down the angles whose sines are:
   (i) 0.3907; (ii) 0.6633; (iii) 0.7604; (iv) 0.4403; (v) 0.4500; (vi) 0.4468; (vii) 0.4504; (viii) 0.2945; (ix) 0.3190; (x) 0.9643.

4. Use Tables to write down the angles whose cosines are:
   (i) 0.5092; (ii) 0.7890; (iii) 0.8712; (iv) 0.1806; (v) 0.1788; (vi) 0.1794; (vii) 0.1802; (viii) 0.7855; (ix) 0.9831; (x) 0.9834.

5. Use Tables to evaluate
   (i) $\sin^{-1}(0.5565)$; (ii) $\cos^{-1}(0.3100)$; (iii) $\tan^{-1}(0.5308)$; (iv) $\cos^{-1}(0.3200)$; (v) $\sin^{-1}(0.0114)$; (vi) $\tan^{-1}(0.0099)$.

6. Use Tables to evaluate the marked angles in Fig. 36.

7. A ladder 12 ft. long leans against the wall, and one end is 3 ft. from the wall. What angle does the ladder make with the wall?

8. What is the angle of slope of a road if a man has risen 30 ft. vertically after walking 100 yards up the road?
9. Use Tables to evaluate the marked angles in Fig. 39.

10. The sides of a parallelogram are 4, 5 inches and its area is 12 sq. in. Calculate its angles.

11. Taking both possible meanings of the term "gradient" (see p. 25) find the inclination to the horizontal of roads whose gradients are (i) 1 in 5, (ii) 1 in 10, (iii) 1 in 30, (iv) 1 in 100. Which meaning gives the greater inclination, and why?

12. A road 800 yards long is represented on a map, scale 1: 20,000, by a line of length 1-42 inches. What is the average inclination of the road to the horizontal?

13. A pencil 6" long casts a shadow 5" long when the sun is vertically overhead. What is the inclination of the pencil to the horizontal?

14. The pole of a bell-tent is 8 ft. high, and the length of the slant side is 11 ft. What angle does the side make with the ground?

15. A soldier's legs are 32" long. At what angle are they inclined when he stands at ease with his feet 10" apart? What difference does this make to his height?

16. In Fig. 45, calculate ∠BAC.

17. The legs of a pair of dividers are each 12 cm. long and are opened so that the points are 5 cm. apart. What is the angle between the legs?

18. With the data of No. 17, the points rest on the surface of a sphere of radius 8 cm. What angle do they subtend at the centre of the sphere?

19. In Fig. 46, calculate ∠CAD.

20. The tops of two vertical poles of heights 20, 15 ft. are joined by a taut wire 12 ft. long. What is the angle of slope of the wire?

21. In Fig. 47, AB is a diameter; AP = 8-5 cm., BQ = 7-5 cm. Calculate (i) ∠PAB, (ii) ∠PQB.

22. The centres of two circles of radii 7, 3 cm. are 12 cm. apart. Calculate the angle between their exterior common tangents PQ, RS.

23. With the data of No. 22, calculate the angle between the interior common tangents.

24. A uniform sphere of radius 6 cm. is suspended by a string AB 9 cm. long from a point A in a smooth vertical wall AD. Calculate ∠BAD. Statical considerations show that AB produced passes through the centre of the sphere.

25. A sphere of radius 8 cm. rests inside a conical funnel whose axis is vertical; the highest point of the sphere is 22 cm. above the vertex of the cone. Find the angle of the cone.
26. The diameter of a cylindrical roller is 30 inches and a handle, OA, 5 ft. long, is attached to its axis O, about which it can rotate. If the roller is stationary on level ground, find the greatest angle through which the handle can swing. [See Fig. 51.]

27. Find a value of $\beta$ if
   (i) $\sin \theta' = 2 \sin \phi'$ and $\phi' = 22^\circ$;
   (ii) $\cos \theta' = 2 \cos \phi'$ and $\phi' = 72^\circ$.

![Fig. 51](image)

28. A chord AB of a circle of radius 6 cm. is 5 cm. long. Calculate $\angle APB$.

29. With the data of No. 28, if $\angle APB = 105^\circ$,
calculate AP.

30. A mechanism (Peaucellier’s cell) consists of four equal rods AB, BC, CD, DA, each of length 10 in., and two other equal rods BE, ED, each of length 7 in., smoothly jointed as shown. If $\angle EAB = 32^\circ$, calculate $\angle BDE$. Find also the maximum angle between AB and AD.

31. In Fig. 54, calculate the length of the perpendicular from B to AC.

![Fig. 54](image)

32. Calculate in Fig. 55 the lengths of AD, BD, BC.

![Fig. 55](image)

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THE SINE AND COSINE

The following exercise may be reserved for a second reading.

EXERCISE II c.

1. What is the distance of a place in latitude $53^\circ$ N. from the axis of the Earth?
   (Radius of Earth = 4000 miles.)

2. Taking the length of the Equator as 25,000 miles, find the length of the parallel of latitude in latitude $53^\circ$ N.

3. A man walks 100 yd. up a slope of $22^\circ$ and then 50 yd. up a slope of $18^\circ$. How far is he (i) vertically, (ii) horizontally, from his starting point?

4. AE, BF are vertical standards; find the heights of C, D above the ground line EF and the length of AB. (Fig. 57.)

5. A mining gallery descends for 100 yd. at an angle of $13^\circ$ to the horizontal and then for 200 yd. at an angle of $7^\circ$ to the horizontal. How far is the point reached below the level of the starting point?

![Fig. 56](image)

6. A man starts at O and walks 1 mile N. $21^\circ$ W. to A, then 2 miles N. $43^\circ$ E. to B. How far (i) North, (ii) East is B from O?

7. The tip of a pendulum 2 ft. long rises 6 inches above its lowest point in each swing. Find the angle of swing.

8. A man starts at O and walks 1 mile N. $27^\circ$ E. to A, then turns to his right through $90^\circ$ and walks half a mile to B. What is the bearing of B from O?

9. A man wishes to row straight across a stream which is running at 1 mile per hour; he can row at $2\frac{1}{2}$ m.p.h., through the water. At what angle to the line of the stream must he point his boat?
10. A pendulum 5 ft. long swings through an angle of 12° on each side of the vertical. How high does its tip rise above its lowest point?

11. ABC represents the path of a bullet fired from A at an angle of 5° to the horizontal AE. If C is its position after 5 seconds, AD = 2000 ft and DC = 160 feet. Find the height of the bullet after (i) 1 sec., (ii) 5 sec. When will it hit the ground?

12. A rectangular block leans against a wall at B with the corner A on the ground. Find the height of C above the ground. [Fig. 56.]

13. A regular pentagon ABCDE is inscribed in a circle of radius 4 inches. Find the length of the perpendicular (i) from A to CD, (ii) from B to AC.

14. Taking a degree of longitude at the Equator as 60 miles, find the latitude of a place where a degree of longitude is 30 miles.

15. A wheel of radius 2 ft. rests at B against an obstacle 6 in. high as shown; the wheel is then pushed on to the top of the obstacle, which is level, turning about B. Through what angle does each spoke of the wheel turn?

16. ABC is a triangle, right-angled at C; CN is the perpendicular from C to AB. Write down cos A in two different forms, and hence prove that AC² = AN. AB. Similarly prove that BC² = EN. BA.

17. A, B are two billiard balls at distances 20, 30 in. from a perfectly elastic cushion CD. The ball A is struck along AP and hits B full on the rebound after travelling altogether 70 in.; neglecting the size of the balls, and assuming that AP, PB make equal angles with CD, calculate ∠APB.

18. A boat sailing against the wind from X to a place Y due East of X takes a course either N. 64° E. or S. 64° E. alternately. What is the distance of Y from X, if the boat has to travel 5000 yards?

19. What can you say about a triangle ABC in which cos A = sin B, if B is acute?

20. One solution of the equation sin x² + cos x² = 1.292 is x = 21. Find another solution.

21. Shew that in any triangle ABC,
   \[ \sin \frac{A}{2} = \pm \sqrt{\frac{B + C}{2}}, \quad \cos \frac{B}{2} = \mp \sqrt{\frac{A + C}{2}}. \]

22. Write down an equation that may connect x and y if \( \cos x^2 = \sin y^2 \).

23. Find a value of x if \( \cos x^2 = \sin (x + 20)^\circ \), \( \sin x^2 = \cos (x - 16)^\circ \).
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24. Fig. 63 represents a section of a rectangular box with its lid DE; a sphere of diameter 10" is placed in the box. What is the least angle DE makes with BC?

25. A uniform rod AB, mid-point C, rests on a smooth peg at D with its end A against a smooth vertical wall. It can be proved by statistical principles that the horizontal line through A cuts the vertical line through C at a point F such that DF is perpendicular to AD. If AB=27 in., and if the peg is 4 inches from the wall, find the angle AB makes with the vertical.

26. Fig. 65 represents the lid AK, pivoted at A, of a hot-water jug AKLM. AB and BC are perpendicular rods attached rigidly to each other and the lid. The upper horizontal surface DE of the handle prevents the lid opening fully by acting as a stop for C; DEF is a straight line. If AB=1 cm., BC=2-5 cm., AF=0-5 cm., and ∠BAK=90°, find the maximum angle through which the lid can turn.

27. A trough has a semi-circular section ACB, diameter 18 inches, and contains water to a depth of 7 inches; initially AB is horizontal; through how large an angle can it be tilted before any water is upset?

CHAPTER III.

COSSECANT, SECANT AND COTANGENT.

The reciprocals of the Sine, Cosine and Tangent of an angle are called respectively the cosecant, secant and cotangent, and are written more shortly as cosec, sec, and cot.

Thus

\[ \text{cosec} \theta = \frac{1}{\sin \theta} \]

\[ \text{sec} \theta = \frac{1}{\cos \theta} \]

\[ \cot \theta = \frac{1}{\tan \theta} \]

Since the sine and cosine of an angle cannot be greater than 1, the cosecant and secant of an angle cannot be less than 1.

Further, since \( \sin \theta \) and \( \tan \theta \) increase as \( \theta \) increases, it follows that \( \text{cosec} \theta \) and \( \cot \theta \) decrease as \( \theta \) increases; but since \( \cos \theta \) decreases as \( \theta \) increases it follows that \( \sec \theta \) increases as \( \theta \) increases. We see, therefore, that the cosine, cosecant, secant and cotangent of an angle all decrease when the angle increases; the common prefix co makes this easy to remember. Consequently, when four-figure Tables are used, the difference columns must be subtracted for an increase of \( \theta \) in the \( \cos \theta \), cosec \( \theta \), \( \cot \theta \) Tables.
Complementary angles.

By definition, with the notation of Fig. 67, we have
\[
\csc \theta^\circ = \frac{z}{y}; \quad \sec \theta^\circ = \frac{z}{x};
\]
and \[
\csc (90^\circ - \theta^\circ) = \frac{z}{y}; \quad \sec (90^\circ - \theta^\circ) = \frac{z}{x};
\]
\[
\therefore \csc \theta^\circ = \sec (90^\circ - \theta^\circ) \quad \text{and} \quad \sec \theta^\circ = \csc (90^\circ - \theta^\circ).
\]

Hence the cosecant of any angle equals the secant of its complement and vice-versa.

Further, \[
\cot \theta^\circ = \frac{y}{x} = \tan (90^\circ - \theta^\circ);
\]
and \[
\cot (90^\circ - \theta^\circ) = \frac{y}{x} = \tan \theta^\circ.
\]
\[
\therefore \text{the cotangent of any angle equals the tangent of its complement and vice-versa.}
\]

In fact, all trigonometrical ratios are equal to the co-ratio of the complementary angle and vice-versa.

One advantage of having all six Trigonometrical ratios defined and tabulated is that numerical work and statements of Trigonometrical facts and formulae can be simplified by using the most suitable ratios. This is illustrated in the following examples.

Example I. In the given triangle, find the length of AC.

\[
AC = \frac{AC \times 10 - 10 \csc 55^\circ}{10} = 10 \times 1.2208 = 12.2 \text{ cm.}
\]

Note. This method is simpler than saying \[
\frac{AC}{\sin 55^\circ} = \frac{10}{0.8192}, \text{ etc.}
\]

Example II. In the given triangle, find the angle A.

\[
\cos \frac{A}{\Theta} = 2.3; \quad A = 64^\circ 14'.
\]

Note. This method makes the calculation simpler than saying \[
\cos \frac{A}{\Theta} = \frac{1}{2.3}, \text{ etc.}
\]

**EXERCISE III. a.**

1. Use Tables to write down the values of the following:
   - (i) cosec 41°; (ii) cosec 41° 36'; (iii) cosec 41° 38';
   - (iv) cosec 75° 22'; (v) sec 28°; (vi) sec 28° 18';
   - (vii) sec 28° 22'; (viii) sec 70° 43'; (ix) cot 44°;
   - (x) cot 45° 18'; (xi) cot 45° 20'; (xii) cot 83° 10'.

2. Use Tables to find the following angles:
   - (i) cosec-1(1.1625); (ii) cosec-1(1.2001); (iii) cosec-1(2.4063);
   - (iv) sec-1(1.3969); (v) sec-1(1.4102); (vi) sec-1(2.2542);
   - (vii) cot-1(0.6694); (viii) cot-1(0.6707); (ix) cot-1(1.5020).
3. Write down the cosecant, secant and cotangent of each of the marked angles in Fig. 36.

4. Using the data of Fig. 37, write the following as trigonometrical ratios in two ways.

5. Using the data and notation of Fig. 37, write down simple expressions for the following:
   (i) \( \text{sec} \ C \);
   (ii) \( \cot \ P \);
   (iii) \( \csc F \);
   (iv) \( \tan X \);
   (v) \( \cot B \);
   (vi) \( \sec R \);
   (vii) \( \cot E \);
   (viii) \( \sec Z \);
   (ix) \( \csc B \);
   (x) \( \sin R \);
   (xi) \( \text{AB} \text{ sec } \text{ABC} \);
   (xii) \( \text{QR} \csc \text{QPR} \);
   (xiii) \( \text{FG} \cot \text{GEF} \).

6. Evaluate as shortly as possible:
   (i) \( \frac{1}{\sin 30^\circ} \);
   (ii) \( \frac{1}{\csc 52^\circ} \);
   (iii) \( \frac{1}{\tan 32^\circ} \);
   (iv) \( \frac{1}{\csc 61^\circ} \);
   (v) \( \frac{\sec 40^\circ}{\csc 50^\circ} \);
   (vi) \( \cot 15^\circ \);
   (vii) \( \tan 40^\circ \cdot \tan 50^\circ \);
   (viii) \( \cot 35^\circ \cdot \csc 55^\circ \).

7. Find the marked angles in the triangles in Fig. 70.

8. Find the remaining sides in the triangles in Fig. 71.

9. Find from the Tables the values of:
   (i) \( \cos 48^\circ \cdot 30^\circ \) and \( \sin 41^\circ \cdot 30^\circ \);
   (ii) \( \tan 16^\circ \cdot 25^\circ \) and \( \cot 73^\circ \cdot 35^\circ \);
   (iii) \( \csc 37^\circ \cdot 10^\circ \) and \( \sec 52^\circ \cdot 50^\circ \).

10. Find a value of \( x \) if:
    (i) \( \cos x^\circ = \sin 62^\circ \);
    (ii) \( \tan x^\circ = \cot 14^\circ \);
    (iii) \( \sin x^\circ = \cos 51^\circ \cdot 25^\circ \);
    (iv) \( \sec x^\circ = \csc 15^\circ \cdot 42^\circ \);
    (v) \( \cot x^\circ = \tan 19^\circ \cdot 47^\circ \);
    (vi) \( \csc x^\circ = \sec 71^\circ \cdot 10^\circ \).

11. Find from a Table of secants the value of \( \csc 64^\circ \cdot 17^\circ \).

12. In Fig. 72, \( \angle \text{BAC} = 90^\circ \) and \( \angle \text{ADB} \);
write the following in terms of the lengths in the figure:
   (i) \( \text{sec} \ A \);
   (ii) \( \cot \ A \text{CB} \);
   (iii) \( \csc \ B \text{AD} \);
   (iv) \( \tan \ B \text{AD} = \cot \ D \text{AC} \);
   (v) \( \csc \ D \text{AC} = \csc \ A \text{CB} \);
   (vi) \( \text{sec} \ B \text{AD} = \sec \ A \text{CB} \).
13. A man on the top of a tower 200 ft. high measures the angle of depression of a milestone as 24°. How far is the milestone from the man?

14. A chord of length 8 cm. subtends an angle of 100° at the centre of the circle; what is the radius?

15. A kite is flying at a height of 300 ft. above the ground at the end of a string which makes 38° with the vertical. What is the length of the string?

16. A quay-side stairway descends from the quay at an angle of 17° with the horizontal; the height of the quay is 26 ft. What is the length of the stairway?

17. Each leg of a tripod is 5 ft. long and makes an angle of 57° with the ground. What is the height of the apex of the tripod?

18. A buoy is attached to the bed of a channel by a chain which makes with the vertical an angle of 14° when the flow of the tide has reduced the depth of water to 18 ft. What is the length of the chain?

19. A boat P is 4 sea-miles due west of a lighthouse Q, and is steaming at 15 knots on a course N. 57° E. After what time will P be due North of Q?

20. The tangents from a point A to a circle are 3.5 in. long and contain an angle of 37°; find the distance of A from the centre.

21. A flagstaff snaps at a point P, 8 ft. above its base A, and the top PB rests at an angle of 17° with the ground. Find the original height of the flagstaff.

22. A portion of road AB which slopes uphill at an angle of 7° is represented on a map of scale 4 inches to the mile by a line of length 3.3 inches. Find the length of AB in yards.

23. A taut elastic string joins two points A, B 30 inches apart and at the same level; when the body is attached to the mid-point C of the string, AC and BC make angles of 28° 20' with the horizontal. How much has the string stretched?

24. One angle of a rhombus is 37°, and the shorter diagonal is 6 cm. Find the length of a side.

25. The diagonals of a rectangle intersect at an angle of 33° 48' and the length of one side is 5 inches. What is the length of a diagonal? [Two possible answers.]

26. In Fig. 70 of No. 12, \(\angle ADB = 90°, \angle ABC = 32°, \angle ACB = 71°\). Calculate AB, AC, BC.

27. A pendulum OA, 6 ft. long, is suspended from O; only that portion of it is visible which is above a horizontal line BC, 4 ft. below O. How much more of the pendulum is visible when it is at an angle of 20° with the vertical than when at an angle of 10°?

28. In Fig. 74, find \(x\).

29. A sphere of radius 6 cm. rests inside a hollow cone of vertical angle 64°, base-radius 10 cm., with axis vertical and apex downwards. Find the distance of the centre of the sphere from the base of the cone.

30. In Fig. 76, \(\angle ABC = 73° = \angle ACB\) Find the diameter of the circle.

(a) if \(AB = 6\) cm.,

(b) if \(BC = 6\) cm.

The following Exercise may be reserved for a second reading.

**Exercise III.**

1. EF is a fence 5 ft. high at a distance of 3 ft. from the wall OD of a house. What is the length of a ladder AB inclined at 72° to the horizontal just grazing the fence?

2. In Fig. 78, the diameter AB is \(d\) inches long and BO is a tangent; prove that PO is \(d (\cos \theta - \cos \theta)\) inches.
3. From the top of a cliff $h$ feet high the angles of depression of two boats in the same vertical plane as the observer are $\theta^\circ$ and $\phi^\circ$. Express the distance between the boats in terms of $h$, $\theta$, $\phi$.

4. In Fig. 79, $ON = a$, $NA = d$; prove that $PQ = \frac{d}{\cos \theta - \cos \phi}$.

5. Fig. 80 represents two semi-circles; $\angle PAB = 35^\circ$, $PQ = 4$ cm. Calculate the length of $BC$. If also $PN = 3$ cm, calculate the length of $AC$.

6. $PN$ is perpendicular to the diameter $AB$; $AP = a$. Find $NB$ in terms of $a$, $\theta$. [Fig. 81.]

7. The cord of a pair of steps is $a$ ft. long and, when taut, is $h$ ft. above the ground. Find the length of each arm of the steps in terms of $h$, $\alpha$, $\theta$. [Fig. 82.]

8. In Fig. 83, $ABCD$ is a rectangle; $PQ = a$. Find $AC$, $PC$ in terms of $a$, $\theta$.

9. In Fig. 84, $AB$ is a diameter and $AQ$, $BP$ are tangents. Calculate $AP$, $BQ$ and $\angle AQP$.

10. In Fig. 85, the triangle $ABC$ is inscribed in the rectangle $APQR$. Calculate the sides and angles of $\triangle ABC$.

11. In Fig. 86, $PBC$ bisects $\angle APD$; $BC = 100$ yd., $\angle APD = 43^\circ$. Find how much further $P$ is from $D$ than from $A$.

12. In Fig. 87, $\angle ABC = 43^\circ$, $\angle ACB = 67^\circ$, and the radius of the circle is 10 inches. Find $BC$.

13. With the data of No. 12, find $AB$.

14. In Fig. 88, $PN = d$, $C$ is the mid-point of $AB$. Express the length of $AB$ in terms of $d$, $\theta$.

15. In Fig. 89, $OM = p$, $ON = x$. Express $PN$ in terms of $x$, $y$, $\theta$.

16. In Fig. 90 represents a circle inscribed in a semicircle; $AP = 10$ cm. Calculate the diameter of each circle.
17. Using only a table of tangents, find the value of cot 72° 16′.
18. What do you know about a triangle ABC, if tan A = cot B?
19. What do you know about a triangle ABC, if sec B = cosec C, and
   if C is acute?
20. One solution of the equation sec x + cosec x = 3.325 is
   x = 27°. Find another solution.
21. One solution of the equation \( \tan x + \cot x = 2.989 \) is
   \( x = 60° \); find another solution.
22. Write down a relation which may connect \( x \) and \( y \), if
   \( \csc^2 x = \sec^2 y \).
   Find a value of \( x \) if \( \csc x = 2 \sqrt{2} \).
23. Find a value of \( x \) if \( \tan 2x = \cot 2x \).
24. Find a value of \( \theta \) if \( \tan (\theta + 20°) = \cot \theta \).
25. Prove that in any triangle ABC,
   \( \frac{\tan \frac{B+C}{2}}{\tan \frac{A}{2}} = \cot A \),
   \( \frac{\sec \frac{A+B}{2}}{\sec \frac{C}{2}} = \csc C \).

REVISION PAPERS. R. 1-6.

R. 1.
1. Find by drawing the values of \( \tan 16°, \tan 32°, \tan 64° \). Write
   down the values obtained from the Tables.
2. In a triangle \( A = 90°, B = 25°, 16', \ b = 10 \text{ cm} \). Find \( c \).
3. The vertical angle of isosceles triangle is \( 67° \), and the base
   is 8 in. long. Find the area of the triangle.
4. Find the length of the shadow of a stick 3 ft. long when the
   sun is at an elevation of 32°. (i) if the stick is held vertically, (ii)
   if the stick is inclined so as to throw the longest shadow possible.
5. The elevation of the top of a tower is \( 20° \) to an observer on the
   ground. What is the elevation of a point half-way up the tower?

R. 2.
1. Find by drawing the values of \( \sin 16°, \cos 16°, \sin 32°, \cos 32° \).
   Write down the values obtained from the Tables.
2. Find the height of a kite when the string is 300 feet long, and
   is inclined at 34° to the horizontal.
3. Find the angles of a triangle whose sides are \( 6 \text{ cm.}, \ 6 \text{ cm.}, \ 5 \text{ cm.} \).

REVISION PAPERS

4. The pilot of an aeroplane flying horizontally at a height of
   3000 feet sees a church at an angle of depression of 65°; 12 seconds
   later the church is vertically below him. Find his speed in feet per
   second.
5. All are the posts of a soccer goal; the ball is at \( P \); the dimen-
   sions in Fig. 91 are in yards. Within what angle must the ball be
   kicked along the ground if it is to enter the goal? The diameter of
   the ball may be neglected.

R. 3.
1. A man sitting at a window with his eye 20 ft. above the ground
   can just see the sun over the top of a roof 45 ft. high, which is 50 yds.
   from him horizontally. Find the elevation of the sun.
2. Two roads meet at \( O \) at an angle of 55°. A man at \( A \) wishes to reach a point \( B \), where
   \( \angle ABO = 90° \), \( AO = 400 \text{ yards} \) (Fig. 92).
   How much distance will be saved by going cross-
   country to \( B \) instead of by the roads \( AO, OB \)?
3. A boy draws the altitude \( AD \) of a triangle \( ABC \). He measures its length correctly, but
   his answer is 1 per cent. too large. At what angle is the line he drew actually inclined to
   the base \( BC \)?
4. Evaluate as shortly as possible:
   \( \frac{1}{\sin 27° 41'}; \) \( \frac{1}{\tan 30° 27'}; \)
   \( \frac{1}{\sec 17° 30'}; \) \( \frac{1}{\cos 20° \csc 70°} \).
5. Find the diameter of a circle in which a chord \( AB, 4 \text{ cm. long} \)
   makes an angle of 35° with the diameter at \( A \).
**TRIGONOMETRY**

**R. 4.**
1. In a triangle ABC, A = 42° 30', B = 90°, c = 10 cm. Find the other two sides.
2. A chord 6 inches long subtends an angle of 140° at the centre of a circle. Find the radius of the circle.
3. (i) Find a value of A if sin A = 2 sin B and B = 17°.
    (ii) Find a value of A if cosec A = 2 cosec B and B = 17°.
4. What is the angle between the tangents to a circle of radius 6 cm. from a point 15 cm. from the centre of the circle?
5. A track zig-zags up a steep slope from A to B; the track is always inclined at 75° to the line AB. If AB = 1000 yd., what is the length of the track?

**R. 5.**
1. In a triangle ABC, A = 90°, a = 17.2 cm., b = 10 cm. Find B and c.
2. The centre of a golf-ball is 2 yd. from the centre of the hole, which is 3 inches in diameter. Within what angle must the ball be struck if it is to drop into the hole?
3. A hill is said to have a gradient of 1 in 6. What is the inclination to the horizontal according to the two possible interpretations of the word gradient?
4. A man walks 1000 yards on a bearing of 25°, and then 800 yards on a bearing of 35°. How far is he North of his starting-point?
5. In Fig. 93, the diameter AB is 5 cm. long. Find the length of PT.

**R. 6.**
1. Find the value of \( \theta + \phi \) if \( \tan \theta = 1 \) and \( \tan \phi = 2 \).
2. The area of the parallelogram ABCD is 10 sq. in.; AB = 3 in., BC = 4 in., calculate \( \angle ABC \).
3. In Fig. 79, p. 46, OQ = 3 cm., NA = 7 cm., \( \angle OQA = 112° 20' \). Calculate QA and ON.
4. A regular heptagon (7 sides) is inscribed in a circle of radius 10 cm. Calculate its perimeter.
5. AB is a diameter and AC is a chord of a circle; E is the midpoint of AC; AB = 10 cm., AC = 8 cm. Calculate \( \angle ABE \).

**CHAPTER IV.**

**THE RIGHT-ANGLED TRIANGLE.**

**Ratios of special angles:** 30°, 45°, 60°.

By using Pythagoras' theorem it is easy to calculate the trigonometric ratios of a few special angles.

**Angle 45°.** Draw a triangle ABC such that \( CA = CB = 1 \) unit of length, \( \angle ACB = 90° \).

![Diagram of a right-angled triangle with angles 45° and 90°](image)

Then \( \angle CAB = 45° \).

Now \( AB^2 = AC^2 + CB^2 = 1^2 + 1^2 = 2 \); \( AB = \sqrt{2} \);

\( \sin 45° = \frac{CB}{AB} = \frac{1}{\sqrt{2}} \); \( \cos 45° = \frac{AC}{AB} = \frac{1}{\sqrt{2}} \); \( \tan 45° = \frac{CB}{AC} = 1 \).

Note. \( \frac{1}{\sqrt{2}} = 1.4142 \approx 0.7071 \).
**TRIGONOMETRY**

*Angles 30°, 60°.* Draw a triangle ABC such that \( AB = BC = CA = 2 \) units of length, and draw BD perpendicular to AC.

![Diagram of triangle ABC with BD perpendicular to AC]

Then \( \angle BAD = 60° \), \( \angle ABD = 30° \); also \( AD = 1 \) unit of length. Now \( BD^2 = AB^2 - AD^2 = 2^2 - 1^2 = 3 \); \( \therefore \) \( BD = \sqrt{3} \).

\[
\sin 60° = \frac{BD}{AB} = \frac{\sqrt{3}}{2} ; \quad \cos 60° = \frac{AD}{AB} = \frac{1}{2} ; \quad \tan 60° = \frac{DB}{AD} = \sqrt{3}.
\]

and

\[
\sin 30° = \frac{AD}{AB} = \frac{1}{2} ; \quad \cos 30° = \frac{BD}{AB} = \frac{\sqrt{3}}{2} ; \quad \tan 30° = \frac{AD}{BD} = \frac{1}{\sqrt{3}}.
\]

*Note.* (i) \( \frac{\sqrt{3}}{2} = \frac{1.73205}{2} = 0.8660 \); \( \frac{1}{\sqrt{3}} = \frac{1.73205}{3} = 0.5774 \)

(ii) These values illustrate the relations between the ratios of complementary angles (p. 24): thus

\[
\sin 45° = \cos(90° - 45°) = \cos 45° = \frac{\sqrt{2}}{2} \]

and

\[
\sin 60° = \cos(90° - 60°) = \cos 30° = \frac{\sqrt{3}}{2}.
\]

(iii) These results should be remembered; this is best done by bearing in mind the two triangles employed.

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**THE RIGHT-ANGLED TRIANGLE**

**EXERCISE IV. a.**

Write down the values of the following, and compare with the values given in the tables:

1. \( \sin 45° \)
2. \( \cos 45° \)
3. \( \cot 45° \)
4. \( \csc 30° \)
5. \( \cot 60° \)
6. \( \csc 60° \)
7. \( \sin 30° \)
8. \( \cos 30° \)
9. \( \sin 60° \)
10. \( \cos 60° \)
11. \( \tan 30° \), \( \tan 60° \)
12. \( \sin 45° \), \( \cos 45° \)

13. The gradient of a mountain side is 1 in 1. What is its inclination to the horizontal? Is there any ambiguity in the data?

14. A climber rises 1 yard vertically for every 2 yards he climbs. What is the inclination of his path to the horizontal?

15. In Fig. 97, MN, the projection of PQ on AB, equals 1/2PQ; what is the inclination of PQ to AB?

16. The shortest side of a 60° set-square is 3 inches. What are the lengths of the other sides?

17. Find the area of an equilateral triangle whose base is 8 cm.

![Diagram of right-angled triangle with sides labeled]

18. A ladder is 20 ft long. How high up a vertical wall will it reach when its inclination to the horizontal is (i) 60°; (ii) 45°; (iii) 30°?

19. In Fig. 98, O is the centre and AQ is a tangent, \( \angle AOP = 60° \); prove \( OP = SQ \).

20. How does the length of shadow of a telegraph pole alter when the sun's elevation decreases from 60° to 30°?
21. An aeroplane flying horizontally passes vertically above a man's head; ten seconds later he notes that its elevation is 60°. When will it be 30°?

22. In Fig. 99, calculate CD, CE, ∠CAD; and prove
\[ \sin 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}}. \]

23. Use Fig. 99 to calculate \( \cos 15° \).

24. Fig. 100 represents a circle, centre O, radius 1 inch; \( \angle FNO = 90° \); AT is a tangent.

(i) Write down the lengths of NP, ON, AT in terms of \( \theta \).

(ii) To what values do these lengths tend when \( \theta \) approaches 90°? Deduce the values of \( \sin 90°, \cos 90°, \tan 90° \).

(iii) To what values do these lengths tend when \( \theta \) approaches 0°? Deduce the values of \( \sin 0°, \cos 0°, \tan 0° \).

25. \( \triangle ABC \) is an equilateral triangle; A is the centre of the arc BEC.

26. \( \angle ADB = 90° \). Use the figure to prove that \( \tan 15° = 2 - \sqrt{3} \).
**Example I.** Given that $\sec \theta = 1.5$, calculate $\sin \theta$ and $\cot \theta$.

![Diagram](image)

Draw the triangle $ABC$, so that

$\angle ABC = 90^\circ$, $BC = 2$, $BA = 3$.

Then

$\frac{\text{hypotenuse}}{\text{adjacent}} = 1.5$.

By Pythagoras, $AC^2 = 3^2 - 2^2 = 9 - 4 = 5$; $\therefore AC = \sqrt{5}$.

$\sin \theta = \sin ABC = \frac{\sqrt{5}}{3}$

and

$\cot \theta = \frac{2}{\sqrt{5}} = \frac{4}{5} = 0.894$.

**Exercise IV. b.**

Evaluate as shortly as possible the expressions in Nos. 1-8:

1. $\sin 50^\circ$
2. $\tan 17^\circ \cos 17^\circ$
3. $\sec 20^\circ \sin 20^\circ$
4. $\cot 65^\circ \sin 65^\circ$
5. $\cosec 23^\circ \cos 23^\circ$
6. $\sec 71^\circ \cot 71^\circ$
7. $\sin 50^\circ \cosec 40^\circ$
8. $\sec 15^\circ \cos 75^\circ$

9. If $\sin \theta = \frac{3}{5}$, calculate $\tan \theta$, $\sec \theta$.
10. If $\cot \theta = 2$, calculate $\cos \theta$, $\cosec \theta$.
11. If $\cos \theta = \frac{1}{2}$, calculate $\sin \theta$, $\sec \theta$.
12. If $\sin \theta = 2 \cos \theta$, calculate $\tan \theta$, $\cosec \theta$.
13. If $\sec \theta = \frac{13}{5}$, calculate $\sin \theta + \cos \theta$ and $\sin^2 \theta + \cos^2 \theta$.
14. Simplify (i) $\tan \theta \cdot \cos \theta$; (ii) $\cosec \theta \cdot \tan \theta$; (iii) $\frac{\cot \theta}{\cosec \theta}$.

**The Right-Angled Triangle**

15. Simplify (i) $\tan \theta \cdot \tan (90^\circ - \theta)$; (ii) $\sin \theta \cdot \cos (90^\circ - \theta)$; (iii) $\cos \theta \cdot \sec (90^\circ - \theta)$.
16. Divide each side of the identity $\sin^2 \theta + \cos^2 \theta = 1$ by $\cos^2 \theta$ and express the result in its simplest form.
17. Divide each side of the identity $\sin^2 \theta + \cos^2 \theta = 1$ by $\sin^2 \theta$, and express the result in its simplest form.
18. Use the notation of Fig. 102 to prove that

$1 + \tan^2 \theta = \sec^2 \theta$.
19. Use the notation of Fig. 102 to prove that

$1 + \cot^2 \theta = \cosec^2 \theta$.
20. Given $\cosec \theta = p$, find (i) $\cot \theta$; (ii) $\cos \theta$ in terms of $p$.
21. If $\theta$ is an acute angle, prove that $\tan \theta$ is greater than $\sin \theta$.
22. Which is the greater $\cot \theta$ or $\cos \theta$, if $\theta$ is an acute angle?
23. Which of the following equations are impossible (for real angles)?

(i) $\sec \theta = 2$; (ii) $\cosec \theta = \frac{1}{2}$; (iii) $\tan \theta = 3$; (iv) $\sin \theta = \frac{1}{2}$; (v) $\cosec \theta = \frac{1}{2}$; (vi) $\cot \theta = 10$; (vii) $\sin \theta = \sec \theta$; (viii) $\cos \theta = \frac{1}{2}$ $\sec \theta$; (ix) $\tan \theta = 2 \sin \theta \sec \theta$; (x) $\sin \theta + \cos \theta = 2$; (xi) $\tan \theta = \cot \theta$; (xii) $\sin \theta + \cos \theta = \frac{1}{2}$.
24. Use the fundamental formulae on p. 55 to deduce the values of $\cos 90^\circ$, $\tan 90^\circ$, given that $\sin 90^\circ = 1$.
25. Use the fundamental formulae on p. 55 to deduce the values of $\sin 0^\circ$, $\tan 0^\circ$, given that $\cos 0^\circ = 1$.
26. Evaluate $\sin^2 20^\circ + \sin^2 70^\circ$.
27. Evaluate $\cos^2 35^\circ + \cos^2 55^\circ$.
28. If $x = 2 \cos \theta$, $y = 3 \sec \theta$, prove that $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
29. The following equations occur in a dynamical problem:

$V^2 \sin \theta \cos \theta = 150$; $\frac{10V^2 \sin^2 \theta}{32} = 75$;

find $V$ and $\theta$.
30. It is proved in Ex. IV. a., No. 22, p. 54, that

$\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$.

(i) calculate the value of $\cos 15^\circ$, and prove that $\tan 15^\circ = 2 - \sqrt{3}$;

(ii) write down similar expressions for $\sin 75^\circ$, $\cos 75^\circ$, $\tan 75^\circ$. 
Problems involving right-angled triangles.

If we have a right-angled triangle, the trigonometrical ratios enable us (i) to find either acute angle if any two sides are given, (ii) to find any side in terms of one other side, and a ratio of either acute angle.

Example II. In Fig. 104 find expressions for (i) \( \angle ACB \), (ii) \( QR \).

![Fig. 104](image)

In Fig. 104 (i), \[ \sin ACB = \frac{c}{a} \quad \therefore \quad \angle ACB = \sin^{-1} \left( \frac{c}{a} \right). \]

In Fig. 104 (ii), \[ \frac{QR}{q} = \csc \theta \quad \therefore \quad QR = q \csc \theta. \]

With a little practice, the reader should be able to write down the second step in each of these two lines without having to write down the first, and he should acquire the habit of doing so.

EXERCISE IV. c.

With the notation of Fig. 105 write down without any preliminary working expressions for the following: Nos. 1-12.

1. \( y \) in terms of \( z, \theta \).
2. \( z \) in terms of \( x, \theta \).
3. \( z \) in terms of \( y, \phi \).
4. \( x \) in terms of \( y, \phi \).
5. \( \phi \) in terms of \( x, y \).
6. \( \theta \) in terms of \( z, x \).
7. \( x \) in terms of \( z, \theta \).
8. \( z \) in terms of \( x, \phi \).
9. \( \theta \) in terms of \( x, y \).
10. \( \phi \) in terms of \( z, x \).
11. \( y \) in terms of \( x, \theta \).
12. \( z \) in terms of \( y, \theta \).

![Fig. 105](image)

With the notation of Fig. 37 write down expressions for the following:

13. \( PR \) in terms of \( PQ, \angle R \).
14. \( EG \) in terms of \( GF, \angle E \).
15. \( XZ \) in terms of \( YZ, \angle X \).
16. \( LR \) in terms of \( QR, PR \).
17. \( LX \) in terms of \( XY, YZ \).
18. \( QR \) in terms of \( PQ, \angle R \).
19. \( XY \) in terms of \( XZ, \angle X \).
20. \( FG \) in terms of \( EF, \angle E \).
21. A Zeppelin is 825 ft. long. What angle does its length subtend at the eye of an observer 5000 ft. vertically below its centre?
22. A captive balloon is held by a rope 400 ft. long which is inclined at 65° to the horizontal. Find the height of the balloon.
23. The eye of an observer is 5 ft. 7 in. above the ground; standing back 4 ft. 6 in. from a 7 ft. wall he can just see a distant aeroplane over the top of the wall. What is its angular elevation?
24. A skylight \( AB \), 18 in. long, is kept open by a stick \( BC \), 8 in. long; through what angle has \( AB \) been opened?

![Fig. 106](image)

25. \( \angle AOB \) is a see-saw, the ground blocks are each 1 ft. high; through what angle can it swing?
26. Paper ruled with parallel lines \( \frac{1}{2} \) inch apart is used to divide a line \( AB \), \( \frac{1}{2} \) inches long, drawn on tracing paper into 7 equal parts. What angle must \( AB \) make with the parallel lines?
27. A pendulum OA, 4 ft. long, hangs inside a case with vertical sides BC, DE and is at a distance of 18 inches from one, and 30 inches from the other. Through what angle can it swing on either side of the vertical?

28. A fleet in line ahead is ordered to maintain a distance of 2 cables between each pair of ships; a midshipman in the bow of one ship knows that the mast of the ship in front is 150 feet from its stern and that its top is 110 ft. above his level. What angular elevation ought he to find for its top for correct stations? (1 cable = 200 yards.)

29. At the top of a tower is a flagstaff 10 ft. high. It throws a shadow 8½ ft. long on the ground. Find the altitude of the sun.

30. Fig. 100 shows a chair with a straight back; DE = 20 in., DB = 10 in., AC = 45 in. The chair is turned over to keep the seat dry. At what angle with the ground are the legs tilted?

31. The sun is due South at elevation 40°; a vertical pole 9 ft. high is 4 ft. away from a vertical wall running East and West. What is the length of the shadow of the pole on the wall?

32. Fig. 110 represents a rectangular protractor. Show that the corners lie between 29° and 36°. Find also the linear distance separating the two 50° graduations.

33. The letter S is formed of two semicircular arcs of radii 5 and 6 cm.; from each extremity of the letter a tangent is drawn to the opposite arc. Find the acute angle between the tangents.

34. Two straight railway lines would if extended intersect at O at an angle of 140°; it is desired to connect them by a circular arc of radius 20 chains; how far from O should the rail begin to curve?

35. In Fig. 111, ABCD is a square; calculate ∠DPC.

36. A straight line is drawn on a map (scale 3 inches to the mile), cutting the contour lines 600, 650, 700 ft. at A, B, C; AB = 0.2 in., BC = 0.3 in.; find the average angle of slope of the hills represented by AB and BC. Find also the greatest height of a vertical flagstaff at C, which is invisible from A.

37. A soldier lying on the ground aims at a mark 60 ft. above the ground on a tower 300 yards away; if the bullet hits the mark it must when leaving the rifle be travelling towards a point on the tower 9 feet above the mark. Find the angle between the axis of the rifle and the line of sight.

38. O is the centre of a circle of radius 7 cm.; PM = 5 cm.; ∠PMA = 90° = ∠QNB. Calculate QN and MN.

39. A marksman under cover can fire in any direction between 15° E. of N. and 9° W. of N. He is 800 yards away from a railway running East and West. What length of the track can he aim at?

40. Standing at the window of a railway carriage travelling at 45 m.p.h. along a straight track, I notice the tower of a cathedral at an angle of 50° to my left. Five minutes later it is 40° to my right. How far away is it on the second occasion?
The following Exercise may be reserved for a second reading.

**EXERCISE IV.**

1. The connecting rod AP of an engine is 6 ft. long; the crank PB of the wheel which the rod drives is 2 ft. long. Calculate the total angle through which AP oscillates.

![Diagram 1](image)

2. Three equal rectangles are described outwards on the sides of an equilateral triangle of side 2 inches. Find their heights if their outward sides form alternate sides of a regular hexagon.

3. A cylinder, radius 5 cm., rests between a vertical wall AB and a wedge ECD. What height will the axis of the cylinder rise when the wedge is pushed up to the wall?

![Diagram 2](image)

4. AB is horizontal and 8 ft. long. C can slide on the cord AB, and always remains vertically below the middle point of AB. Initially its depth is 5 ft. Calculate $\angle ACB$. If the end P is pulled down 3 ft., find the change in $\angle ACB$.

![Diagram 3](image)

5. A boy tries to find the area of a parallelogram by multiplying together the lengths of two adjacent sides; his answer is 29 per cent. too large. Find the acute angle of the figure.

![Diagram 4](image)

6. In $\triangle ABC$, $\angle ABC = 90^\circ$, $AB = 5$ cm., $\angle ACB = 34^\circ$; the bisector of $\angle BAC$ cuts BC at D. Calculate CD.

7. A garden gate ABCD, $3\frac{1}{2}$ ft. wide, is kept shut by a cord attached to B, passing over a fixed pulley F and carrying a heavy weight W. When shut, B is at F. How far does W rise when a man going through the gate opens it to an angle of $75^\circ$?

![Diagram 5](image)

8. Fig. 117 represents in plan the equal doors of a gramophone; the hinges are at A, E, and to prevent jamming, when opening, the faces DC, DG are cut away as shown; $AB = 6$ in., $AD = 6$ in. Find the greatest size of $\angle ADG$ compatible with clearance.

9. Fig. 118 represents a skylight BP, 18 in. long, in a roof ABCD sloping at $48^\circ$ to the horizontal. BP exactly fits the opening BC, and is kept shut by a string attached to P, passing over a small pulley at C and carrying a weight W. How high does W rise when BP is opened through $20^\circ$, and what is then the vertical height of P above G?

![Diagram 6](image)

10. A man standing on a bank of a river with his eye 8 ft. above the water observes that the angle of elevation of the top of a tree on the opposite bank is $23^\circ 45'$, and the angle of depression of its image in the water is $37^\circ 15'$. What is the height of the tree top above the water and the breadth of the river?
11. In Fig. 119, $\angle PMO = 90^\circ = \angle MQO = \angle QNO$. Express terms of $\theta$.

12. A symmetrical cross is tilted as shown in Fig. 120:

$DE=EF=2 \text{ ft.}, \ AB=CD=1 \text{ ft.}, \ \angle BAX=20^\circ$.
Calculate OA and the distances of D, E, F from OX and OY.

13. Fig. 121 represents a loosely fitting drawer tilted at $\theta^\circ$ to the base. Find an equation connecting $\theta^\circ$ with the given measurements $a, b, c, d$.

14. The vertical angle of an isosceles triangle is $2\theta^\circ$ and the radius of its circumcircle is R inches; prove that the area of the triangle is $4R^2 \sin \theta \cos \theta \text{ sq. inches}$.

15. O is the centre of the rectangular top ABCD of a billiard table; a ball struck from O moves along OXY; $AB=2a$, $BC=2b$, $\angle OXA=\angle OXY=\theta$. Find BY in terms of $a, b, \theta$.

16. ABCD is the base of a rectangular tank, full of water; $AB=1 \text{ ft.}, \ BC=2 \text{ ft.}, \ \angle BAX=20^\circ$. The tank is tilted about an angle $\theta^\circ$, so that some water runs out, and then AB through an angle $\theta^\circ$, so that some water returns to its original position. Prove that the new depth of water in the trough is $(1 - \tan \theta^\circ)$ feet, if $\tan \theta^\circ < 1$.

17. With the data of No. 16, prove that the new depth is $\frac{1}{4} \cot \theta$, if $\tan \theta^\circ = \frac{1}{2}$. Find the value of $\theta$ if (i) one-third, (ii) two-thirds of the water overflows.

18. Fig. 123 represents a roof-frame; ADE is a circular arc with its centre on the vertical line COD, and the tangents at A, E are parallel to CB, CO; $OA=a$, $OB=b$. Prove that $CD=b \tan \theta - a(\csc \theta - \cot \theta)$.

19. P is a point on a semicircle whose diameter is AB; $PQ$ is the perpendicular from P to the tangent at B; QR is the perpendicular from Q to PB, RN is the perpendicular from R to AB; $AB=d$, $\angle PAB=\theta$. Express QR and AN in terms of $d, \theta$.

20. Prove that the equation $a \sin \theta + b \cos \theta = c$ can be solved graphically for $\theta$, as follows: Draw two perpendicular lines OA, OB such that $OA=\theta$, $OB=c$; draw a circle, centre O, radius unity, and let it cut AB at P, Q; then $\theta = \angle AOP$ or $\angle AQO$.

[To prove this, draw PM perpendicular to OA.] Solve graphically $2 \sin \theta + \cos \theta = 2$.

21. In Fig. 124, C is the mid-point of the semi-circular arc AB, centre O, radius $a$; $\angle OCP=\theta$, $\angle QPB=90^\circ$; show that

(i) $\angle QPA=\angle PCA=45^\circ + \theta$;

(ii) $AP=1 - \tan \theta$;

(iii) $PQ=PB=1 + \tan \theta$;

(iv) $\tan (45^\circ + \theta) = 1 + \tan \theta$.
CHAPTER V.

THREE DIMENSIONAL PROBLEMS.

Intersecting planes.

If we open a book at any angle, the two pages, if flat, form two planes intersecting in a straight line, the line of the binding. Draw a straight line on each page at right angles to the line of the binding and intersecting on that line. Then the angle between the two planes formed by the pages is defined as the angle between these two straight lines.

The following statements are important:

(i) Any two planes intersect in a straight line unless they are parallel.

(ii) If AB is the line of intersection of two planes ABCD, ABEF and from any point O on AB lines OP, OQ are drawn in the two planes perpendicular to AB, then the angle POQ is by definition equal to the angle between the two planes.

(iii) If the plane ABEF is horizontal, the line OP is called a line of greatest slope of the plane ABCD.

Note: all lines of greatest slope in a plane are parallel; each represents the steepest and shortest path up hill.

Intersecting line and plane.

Suppose any line OP cuts a plane ABCD at O; draw PN perpendicular to the plane ABCD to cut it at N; join ON and produce to Q. Then the angle POQ is defined as the angle between the straight line OP and the plane ABCD.

Note: (i) Since PN is perpendicular to the plane ABCD, it is perpendicular to every line in the plane; thus $\angle PNO = 90^\circ$. For example, if ABCD is a horizontal plane, PN is a vertical line, and every line in ABCD is a horizontal line and so is perpendicular to PN.

(ii) If $\angle POQ = \theta$, ON = OP $\cos \theta$ and NP = OP $\sin \theta$.

(iii) ON is the projection of OP on the plane ABCD.

General procedure. Three-dimensional problems are usually solved by taking a succession of triangles in different planes and applying to each separately the results which have already been established.

In order to calculate the angle between two planes, it is usually necessary to take (or construct) two lines perpendicular to the
line of intersection of the planes and consider some triangle to which these two lines belong.

In order to calculate the angle between a line and a plane, it is usually necessary to take (or construct) the projection of the line on the plane and consider some triangle to which the line and its projection belong.

Part of the initial difficulty occurs in drawing suitable figures. A perspective figure should first be drawn with the dimensions clearly marked. The student may then draw separately the triangles used in the working, as in Example I, below, so as to see more clearly which angles in the perspective figure are right angles. But he should as soon as possible acquire the habit of working only with a perspective figure, as in Examples II, and III, below.

When possible the problem should be illustrated by a simple model; e.g., the cover of a book can be tilted to represent an inclined plane, a match box can be used to represent a room, etc.; skeleton solids made from thin rods are very instructive.

Example I. A hall is 16 ft. long, 12 ft. wide, 8 ft. high.

Calculate the angle which a cord stretched from the centre of the ceiling to one corner of the floor makes with the floor.

\[
\text{Fig. 127.}
\]

\[
\text{O is the centre of the ceiling and } OA \text{ is the cord. } N \text{ is the centre of the floor and } ON \text{ is perpendicular to the floor.}
\]

\[
\therefore \angle OAN \text{ is the projection of } AO.
\]

Then \( \angle OAN \) is the required angle.

By Pythagoras, \( AC^2 = 16^2 + 12^2 = 256 + 144 = 400; \)

\[
\therefore AC = 20 \text{ ft. ;}
\]

\[
\therefore AN = 10 \text{ ft., also } ON = 8 \text{ ft. ;}
\]

\[
\therefore \tan OAN = \frac{8}{10} = 0.8 ;
\]

\[
\therefore \angle OAN = 53^\circ 38'.
\]

Example II. With the data of the above Example, calculate the angle between the planes OAB, ABCD.

\( \angle OAB \) is the line of intersection of the two planes: we therefore look for two lines perpendicular to \( AB \) in the two given planes.

\[
\text{Fig. 128.}
\]

Let \( P \) be the mid-point of \( AB \).

\( \therefore OPN \) is perpendicular to \( AB \), because \( OAB \) is an isosceles triangle; also \( PN \) is perpendicular to \( AB \);

\[
\therefore \angle OPN \text{ is the required angle.}
\]

Now \( ON = 8 \text{ ft. , } PN = \frac{1}{2} BC = 6 \text{ ft. ; } \therefore \tan OPN = \frac{8}{6} = 1.3333 ;
\]

\[
\therefore \angle OPN = 53^\circ 8'.
\]
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EXERCISE V. a.

1. A man holds one end of a pole 8 ft. long in his hand and the other end rests on level ground. His hand is 3 ft. above the ground. What angle does the pole make with the ground?

2. AG and plane ABCD.
3. HB and plane HDAE.
4. HB and plane DHC.
5. HK and plane ABCD.
6. Planes ABCD and ABGH.
7. Planes HECB and FBCG.
8. Planes HDX and HDAE.
9. Planes HEX and ABCD.
10. Planes HDX and HDB.
11. Lines BH and AG.
12. Lines HX and GX.

Fig. 129.

With the notation of Fig. 129, which represents a box with rectangular faces, find the angle of inclination in Nos. 2-12.

21. In Fig. 130, draw CE perpendicular to OB; calculate the length of CE; hence find the angle between the planes OEC and OSA.

22. The edges of a box are 5, 6, 7 inches; find the angle of a diagonal of the box makes with the longest face.

23. Two vertical poles 6 ft., 10 ft. high stand on level ground 5 ft. apart on an East-West line; a tight rope connects their tops; when the sun is due South the shadow of the shorter pole is 4 1/2 ft. long. Find the bearing of the shadow cast by the rope.

24. A flagstaff AP stands at the corner A of a rectangular court ABCD; AB = 80 yd., BC = 60 yd.; the angle of elevation of P from B is 12° 30'. Find the angle of elevation of P from C and D.

25. A wall 12 ft. high runs east and west; the sun is 8° 80' W., at an elevation of 35°. Calculate the breadth of the shadow of the wall on the ground.

26. Find the breadth of the shadow in No. 25, if, further on, the wall runs north and south.

27. A ring, radius 2 ft., is suspended from a point by eight equal strings, each 3 ft. long, attached symmetrically to the ring. Find the angle between two consecutive strings.

28. ABC is an equilateral triangle inscribed in a circle, centre O, radius 30 ft., on a horizontal plane; a mast OE of length 60 ft. is fixed vertically at O, and stayed by wires from E to A, B, C. Calculate θ. Find also the angle between the planes SEC and EAB.

29. The elevation of the top of a tower is 50° from each of two points on the ground 200 ft. apart, one due South and the other due East of the tower. What is the height of the tower?

30. From Tal-y-foel, due South of the Sugar Loaf Mountain, the elevation of the peak is 9° 26'; from Llangrywne 2 2/3 miles due West of Tal-y-foel and at the same level, 200 ft. above the sea, the angle of elevation of the peak is 6° 19'. Find the height of the peak above sea level.

31. The base of a pyramid is an equilateral triangle ABC of side 2 inches, and one of the faces is also an equilateral triangle OAB at right angles to the base. Find the sides and angles of the other two triangular faces OAC, OBC.

Find also the length of the perpendiculars from O to AB and AC, and hence find the angle between the planes BAC and OAC.
32. A man observe that the bearing of a chimney is N. 70° W.; after walking 100 yds. S.W., he finds that the bearing is N.W., and that the angle of elevation of its summit is 60° 20'. Find the height of the chimney.

33. The base of a pyramid is a regular hexagon of side 8 cm. and its height is 6 cm. Find (i) the inclination of each slant edge to the base, (ii) the angle between each face and the base, (iii) the angle between two adjacent faces.

34. The base of a right pyramid is a square of side 4 in.; each face makes an angle of 33° with the base. Find (i) the height of the pyramid, (ii) the angle each slant edge makes with the base.

35. The base of a right pyramid is a regular pentagon; each face is an equilateral triangle. Find the angle which (i) each face, (ii) each edge makes with the base.

Example III. A hill-side is a plane sloping at 27° to the horizontal; a straight track runs up the hill at an angle of 34° with a line of greatest slope. What angle does the track make with the horizontal?

![Figure 131](image)

**Figure 131.**

AB is the line of intersection of the hill-side and a horizontal plane ABCD; AF, BE are lines of greatest slope meeting a horizontal line at F, E.

Let the track AP cut EF at P; draw PN, EC perpendicular to the horizontal plane ABCD.

Then AN is the projection of AP on ABCD; it is required to find ∠PAN = θ, say.

We have

∠FAP = 34°, ∠AFP = 90°, ∠EBC = 27°.

Let PN = h ft., then EC = h ft.

From the right-angled triangle EBC,

\[ BC = h \text{ sec } 27° \text{ ft.} \]

\[ AF = BE = h \text{ sec } 27° \text{ ft.} \]

From the right-angled triangle AFP,

\[ AP = AF \sec 34° = h \text{ sec } 27° \sec 34°. \]

From the right-angled triangle ANP,

\[ \sin \theta = \frac{PN}{AP} = \frac{h}{h \text{ sec } 27° \sec 34°} = \sin 27° \cos 34°. \]

\[ \sin \theta = 0.4540 \times 0.8290 \approx 0.3764; \]

\[ \therefore \theta = 22.7°. \]

Compass bearings. The "bearing" of a horizontal line has already been defined (p. 3). A further definition is required for lines which are not horizontal.

Suppose in Fig. 131, where ABCD is a horizontal plane, that AD points due North; then AFD is the vertical plane which contains the line through A pointing North, and APN is the vertical plane containing AP. The bearing of AP is defined as the angle between the vertical plane containing AP and the vertical plane containing the line through A pointing North, i.e. the angle between the planes APN and AFD, and this is equal to ∠NAD.

It is important to notice that it is not equal to ∠PAF, see Example IV. The reader will see that the bearing of the line AP is the angle which the projection of AP on a horizontal plane makes with a line in the plane pointing North.

Example IV. If, with the data of Example III above, the lines of greatest slope of the plane ABEF bear due North, find the bearing of the track AP.

Let the required angle ∠NAD = φ.

Then \[ \tan \phi = \frac{FP}{FA} \text{ tan } 34°, \text{ since } \angle PFA = 90°. \]
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But FA = DA sec 27°, since ∠FDA = 90°, ∠FAD = 27°;

DN = DA sec 27° tan 34°;

\[ \tan \phi = \frac{DN}{DA} = \text{sec 27° tan 34°}, \text{since } \angle NDA = 90° \]

= 1.1223 \times 0.6745 = 0.7570;

\[ \phi = 37° 7' \]

the bearing of AP is 37° 7'.

Note that this is not equal to ∠PAF, which is 34°.

EXERCISE V. h.

1. A rectangular sheet of paper ABCD lies flat on the face of a desk which slopes at 20° to the horizontal; the lower edge AB is horizontal; AB = 8 in., BC = 6 in. (i) What is the slope of AC?
(ii) What is the slope of a line on the paper making 72° with AB?
(iii) A line AP is drawn on the paper with a slope of 15°; what is ∠PAB?

2. XY is the axis and AB a generator of a circular cylinder, diameter 4 in., height 12 in.; YP is a radius of the base, such that ∠BYP = 50°. Calculate the angle which AP makes with the base.

3. Fig. 133 represents a door opening through an angle of 38°; find the angle between AC and AF.

4. With the data of No. 3, find the angle which AD makes with the plane AGF.

5. BC, the base of the isosceles triangle ABC, is horizontal; ∠ABC makes an angle of 70° with the horizontal; ∠BAC = 34°.
Find the angle of slope of AB.

6. A book is 4 in. wide, 7 in. high; it is placed flat on a level table and the cover is opened so as to make an angle of 125° with the first page. What is the angle of slope of a diagonal of the cover?

7. With the data of No. 6, find the angle through which the cover is opened if the angle of slope of a diagonal of the cover is 25°.

8. A blackboard on an easel slopes at 72° with the ground; a line is drawn on the board making 28° with a horizontal edge of the board. What angle does this line make with the ground?

9. A watch lies on a stand which makes 70° with the horizontal; the hour hand is horizontal at 3 o'clock. What is its angle of slope at (i) 11 o'clock, (ii) 8 o'clock?

10. A man zig-zags in ascending a road of gradient 1 in 10 (i.e., \( \sin^{-1}\left(\frac{1}{10}\right)\)); his path makes an angle of 40° with a line of greatest slope. What is the gradient of the path he follows?

11. In ∆ABC, AB = 5, BC = 4, CA = 3; the triangle is rotated about AB through 35°. Find the angle between the old and new positions of AC.

12. With the data of No. 11, find the angle through which the triangle is rotated about AB if the new and old positions of BC are inclined to each other at 50°.

13. A rectangular sheet of paper ABCD is folded about BO so that the new position BED of BCD is perpendicular to its old position; AB = 6 in., BC = 8 in. Find the angle which AE makes with the plane ABD.

14. A hill, facing due North, slopes at an angle of 18° with the horizontal, and a road is made on its face bearing N. 57° E. Find the angle of slope of the road.

15. With the data of No. 14, find the bearing of a path up the hill if the gradient of the path is 1 in 5 (i.e., \( \sin^{-1}\left(\frac{1}{5}\right)\)).

16. The angle of a line of greatest slope of a hill is N. 72° E., and its angle of slope is 21°; a track running South of East makes 30° with a line of greatest slope. What is its bearing?

17. Four equal panes of glass in the shape of trapeziums, with parallel sides 14 in., 4 in. long and slant sides each 10 in. long, are fitted together to form the cover of a street-lamp. What is the angle between (i) each pane and the vertical, (ii) two adjacent panes?
18. A hill slopes up at an angle of 25° with the horizontal. A skier with skis can climb at an angle of 12° with the horizontal, and one without skis only at an angle of 5° with the horizontal. What is the angle between their tracks on the hill?

19. The roofs of the buildings along two adjacent sides of a rectangular court make angles 30°, 45° with the horizontal. What is the angle of slope of the gutter running down the line of intersection of the roofs?

20. ABC is an isosceles triangle in a vertical plane with its base BC horizontal; its shadow on the horizontal plane is the triangle PBC; if PB = PC and \( \angle BPC = 2\beta \) and \( \angle BAC = 2\alpha \), prove that the sun's elevation is \( \tan^{-1}(\cot \alpha \cdot \tan \beta) \).

21. A rod AB, 6 inches long, is suspended by two equal vertical strings EA, FB, each 10 inches long from fixed points E, F, at the same level. The rod is now twisted so that its midpoint O rises vertically one inch, the rod remaining horizontal. What is the angle of twist? [\( \angle CDP \) is the new position of \( \angle AOB \); if QPH is drawn parallel to AB, \( \angle EQC = 90° \); required to find \( \angle QPC \).]

---

CHAPTER VI.

GRAPHICAL METHODS.

Limit ratios. Draw a circle, centre O, of unit radius OA; draw a radius OP and produce it to meet the tangent at A in T; draw PN perpendicular to OA.

Suppose \( \angle AOP = \alpha \).

Then we have seen that:

\[
\begin{align*}
\sin \alpha & = \frac{NP}{OP} = \frac{NP}{T} \\
(\text{i.e. the number of units of length in } NP) \\
\cos \alpha & = \frac{ON}{OP} = \frac{ON}{T} \\
\tan \alpha & = \frac{AT}{OA} = \frac{AT}{T} \\
\end{align*}
\]
The Angle 0°.

Draw a figure similar to Fig. 135, making \( x \) very small. [The reader should draw such a figure and consider what happens to the lengths of NP, ON, AT.]

We see that the nearer \( x \) approaches the value 0, the smaller NP and AT become, while ON approaches the value 1.

![Figure 135](image)

And we say that, in the limit,

\[
\sin 0° = 0 \quad \cos 0° = 1 \quad \tan 0° = 0.
\]

**Note.** (i) The formula \( \sin^2 x + \cos^2 x = 1 \) shows that if \( \cos 0° = 1 \), then \( \sin 0° \) must equal 0.

(ii) The formula \( \tan x = \frac{\sin x}{\cos x} \) shows that \( \tan 0° = \frac{\sin 0°}{\cos 0°} = \frac{0}{1} = 0 \).

The Angle 90°.

Now draw a figure similar to Fig. 135, making \( x \) very nearly 90. [The reader should draw such a figure and make his own deductions as before.]

We see that the nearer \( x \) approaches the value 90, the smaller ON becomes, while NP approximates to OC and AT increases indefinitely. When \( x = 90° \), O coincides with C, P coincides with T.

With O, and OP has become parallel to AT. We therefore say that, in the limit,

\[
\sin 90° = 1 \quad \cos 90° = 0 \quad \tan 90° = \infty.
\]

**Note.** (i) The symbol \( \infty \) is used for the word "infinity"; the statement \( \tan 90° = \infty \) is conventional, i.e. it does not imply any numerical equality; it merely means that the tangent of an acute angle can be made to exceed any named value by taking \( x \) sufficiently near 90°. Thus

\[
\tan 84°18' > 10 \quad \tan 87°12' > 20 \quad \tan 88°54' > 50 \quad \tan 89°30' > 100 \quad \tan 89°54' > 500 \quad \text{etc.}
\]

(ii) It is suggested that some oral work should be based on the application of the formulae

\[
\sin^2 x + \cos^2 x = 1 \quad \tan x = \frac{\sin x}{\cos x} \quad \sin x = \cos (90° - x) \quad \cos x = \sin (90° - x) \quad \tan (90° - x) = \cot x = \frac{1}{\tan x}
\]

to the six results obtained above.
TRIGONOMETRY

EXERCISE VI. a. (Oral.)

Obtain from the Tables the values of the ratios in examples Nos. 1-12.
1. sin 89°;  2. cos 89°;  3. cos 89°;  4. sec 89°;
5. tan 89°;  6. cot 89°;  7. sin 30°;  8. sec 30°;

What can you say about an acute angle $x^o$ in the following examples, Nos. 13-24.
13. sin $x^o > 0.9999$.  14. cos $x^o < 0.01$.  15. tan $x^o > 25$.
16. cos $x^o > 0.9999$.  17. sin $x^o < 0.01$.  18. cot $x^o > 25$.
19. cosec $x^o > 100$.  20. sec $x^o < 1.0001$.  21. cot $x^o < 0.01$.
22. cosec $x^o < 1.0001$.  23. sec $x^o > 100$.  24. tan $x^o < 0.01$.

25. Make a Table similar to the given Table showing values of cosec $x^o$, sec $x^o$, cot $x^o$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>sin $x^o$</th>
<th>cos $x^o$</th>
<th>tan $x^o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>90°</td>
<td>1</td>
<td>0</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

26. Using the table in No. 25, what do you deduce from cot $x^o = \tan (90° - x)$

if (i) $x = 0$, (ii) $x = 90°$?

Graphs of sin $x^o$ and cos $x^o$.

The variation in value of sin $x^o$ and cos $x^o$, as $x$ varies from 0 to 90, can be illustrated by drawing their graphs. The Tables may be used to give the necessary values:

<table>
<thead>
<tr>
<th>$x$ (to 2 figures)</th>
<th>sin $x^o$ (to 2 figures)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.17</td>
</tr>
<tr>
<td>10</td>
<td>0.34</td>
</tr>
<tr>
<td>20</td>
<td>0.50</td>
</tr>
<tr>
<td>30</td>
<td>0.64</td>
</tr>
<tr>
<td>40</td>
<td>0.77</td>
</tr>
<tr>
<td>50</td>
<td>0.87</td>
</tr>
<tr>
<td>60</td>
<td>0.94</td>
</tr>
<tr>
<td>70</td>
<td>0.98</td>
</tr>
<tr>
<td>80</td>
<td>1</td>
</tr>
</tbody>
</table>

Note. When plotting a graph of any function of $x$, always draw the x-axis from left to right across the page.

EXERCISE VI. b. (Oral.)

1. If you hold Fig. 138 in front of a looking-glass what can you say about the image of the graph of (i) sin $x^o$, (ii) cos $x^o$?

2. Use Fig. 138 to read off the values of (i) sin 18°; (ii) sin 36°; (iii) sin 72°; (iv) cos 18°; (v) cos 36°; (vi) cos 72°.

D.W.T.L.
3. Use Fig. 138 to solve the following equations:
   (i) \( \sin x = 0.56 \);  (ii) \( \cos x = 0.24 \);  (iii) \( \sin x = 0.83 \);
   (iv) \( \cos x = 0.90 \);  (v) \( \sin x = \cos x \).

4. Can you use the graph of \( \sin x \) to solve the equation \( \cos x = 0.60 \)?

5. Can you use the graph of \( \cos x \) to solve the equation \( \sin x = 0.74 \)?

Graphs of \( \tan x \) and \( \cot x \).

The variation in value of \( \tan x \) and \( \cot x \) as \( x \) increases from 0 may also be illustrated graphically, but the values of \( \tan x \), when \( x \) approaches 80 from below, and the values of \( \cot x \) when \( x \) approaches 10 from above, cannot be shown on the graph without making the scale very small. As before, the Tables may be used to give the necessary values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>12</th>
<th>15</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>75</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan x )</td>
<td>0.2</td>
<td>0.27</td>
<td>0.36</td>
<td>0.56</td>
<td>0.84</td>
<td>1.19</td>
<td>1.73</td>
<td>2.75</td>
<td>3.73</td>
<td>4.70</td>
<td>5.70</td>
<td>7.00</td>
</tr>
<tr>
<td>( \cot x )</td>
<td>4.70</td>
<td>3.73</td>
<td>2.75</td>
<td>1.73</td>
<td>1.19</td>
<td>0.84</td>
<td>0.56</td>
<td>0.36</td>
<td>0.27</td>
<td>0.21</td>
<td>0.15</td>
<td>0.04</td>
</tr>
</tbody>
</table>

**EXERCISE VI. c. (Oral.)**

1. Suppose the graph of \( \tan x \) is drawn on transparent paper. If you hold it up to the light with the back of the paper towards you what can you say about the view you get of the graph?

2. What is the reflection of the graph of \( \cot x \) in a looking-glass if the paper is parallel to the face of the glass?

3. Why are there two blank spaces in the table of values for \( \tan x \) and \( \cot x \) above?

4. Use Fig. 139 to read off the values of:
   (i) \( \tan 24^\circ \);  (ii) \( \tan 56^\circ \);  (iii) \( \tan 74^\circ \);
   (iv) \( \cot 16^\circ \);  (v) \( \cot 55^\circ \);  (vi) \( \cot 76^\circ \).

5. Use Fig. 139 to solve the following equations:
   (i) \( \tan x = 0.23 \);  (ii) \( \cot x = 0.51 \);  (iii) \( \tan x = 3 \);
   (iv) \( \cot x = 4 \);  (v) \( \tan x = \cot x \).

6. Can you use the graph of \( \tan x \) to solve the equation \( \cot x = 0.70 \)?

7. Can you use the graph of \( \cot x \) to solve the equation \( \tan x = 1.1 \)?

**Graphical applications.**

As in Algebra, the applications of graphical methods which arise most frequently are (i) the solutions of equations, (ii) the determination of maxima and minima values of given functions. These are best illustrated by an example.
Example. Draw the graph of \( 3 \sin x^2 + 2 \cos x^2 \) for values of \( x \) from 30 to 80, and use it to find (i) a maximum value of \( 3 \sin x^2 + 2 \cos x^2 \), (ii) any solutions of the equation

\[
3 \sin x^2 + 2 \cos x^2 = 3.4
\]
in that range.

\[
\begin{array}{c|cccccccc}
\hline
\text{VALUES OF } x^2 & 30^2 & 40^2 & 50^2 & 60^2 & 70^2 & 80^2 \\
\hline
3 \sin x^2 & 3.382 & 3.436 & 3.598 & 3.398 & 3.501 & 3.303 \\
2 \cos x^2 & 3.232 & 3.481 & 3.584 & 3.396 & 3.501 & 3.303 \\
\hline
\end{array}
\]

Using the Tables, we obtain the following values:

1. Use Fig. 140 to read off the values of \( 3 \sin x^2 + 2 \cos x^2 \), when
   (i) \( x=34^\circ \); (ii) \( x=45^\circ \); (iii) \( x=67^\circ \); (iv) \( x=78^\circ \).
2. Use Fig. 140 to obtain solutions of the equations:
   (i) \( 3 \sin x^2 + 2 \cos x^2 = 3.5 \); (ii) \( 3 \sin x^2 + 2 \cos x^2 = 3\frac{1}{2} \);
   (iii) \( 3 \sin x^2 + 4 \cos x^2 = 6.9 \); (iv) \( 3 \sin x^2 + 2 \cos x^2 = 3.25 \).
3. Draw in one figure (as on p. 82), the graphs of
   (1) cosec \( x^2 \) for values of \( x \) from 10 to 90,
   (2) sec \( x^2 \) for values of \( x \) from 0 to 80.

   (a) Read off the values of
       (i) cosec \( 26^\circ \); (ii) cosec \( 44^\circ \);
       (iii) cosec \( 65^\circ \); (iv) sec \( 25^\circ \);
       (v) sec \( 36^\circ \); (vi) sec \( 75^\circ \).
   (b) Use your graphs to solve the equations
       (i) cosec \( x^2 = 3.9 \); (ii) sec \( x^2 = 3.4 \);
       (iii) cosec \( x^2 = 1.1 \); (iv) sec \( x^2 = 1.2 \);
       (v) cosec \( x^2 = \text{cosec} x^2 \).
4. Draw the graph of \( \sin (2x^2) \) for values of \( x \) from 0 to 45. Compare the results with Fig. 138, p. 81.
5. Draw the graph of \( \cos (x^2 + 30^\circ) \) for values of \( x \) from 0 to 90. Compare the results with Fig. 138, p. 81.
6. Draw the graph of \( 3 \sin x^2 + 4 \cos x^2 \) for values of \( x \) from 0 to 90. (i) Find a maximum value of the function; (ii) find solu-
7. Draw the graph of \( \sin x^2 - \cos x^2 \) for values of \( x \) from 0 to 90. What is the acute angle whose sine exceeds its cosine by 0.3?

8. Draw the graph of \( \tan x^2 + \cot 2x^2 \) for values of \( x \) from 10 to 20. Find a solution of the equation \( \tan 3x + \cos 2x = 2.9 \).

9. Find the maximum values of \( \frac{d^2}{dx^2} - \sin \theta \) if \( \theta + \theta' = \frac{\pi}{2} \), and the value of \( \theta \) for which the expression is a maximum.

10. Draw with the same scale and axes the graphs of \( \cot (x^2 + 15^\circ) \) and \( \sec x^2 \) for values of \( x \) from 0 to 60. Find a solution of the equation \( \cot (x^2 + 15^\circ) = \sec x^2 \). Prove that the correct answer is given by \( x^2 = 20 - x \).

11. Draw the graph of \( \sin x^2 + \sin (2x^2) \) for values of \( x \) from 0 to 45; hence find a solution of the equation \( \sin x^2 + \sin (2x^2) = 1 \).

12. Draw the graph of \( x \sin x^2 \) for values of \( x \) from 0 to 90; hence find a solution of the equation \( x \sin x^2 = 1 \).

13. Tabulate the values of \( \sin x^2 \) when \( x \) has the values 31, 31.1, 31.2, ... 51.0, 51.2. Plot the results and read off from the graph the difference between (i) \( \sin 31^\circ \theta^\prime \) and \( \sin 31^\circ \theta \); (ii) \( \sin 31^\circ \theta^\prime \) and \( \sin 31^\circ \theta \).

(a) What do the difference columns in the Tables give for a difference of \( 3^\circ \)?

(b) What is the general effect if a small portion of Fig. 139 in the neighborhood of \( x^2 = 31 \) is examined with a magnifying glass?

14. Tabulate the values of \( \tan x^2 \), (i) when \( x \) has the values 31, 31.1, 31.2, ... 31.9, 32; (ii) when \( x \) has the values 87, 87.1, 87.2, ... 87.9, 88. Plot the results in two separate graphs, choosing for each the most suitable scale.

(a) What is the chief difference between the graphs?

(b) Read off the values of

(i) \( \tan 31^\circ \theta^\prime \) - \( \tan 31^\circ \theta \)

(ii) \( \tan 87^\circ \theta^\prime \) - \( \tan 87^\circ \theta \)

(c) What do the difference columns in the Tables give for a difference of \( 3^\circ \), and why?

15. Find graphically a solution of the following equations:

16. Find graphically the length of the longest pole that can be carried round the corner without tilting it.

17. Find graphically the value of \( x \) for which \( \frac{60}{x} + \tan x^2 \) is a minimum.

21. Draw the graph of \( \sin x^2 = \sin 3x^2 \).

18. 2 \sin x^2 = \sin 3x^2 \).

19. 2 \tan x^2 = \tan (x^2 + 36^\circ) \).

20. 1 + \sec (2x^2) = \frac{z}{10} \).

22. Find Graphically the value of \( x \) for which \( \frac{60}{x} + \tan x^2 \) is a minimum.

23. A uniform rod AB, 8 inches long, rests on a peg P with the end B against a smooth vertical wall; \( P \) is 1 inch from the wall. If the rod makes an angle \( \theta^\circ \) with the horizontal, show that the height of its mid-point G above the level of \( P \) is \( (4 \sin \theta^\circ - \tan \theta^\circ) \) inches. Find graphically the value of \( \theta \) for which this is greatest. (This corresponds to the position of equilibrium.)

24. A string 6 ft. long rests over two pegs A, B at the same level 2 ft. apart; a body of weight 4 lb. is attached to its mid-point and bodies each of weight 3 lb. are attached to its ends. [See Fig. 143.]
TRIGONOMETRY

In the position shown in Fig. 143 the depth of the centre of gravity of the system below AB is \( \frac{1}{2} (2 \tan \theta - 3 \sec \theta' + 9) \) feet. Find graphically the value of \( \theta \) for which this depth is a maximum. [This corresponds to the position of equilibrium.]

REVISION PAPERS. R. 7-18.

R. 7.

1. A tower 300 ft. away subtends an angle of 25° at a point 25 ft. above the foot of the tower. Calculate the height of the tower.
2. ABCD is a rectangle. Calculate \( \angle APB \).

R. 8.

3. Evaluate (i) \( \sin^2 45° \); (ii) \( \tan 60° \cot 30° \).
4. Find a value of \( x \) such that (i) \( 4x^3 = \sin 5x° \); (ii) \( \sec (x + 10°) = \csc (2x - 10°) \).
5. In \( \triangle ABC \), \( AB = 3, BC = 4, CA = 5 \); the triangle is rotated about \( AB \) through 60°. Find the angle between the old and new positions of AC.

R. 9.

1. A man walks 3 miles N.E., then 5 miles N., then 2 miles N. 25° E. Find how far he then is (i) East, (ii) North of his starting point.
2. Evaluate (i) \( \cot 45° \sec 45° \); (ii) \( \sin 30° \).
3. In Fig. 147 \( AP \) is parallel to \( BQ \). Find \( AB \).
4. Two circles, radii 6 cm. and 10 cm., have their centres 20 cm. apart. Find the angle made with the line of centres, (i) by their direct common tangents, (ii) by their transverse common tangents.

5. A pyramid 6 inches high has a square base, side 4"; its faces are equal. Find the inclination of each face to the base.

R. 10.

1. In Fig. 148 find AN and NC.

![Figure 148](image)

2. What can you say about an acute angle $x^\circ$?

(i) tan $x^\circ > 2$; (ii) sec $x^\circ > 2$; (iii) sin $x^\circ > \cos x^\circ$?

3. A man standing on a cliff 120 ft. high observed two boats in the same vertical plane as himself. The angles of depression are $15^\circ$, $35^\circ$. How far apart are the boats?

4. Draw with the same scale and axes the graphs of sin $x^\circ$ and tan $x^\circ$ for values of $x$ from 0 to 15. What can you deduce from these graphs?

5. OA, OB are edges of the rectangular floor of a room and OC is vertical. P is a point on the floor 2 ft. from OA and 3 ft. from OB, Q is a point on CO 4 ft. above O. What angle does the line PQ make with the floor?

R. 11.

1. If cos $\theta = \frac{1}{2}$, calculate sin $\theta$ without using Trigonometric Tables. Compare your result with that obtained direct from the Tables.

2. The sun is due South at elevation $47^\circ$; a telegraph pole 20 ft. high is 12 ft. away from a vertical wall running East and West, and on the south side of it. What is the length of the shadow of the pole on the wall?

3. In Fig. 154 calculate $\angle PCQ$.

4. Draw the graph of $\sin x^\circ + 3 \cos x^\circ$ for values of $x$ from 10 to 40, and find from it the maximum value of the expression.
5. ABCD is a rectangular court-yard surrounded by buildings 60 ft. high. When the sun is due South the shadow is represented by the shaded portion of the figure. Find (i) the elevation of the sun, (ii) the bearing of B from A.

R. 13.

1. Find values for $r$ and $\theta$, given that
   \[ r \sin \theta^\circ = 7 \quad \text{and} \quad r \cos \theta^\circ = 11. \]

2. A case is being raised vertically through a hatchway CB as shown; the trap-door, CE = DB = 5 ft., rests against it. To what angle is CE opened?

3. A regular pentagon is inscribed in a circle of radius 10 cm. Calculate the height of the minor segment cut off by one side.

4. With the notation and figure of Ex. IV. d., No. 13, p. 64, calculate $c$, given that $d = 10 + 75^\circ$, $d = 10^\circ$, $b = 30^\circ$, $\theta = 10^\circ$.

R. 14.

1. Evaluate as shortly as possible
   \[ \frac{1}{\sin 15^\circ 35' \cos 71^\circ 20'}; \quad (ii) \cos 63^\circ + \cos 27^\circ. \]

2. The tangents from a point A to a circle are each 7 cm. long and contain an angle of 152°. Find the distance of A from the centre of the circle.

3. A rod AB pivoted at A rests with B on a horizontal plane CD as shown. Find the height of B above the plane if AB is rotated about A through 100° in a vertical plane.

4. Draw the graph of $\tan x + 2 \cot x$ for values of $x$ from 35° to 70°. Find from the graph (i) the value of $x$ for which $\tan x + 2 \cot x$ is a minimum; (ii) two solutions of $\tan x + 2 \cot x = 3.2$.

5. A book lies on a level table; its cover 7° by 5° is open at an angle of 20° to the horizontal. What angle does a diagonal of the cover make with the horizontal if the cover turns about one of its longer sides?

R. 15.

1. Given $\sec \theta = b$, express $\sin \theta$ and $\tan \theta$ in terms of $b$.

2. A ball falls vertically from P and strikes a projecting ledge AB at Q and rebounds in the direction QR. If $\tan \theta = 2$ and $\theta = 32^\circ$, find the angle QR makes with the horizontal.

3. With the notation and figure of Ex. IV. d., No. 4, p. 62, if AB = 8 ft. and $\angle AGB = 100^\circ$, find the distance the end P must be pulled down to increase $\angle AGB$ by 40°.
4. A rectangular box is tilted as shown, so that the base makes an angle $\theta$ with the horizontal AE. Show that the height of C above AE is $(p \sin \theta + q \cos \theta)$ inches.

Hence determine the maximum value of $p \sin \theta + q \cos \theta$ for values of $\theta$ between 0 and 90.

If $p = 5, q = 2$, find the value of $\theta$ which gives the maximum value.

5. The greatest slope of a hill is $20^\circ$ to the horizontal. What will be the slope of a path on the hill which makes an angle of $42^\circ$ with a line of greatest slope?

R. 16.

1. If $\sin \theta = \frac{m^2 - 1}{m^2 + 1}$, express $\tan \theta$ and $\cos \theta$ in terms of $m$.

2. A ship A steams at 20 knots on a bearing $330^\circ$, and a ship B at 18 knots on a bearing of $250^\circ$. Find the distance of A, (i) North, (ii) East of B, 3 hours after they parted company. Find also the bearing of A from B at this time.

3. Fig. 156 represents a lamina which when suspended from A hangs so that G is vertically below A. Find the angle which BG makes with the horizontal.

4. With the data of Ex. IV. d., No. 7, p. 63, find the greatest angle to which the gate can be opened if the weight jams at the top when it has risen $4\frac{1}{2}$ feet.

5. A rod 3 ft. long is suspended from the ceiling in a horizontal position by two equal vertical strings 5 ft. long attached to its ends.

R. 17.

1. Write down a value of $\beta$, if
   
   (i) $\sin \beta = \frac{1}{2}$; (ii) $\cos \beta = \frac{1}{\sqrt{3}}$; (iii) $\tan \beta = 2$; (iii) $\cot \beta = 3$.

2. A piece of wire 3 ft. long is bent into the form of an isosceles triangle, $ACB$, one angle of which is $10^\circ$. Find the longest side.

3. A car is made for a cliff railway with wheels of diameters 60 in. and 20 in.; the centres of the wheels are 10 ft. apart, and the line joining them is horizontal when the car is on the rails. Find the inclination of the rails to the horizontal.

4. Draw the graph of cosec $2x + \sec 3x$ for values of $x$ from 10 to 25, and find the solution of $\cos 2x + \sec 3x = 3$.

5. The funnel of a steamer makes an angle of $80^\circ$ with the deck. The steamer rolls, without pitching, through $10^\circ$ on either side of the vertical. Find the extreme inclination of the funnel to the vertical.

R. 18.

1. If $x \cos \theta + y \sin \theta = 4$ and $x \sin \theta - y \cos \theta = 3$, find by squaring and adding a relation between $x$ and $y$, independent of $\theta$.

2. In Fig. 157, ABCD is a square of side 3 inches; find the distance of C from AP in two different ways. Hence prove that

   $$\sin 20^\circ + \cos 20^\circ = \sqrt{3} \cdot \sin 65^\circ.$$
4. From an upper window in a house which is 100 ft. from a church tower the angle of elevation of the top of the tower is 41°, and the angle of depression of the bottom is 16°. How high is the tower?

5. A billiard-ball, diameter 5 cm., moves on a horizontal table along a line making 18° with a cushion, which overhangs so that the point at which the ball strikes it is 4 cm. above the table. Find the distance along the cushion between the point of contact and the point at the same height apparently aimed at.

CHAPTER VII.
ANGLES GREATER THAN A RIGHT ANGLE.

Coordinates. Fig. 158 represents two rectangular axes Ox, Oy drawn on inch-paper, i.e. unit of length one inch. The position of any point in the plane is fixed by its coordinates.

Thus \( A \) is \((+0.8, +0.6)\); \( C \) is \((-0.5, +0.8)\);
\( E \) is \((-0.8, -0.4)\); \( H \) is \((+0.7, -0.8)\).

The axes divide the plane into four quadrants; the object of attaching positive and negative signs to the coordinates is solely to distinguish between the quadrants; it is merely a convention, but a very useful one. It is therefore
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necessary to distinguish between the coordinates of a point and the number of units of length in the distances of the point from the axes Ox, Oy. Thus, although the coordinates of C are (−0.5, +0.8), the number of units of length in OS is 0.5, for OS = 0.5 in., and although the coordinates of E are (−0.8, −0.4), the number of units of length in OT is 0.8, for OT = 0.8 in. and the number of units of length in TE is 0.4, for TE = 0.4 in.

EXERCISE VII. a. (Oral)

Suppose the perpendiculars from B, D, F, G to Ox are BB', DD', FF', GG'. See Fig. 158.
1. What are the coordinates of B and D? What are the lengths of the lines OB', B'B, OD', D'D?
2. Repeat No. 1 for the points F and G.
3. A point is 0.3 inch from Ox and 0.4 inch from Oy. Does this fix its position? What can you say about its position?
4. Mark the point (−0.3, −0.4) on Fig. 158. Name, by a letter, some other point shown in the same quadrant.
5. Repeat No. 4 for the points (−0.3, +0.4) and (+0.3, −0.4).
6. The coordinates of a point K are (x, y); what do you know about x and y if K lies in the same quadrant as (i) C, (ii) H, (iii) A, (iv) E?

Note. The quadrants in which A, C, E, H lie are called the first, second, third and fourth quadrants respectively.

Trigonometrical ratios.

Take two rectangular axes Ox, Oy and imagine a line OP of fixed length r inches to rotate anti-clockwise about O, starting from the position Ox.

Suppose that at any time \( \angle xO P = \theta \).

Draw PN perpendicular to Ozy; let \( ON = a \) in., \( NP = b \) in.

If \( \theta^0 < 90^0 \), we have by previous definitions,
\[
\sin \theta^0 = \frac{b}{r}, \quad \cos \theta^0 = \frac{a}{r}, \quad \tan \theta^0 = \frac{b}{a}.
\]

Now the previous definitions apply only to acute angles. For angles greater than 90°, new definitions are necessary.

VII. ANGLES GREATER THAN A RIGHT ANGLE

The fact that the coordinates of P are (+a, +b) suggests the form these new definitions should take.

If \( \theta^0 > 90^0 \), we define the trigonometrical ratios of \( \theta \) as follows:
\[
\sin \theta^0 = \frac{y}\text{-coordinate of } P, \quad \cos \theta^0 = \frac{x}\text{-coordinate of } P, \quad \tan \theta^0 = \frac{y}\text{-coordinate of } P.
\]

This definition evidently includes the original definition as a particular case and extends it.

(i) Suppose \( \angle xOP \) is obtuse, i.e. \( 180^0 > \theta^0 > 90^0 \), see Fig. 160.
TRIGONOMETRY

As before, let ON = a inches, NP = b inches.
Then the coordinates of P are (-a, +b);
\[
\begin{align*}
\sin \theta &= \frac{b}{r} = \frac{b}{r} \\
\cos \theta &= -\frac{a}{r} = -\frac{a}{r} \\
\tan \theta &= \frac{b}{-a} = -\frac{b}{a}
\end{align*}
\]

\therefore \text{ if } \theta^\circ \text{ is obtuse, } \sin \theta^\circ \text{ is positive, but } \cos \theta^\circ \text{ and } \tan \theta^\circ \text{ are each negative.}

Further, in Fig. 160, \angle PON = 180^\circ - \theta^\circ \text{ and is acute.}
\[
\sin \theta^\circ = \frac{b}{r} = \sin \theta^\circ = \sin (180^\circ - \theta^\circ),
\]
\[
\cos \theta^\circ = -\frac{a}{r} = -\cos \theta^\circ = \cos (180^\circ - \theta^\circ),
\]
\[
\tan \theta^\circ = -\frac{b}{a} = -\tan \theta^\circ = \tan (180^\circ - \theta^\circ).
\]

For example,
\[
\sin 138^\circ = \sin (180^\circ - 138^\circ) = \sin 42^\circ = 0.6691,
\]
and \[
\cos 138^\circ = -\cos (180^\circ - 138^\circ) = -\cos 42^\circ = -0.7431,
\]
and \[
\tan 138^\circ = -\tan (180^\circ - 138^\circ) = -\tan 42^\circ = -0.9009.
\]

(ii) Suppose 270° > \theta^\circ > 180°, see Fig. 161.
As before, let ON = a inches, NP = b inches.

\begin{align*}
\sin \theta &= -\frac{b}{r} = -\frac{b}{r} \\
\cos \theta &= +\frac{a}{r} = +\frac{a}{r} \\
\tan \theta &= -\frac{b}{a} = -\frac{b}{a}
\end{align*}

\therefore \text{ if } 270^\circ > \theta^\circ > 180^\circ, \text{ tan } \theta^\circ \text{ is positive, but } \sin \theta^\circ \text{ and } \cos \theta^\circ \text{ are each negative.}

(vii) ANGLES GREATER THAN A RIGHT ANGLE 101

Then the coordinates of P are (-a, -b);
\[
\begin{align*}
\sin \theta &= -\frac{b}{r} = -\frac{b}{r} \\
\cos \theta &= +\frac{a}{r} = +\frac{a}{r} \\
\tan \theta &= -\frac{b}{a} = -\frac{b}{a}
\end{align*}
\]

\therefore \text{ if } 360^\circ > \theta^\circ > 270^\circ, \cos \theta^\circ \text{ is positive, but } \sin \theta^\circ \text{ and } \tan \theta^\circ \text{ are each negative.}
Numerical values of ratios. Suppose that in the four figures, Figs. 159-162, the four triangles ONP are congruent; then, apart from sign, the numerical values of any one trigonometrical ratio are all equal.

For example, suppose, in Fig. 159, $\theta^\circ=63^\circ$; $\sin 63^\circ=0.8910$.

Then in Fig. 160, $\theta^\circ=180^\circ-63^\circ=117^\circ$,

$\therefore \sin 117^\circ=0.8910$;

and in Fig. 161, $\theta^\circ=180^\circ+63^\circ=243^\circ$,

$\therefore \sin 243^\circ=-0.8910$;

and in Fig. 162, $\theta^\circ=360^\circ-63^\circ=297^\circ$,

$\therefore \sin 297^\circ=-0.8910$.

Similarly, since $\cos 63^\circ=0.4540$, we have

$\cos 117^\circ=-0.4540$; $\cos 243^\circ=-0.4540$; $\cos 297^\circ=0.4540$.

And, since $\tan 63^\circ=1.9626$, we have

$tan 117^\circ=-1.9626$; $tan 243^\circ=1.9626$; $tan 297^\circ=-1.9626$.

We may state these results as follows:

The ratio of any angle $zOP=\theta^\circ$ is numerically equal to the same ratio of any angle whose sum with $\theta^\circ$ or difference from $\theta^\circ$ is a multiple of $180^\circ$; the sign of the value of the ratio is determined by the quadrant in which $OP$ lies.

The reader should determine this sign by drawing a figure as above. It may, however, be of interest to give a mnemonic; write the letters of the word CAST in the quadrants; these indicate which ratio is positive for the marked quadrant, Cosine, All, Sine, Tangent. Obviously all the ratios are positive in the first quadrant; this fixes the position of $A$; and the letters are written the same way round as $OP$ rotates.

The following is a summary of the results established:

$\sin (180^\circ-\theta^\circ)=-\sin \theta^\circ$; $\sin (180^\circ+\theta^\circ)=-\sin \theta^\circ$;

$\sin (360^\circ-\theta^\circ)=-\sin \theta^\circ$.

EXERCISE VII. b.

The radius of each circle in Fig. 164 is 5 cm.; $\angle AOP$, swept out anti-clockwise, equals $\theta^\circ$. Write down the values of $\sin \theta^\circ$, $\cos \theta^\circ$, $\tan \theta^\circ$ corresponding to the data in Nos. 1-9.

1. OM = 3 cm. 2. ON = 3 cm. 3. OR = 3 cm.
4. ON = 4 cm. 5. OR = 4 cm. 6. OM = 4 cm.
7. NP = 4 cm. 8. RP = 3 cm. 9. OM = 2 cm.

10. Draw on squared paper a circle of radius 1 inch, and, by drawing the actual angles involved, use it to find approximate values of

(i) $\sin 15^\circ$; (ii) $\cos 15^\circ$; (iii) $\cos 220^\circ$; (iv) $\tan 220^\circ$;

(v) $\sin 300^\circ$; (vi) $\cos 300^\circ$; (vii) $\cos 170^\circ$; (viii) $\tan 140^\circ$.

What can you say about $\theta$ in the following cases, Nos. 11-16?

11. $\sin \theta^\circ$ is positive, $\cos \theta^\circ$ is negative.
12. $\cos \theta^\circ$ is positive, $\sin \theta^\circ$ is negative.
13. $\tan \theta^\circ$ and $\cos \theta^\circ$ are both negative.
14. $\cos \theta^\circ$ is negative, $\tan \theta^\circ$ is positive.
15. $\cos \theta^\circ$ is positive, $\sin \theta^\circ$ is negative.
16. $\sin \theta^\circ$ and $\tan \theta^\circ$ are both negative.

Express each of the ratios in Nos. 17-28 as the ratio of an acute angle with the appropriate sign (thus: $\sin 290^\circ=-\sin 70^\circ$):

17. $\cos 200^\circ$. 18. $\sin 170^\circ$. 19. $\sin 240^\circ$. 20. $\cos 280^\circ$.
21. $\sin 190^\circ$. 22. $\cos 165^\circ$. 23. $\sin 260^\circ$. 24. $\cos 260^\circ$.
25. $\tan 145^\circ$. 26. $\sin 90^\circ$. 27. $\tan 230^\circ$. 28. $\tan 335^\circ$. 

Fig. 164.
Construct on squared paper the following angles and measure them (two answers in each case):

29. \( \cos^{-1}(-\frac{1}{2}) \)  
30. \( \sin^{-1}(-\frac{1}{2}) \)  
31. \( \tan^{-1}(-\frac{1}{2}) \)  
32. \( \tan^{-1}(\frac{1}{2}) \)

Find from the tables the values of the following:

33. \( \sin 160^\circ \)  
34. \( \cos 105^\circ \)  
35. \( \tan 300^\circ \)  
36. \( \cos 153^\circ \)  
37. \( \tan 210^\circ \)  
38. \( \sin 317^\circ \)  
39. \( \cos 317^\circ \)  
40. \( \sin 315^\circ \)  
41. \( \sin 123^\circ 40' \)  
42. \( \cos 123^\circ 40' \)  
43. \( \tan 216^\circ 26' \)  
44. \( \cos 308^\circ 35' \)

**Example I.** Draw with the same axes and scale the graphs of \( \sin x^\circ \) and \( \cos x^\circ \) for values of \( x \) from 0 to 360.

We have from the tables the following values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>45</th>
<th>90</th>
<th>135</th>
<th>180</th>
<th>225</th>
<th>270</th>
<th>315</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin x^\circ )</td>
<td>0-71</td>
<td>1</td>
<td>0-71</td>
<td>0</td>
<td>-0-71</td>
<td>-1</td>
<td>0-71</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \cos x^\circ )</td>
<td>1</td>
<td>0-71</td>
<td>0</td>
<td>-0-71</td>
<td>-1</td>
<td>0-71</td>
<td>0</td>
<td>0-71</td>
<td>1</td>
</tr>
</tbody>
</table>

In Figure 165, the graph of \( \sin x^\circ \) is represented by a continuous curve and the graph of \( \cos x^\circ \) by a broken curve.

**Note.** 1. In order to draw a reasonably accurate graph, it is necessary to take a larger number of values of \( x \) than are

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shown in the table above. (Compare the graph and table of values on p. 80.)

2. The definitions for sines and cosines of angles of any magnitude give smooth curves for each graph; further, if the graph of \( \cos x^\circ \) is moved in the direction of the x-axis through 90 units, it coincides with the graph of \( \sin x^\circ \); this is equivalent to saying that \( \sin (x^\circ + 90^\circ) = \cos x^\circ \).

3. For oral questions on this graph, see Ex. VII. c., Nos. 1-8.

The remaining trigonometrical ratios of \( \theta \), when \( \theta^\circ > 90^\circ \), are defined in accordance with the definitions already given on p. 59.

Thus _or all values of \( \theta \), we say that_

\[
\csc \theta^\circ = \frac{1}{\sin \theta^\circ} \quad \sec \theta^\circ = \frac{1}{\cos \theta^\circ} \quad \cot \theta^\circ = \frac{1}{\tan \theta^\circ}
\]

Hence

\[
\csc 117^\circ = \frac{1}{\sin 117^\circ} = \frac{1}{\sin 63^\circ} = \csc 63^\circ,
\]

\[
\sec 117^\circ = \frac{1}{\cos 117^\circ} = \frac{1}{-\cos 63^\circ} = -\sec 63^\circ,
\]

\[
\cot 117^\circ = \frac{1}{\tan 117^\circ} = \frac{1}{-\tan 63^\circ} = -\cot 63^\circ.
\]

And, in general,

\[
\csc \theta^\circ = \csc (180^\circ - \theta^\circ) \quad \sec \theta^\circ = -\sec (180^\circ - \theta^\circ) \quad \cot \theta^\circ = -\cot (180^\circ - \theta^\circ)
\]

**EXERCISE VII. c.**

Use the graphs in Fig. 165 for Nos. 1-8.

1. What are the values of \( \cos 27^\circ \), \( \cos 153^\circ \), \( \cos 207^\circ \), \( \cos 333^\circ \); \( \sin 117^\circ \), \( \sin 243^\circ \), \( \sin 297^\circ \)?

2. What is \( x \) if \( \cos x^\circ = -0-3 \); \( \sin x^\circ = -0-3 \)?

3. What is \( x \) if \( \sin x^\circ = 0-8 \); \( \cos x^\circ = 0-8 \)?

4. For what range of values of \( x \) between 0 and 360 is \( \sin x^\circ \) negative; \( \cos x^\circ \) negative?
5. What can you say about $x$ if (i) $\sin x^\circ > 0.4$; (ii) $\sin x^\circ < -0.4$?
6. What can you say about $x$ if (i) $\cos x^\circ > 0.4$; (ii) $\cos x^\circ < -0.4$?
7. For what values of $x$ is $\sin x^\circ = \cos x^\circ$?
8. Make a rough copy of Fig. 165, and show how you think the graphs continue for values of $x$ beyond 300.

What would you expect the graphs to be for negative values of $x$?

9. Sketch the graphs of $\sin (2x^\circ)$ and $\cos (2x^\circ)$ for values of $x$ from 0 to 180.

Use the tables to find two solutions of each of the following equations:
10. $\cos \theta^\circ = 0.5$. 11. $\sin \theta^\circ = 0.342$. 12. $\tan \theta^\circ = 1.6$.
13. $\sin \theta^\circ = -0.766$. 14. $\cos \theta^\circ = -0.454$. 15. $\tan \theta^\circ = -0.404$.

Find the solutions of the following from the tables values of the following:
20. cot 265°. 21. cosec 100°. 22. sec 220°. 23. cot 310°.

Use the tables to solve the following equations:
24. $\cot x^\circ = \frac{1}{2}$. 25. $\cosec x^\circ = 2.5$. 26. $\sec x^\circ = -2.4$.
27. $\cosec x^\circ = -2.4$. 28. $\tan x^\circ = 4$. 29. $\sec x^\circ = 3$.

30. Draw the same figure
(i) the graph of $\tan x^\circ$ for values of $x$ from 0 to 75, and from 105 to 255, and from 285 to 360;
(ii) the graph of $\cot x^\circ$ for values of $x$ from 15 to 105, and from 195 to 345.

31. Draw the same figure
(i) the graph of $\cosec x^\circ$ for values of $x$, from 15 to 105, and from 106 to 345;
(ii) the graph of $\sec x^\circ$ for values of $x$ from 0 to 75, and from 105 to 255, and from 285 to 360.

32. Find $x$ if
(i) $\sin x^\circ = 0.6$ and $\cos x^\circ = -0.8$;
(ii) $\cos x^\circ = 0.6$ and $\sin x^\circ = -0.8$;
(iii) $\cos x^\circ = 0.8$ and $\tan x^\circ = -0.75$.

### ANGLES GREATER THAN A RIGHT ANGLE

Simplify the following:
33. $\cosec (180^\circ - \theta)$. 34. $\sec (180^\circ - \theta)$. 35. $\cot (300^\circ - \theta)$.
36. $\sec (260^\circ - \theta)$. 37. $\cosec (180^\circ + \theta)$. 38. $\cot (180^\circ + \theta)$.
39. The angles of a triangle are $A'$, $B'$, $C'$; express $\sin (B' + C')$ and $\cos (B' + C')$ in terms of $A$.

**Generalisations.** The introduction of ratios of angles of any magnitude enables many results to be stated in a more general form than would otherwise be possible. The definitions on p. 99 have been so chosen that in general formulas which are established for acute angles hold equally for angles of any magnitude.

**Example II.** A man walks 3 miles along a straight road whose true bearing is $\theta^\circ$. How far (i) north, (ii) east is he of his starting point? Consider also the special cases when the true bearings are 160° and 245°.

![Diagram](image1)

In Fig. 166, where $\angle NOP = 0^\circ < 90^\circ$, we see that $P$ is 3 cos $\theta^\circ$ miles North of $O$ and 3 sin $\theta^\circ$ miles East of $O$. These statements remain true whatever size the angle $\theta^\circ$ is.

Thus, in Fig. 167, $\theta^\circ = 160^\circ$, and these results become 3 cos 160° miles North of $O$ and 3 sin 160° miles East of $O$. But, since $\angle KOP = 180^\circ - 160^\circ = 20^\circ$, it is clear that $P$ is 3 cos 20° miles South of $O$ and 3 sin 20° miles East of $O$; or we may say that $P$ is $-3$ cos 20° miles North of $O$. 

![Diagram](image2)

![Diagram](image3)
The two sets of results are therefore consistent if
\[ \cos 160^\circ = -\cos 20^\circ \quad \text{and} \quad \sin 160^\circ = -\sin 20^\circ ; \]
and we have already seen (pp. 102-103) that this is so.

Again, in Fig. 168, \( \theta = 245^\circ \), and the general results become
3 \cos 245^\circ \text{ miles North of } O \text{ and } 3 \sin 245^\circ \text{ miles East of } O.

But, since \( \triangle KOP = 245^\circ - 180^\circ = 65^\circ \), it is clear that \( P \) is
3 \cos 65^\circ \text{ miles South of } O \text{ and } 3 \sin 65^\circ \text{ miles West of } O; \text{ or we may say that } P \text{ is}
-3 \cos 65^\circ \text{ miles North of } O \text{ and } -3 \sin 65^\circ \text{ miles East of } O.

The two sets of results are therefore consistent if
\[ \cos 245^\circ = -\cos 65^\circ \quad \text{and} \quad \sin 245^\circ = -\sin 65^\circ ; \]
and we have already seen (pp. 102-103) that this is so.

**Example III.** Find the area of \( \triangle ABC \) in terms of two sides
and the included angle, say, \( b, c, \theta \).

![Diagram of \( \triangle ABC \)](fig. 169)

Draw \( CF \) perpendicular to \( AB \) or \( AB \) produced.

The area of \( \triangle ABC = \frac{1}{2} b \cdot c \cdot \sin \theta \).

In Fig. 169, \( FC = b \sin \theta \).

In Fig. 170, \( FC = b \sin \frac{\angle BAC}{2} = b \sin \left( 180^\circ - \theta \right) = b \sin \theta \);
\( \therefore \text{in each case, area of } \triangle ABC = \frac{1}{2} b \cdot c \cdot \sin \theta \).

**Note.** This formula for the area of a triangle is therefore the same whether the included angle is acute or obtuse; obviously this is a great convenience. It is a direct consequence of the definition chosen above for the sine of an obtuse angle.

This result was first given by Snell (1627), Professor of Mathematics at Leyden.

**EXERCISE VII.**

1. A rod \( OP \), 5 ft. long, rotates about \( O \) through 10° per second
in a vertical plane, anti-clockwise, starting from the horizontal \( OA \).

![Diagram of \( \triangle OPA \)](fig. 173)

Show that after \( t \) seconds, the height of \( P \) above \( OA \) is \( 5 \sin (10t) \) feet.
Evaluate this expression for \( t = 3, 15, 21, 33 \), and show roughly on a figure the position corresponding to each case.
2. With the data of No. 1, find the height of P above OA after 5 seconds and the time when it is next at the same height.

3. With the data of No. 1, find how far P is to the right of the vertical line OY after 10 seconds. Evaluate this expression for \( t = 3, 15, 21, 30 \), and show roughly on a figure the corresponding positions.

4. With the data of No. 1, find after what times P will be (i) 2 feet to the right of the vertical line OY, (ii) 2 feet to the left of OY.

5. In a certain tidal channel, in \( t \) hours the water rises \( 12 \sin \left( \frac{\pi t}{6} \right) \) feet above mean level. If it is mean level at 2 a.m., find the height above mean level at 7 a.m., 12 noon, 5 p.m., 10 p.m. on the same day.

6. With the data of No. 5, find the times of (i) high-water, (ii) low-water during the 24 hours after 2 a.m.

7. A buoy in the sea is rising and falling vertically with the waves; its height above the mean level after 10 seconds from the time when first observed is 6 cos (30°) feet. Find this height for \( t = 2, 4, 6, 8, 10, 12 \). Through what distance does it oscillate? What is the time of one complete oscillation?

8. There is a steady wind of 24 m.p.h. blowing from S East of North; an aeroplane starts from O and heads due South; if there were no wind it would be travelling at 90 m.p.h.; but owing to the wind its position after 10 minutes is shown by the point P in Fig. 174. How far, in terms of \( \theta \), is P (i) north, (ii) west of O? Evaluate these expressions if \( \theta \) equals \( a) \ 20^\circ, \ b) \ 160^\circ, \ c) \ 0^\circ, \ d) \ 180^\circ, \ e) \ 360^\circ, \ f) \ 200^\circ, \ g) \ 340^\circ \), and illustrate your answers by rough figures, showing the various positions.

9. The legs of a compass, each \( l \) inches long, are opened to an angle \( 2\theta \). Show that the distance between the points is \( 2l \sin \theta \) inches. Is this true if \( \theta > 90^\circ \)? What happens if \( \theta = 180^\circ \) ?

10. A rod OP, \( l \) inches long, swings about O in a vertical plane; OP is perpendicular to the vertical OA; if \( \angle AOP = \theta \), show that \( AN = l(1 - \cos \theta) \) inches. Is this still true if \( \theta > 90^\circ \)? What happens if (i) \( \theta = 180^\circ \), (ii) \( \theta = 270^\circ \), (iii) \( \theta = 360^\circ \)?

11. Prove that the area of the parallelogram in Fig. 177 is \( ab \sin \theta \).

12. In Fig. 177, express the length of AD in two different ways, and show that

\[
\frac{b}{\sin B} = \frac{c}{\sin C}
\]

Is this result also true for Fig. 180?

13. It has been proved (p. 65) that, if \( \theta \) is any acute angle, then

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad \text{and} \quad \sin \theta = \tan \theta.
\]

Prove from the definitions on p. 99 that these formulae are always true.
14. In Fig. 181, \(a, b, c, d\) are the feet of the perpendiculars from \(A, B, C, D\) to a given line \(OX\); \(AB, BC, CD\) make angles \(\alpha, \beta, \gamma\) with 
\[\text{OX, measured anti-clockwise from } OX; \ AB = p, BC = q, CD = r.\]
Prove that \(ad = p \cos \alpha - q \cos \beta + r \cos \gamma\); and find an expression for the height of \(D\) above \(A\), if \(OX\) is regarded as horizontal and \(Aa, Bb, \text{ etc.}, \) as vertical.
Evaluate these expressions, taking \(p = 1, q = 1, r = 2\), in the following cases, illustrating each answer by a rough figure:
(i) \(\alpha = 30^\circ, \quad \beta = 110^\circ, \quad \gamma = 200^\circ;\)
(ii) \(\alpha = 140^\circ, \quad \beta = 50^\circ, \quad \gamma = 310^\circ;\)
(iii) \(\alpha = 110^\circ, \quad \beta = 320^\circ, \quad \gamma = 200^\circ.\)

15. With the data of No. 14, show that the angle \(\theta^\circ\) which the line joining \(A\) to \(C\) makes with \(OX\) is given by 
\[\tan \theta = \frac{p \cos \alpha - q \cos \beta + r \cos \gamma}{p \sin \alpha + q \sin \beta}.\]
Taking \(p = 1, q = 1, \) and \(\beta = \theta\), find \(\theta\) in the following cases, illustrating each answer by a rough figure:
(i) \(\alpha = 50^\circ, \quad \beta = 130^\circ;\)
(ii) \(\alpha = 30^\circ, \quad \beta = 30^\circ;\)
(iii) \(\alpha = 110^\circ, \quad \beta = 10^\circ;\)
(iv) \(\alpha = 220^\circ, \quad \beta = 100^\circ.\)

16. (i) If in Fig. 182, \(PQ = QR,\) prove that \(\cot \beta = (\cot \alpha + \cot \gamma).\)

(ii) If in Fig. 183, \(AD\) is a median of \(\triangle ABC,\) use (i) to express \(\cot \theta\) in terms of \(B, C,\) and evaluate \(\theta\) in the following cases, illustrating each by a rough figure:
(1) \(B = 30^\circ, \quad C = 50^\circ;\)  
(2) \(B = 35^\circ, \quad C = 105^\circ;\)  
(3) \(B = 50^\circ, \quad C = 35^\circ.\)

CHAPTER VIII.

USE OF LOGARITHM TABLES.

The numerical work necessary in applications of Trigonometry can generally be shortened by using logarithms, and, to save time, tables of logarithms of each trigonometrical ratio have been constructed. For example, we find on one page that \(\sin 50^\circ = 0.7660\) and on another page that \(\log 0.7660 = -1.8842;\) therefore \(\log \sin 50^\circ = -1.8842;\) but if we use the table of log-sines we obtain this result by a single reading.

It is not always easy to fix the characteristic by common sense; it is therefore always printed, but only at the beginning of each line:
\[\text{e.g. } \log \tan 10^\circ 30' = 1.2680; \quad \log \tan 84^\circ 30' = 1.0164.\]

As in the case of the natural ratios, the figures in the difference columns must be \textit{subtracted} for log cosines, log cosecants and log cotangents.

The figures in the difference columns can only be \textit{average differences}, and, usually, sufficient accuracy is secured by taking the average over an interval of one degree. When, however, this introduces an appreciable error, the difference columns in the tables at the end of the book give \textit{average differences for one minute}, calculated over "12 minute intervals."

\[\text{e.g. To find } \log \tan 84^\circ 56' \text{ and } \log \tan 84^\circ 58'.\]
For the interval 48' to 60', the difference for 1' is given as 14.
\[ \therefore \text{the difference for } 2' \text{ is } 28, \text{ i.e. } 0.028. \]
\[ \log \tan 84^\circ 54' = 1.0494, \]
\[ \therefore \log \tan 84^\circ 56' = 1.0494 + 0.0028 = 1.0522, \]
\[ \log \tan 85^\circ = 1.0590, \]
\[ \therefore \log \tan 84^\circ 58' = 1.0590 - 0.0028 = 1.0562. \]

**Note.** The difference correction is applied to the nearest angle given in the tables.

**Example.** It is proved in Chapter IX. (see p. 119) that, in any triangle ABC, \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \).

![Diagram of a triangle with sides A, B, C and angles A, B, C.](Image)

(i) Given \( b = 4.618, B = 41^\circ 29', C = 57^\circ 37', \) find \( c \).

(ii) Given \( b = 67.24, c = 89.69, C = 62^\circ 46', \) find \( B \).

(i) \( \frac{c}{\sin 57^\circ 37'} = \frac{4.618}{\sin 41^\circ 29'} \)

\[ 4.618 \sin 57^\circ 37' = 5.887 \approx 5.89 \]

(ii) \( \sin B = \sin 62^\circ 46' \)

\[ 67.24 = 89.69 \]

\[ 67.24 \sin 62^\circ 46' = 89.69 \]

\[ \therefore B = 41^\circ 50'. \]

**Note.** (i) The working should at first be set out in the way shown above, but after a reasonable standard of accuracy has been attained the side-columns which show the numbers may be omitted. Logarithms should never be allowed to appear in the middle of the page unless expressed either in the index form or in some other unambiguous fashion.

(ii) When using four-figure tables, results should not be given correct to 3 significant figures; although this applies also to angles, it is convenient to give angles to the nearest minute, but it should be realised that the fourth figure is not reliable.

**Exercise VIII.**

1. Look up \( \sin 56^\circ \) and then look up the logarithm of this number. What does the "Log Sine" table give for \( \log \sin 55^\circ \)?

Repeat this process for (i) \( \cos 37^\circ 24' \); (ii) \( \tan 48^\circ 30' \);

(iii) \( \cot 85^\circ 18' \); (iv) \( \sec 79^\circ 57' \); (v) \( \cosec 23^\circ 20' \).

2. Find from the tables the values of the following:

(i) \( \log \sin 17^\circ 36' \); (ii) \( \log \cos 63^\circ 32' \); (iii) \( \log \cot 65^\circ 26' \);

(iv) \( \log \tan 17^\circ 16' \); (v) \( \log \cosec 5^\circ 36' \); (vi) \( \log \sec 84^\circ 24' \).

Why are (v) and (vi) equal?

3. Find \( x \), given that

(i) \( \log \sin x = 1.9023 \); (ii) \( \log \cos x = 1.9435 \);

(iii) \( \log \tan x = 0.478 \); (iv) \( \log \cot x = 1.0006 \);

(v) \( \log \sec x = 0.0573 \); (vi) \( \log \cosec x = 1.403 \);

(vii) \( \log \sin x = 1.5491 \); (viii) \( \log \tan x = 1.9 \);

(ix) \( \log \cos x = 1.531 \); (x) \( \log \cot x = 1.55 \);

(xi) \( \log \sec x = 1.501 \); (xii) \( \log \cosec x = 1.6005 \).

Evaluate the following, Nos. 4-15.

4. \( \sin 79^\circ \)

5. \( \sin 18^\circ \cos 18^\circ \)

6. \( \cos^2 37^\circ 15' \)

7. \( \sin 37^\circ 15' \tan 40^\circ 24' \)

8. \( \sin 25^\circ 32' \cos 33^\circ 15' \)

9. \( \tan 27^\circ 11' \tan 75^\circ 15' \cos 18^\circ 20' \)

10. \( \sin 38^\circ 17' \tan 15^\circ 20' \cos 18^\circ 20' \)

11. \( \cos^2 54^\circ 40' \)

12. \( 3-64 \tan 22^\circ 17' \)

13. \( 3-64 \cos 32^\circ 29' \cos 55^\circ 25' \)

14. \( \cos 97^\circ 57' \)

15. \( 3-32 \cosec 18^\circ 11' \sec 39^\circ 16' \)

Find a value of \( \theta \) satisfying the following equations, Nos. 16-24.

16. \( \sin \theta = 0.27 \)

17. \( \cos \theta = 0.8 \)

18. \( \tan \theta = 0.907 \)
19. \[ \sin \theta = \frac{11 - 3 \sin 17^\circ 45' \cdot 10.87}{29.71 \sin 69^\circ 17'}. \]
20. \[ \sin \theta = \frac{29 - 71 \sin 60^\circ 17'}{27.18}. \]
21. \[ \cos \theta = \sqrt{\frac{18 - 02 \times 3.49}{22.71}}. \]
22. \[ \tan \theta = \frac{3 - 72 \tan 17^\circ 52'}{2.696}. \]
23. \[ \sec \theta = \frac{2.4673 \times 2.497}{17.62}. \]
24. \[ \cos \theta = \frac{1 - 86 \cos 63^\circ 10'}{2.271}. \]
25. If \( \cos \phi = \sin \alpha \sin \beta \), find \( \phi \) given \( \alpha = 14^\circ 50' \), \( \beta = 67^\circ 26' \).
26. If \( r = 4 \sqrt{\sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}} \), find \( r \), given that \( R = 3.23 \), \( A = 57^\circ \), \( B = 49^\circ \), \( C = 74^\circ \).
27. Evaluate \( \frac{\sin \alpha \sin \beta}{\cos \alpha + \cos \beta} \), when \( \alpha = 147^\circ \), \( \alpha = 42^\circ 17^\prime \), \( \beta = 51^\circ 43^\prime \).
28. From the formula for the volume \( V \) of a cone, \( V = \frac{1}{3} \pi h^2 \tan^2 \alpha \), where \( h \) is the height and \( \alpha \) is the semi-vertical angle, find \( V \), given \( h = 11 \), \( \alpha = 14^\circ \).
29. With the notation of No. 28, find \( \alpha \), given \( V = 60 \), \( h = 6 \).
30. In any \( \triangle ABC \), find \( B, C \) given that \( b = 812 \), \( c = 639 \), \( A = 37^\circ \).
31. Find \( A \) from the formula \( \sin \frac{A}{2} = \sqrt{\frac{(b - c)(a - c)}{bc}} \), where \( b = 43 \), \( c = 37 \), \( a = 59 \).
32. Find \( \theta \) from the formula \( \tan \frac{\theta}{2} = \tan \frac{a}{2} \sqrt{\frac{1 + e}{1 - e}} \), where \( e = 0.43 \) and \( a = 23^\circ 20' \).
33. The volume of a triangular pyramid is given by the formula \( V = \frac{1}{6} \sqrt{(\sigma - \alpha)(\sigma - \beta)(\sigma - \gamma)} \); find \( V \), given \( \sigma = 37^\circ \), \( a = 41^\circ \), \( c = 29^\circ \), \( \beta = 51^\circ \), \( \gamma = 78^\circ \), \( \sigma = 0.43 \) and \( \alpha + \beta + \gamma \).
34. Find \( A \) from the formula for a spherical triangle: \( \sin \frac{A}{2} = \sqrt{\frac{\sin (a - b) \sin (a - c)}{\sin a}} \), where \( a = 56^\circ 20' \), \( b = 50^\circ 40' \), \( c = 47^\circ 33' \) and \( \sigma = 0.43 \).
35. From the formula \( \cos a = \cos b \cos c + \sin b \sin c \cos A \), find \( a \), given \( b = 73^\circ 35' \), \( c = 61^\circ 20' \), \( A = 22^\circ 30' \).

CHAPTER IX.

SOLUTION OF TRIANGLES.

In Elementary Geometry, the tests for congruent triangles are ascertained by enquiring what various sets of data are necessary and sufficient for copying a triangle. This work should be revised orally before proceeding to the formal methods of solution.

EXERCISE IX. a. (Oral.)

Is it possible to draw several, one or no triangle subject to the following conditions? Draw rough figures in each case.

1. \( b = 1 \), \( c = 2 \), \( A = 52^\circ \).
2. \( b = 3 \), \( c = 4 \), \( a = 5 \), \( B = 127^\circ \).
3. \( a = 5 \), \( b = 7 \), \( c = 9 \).
4. \( b = 4^\circ 40^\prime \), \( A = 60^\circ \), \( C = 80^\circ \).
5. \( b = 8 \), \( c = 7 \), \( C = 30^\circ \).
6. \( a = 5 \), \( c = 1 \), \( C = 30^\circ \).
7. \( b = 8 \), \( c = 9 \), \( C = 30^\circ \).
8. \( b = 8 \), \( c = 4 \), \( C = 30^\circ \).
9. \( a = 45^\circ \), \( B = 65^\circ \), \( C = 60^\circ \).
10. \( a = 2 \), \( B = 100^\circ \), \( A = 44^\circ \).
11. \( a = 2 \), \( B = 116^\circ \), \( C = 70^\circ \).
12. \( a = 10 \), \( A = 20^\circ \), \( B = 30^\circ \).
13. \( a = 8 \), \( A = 40^\circ \).
16. Are the values of \( a, b, c \) subject to any conditions? If so, what are they?

17. Name one set of three measurements which fixes a triangle uniquely. How many other such sets of a different kind are there, and what are they?

18. Invent a numerical example in which \( a, b, A \) are all given and in consequence of which it is found that it is possible to draw (i) two triangles of different size, (ii) only one triangle, (iii) no triangle at all, satisfying the data. Illustrate by rough figures.

Noting that there are six fundamental elements of a triangle, 3 sides and 3 angles, we may summarise the ideas of the last exercise as follows:

1. The triangle is determined uniquely if we are given (i) the 3 sides, (ii) 2 sides and the included angle, (iii) 1 side and 2 angles.

Note. In (i) any side must be less than the sum of the other two; and in (iii) the sum of the two angles must be less than 180°.

2. There may be two, one or no possible solution, if we are given two sides and the angle opposite one of them. [Ex. IX. a. Nos. 7-10.]

3. The triangle cannot be determined unless the data include the length of at least one side.

Since the necessary data include 3 elements, one of which at least is a length, the remaining elements may be calculated from formulae connecting together four of the six elements; and two at least of these four must be lengths of sides.

**The sine formula.**

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.
\]

In any triangle \( ABC \),

\[
a = \frac{b}{\sin B} = \frac{c}{\sin C}.
\]

**SOLUTION OF TRIANGLES**

Draw \( AD \) perpendicular to \( BC \), produced if necessary.

In Fig. 185, \( AD = e \sin B \) and \( AD = b \sin C \).

In Fig. 186, \( AD = c \sin B \) and \( AD = b \sin(180° - C) = b \sin C \).

\[
\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C}.
\]

Similarly, by drawing a perpendicular from \( C \) to \( AB \), produced if necessary, we can prove that \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \).

\[
\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.
\]

Note. Never apply the sine formula to a right-angled triangle. It is of course true; but it is mere waste of time to use it, when all that is necessary is the use of the definition of a sine or cosine.

A straightforward illustrative example on the uses of the sine formula is given on p. 114.

**The ambiguous case.** To construct the triangle \( ABC \), given \( a, b \) and the angle \( A \), which is acute.

**Fig. 187.**

(i) (ii) (iii) (iv)

Draw the angle \( \angle AK \) equal to the given \( \angle A \) and make \( AC \) equal to \( b \). With centre \( C \) and radius \( a \) describe a circle. There are various possibilities:

1. This circle may cut \( AK \) at all, Fig. (i); then no triangle can be drawn to fit the data.

2. This circle may touch \( AK \) at \( B \), Fig. (ii); then there is one triangle \( ABC \) and it is right angled at \( B \).
(3) This circle may cut $AK$ at points $B, B'$ on the same side of $A$, Fig. (iii); then there are two different triangles $ABC$ and $AB'C$, which fit the data.

(4) This circle may cut $AK$ at points $B, B'$ on opposite sides of $A$, Fig. (iv); then $\triangle ABC$ fits the data and $\triangle AB'C$ does not.

We may state these results as follows, using the fact that the length of the perpendicular from $C$ to $AK$ equals $b \sin A$:

1. If $a < b \sin A$, there is no solution, Fig. 187 (i).
2. If $a = b \sin A$, one triangle exists and it is right angled, Fig. 187 (ii).
3. If $b > a > b \sin A$, there are two distinct solutions, Fig. 187 (iii).
4. If $a > b$, there is one and only one solution, Fig. 187 (iv).

Note. (i) The case of two distinct solutions arises only when the given angle is opposite the shorter of the two given sides.

(ii) If there are two distinct solutions, Fig. 187 (iii), the angles $ABC$, $AB'C$ are supplementary; for $CB = CB'$;

$$\therefore \angle CBA = \angle CB'B = \angle CB'B = 180^\circ - \angle CB'A.$$

(iii) If the given angle $A$ is obtuse, there cannot be more than one solution, and there may not be any.

In Fig. 188 (i), $a > b$; one solution exists; $\triangle ABC$ fits the data, but $\triangle AB'C$ does not.

In Fig. 188 (ii), $a < b$; no solution exists; neither $\triangle AB'C$ nor $\triangle AB'C$ fit the data.

(iiv) If the given angle $A$ is a right angle, there is one solution if $a > b$ and no solution if $a < b$.

**Example 1.** Solve $\triangle ABC$, given $a = 8$, $b - 10$, $A = 40^\circ$.

From the formula $\frac{\sin A}{a} = \frac{\sin B}{b}$, we have

$$\frac{\sin B}{\sin 40^\circ} = \frac{10}{8}; \therefore \sin B = \frac{10 \sin 40^\circ}{8} = \frac{6.428}{8} = 0.8035.$$

From the tables, $\sin 53^\circ = 0.8035$; $\therefore \sin 126^\circ = 0.8035$;

$$\therefore B = 53^\circ 28' \text{ or } 126^\circ 32'.$$

In Fig. 189, $\angle ABC = 53^\circ 28'$ and $\angle AB'C = 126^\circ 32'$.

$$\therefore \angle ACB = 180^\circ - 40^\circ - 53^\circ 28' = 86^\circ 32';$$

and $$\angle AC'B = 180^\circ - 40^\circ - 126^\circ 32' = 13^\circ 28'.
$$

$$\therefore \frac{AB}{\sin 86^\circ 32'} = \frac{8}{\sin 40^\circ};$$

$$\therefore AB = \frac{8 \sin 86^\circ 32'}{\sin 40^\circ} = 12.4.$$

And $$\frac{AB'}{\sin 13^\circ 28'} = \frac{8}{\sin 40^\circ};$$

$$\therefore AB' = \frac{8 \sin 13^\circ 28'}{\sin 40^\circ} = 2.90.$$

Note. There are two solutions, because the given angle is opposite the shorter of the two given sides.
Example II. In the \( \triangle ABC \), given \( a = 12 \), \( b = 10 \), \( A = 40^\circ \), find \( B \).

From the formula,
\[
\sin B = \frac{\sin 40^\circ}{10} \cdot 12.
\]
\[
\therefore \sin B = \frac{10 \sin 40^\circ}{12} = \frac{6.428}{12} = 0.5357.
\]

From the tables, \( \sin 32^\circ 24' = 0.5357 \);
\[
\therefore \sin 147^\circ 36' = 0.5357;
\]
\[
\therefore B = 32^\circ 24' \text{ or } 147^\circ 36'.
\]

But \( B = 147^\circ 36' \) is impossible since
\[ A = 40^\circ \text{ and } A + B + C = 180^\circ; \]
\[
\therefore B = 32^\circ 24' \text{ is the only solution.}
\]

Note. There is only one solution, because the given angle is opposite the larger of the two given sides.

EXERCISE IX. b.

Illustrate your answers for Nos. 1-12 by rough figures.
1. If \( B = 47^\circ \), \( C = 53^\circ \), \( b = 8.61 \), find \( c \).
2. If \( A = 25^\circ \), \( B = 79^\circ 30' \), \( a = 15.6 \), find \( b \).
3. If \( A = 116^\circ \), \( B = 29^\circ \), \( a = 12.4 \), find \( b \).
4. If \( B = 32^\circ \), \( C = 99^\circ 20' \), \( b = 4.28 \), find \( c \).
5. If \( A = 49^\circ \), \( B = 77^\circ \), \( c = 7.48 \), find \( a \).

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8. If \( b = 19^\circ \), \( C = 43^\circ \), \( c = 9.36 \), find \( a \).
9. If \( b = 11.2 \), \( a = 8.3 \), \( B = 52^\circ \), find \( C \).
10. If \( a = 8.45 \), \( b = 6.73 \), \( A = 67^\circ 45' \), find \( B \).
11. If \( \angle a = 15^\circ \), \( b = 21.7 \), \( B = 112^\circ \), find \( A \).
12. If \( a = 9.45 \), \( b = 7.32 \), \( A = 121^\circ 24' \), find \( B \).
13. If \( a = 6.31 \), \( c = 8.45 \), \( C = 73^\circ 15' \), find \( B \).
14. If \( a = 9.24 \), \( c = 7.48 \), \( A = 37^\circ 40' \), find \( C \).

By drawing rough figures, find out whether two, one or no triangle can be drawn to fit the following data, Nos. 15-22.

15. \( c = 4 \), \( b = 5 \), \( C = 25^\circ \).
16. \( c = 3 \), \( b = 10 \), \( C = 30^\circ \).
17. \( c = 7 \), \( a = 10 \), \( C = 30^\circ \).
18. \( a = 12 \), \( b = 10 \), \( C = 30^\circ \).
19. \( a = 5 \), \( c = 4 \), \( A = 70^\circ \).
20. \( a = 5 \), \( c = 4 \), \( A = 110^\circ \).
21. \( a = 1 \), \( c = 4 \), \( A = 110^\circ \).
22. \( a = 8 \), \( b = 7 \), \( B = 50^\circ \).
23. \( a = 10 \), \( B = 53^\circ \); \( b \) is also given; what can you say about the value of \( b \), if (i) two distinct triangles, (ii) only one triangle, (iii) no triangle can be drawn to fit the data?
24. \( a = 5 \), \( b = 6 \); in addition either \( A \) or \( B \) is given; in which case will the triangle be uniquely determined?
25. \( b = 10 \), \( C = 115^\circ \); in addition \( c \) is given; what can you say about \( a \), if a solution is possible? Is there more than one solution possible for any one given value of \( c \)?

Find all possible answers in Nos. 26-34; if there is no possible answer, say so.

26. If \( a = 6.32 \), \( b = 8.47 \), \( A = 43^\circ \), find \( B \).
27. If \( b = 12.3 \), \( c = 16.9 \), \( B = 51^\circ \), find \( C \).
28. If \( a = 3.48 \), \( c = 3.57 \), \( C = 68^\circ \), find \( B \).
29. If \( a = 7.14 \), \( b = 10.3 \), \( A = 57^\circ \), find \( C \).
30. If \( b = 5.92 \), \( a = 4.73 \), \( C = 53^\circ 3' \), find \( B \).
31. If \( b = 8.46 \), \( a = 7.15 \), \( B = 41^\circ 24' \), find \( A \).
32. If \( b = 7.2 \), \( c = 8.1 \), \( B = 127^\circ \), find \( C \).
33. If \( b = 3.8 \), \( c = 2.9 \), \( B = 117^\circ 45' \), find \( A \).
34. If \( a = 5.61 \), \( c = 4.73 \), \( C = 52^\circ 27' \), find \( B \).
Find the remaining sides and angles of the following triangles:
Nos. 33-45.
35. \( A = 75^\circ 20', C = 42^\circ 50', a = 8.23. \)
36. \( a = 7.81, b = 6.24, B = 51^\circ 15'. \)
37. \( A = 29^\circ 17', B = 32^\circ 48', c = 3.64. \)
38. \( b = 8.46, c = 6.38, B = 127^\circ 20'. \)
39. \( a = 5.18, b = 6.96, A = 54^\circ 35'. \)
40. \( B = 32^\circ 46', C = 111^\circ 25', a = 4.35. \)
41. \( a = 7.64, c = 8.23, C = 63^\circ 30'. \)
42. \( a = 9.92, b = 7.23, A = 90^\circ. \)
43. \( a = 4.87, c = 9.14, B = 90^\circ. \)
44. \( a = 513, c = 724, C = 132^\circ 30'. \)
45. \( b = 804, c = 640, C = 39^\circ 20'. \)

The cosine formula.
In any triangle \( ABC, \)
\[ c^2 = a^2 + b^2 - 2ab \cos C. \]

Draw \( AD \) perpendicular to \( BC. \)
In Fig. 191, \[ c^2 = BD^2 + DA^2 = (a - x)^2 + DA^2 \]
\[ = a^2 - 2ax + x^2 + DA^2 \]
\[ = a^2 - 2ax + b^2. \]
But \( x = b \cos C; \quad \therefore \quad c^2 = a^2 + b^2 - 2ab \cos C. \)
In Fig. 192, \[ c^2 = BD^2 + DA^2 = (a + y)^2 + DA^2 \]
\[ = a^2 + 2ay + y^2 + DA^2 \]
\[ = a^2 + 2ay + b^2. \]
But \( y = 5 \cos DCA = b \cos (180^\circ - C) = -b \cos C; \quad \therefore \quad c^2 = a^2 + b^2 - 2ab \cos C. \)

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Note. It is worth while pointing out three special cases of this formula:

Consider two rods \( CA, CB \) joined at \( C, \) with their ends \( A, B \) connected by an elastic string. Keep \( CB \) fixed and rotate \( CA. \)

Fig. 193 represents \( \triangle A CB \) when \( C = 0^\circ; \quad \cos 0^\circ = 1, \) and the formula gives \( c^2 = a^2 + b^2 - 2ab = (a - b)^2; \quad \therefore \quad c = a - b. \)

In Fig. 194, \( C = 90^\circ; \quad \cos 90^\circ = 0; \) the formula gives \( c^2 = a^2 + b^2. \)

Fig. 195 represents \( \triangle A CB \) when \( C = 180^\circ; \quad \cos 180^\circ = -1, \) and the formula gives
\[ c^2 = a^2 + b^2 - 2ab(-1) = (a + b)^2; \quad \therefore \quad c = a + b. \]

Note. We have proved that the formula
\[ c^2 = a^2 + b^2 - 2ab \cos C \]
is true in every triangle, whether \( C \) is acute or obtuse.

If \( C = 90^\circ, \) the formula is equivalent to Pythagoras' theorem.

If \( C \) is acute, \( \cos C \) is positive; if \( C \) is obtuse, \( \cos C \) is negative.

Consequently \( c^2 < a^2 + b^2 \) if \( C \) is acute, and \( c^2 > a^2 + b^2 \) if \( C \) is obtuse.

In the same way, we can express \( a \) in terms of \( b, c, A \) and \( b \) in terms of \( c, a, B. \) We have, therefore, the following results, which must be committed to memory:

\[ a^2 - b^2 + c^2 - 2bc \cos A \quad \text{or} \quad a^2 = b^2 + c^2 - 2bc \cos A = \frac{b^2 + c^2 - a^2}{2bc}; \]

\[ b^2 - c^2 + a^2 - 2ac \cos B \quad \text{or} \quad b^2 = c^2 + a^2 - 2ac \cos B = \frac{c^2 + a^2 - b^2}{2ca}; \]

\[ c^2 - a^2 + b^2 - 2ab \cos C \quad \text{or} \quad c^2 = a^2 + b^2 - 2ab \cos C = \frac{a^2 + b^2 - c^2}{2ab}. \]

By means of the cosine formula, we can solve a triangle, given either two sides and the included angle or three sides.

It often saves time to use a Table of Squares.
TRIGONOMETRY

Example III. Given \( b = 4, c = 5, A = 115^\circ \), find \( a \).
\[ a^2 = b^2 + c^2 - 2bc \cos A. \]
\[ \cos 115^\circ = -\cos 65^\circ = -0.4226; \]
\[ \therefore a^2 = 4^2 + 5^2 - 2(4)(5)(-0.4226) = 16 + 25 + 40 \times 0.4226 \]
\[ = 41 + 16.90 = 57.90; \]
\[ \therefore a = 7.61. \]

Example IV. Given \( a = 3.82, c = 5.46, B = 37^\circ 25' \), solve \( \Delta ABC \).
\[ b^2 = c^2 + a^2 - 2ac \cos B = (5.46)^2 + (3.82)^2 \]
\[ - 2(5.46)(3.82) \cos 37^\circ 25' \]
\[ = 29.81 + 14.59 - 33.13 = 44.40 - 33.13 \]
\[ = 11.27; \]
\[ \therefore b = 3.367 \approx 3.36. \]

From the sine formula,
\[ \frac{\sin A}{\sin 37^\circ 25'} = \frac{3.82}{3.367} ; \]
\[ \therefore \sin A = \frac{3.82 \sin 37^\circ 25'}{3.367} ; \]
\[ \therefore A = 43^\circ 46'. \]

[Since \( a < c, A < C; \therefore A \) cannot be obtuse; see Note (ii) below.]

Lastly,
\[ C = 180^\circ - A - B = 180^\circ - 43^\circ 46' - 37^\circ 25' = 180^\circ - 81^\circ 11'; \]
\[ \therefore C = 98^\circ 49'. \]

Note. (i) If the data consist of either two sides and the included angle or three sides, it is necessary to use the cosine formula for the first operation; but it is never necessary to use it twice. Always continue with the sine formula, because it is quicker.

(ii) In the second operation, always find the smaller of the two unknown angles: this must be acute, and so there is no possibility of any ambiguity.

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Example V. Given \( a = 3.46, b = 5.39, c = 7.12 \), find \( C \).
\[ \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(3.46)^2 + (5.39)^2 - (7.12)^2}{2(3.46)(5.39)} \]
\[ = \frac{11.97 + 29.05 - 50.69}{6.92 \times 5.39} \]
\[ = \frac{10.32}{6.92 \times 5.39} \]
\[ \therefore \cos C = 0.6544 \approx 0.654; \]
\[ \therefore C = 180^\circ - 51^\circ 59' \approx 128^\circ 2'. \]

[As \( \cos \theta = 0.654 \), \( \theta = 51^\circ 59' \) and \( \theta = 128^\circ 2' \).]

(iv) Four figures should be retained throughout the working, in order to secure as high a degree of accuracy in the answer as the tables permit.

Example V. Given \( a = 3.46, b = 5.39, c = 7.12 \), find \( C \).
\[ \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{(3.46)^2 + (5.39)^2 - (7.12)^2}{2(3.46)(5.39)} \]
\[ = \frac{11.97 + 29.05 - 50.69}{6.92 \times 5.39} \]
\[ = \frac{10.32}{6.92 \times 5.39} \]
\[ \therefore \cos C = 0.6544 \approx 0.654; \]
\[ \therefore C = 180^\circ - 51^\circ 59' \approx 128^\circ 2'. \]

Note. (i) If you are asked to solve a triangle, given all three sides, start by finding the smallest angle; this avoids any difficulty arising from the cosine being negative. Then, as before, continue with the sine formula and find the smaller of the two remaining angles; this avoids any possibility of ambiguity.

(ii) The portion in brackets is inserted to explain the argument; it would not appear in a formal solution.

(iii) If among the data there are either two equal sides or two equal angles, it is a waste of time to use either sine or cosine formula; the triangle is isosceles, and should be solved by drawing a perpendicular from the vertex to the base.

In early times, triangles were solved by inscribing them in circles and calculating the sides in terms of the radius \( r = 2\sin A \); the sine formula was known to Ptolemy, although not of course as expressed in modern notation. The cosine formula is equivalent to a theorem of Euclid, but its first explicit statement is due to Vieta (1593).
EXERCISE IX. c.

1. If \( a = 2, b = 3, C = 15^\circ \), find \( c \).
2. If \( b = 10, c = 5, A = 41^\circ 27^\prime \), find \( a \).
3. If \( b = 5, c = 5, A = 124^\circ 15^\prime \), find \( a \).
4. If \( a = 2, b = 1, B = 164^\circ 18^\prime \), find \( b \).
5. If \( a = 4, b = 3, c = 2 \), find \( B \).
6. If \( a = 7, b = 6, c = 10 \), find \( A \).
7. If \( a = 5, b = 6, c = 9 \), find \( C \).
8. If \( a = 7, b = 5, c = 3 \), find \( A \).

Solve the following triangles, Nos. 9-26.

9. \( a = 2, b = 5, C = 21^\circ 30^\prime \).
10. \( b = 4, c = 5, A = 102^\circ 8^\prime \).
11. \( a = 6, c = 10, B = 15^\circ 24^\prime \).
12. \( a = 3, b = 5, C = 130^\circ 33^\prime \).
13. \( a = 6, b = 5, c = 3 \).
14. \( a = 10, b = 7, c = 6 \).
15. \( a = 100, b = 80, c = 50 \).
16. \( a = 11, b = 18, c = 11 \).
17. \( a = 8.63, b = 7.42, C = 37^\circ 20^\prime \).
18. \( a = 4.17, b = 5.85, C = 141^\circ 25^\prime \).
19. \( a = 114, b = 137, c = 184 \).
20. \( a = 38.2, b = 21.7, c = 29.3 \).
21. \( a = 321, c = 436, A = 110^\circ 15^\prime \).
22. \( a = 8.07, c = 3.14, B = 22^\circ 30^\prime \).
23. \( a = 97, b = 86, c = 74 \).
24. \( a = 4.35, b = 11.91, c = 6.6 \).
25. \( a = 73, b = 89, c = 73 \).
26. \( a = 6.8, c = 6.8, B = 111^\circ 30^\prime \).

General procedure. Use the sine formula whenever possible. If you are given either 3 sides or 2 sides and the included angle, you must start with the cosine formula, but you should continue with the sine formula, using it to find the smaller of the two remaining angles. If you are given either 2 angles and one side or 2 sides and a non-included angle, the sine formula gives all that is required; in the latter case a rough figure should be drawn as a guide to the nature of the solution.

Half-angle formulae. The half-angle formulae obtained in Chapter XVII. may be used for the solution of triangles, instead of the cosine formula. Geometrical proofs of these formulae are therefore given on p. 136a., etc., together with illustrative examples and an additional exercise.

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16. An officer at \( O \) (see Fig. 199) is observing for a battery \( B \) firing on a target at \( T \); \( SO = 1500 \) yd., \( OT = 2000 \) yd.; the magnetic bearing of \( O \) from \( B \) and \( T \) from \( O \) are 70°, 76° respectively. Find the range BT.

17. Two ships leave harbour at noon in directions S. 62° W., S. 38° E. at 10, 12 knots respectively. How far apart are they at 12.45 p.m. ?

18. B, O, T represent the positions of a battery, an observer and the target respectively; \( \angle BTO = 160^\circ 35^\prime \); \( SO = 1850 \) yd.; \( OT = 3113 \) yd. Find the range BT and the bearing of the line of fire if the bearing of \( B \) from \( O \) is 212° and if \( T \) is north of the line BO.

19. A, B, are two observers; A is 7000 yd. due west of B; A locates a battery on a bearing 52° 40′, and B locates it on a bearing 10°; how far is A from the battery?

20. In \( \triangle ABC \), \( a = 3, b = 5, c = 7 \); prove \( C = 120^\circ \).

21. Two searchlights A, B, 14 miles apart, are both directed on a Zeppelin C, vertically over the line AB; the elevations of the beams AC, BC are 76°, 46°. Find the height of the Zeppelin in feet.

22. A boat steaming due East is 3 miles away in a direction N. 30° E.; 5 minutes later, her direction is N. 50° E. What is her speed?

23. A road rises from A for 1 mile at an angle of 5° to the horizontal, and then descends at an angle of 7° to the horizontal to the same level as A. How much longer to the nearest 10 yards is the uphill portion than the downhill portion?

24. A man AB, 6 ft. high, stands vertically on a hill-side (see Fig. 200), and his shadow BC falls on a slope of 25° when the sun's elevation is 67°. What is the length of BC?

25. A crane ABC (see Fig. 201) carries a load D, as shown; AB is vertical. Find \( \angle BAC \) and the height of D above the level of A.

26. ABCD is a cyclic quadrilateral; \( AB = 5, BC = 4, CD = 7 \). DA = \( \sqrt{12} \). Calculate \( \angle ABC \).

D.W.T.H.
Harder applications of the sine and cosine formulae.

**Exercise IX. f.**

1. An observer O is 100 ft. away from the base A of a tower AB, and is on the same level as A; the tower has a spire BC; AB and BC subtend angles 45° and 12° at O. Find BC.

2. A road stretches from A 100 yards uphill at a slope of 5° to B; P is an object beyond B, and is in the same vertical plane as AB; the elevations of P from A, B are 32°, 38°. Find the height of P above A.

3. A rectangular block (see Fig. 202) rests against an inclined plane OX; AOX is the ground line; OB = 2'. Find the heights of C, D above the ground.

4. The radii of two circles (see Fig. 203) are 8, 6 inches, and their centres are 12 inches apart; AP, AQ are tangents. Calculate ∠PAQ.

5. A window AB (see Fig. 204), pivoted at A, is held in position by a bar CD attached to it at D; small holes are punched in CD at intervals of 2 inches from D, and a peg P, fixed to the sill, passes through one of these holes. Find ∠ABP through which the window is opened if (i) DP = 2", (ii) DP = 4".

6. With the data of No. 5, find the least length of DC which enables the window to be fixed, when open at an angle of 90°.

7. In the triangle ABC, a = 6, b = 8, c = 4. Find the length of the line joining A to a point on BC 2 inches from C.

8. In a convex quadrilateral ABCD, AB = 5, BC = 3, CD = 3, DA = 4, AC = 6. Find the length of BD.

9. On a map, scale 1 inch to the mile, the distance between two villages A, B by road is shown as 674 inches; the road from A rises to B at a gradient of 1 in 10 for the first 2 miles and then descends at a steady gradient to B, at the same level as A. Find, to the nearest yard, how much further the distance is from A to B by road than it appears to be on the map.

10. The crank OA (see Fig. 205) is free to turn about O, and the end P of the connecting rod AP is constrained to move along a line through O. Find the distances of P from its extreme positions when (i) ∠AOP = 25°; (ii) ∠AOP = 155°.

11. An arm OP (see Fig. 206) can rotate about O, and is hinged at P to a rod PQ, which can slide through a small fixed ring at A. Show that there are two positions of the mechanism for which ∠AOP = 25°. Find the distance between these two positions of P and the angle between the corresponding positions of OP.

12. Figure 207 represents the framework of a deck chair, whose shape is controlled by an adjustable arm SP. If OA, OC make angles of 30° with the ground, and if OB = 16", BP = 20", find the distance of P from O.

13. The mechanism in Fig. 208 consists of a fixed vertical bar AB, 30 inches long, with a bar BFC pivoted to AB at B and carrying a circular disc attached rigidly to it with its centre on FB. BC = 30 inches, BD = 10 inches, DE = 12 inches. Find the distance of A from C.
14. A triangular wedge ABC (see Fig. 209) is standing on an incline plane as shown: it would topple over if the median through A and the vertex C were on the same side of the vertical through A. AB = 10 inches; \( \angle ABC = 35^\circ \). What is the greatest length of BC?

15. In the framework in Fig. 210, AB = 6 ft. Calculate BD.

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16. In the framework in Fig. 211, AB = 12 ft. Calculate BC.

17. In the framework in Fig. 212, BE = 20 ft. Calculate AC.

18. In the framework in Fig. 213, AB = 10 ft. Calculate BC.

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19. A is North of B; P is East of A and bears N. 57° 30' E. from B; Q is East of B and bears N. 72° 20' E. from A. What is the bearing of Q from P?

20. In Fig. 214, prove that BC = 2x sin \( \theta \); then use the cosine formula, and so obtain cos 2\( \theta \) in terms of sin \( \theta \).

21. In Fig. 215, the tangent at C cuts AB at T. Calculate CT, given AB = 3.4 inches.

22. If, in Fig. 216, \( \phi = 29 \), calculate the ratio \( \frac{BC}{CD} \).

23. The sines of the angles of a triangle are in the ratio 5:6:7; prove that the cosines are in the ratio 25:19:7.

24. The extremities A, D of the mechanism in Fig. 217 are fixed, and B, C are free joints. Calculate the angle ADC when AB is perpendicular to AD.

25. From a certain point two roads OA, OB run due West and West-Southwest. A boy-scout at C is ordered to go to a farm F 2 miles away down the western road, but by mistake goes along OB: after walking 2 miles he realises he is wrong, and taking a line across country arrives on the road OA after walking another mile. Should he now turn right or left to find F, and how much further must he walk?
26. The mechanism in Fig. 218 consists of three rods; AB and CD can turn about their ends A and D, which are fixed. Calculate the total angle through which AB can oscillate.

![Diagram](image)

27. In Fig. 219, AB is a window which pivots about its centre E; CED is the window frame. The window is held open by two cords, one from B passing over a pulley at D and the other from A; the cords are attached to pegs at F. FE = 20 inches and∠FED = 25°. Find the length of each cord AF, BDF when the window is opened to an angle of 55°.

28. From an observation balloon at A, at an altitude of 10,000 feet, the angle of depression of a peak P (see Fig. 220) is 35°; the balloon sinks vertically 3000 feet to B, where the angle of depression of P is found to be 12°. What is the height of P above C?

![Diagram](image)

29. Fig. 221 represents an ellipse; S is a fixed point and ASA' is a fixed line; if P is a variable point on the curve, the length of SP, r inches, is given by \( r = 2a \cos \theta \), where \( \angle ASP = \theta \); B is a point on the curve such that SB = 1AA'. Calculate the lengths of SA, SA', AB, A'B, and the angle ASB.

30. A flat triangular piece of wood has sides 5°, 6°, 7°. It is placed flat on a horizontal table and is then rotated through 30° about the 7° side. What is the height of the opposite vertex above the table?
Check: \( A + B + C = 180^\circ \).

Note. (i) As previously explained, if 4-figure tables are used, the result will not necessarily be correct to the nearest minute.

(ii) It is useful to check the values of \( s - a \), \( s - b \), \( s - c \) by adding them up, as above:

\[
s - a + s - b + s - c = 3s - (a + b + c) = 3s - 2s = s.
\]

(iii) When \( A \) and \( B \) have been found, we can of course at once write down the value of \( C \), since \( C = 180^\circ - (A + B) \).

The same method is used for finding the remaining angles and sides from the known sides and included angle.

To prove that

\[
\tan \frac{B-C}{2} = \frac{b-c}{\tan \frac{B+C}{2} b+c}.
\]

Suppose \( AC > AB \). With centre \( A \) and radius \( AC \), describe a circle, and let it cut \( AB \) produced at \( P, Q \); join \( CP, CQ \); draw \( BR \) perpendicular to \( CP \).

Example. Solve the triangle \( ABC \), given that

\[
a = 24.76, \quad b = 16.38, \quad c = 36^\circ 26'.
\]
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\[ \tan \frac{A-B}{2} = \frac{8.38 \tan 71^\circ 47'}{41.14} \]

\[ \therefore \frac{A-B}{2} = 51^\circ 45' \]

\[ \therefore \frac{A+B}{2} = 71^\circ 47' \]

\[ \therefore \text{adding, } A = 103^\circ 32', \text{ and subtracting, } B = 40^\circ 2' \]

Further,

\[ c = 16.38 \]

\[ \frac{\sin 36^\circ 26'}{\sin 40^\circ 2'} = \frac{16.38 \sin 36^\circ 26'}{\sin 40^\circ 2'} = 15.1 \]

The examples in Ex. IX. c. may now be solved by using the half-angle formulae. Further practice in their use is given in Ex. XVII. a. (p. 249); for the convenience of the reader, part of that exercise is reprinted below.

**EXERCISE XVII. a.**

Solve the following triangles, Nos. 1-16:

1. \( a = 63.4 \), \( b = 58.7 \), \( c = 78.4 \).
2. \( a = 134 \), \( b = 347 \), \( c = 269 \).
3. \( a = 5612 \), \( b = 4381 \), \( c = 7165 \).
4. \( a = 1180 \), \( b = 1413 \), \( c = 1977 \).
5. \( a = 694 \), \( b = 732 \), \( c = 65282' \).
6. \( b = 314 \), \( c = 415 \), \( A = 72^\circ 44' \).
7. \( a = 836 \), \( b = 478 \), \( B = 124^\circ 26' \).
8. \( a = 1173 \), \( b = 1564 \), \( C = 104^\circ 48' \).
9. \( b = 7296 \), \( c = 9814 \), \( A = 49^\circ 40' \).
10. \( a = 5614 \), \( A = 41^\circ 20' \), \( B = 50^\circ 17' \).
11. \( a = 5094 \), \( b = 8613 \), \( C = 56^\circ 45' \).
12. \( a = 1478 \), \( B = 110^\circ 32' \), \( C = 47^\circ 10' \).
13. \( a = 1273 \), \( b = 1585 \), \( c = 1933 \).
14. \( a = 1714 \), \( b = 1065 \), \( A = 112^\circ 17' \).
15. \( a = 1740 \), \( b = 2125 \), \( c = 1435 \).
16. \( a = 2987 \), \( b = 1718 \), \( A = 49^\circ 31' \).

**REVISION PAPERS. R. 19-26.**

**R. 19.**

1. (i) What angle does the line joining the origin to the point \( (2, 5) \) make with the positive direction of the \( z \)-axis?
2. (ii) Repeat part (i) for the point \( (-2, 5) \).
3. What can you say about \( \theta \), (i) if \( \sin \theta \) is positive and \( \sec \theta \) is negative, (ii) if \( \tan \theta \) is greater than 1 and \( \sin \theta \) is negative?
4. Find the value of \( \tan 75^\circ 28' \cos 14^\circ 15' \).
5. The top of a sloping desk is a rectangle 40 in. by 24 in., and the 24 in. sides are inclined at \( 10^\circ \) to the horizontal. Find the inclination of a diagonal to the horizontal.
6. In a triangle \( a = 10 \) cm., \( B = 47^\circ \), \( C = 73^\circ \). Find \( b \).

**R. 20.**

1. What is the angle between the line joining \( (1, 2) \) to \( (3, 5) \), and the line joining \( (1, 2) \) to \( (5, 6) \)?
2. The minute-hand of a clock, whose face is in a vertical plane, is 4 in. long. Find a formula for the distance of the tip from the central vertical line of the clock at \( t \) minutes past the hour. Evaluate the result when \( t = 10, 20, 30, 40, 50 \), and interpret your answers.
3. If \( \sin \theta = \mu \sin \phi \), find \( \phi \) when \( \mu = 1.12 \) and \( \phi = 47^\circ 20' \).
4. In the jointed mechanism in Fig. 222, \( AP = AQ = 3 \) in., \( PB = QC = 6 \) in. and \( PO = OQ = 2 \) in. Find the greatest value of

![Diagram](image-url)
5. Find the smallest angle of a triangle whose sides are 5 cm., 7 cm., and 8 cm.

R. 21.

1. Find the values of \( \theta \) less than 300° if
   - (i) \( \sin \theta = 0.432 \); (ii) \( \cos \theta = 0.417 \); (iii) \( \tan \theta = 4 \).

2. The ends of the link AB in Fig. 223 move on fixed lines OX, OY. Find the distance of P from these lines when \( \angle OAB = 70^\circ \).

3. A stone thrown into the air with velocity \( u \) ft. per sec. at an angle of \( \alpha \)° to the horizontal will hit the ground at a distance \( \frac{u^2 \sin 2\alpha}{g} \) ft., where \( g = 32 \).

   If \( u = 80 \), draw a graph to show the distance reached for values of \( \alpha \) from 0 to 90, and read from it (i) the greatest distance that can be reached, (ii) the values of \( \alpha \) for which the distance is 50 ft.

4. Find the angles of a triangle whose sides are 9 cm., 9 cm., 10 cm.

5. In the shear-legs shown in Fig. 224, find the height of the load D above the horizontal level of AB. Find also the length of AB.

R. 22.

1. Find the area of a triangle in which \( b = 10.4 \) cm., \( c = 8.9 \) cm., \( A = 114^\circ \ 27' \).
R. 24.

1. In Fig. 228, AB is a diameter; TB, TP are tangents. Find the length of AP.

2. (i) What equation connects the acute angles $\theta$ and $\phi$, if $\sin^2\theta + \sin^2\phi = 1$?

(ii) Find a value of $\theta$ for which $\sin 3\theta = \cos 5\theta$.

3. A sphere of radius 4-32 inches rests in a conical funnel of vertical angle $60^\circ$, height 3-81 inches; the axis of the funnel is vertical and its rim is uppermost. How far does the top of the sphere project above the plane of the rim?

4. ABCD is a trapezium with AB and CD as parallel sides; AB = 3 in., BC = 4 in., CD = 8 in., $\angle BCD = 123^\circ$. Find the length of AD.

5. In $\triangle ABC$, $AB = 5$ in., $AC = 4$ in., $\angle BAC = 108^\circ$; the altitudes BE, CF of $\triangle ABC$ intersect at H. Find the length of AH.

R. 25.

1. ABCD is a rhombus; AC = 7-3 in., $\angle ABC = 162^\circ$. Find the length of BD.

2. Find a relation between $x$ and $y$, independent of $\theta$, given that $x = 1 + 2 \tan \theta$, $y = 1 - 3 \cot \theta$.

3. Two dice, centres A, B, rest in contact with each other, and a vertical wall CF, see Fig. 229; and the line AB makes an angle of $70^\circ$ with the horizontal CF. If B, whose radius is 10 cm., now rolls away from the wall and allows A to fall vertically. How far has B rolled when AB is inclined at $30^\circ$ to the horizontal?

4. $\cos 2\theta = -0.67$. Find from the tables a value of $\sin 2\theta$. Is there more than one possible value?

Find also from the tables the possible values of $\sin \left(\frac{\theta}{3}\right)$.

5. In Fig. 230, find $\theta$, if AC = BD.


1. If sec $a = \frac{1}{2}$, prove that $\sec \alpha + \tan \alpha = m$ or $\frac{1}{m}$.

What is $\sec \alpha - \tan \alpha$?

2. The range $R$ feet of a projectile up an inclined plane is given by the formula

$$R = \frac{V^2 \cos \alpha \cdot \sin (\alpha - \beta)}{2g \cos^2 \beta}$$

Calculate $R$ when $V = 80$, $g = 32-2$, $\alpha = 45^\circ$ and $\beta = 27^\circ$.

3. Find expressions for the coordinates of P (Fig. 229) referred to OX, OY as axes in terms of $\angle OAB = \theta$; and prove that as $\theta$ varies, these coordinates are connected by the equation $x^2 + y^2 = 1$.

4. In a triangle ABC, a, b and B are known, and there are two possible values $c_1$ and $c_2$ for the third side. Prove that $c_1 + c_2 = 2a \cos B$.

5. A circular cam, radius 2", rotates about a fixed axis C which is 1" from O, the centre of the cam. As the cam rotates the bar AB is raised and lowered, but remains horizontal. Find how far AB has descended when the cam has rotated through 100° from the position shown in the figure, where O is vertically above C.
CHAPTER X.

MENSURATION OF THE CIRCLE.

Circumference of circle.

All circles are of the same shape and are similar figures. Therefore the ratio of circumference to diameter is the same in all circles.

A rough idea of the value of this ratio can be obtained as follows:

![Diagram of a circle with a hexagon inscribed]

Fig. 282.

Take a circle, diameter \( d \) inches, and circumscribe a square about it, and inscribe a regular hexagon in it. Each side of the square is \( d/\sqrt{2} \) in. long; \( \therefore \) the perimeter of the square is \( 4d \) in.; each side of the hexagon is equal to the radius and is \( d/2 \) in. long; \( \therefore \) the perimeter of the hexagon is \( 6 \times d/2 = 3d \) in.

Hence the circumference is less than \( 4d/d \) and greater than \( 3d/d \), and therefore equals some number between 3 and 4.

We can find an approximate value of this number by experiment and measurement: and its value can be calculated to any required degree of accuracy: it is denoted by \( \pi \); calculation gives

\[ \pi = 3.14159 \ldots \]

We therefore have

\[ \frac{\text{circumference}}{\text{diameter}} = \pi. \]

Then area of polygon = \( \Delta OAB + \Delta OBC + \Delta OCD + \ldots \)

\[ = \frac{1}{2} r \cdot AB + \frac{1}{2} r \cdot BC + \frac{1}{2} r \cdot CD + \ldots \]

\[ = \frac{1}{2} r(AB + BC + CD + \ldots) \]

\[ = \frac{1}{2} r \times \text{perimeter}. \]

By increasing the number of sides of the polygon, it is possible to make the difference between the area of the polygon and the area of the circle as small as we please; and we say that in the limit, the area of the circle = \( \frac{1}{2} r \times \text{perimeter of circle} \)

\[ = \frac{1}{2} r \times 2\pi r = \pi r^2 \text{ sq. inches.} \]

Note. A rigorous statement of the argument used above and rigorous definitions of what is meant by the length of a curved line or the area enclosed by a curved line are obviously unsuitable at this stage of the work.

Length of circular arc.

Equal arcs of a circle subtend equal angles at the centre \( O \) of the circle. Suppose, for example, \( \text{arc} \ CD = 3 \text{ arc} \ AB \); then
\[ \angle COD = 3 \angle AOB, \text{ because } CD \text{ can be divided into three areas each equal to } AB. \text{ And in general} \]

\[
\frac{\text{arc } PQ}{\text{arc } AB} = \frac{\angle POQ}{\angle AOB}
\]

\[ \text{If the radius of the circle is } r \text{ inches and } \angle POQ = x^\circ, \text{ then} \]

\[
\frac{\text{arc } PQ}{\text{circumference}} = \frac{x^\circ}{360^\circ};
\]

\[ \therefore \text{arc } PQ = \frac{x}{360} \times 2\pi r = \frac{\pi x}{180} \text{ inches}. \]

**Area of circular sector.**

By the same argument,

\[
\frac{\text{area of sector } POQ}{\text{area of circle}} = \frac{\angle POQ}{360^\circ};
\]

\[ \therefore \text{area of sector } POQ = \frac{x}{360} \times \pi r^2 = \frac{\pi x}{360} \times r^2 \text{ sq. inches}. \]

*Note that the area of sector } POQ = \frac{1}{2} r \times \angle POQ \]

\[ = \frac{1}{2} \text{ radius } \times \text{arc } PQ. \]

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**MENSURATION OF THE CIRCLE**

**Curved surface of circular cylinder.**

Suppose a circular cylinder (e.g., a round tin or a pencil with circular section) is of height \( h \) inches and radius \( r \) inches.

![Curved surface of a cylinder](image)

\[ \text{Take a sheet of paper the same height as the cylinder and wrap it round the curved surface and crease it so that the sheet just covers the cylinder without overlapping. When we unwrap it and fold it flat we obtain a rectangle of height } h \text{ in. and breadth } 2\pi r \text{ in.; } \]

\[ \therefore \text{the area}=2\pi rh \text{ sq. in.; } \]

\[ \therefore \text{the area of the curved surface of the cylinder= } 2\pi rh \text{ sq. in.} \]

**Volume of circular cylinder.**

The volume of the cylinder= base-area \times height

\[ = \pi r^2 \times h = \pi rh \text{ cu. in.} \]

**Example I.** Find the area of the minor segment cut off from a circle of radius 4 in. by a chord of length 6 in.

O is the centre of the circle; chord \( AB = 6 \) in.

Draw ON perp. to \( AB \); let \( \angle AON = x^\circ \).

![Diagram of Example I](image)

Then \[ \sin x^\circ = \frac{AN}{AO} = \frac{3}{5} = 0.75; \therefore x^\circ = 48.67^\circ; \]

\[ \therefore \angle AOB = 2x^\circ = 97^\circ 12' = 97.2^\circ; \]

\[ \therefore \text{area of sector } AOB = \frac{97.2}{360} \times \pi \times 4^2 = 1.92 \text{ sq. in.} \]

**D.W.T.H.**

\[ \therefore \]
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Area of $\triangle AOB = \frac{1}{2} OA \cdot OB \sin AOB = 8 \sin 97^\circ 12'$
\[= 8 \sin 82^\circ 48' = 8 \times 0.9921\]
\[= 7.937;\]
minor segment $ACB = 13.57 - 7.94 = 5.63$ sq. in.
\[= 5.64 \text{ sq. in.}\]

Note. (i) Using 4-figure tables, we cannot, owing to the subtraction, rely on more than two significant figures in the answer.

(ii) It is sometimes convenient to use $\sqrt{2}$ as a rough approximation for $\pi$. This value is correct to 3 figures, and results obtained from it are likely to be correct to 2 figures.

EXERCISE X. a.

1. A piece of fine cotton is wound 20 times round a cylinder and is then unwrapped and measured; its length is found to be 188.5 cm, the diameter of the cylinder is measured and found to be 3 cm. Find the value of circumference/diameter.

2. A circle of radius 8 cm. is drawn; steps of 1 cm. are taken round the circumference with a pair of dividers opened to 1 cm, and it is found that 50 such steps are required. Find the value of circumference/diameter from this experiment.

3. A small wheel, radius 1.2", is rolled along a straight line on a piece of paper and is found to travel a distance of 7.6" in one revolution. Find the value of circumference/diameter given by this experiment.

4. Fig. 237 represents two squares, one circumscribing the circle and the other inscribed in it. If the radius of the circle is $r$ cm, what are the areas of these squares? What does this tell you about the area of the circle?

5. Draw on squared paper two circles, one of radius 2 inches, the other of radius 3 inches. Find the area of each by counting the squares enclosed. What is the value of the ratio—area of circle square of radius in each instance? If the answers disagree, which is likely to be the more accurate?

In the following questions $\pi$ may be taken either as $\frac{22}{7}$ or as $3.142$, whichever is more convenient; log $\pi$ may be taken to be 0.4971. Answers should never be given to more than 3 significant figures.

6. Find the circumference and area of a circle (i) of radius 7 cm.; (ii) of diameter 4.7 cm.

7. Find the diameter of a circle (i) whose circumference is 11.34 in.; (ii) whose area is 15.8 sq. in.

8. How many revolutions per mile are made by a wheel of diameter 3.1 ft.?

9. Find the speed of the earth in its orbit round the sun, in miles per sec., taking the orbit as a circle of radius 93,000,000 miles.

10. What is the area of the ring between two concentric circles of radii 7, 5.4 cm.?

11. The minute-hand of a church clock is 1 ft. 9 in. long. Find the distance its tip moves in 35 minutes.

12. An arc PQ of a circle of radius 8 cm. subtends 50° at the centre O. What is the length of the arc?

13. With the data of No. 12, find the area of the sector OPQ.

14. An arc of a circle of radius 7.3 cm. is 7.3 cm. long. What angle does the arc subtend at the centre?

15. A piece of flexible wire in the form of an arc of a circle of radius 4.2 in. subtends an angle of 30° at the centre of the circle; it is bent so as to form a complete circle. What is the radius of this circle?

16. The arcs in Fig. 238 are quadrants of circles. Prove that if the squares are equal, the shaded areas are equal.

17. Fig. 239 shows an equilateral triangle, side 4'. Find the length of the curve, if it is composed of arcs of the same radius 1'.
18. A piece of wire 4 ft. long is bent into an arc of radius 1 ft. How far apart are the ends of the wire?

19. A swing has ropes 14 ft. long, and when at rest the seat is 2 ft. above the ground; the seat is prevented from rising more than 10 ft. above the ground. What is the length of the arc in which it can swing?

20. What length of arc of a circle of radius 5 cm. is cut off by a chord of length 7 cm.?

21. What is the area of the minor segment of a circle of radius 6 inches cut off by a chord of length 5 inches? Also find the area from the approximate rule that
   \[ \text{area of segment} = \frac{1}{2} \times \text{base} \times \text{height}. \]

22. What is the area of the major segment of a circle of radius 10 cm. cut off by a chord of length 12 cm.?

23. A window consists of a rectangle surmounted by a semi-circle; its width is 5 ft. and its greatest height is 8 ft. Find (i) its area (ii) its perimeter.

24. Find (i) the volume, (ii) the total surface of a closed cylinder of height 6 in. and radius 4 in.

25. Find the diameter of a cylinder whose length is 10 feet and volume 300 cu. inches.

26. How many cylindrical glasses 3 in. in diameter can be filled to a depth of 4 in. from a cylindrical jug of diameter 6 in. and height 12 in.?

27. A garden roller is 3 ft. in diameter and is 4 ft. wide. What area does it roll in 60 revolutions?

28. A regular polygon of nine sides is inscribed in a circle of radius 10 cm. Calculate the difference between the area of the circle and the area of the polygon.

29. TA, TB are tangents to a circle of radius 4 in.; \( \angle ATB = 50^\circ \).

Calculate the area bounded by TA, TB and the arc AB. (Fig. 240.)

30. AB, AC, BC are arcs of circles of radii 4, 4, 5 inches, touching each other. Calculate (i) the area, (ii) the perimeter of Fig. 241.

31. Two wheels of radii 1 ft., 3 ft., with their centres 5 ft. apart, are connected by a belt, see Fig. 242. Calculate the total length of the belt.

32. A screw thread is cut on the surface of a cylinder of diameter 4 cm.; the thread makes an angle of 75° with the axis of the cylinder. Find the length of thread if the cylinder is 50 cm. long; find also the number of turns it makes round the axis.

33. TP, SQ are tangents to the circle, centre O; OA = 4 cm., AT = TS = 2 cm. Calculate the length of the arc PQ. (Fig. 244.)

34. CAD is a tangent to the circle, centre O; \( \angle COA = 30^\circ \); CD = 3AO. Prove that \( \angle BDO \) is a close approximation for the length of the circumference of the circle. (Fig. 245.)

35. The two wheels of a cart are fixed 6 feet apart on an axle; the cart describes a circular course such that the diameter of the outer rut is 5000 ft. Find the difference between the lengths of the outer and inner ruts. Is any part of the data superfluous?
36. A circular disc, centre O, diameter 1 ft., is fixed flat on a table; a taut string AEFB joins two tacks A, E driven into the table; AC=DC=DB=3 in. and AOB is a straight line. Find the length of the string. (Fig. 246.)

37. The nut of a screw rises 3 inches in 10 turns; the diameter is 1 inch. Find the angle which the thread of the screw makes with the axis.

**Latitude and longitude.**

Let N, S represent the North and South Poles of the Earth and O its centre; the **Equator** is the section of the Earth's surface made by a plane through O perpendicular to NS. Any section of the Earth's surface by a plane through O is called a **Great Circle.** The great circles which pass through N and S are called **Meridians, and** the particular meridian through Greenwich (O) is called the "Greenwich meridian." Let it cut the Equator at A, as shown.

Take any point P on the Earth's surface and draw the meridian through it, cutting the equator at Q. Let $\angle AOQ = F$ and $\angle QOP = \lambda^\circ$.

Then, with the notation of the figure, P is said to have latitude $\lambda^\circ$ and longitude $1^\circ$ West.

**North and longitude 1° West.**

Latitudes vary from 90° S. (at the South Pole) to 0° (on the Equator) to 90° N. (at the North Pole).

Longitudes vary from 180° W. to 0° (on the Greenwich meridian) to 180° E.

Any section of the Earth's surface by a plane parallel to the Equator is called a **Parallel of Latitude**; all points on that small circle have equal latitudes.

In the figure K is the centre of the small circle PD, which is the parallel of latitude through P and cuts the Greenwich meridian at D.

It is important to notice that (i) $\angle PKD = \angle QOA = F$, and that (ii) $\angle KPO = \angle POQ = \lambda^\circ$.

Hence, if the radius of the Earth = a miles,

$$PK = a \cos \lambda^\circ$$

and

$$\text{arc PD} = \frac{1}{360} \times 2\pi a \cos \lambda^\circ$$

A **nautical mile** is the length of an arc of the meridian which subtends an angle of 1' at the centre of the Earth.

Taking the radius of the Earth as 3960 statute miles, we see that

$$1 \text{ nautical mile} = \frac{1}{60} \times \frac{1}{360} \times 2\pi \times 3960$$

$$= 1.15 \text{ statute miles} = 6080 \text{ feet}.$$
TRIGONOMETRY

Local time. The local time at any place \(P\) on the Earth's surface is 12 noon at the moment when the Sun appears to cross the meridian plane NP of \(P\). Thus if \(P\) is west of Greenwich, noon at \(P\) occurs after noon at Greenwich. If we know the correct local time at any given place and also know the corresponding Greenwich time, we can calculate the longitude of that place.

EXERCISE X. b.

[Take the radius of the Earth as 3960 statute miles.]

All distances are to be taken as measured along the earth's surface unless otherwise stated.

1. Two places on the Equator are 150 nautical miles apart. What is the difference (i) in their longitudes, (ii) in their local times?

2. Two places on the same meridian have latitudes (i) 20° N., 33° N.; (ii) 10° N., 25° S. What is their distance apart (statute miles)?

3. What is the length of the Equator in nautical miles?

4. Reading and Greenwich have equal latitudes, 51° 28' N., and the longitude of Reading is 5° W. How far is Reading from Greenwich (statute miles)?

5. What is the difference of local time between Paris (lat. 48° 50' N., long. 2° 20' E.) and Bombay (lat. 18° 55' N., long. 72° 54' E.)?

6. At the equinox, when the Sun is vertical at the Equator, find the length of the shadow at mid-day in Winchester (lat. 51° 55' N., long. 1° 18' W.) of a vertical pole 10 feet high.

7. Eratosthenes found that the sun was in the zenith at Syene when it was 7° 12' South of the zenith at Alexandria, which was known to be 6900 stadia North of Syene. What result did Eratosthenes deduce for the radius of the Earth (i) in stadia, (ii) in English miles, taking 1 stadium = 6064 ft.?

8. In what latitude does a distance of 1 nautical mile measured along a parallel of latitude correspond to a difference of 3° in longitude?

9. A ship after sailing 200 (nautical) miles due West finds that her longitude has altered by 5°. What is her latitude?

10. Find the distance travelled by the Eiffel Tower (lat. 48° 50' N.) in 15 minutes, due to the Earth's rotation.

11. \(A, B\) are two points in latitude 52° N., whose longitudes differ by 20°. Find (i) the distance between \(A\) and \(B\) measured along the parallel of latitude, (ii) the angle which \(AB\) subtends at the centre of the Earth, (iii) the distance between \(A\) and \(B\) along a great circle.

MENSURATION OF THE CIRCLE

Circular cone.

Let the base-radius of the cone be \(r\) in., the slant height \(h\) in., the slant height \(l\) in., and the semi-vertical angle \(x\), then

\[ \tan x = \frac{h}{r} \]

and

\[ x = \tan^{-1} \left( \frac{h}{r} \right) \]

The perimeter of the base \(= 2\pi r\) in.

If we make a cut along a slant edge and unwrap the curved surface, we obtain a circular sector, radius \(l\) in., and bounded by an arc of length \(2\pi r\) in.;

\[ \text{area of sector} = \frac{\pi}{2} \times 2\pi r = \pi rl \text{ sq. in.} \]

\[ \text{area of curved surface of cone} = \pi rl \text{ sq. in.} \]

It can be proved by the methods of the calculus that the volume of any pyramid \(= \frac{1}{3} \text{base-area} \times \text{height} \;

\[ \text{volume of cone} = \frac{1}{3} \pi rh \text{ cu. in.} \]

Frustum of a cone.

(i) Volume. Let the radii of the end-faces, centres \(E, F\), of the frustum be \(a, b\), and let the distance between the faces, be \(h\). Figure

(ii) Surface. If \(h\) is the slant height of the frustum, the lateral surface is composed of two right-angled triangles, one on each base.

\[ \text{surface area} = \pi (a + b) h \]

\[ \text{volume of frustum} = \frac{1}{3} \pi h (a^2 + ab + b^2) \]

249 is a section through the axis; \(AB, CD, \) meet at the vertex \(O\) of the cone from which the frustum is cut; let \(OF = x\).
TRIGONOMETRY

Volume of frustum = \( \frac{1}{3} \pi (a^2 + ab + b^2) \).

By similar triangles, \( \frac{x}{b} = \frac{h}{a-b} \); \( \therefore \frac{x}{a-b} = \frac{bh}{a-b} \);

\[ x = \frac{h(b)}{a-b} + \frac{b}{a-b} + \frac{a}{a-b} \]

\[ x = \frac{h(b) + b(a-b) + a(b)}{a-b} \]

\[ x = \frac{h^2}{3(a-b)} + \frac{a}{a-b} \]

\[ \therefore \text{volume of frustum} = \frac{\pi}{3} \left( \frac{a(b^2) + b(h^2)}{a-b} \right) = \frac{\pi}{3} \left( \frac{a^2 + b^2}{a-b} \right) \]

Note. If \( s_1 \), \( s_2 \) are the areas of the plane faces of the frustum, this formula for the volume may be written \( \frac{h}{3} (s_1 + \sqrt{s_1 s_2} + s_2) \).

In this form, it is true for the frustum of any pyramid.

(ii) Area of curved surface. Let length of slant edge \( AB \) of frustum = \( l \).

Let \( OB = y \).

When unwrapped, the surface becomes a plane figure bounded by two concentric area of lengths \( 2\pi a, 2\pi b \) and equal portions \( l \) of two radials.

\[ \text{area} = \pi l \left( \frac{2\pi a}{2\pi b} + 1 \right) = \pi l \left( \frac{a}{b} + 1 \right) \]

\( \therefore \text{area of curved surface} = \pi l \left( \frac{a^2 + b^2}{a-b} \right) \)

\[ = r l \left( a + b \right) \]

Note. This area can also be obtained by regarding the surface as the limit of the sum of a number of trapezoids with the same height \( l \). Thus, by applying the formula \( \frac{1}{2} (a + b) \) for the area of a trapezium (see p. 178), the result \( \frac{1}{2} (2\pi a + 2\pi b) \) can be obtained. This is the simplest way of remembering the result.

Spheroid.

Suppose a sphere rests on the base of a cylindrical vessel which it just fits, i.e., the diameter of the sphere is the internal diameter of cylinder.

Suppose any two planes are drawn parallel to the base; the surface of the sphere intercepted between these two planes is called a zone, and Archimedes proved that the area of the zone is equal to the area of the surface intercepted between these two planes on the (inner) surface of the cylinder, circumserching the sphere.

If the sphere is of radius \( r \) in., and if the distance between the parallel planes is \( a \) in., then

\[ \text{area of zone of sphere} = 2\pi a r \text{ sq. in.} \]

By taking \( d = 2r \), we obtain

\[ \text{total area of surface of sphere} = 2\pi \times 2r = 4\pi r^2 \text{ sq. in.} \]

We can regard the solid sphere as composed of a large number of pyramids, each with its vertex at the centre of the sphere and with a small portion of the surface of the sphere as base. We therefore can say that

\[ \text{Volume of sphere} = \frac{4}{3} \times \text{total area of surface of sphere} \]

\[ = \frac{4}{3} \times 4\pi r^2 = \frac{4}{3} \pi r^2 \text{ cu. in.} \]

MENSURATION OF THE CIRCLE

Also, from Fig. 249,

\[ \frac{y}{a} = \frac{y}{b} \]

\[ \therefore y = \frac{a}{b} \]

\[ \text{area of a circular surface} = \pi a^2 \]

\[ \therefore \text{area of a spherical surface} = \pi (a^2 + b^2) \]

\[ = \pi l (a + b) \]
Example II. A sector of a circle of radius 5 in., angle of sector $110^\circ$, is bent into the form of a circular cone. Calculate the height, $h$ in., and semi-vertical angle, $x^\circ$, of the cone.

The arc of the sector $= \frac{110\times2\pi}{180} \times 5$ in.

Let the radius of the base of the cone be $r$ in.

Then $2\pi r = \frac{110\times2\pi}{180} \times 5$;

$\therefore r = \frac{11}{36}$ in.

The slant edge of the cone $= 5$ in.;

$\therefore \sin x^\circ = \frac{5}{r} = 0.3056$;

$\therefore x^\circ = 17^\circ 48'$.

$\therefore$ height of cone $= 5 \cos x^\circ = 5 \cos 17^\circ 48' = 5 \times 0.9621 = 4.81$ in.

EXERCISE X. a.

1. Find (i) the volume, (ii) the area of the curved surface of a cone, height 8 cm., base-diameter 5 cm.

2. The base of a conical tent is 14 ft. in diameter and its height is 8 ft. Find (i) the volume of the tent, (ii) the area of canvas required for making it.

3. Find the volume of a cone, vertical angle $54^\circ$, base-diameter 4 inches.

4. The curved surface of a cone of height 5', base-radius 6', is folded out flat. What is the angle of the sector so formed?

5. A sector of angle $80^\circ$ is bent into the form of a circular cone. Find the vertical angle of the cone.

6. Find (i) the volume, (ii) the area of the surface of a sphere of diameter 3 inches.

7. Taking the radius of the Earth as 3960 miles, find (i) the area of the Earth's surface, (ii) the area between the parallels of latitude, $30^\circ$ N. and $50^\circ$ N.

8. Find (in inches) the diameter of a sphere of volume 1 cu. ft.

9. Find (in inches) the diameter of a sphere whose surface area is 1 sq. ft.

10. A cylinder of diameter $d$ inches contains water. Three spheres each of diameter $d$ inches are placed in the cylinder. If all are submerged and no water overflows, find the height the water-level rises.

11. Find the area of the Earth's surface within the Arctic Circle, i.e. in latitudes North of $66^\circ 33'$ N.

12. A cylindrical boiler has hemi-spherical ends; its diameter is 4 ft. and its total internal length is 12 ft. Find its volume and its internal surface.

13. Spherical balls, each of diameter 11 in., are packed in a box measuring 8 in. by 8 in. by 3 in.; how much free space is there in the box, if as many are packed as possible?

14. Fig. 253 represents a hemi-spherical bowl of radius 8 inches, containing water to a depth of 3 inches. Find

(i) $\angle POQ$;

(ii) the area of the wetted surface $POQ$;

(iii) the volume of the sector of the sphere whose base is the wetted surface;

(iv) the volume of the cone, vertex $O$, base $PQ$;

(v) the volume of the water in the bowl.

15. Fig. 254 represents a frustum of a cone whose faces have diameters 6 cm., 8 cm., and are 6 cm. apart. Find (i) the volume of the frustum, (ii) the vertical angle of the cone of which it forms part, (iii) the area of the curved surface of the frustum.

16. A solid consists of a cone mounted on a hemi-spherical base. Find the vertical angle of the cone if the volumes of the conical and spherical portions are equal. Fig. 255.

17. A chemist's measuring glass is conical in shape; it is 8 cm. deep and 8 cm. across the mouth. Calculate the distance on the slant edge between the markings for 1 c.c. and 2 c.c.

18. Find the radius of the sphere inscribed in a circular cone whose slant side is 9 cm. and semi-vertical angle $25^\circ$.

19. A sector of a circle of angle $\theta^\circ$ is bent into the form of a cone of semi-vertical angle $\phi^\circ$; obtain a relation between $\theta$ and $\phi$. 
20. A, B are diametrically opposite points on the base of a cone of semi-vertical angle 35° and slant edge 6 inches. Find the difference between the distances of B from A, (i) measured round the rim of the base, (ii) measured along the shortest path across the curved surface of the cone. (Unwrap the curved surface, as on p. 152.)

21. The section of a tube railway is the major segment of a circle of radius 6 ft. cut off by a chord of length 8 ft. What is the area of the inner curved surface of a tunnel 400 yd. long?

22. A rumble is a frustum of a cone 4 in. deep: the diameters of its upper and lower ends are 3 in., 2 in., and it contains water to a depth of 1 in. How many spherical shot of diameter 7\(\frac{1}{2}\) inch must be poured in to raise the level half an inch?

23. A solid sphere of diameter 10 cm. is melted down and recast as a hollow sphere whose thickness is \(\frac{1}{4}\) of its outside radius. Find the area of its outer surface.

24. The thickness of a metal spherical shell is \(t\) in., and its mean radius is \(r\) in.; prove that the volume of the metal is \(\pi r^2 t\).

25. Find the parallel of latitude which divides the surface of the Earth in the ratio 4:1.

26. In Fig. 256, PQ represents a segment of height CN = \(a\) in. of a sphere of radius \(R\) in.; O is the centre of the sphere. Prove that (i) the volume of the cone OPQ is \(\frac{1}{3}\pi R^2(a - d)(2a - d)\) cu. in., (ii) the volume of the spherical sector OPQ is \(\frac{2\pi aR}{3}\) cu. in., (iii) the volume of the spherical segment PCQ is \(\frac{1}{3}\pi R^2(3a - d)\) cu. in.

27. Fig. 257 represents two concentric spheres of radii \(a\) and \(a - d\) inches. PQ and RS are parallel tangent planes. A variable plane HK parallel to PQ and RS is drawn between them; prove that the area intercepted on HK between the spheres is constant. Hence show that the space between the spheres and PQ and RS is 2\(\pi\)(\(a - d\))(2\(a - d\)).

Hence prove that the volume of the spherical segment PCQ is \(\frac{1}{3}\pi R^2(3a - d)\) cu. in., as in No. 26.

CHAPTER XI.

CIRCULAR MEASURE.

Radius of small angles. AB is a diameter of a circle, centre O, radius \(r\) in.; AP is an arc subtending \(\theta\) at O; the tangent at P meets BA produced at T.

\[
\text{Area of } \Delta OAP = \frac{1}{2} OA \cdot OP \sin AOP = \frac{1}{2} r^2 \sin \theta.
\]

Area of sector \(\Delta OAP = \frac{2\theta}{360} \cdot \pi r^2 = \frac{\pi r^2 \theta}{360}\).

Area of \(\Delta TOP = \frac{1}{2} TP \cdot OP = \frac{1}{2} r^2 \tan \theta, r = \frac{1}{2} \tan \theta \cdot r^2\).

But \(\Delta OAP < \text{sector } \Delta OAP < \Delta TOP\):

\[
\frac{\frac{r^2 \sin \theta}{360}}{\frac{4\theta}{360} \cdot \frac{\pi r^2}{360}} \leq \frac{1}{\cos \theta^2}.
\]

Divide by \(\frac{1}{\cos \theta^2}\); \(1 < \frac{r^2}{180 \sin \theta^2} \leq \frac{1}{\cos \theta^2}\).

Now we can make \(\theta^2\) take a value as near \(\pi\) as we like by making \(\varphi\) a sufficiently small angle, for \(\cos \theta^2 = 1\).

\[
\frac{180 \times \sin \theta^2}{\varphi^2} \leq 1.
\]

\(\frac{\pi}{180} < \sin \theta^2 < \frac{1}{\varphi^2}\).

\(\cdot \cdot \cdot \sin \theta^2 \approx \frac{\pi}{180} \times \theta^2\).
TRIGONOMETRY

and, the smaller \(x\) is, the less is the percentage error of this approximation.

The Tables may be used to illustrate this result:

From the Tables, \(\sin 6^\circ = 0.10472\); also \(\frac{\pi}{180} \times 5 = \frac{\pi}{36} = 0.10472\).

The awkward numerical factor, \(\frac{\pi}{180}\), which occurs in this formula, also appeared in some of the formulae of the last chapter;

\[\text{e.g.} \quad \text{the length of an arc (p. 144)} = \frac{\pi}{180} \times xx,\]

and the area of a circular sector = \(\frac{\pi}{180} \times \frac{1}{2}xx\).

There are many other formulae of the same kind in which this factor appears; it is due to the unit (degrees) in which the angle \(x\) is measured. We can simplify these formulae, making them easier to remember and easier to work with, by choosing a new unit for measuring angles.

A radian.

\(O\) is the centre of a circle of radius \(r\) inches; \(AP\) is an arc of length equal to the radius, i.e. \(\text{arc } AP = r\) inches.

![Diagram of a circle with a radian angle](image)

Then \(\angle POA\) is said to equal 1 radian (often written \(1^\text{r}\)).

Let \(AB\) be a diameter; suppose \(\angle AOP = x^\circ\).

Then

\[
\frac{x}{180} = \frac{\text{arc } AP}{\text{semicircle } APB} = \frac{r}{r} = \frac{1}{2}
\]

\[
\therefore \quad \frac{x}{180} = \frac{r}{\pi} \cdot \pi = \frac{x}{\pi}
\]

\[
\therefore \quad 1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.3^\circ 17.7',
\]

and \(\pi\) radians = 180 degrees.

Note. The size of a radian does not depend on the radius \(r\) of the circle, used in the definition; if it did so, it would be little use as a unit for angle-measure.

CIRCULAR MEASURE

The system of measuring angles in radians is called "Circular Measure," and the number of radians in an angle is often called the "circular measure" of the angle.

Radians and degrees. The fundamental relation

\[\pi \text{ radians} = 180 \text{ degrees}\]

enables any angle expressed in either unit to be converted to the other. We have at once

\[
\frac{\theta}{\pi} \text{ radians and } \frac{\pi \theta}{180} \text{ degrees}
\]

Tables have, however, been constructed to save the time which this arithmetical calculation requires. (See end of book.)

The following special relations should be noted:

\[
360^\circ = 2\pi; \quad 90^\circ = \frac{\pi}{2}; \quad 45^\circ = \frac{\pi}{4}; \quad 120^\circ = \frac{2\pi}{3}; \quad 60^\circ = \frac{\pi}{3}; \quad 30^\circ = \frac{\pi}{6}.
\]

The reader should make himself familiar with these results.

It is customary to speak of an "angle \(\pi/2"\) or an "angle \(\pi/4"\) etc., as shorthand for \(\pi\) radians (\(\pi/2\)) or \(\pi/4\) radians, etc. When the unit is not named explicitly, it is usually implied that the angle is measured in radians.

We shall now show how the formulae mentioned above are simplified by working in radians instead of in degrees.

Length of arc : area of sector.

Let \(O\) be the centre of a circle of radius \(r\) inches; let \(PQ\) be an arc subtending \(\theta\) radians (\(\theta\)) at \(O\).

![Diagram of a circle with a radian angle](image)

Let \(\angle AOB = 1\) radian; \(\therefore \text{arc } AB = r\) in.
TRIGONOMETRY

But
\[ \frac{\text{arc } PQ}{\text{arc } AB} = \frac{\theta}{1} = \frac{\text{arc } PQ}{r} = \theta; \]
\[ \therefore \text{arc } PQ = \pi \text{ inches.} \]

Again, since \(2\pi\) radians = 4 right angles,
\[ \frac{\text{sector } POQ}{\text{area of circle}} = \frac{\theta}{2\pi}; \]
\[ \therefore \text{sector } POQ = \pi r^2 \times \frac{\theta}{2\pi} \text{ sq. in.}; \]
\[ \therefore \text{sector } POQ = \frac{\pi r^2}{2} \text{ sq. in.} \]

Ratios of small angles.
Use the same figure as on p. 150, taking \( \angle AOP = \theta \).

Since area of sector \( AOP = \frac{1}{2} r^2 \theta \), we have as before
\[ \frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \cos \theta; \]
\[ \therefore 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}; \]
\[ \therefore \text{ as before, if } \theta \text{ is a small angle, } \]
\[ \sin \theta \approx \theta. \]

The Tables may be used to illustrate this approximation.
If \( \theta = \frac{1}{2} \), \( \sin \frac{1}{2} = \sin 11^\circ 28' \approx 0.199; \) while \( \frac{1}{2} = 0.2. \)
Further, if \( \theta \) is a small angle, \( \cos \theta \approx 1, \) and so
\[ \tan \theta \approx \frac{\sin \theta}{\cos \theta} \approx \sin \theta \approx \theta; \]
\[ \therefore \tan \theta \approx \theta, \text{ if } \theta \text{ is a small angle.} \]

From the Tables, if \( \theta = \frac{1}{2}, \)
\[ \tan \frac{1}{2} \approx \tan 11^\circ 28' \approx 0.203; \] while \( \frac{1}{2} = 0.2. \)

Note. (i) These two numerical examples illustrate the facts proved above that \( \sin \theta < \theta \) and \( \tan \theta > \theta. \)

CIRCULAR MEASURE

(i) On p. 160 we took \( x = 5 \) as an example of a small angle because it corresponded to \( 5^\circ; \) but the angle corresponding to \( \theta = 5 \) would be \( 5 \text{ radians} \approx 286^\circ, \) which is not small.

Example I. Express \( 0.35^\circ \) in degrees and \( 37^\circ 20' \) in radians.

(i) \( 0.35^\circ = \frac{0.35 \times 180}{\pi} \text{ degrees} = 20^\circ 6' \)
\[ = 20^\circ 4'. \]

(ii) \( 37^\circ 20' = \frac{37.33 \times \pi}{180} \text{ radians} \approx 0.651. \)

Note. These results can be obtained direct from the printed conversion tables.

Example II. Find the length of the chord \( PQ \) which cuts off an arc 12 cm. long from a circle, centre \( O, \) radius 5 cm.

\[ \text{arc } PAQ = 12 \text{ cm.}; \]
\[ \therefore \text{arc } POQ = \frac{1}{2} \text{ arc } POQ = \frac{1}{2} = 2.4 \text{ radians.} \]

From the Tables, \( 1^\circ = 57^\circ 15', 1^\circ = 80^\circ 13'; \)
\[ \therefore \text{arc } POQ = 137^\circ 31'. \]

Draw \( ON \) perpendicular to \( PQ; \) then \( \angle PON = \frac{1}{2} \angle POQ = 65^\circ 45'; \)
\[ \therefore PQ = 2PN = 2 \times 5 \sin 65^\circ 45' \times 0.9320 \]
\[ = 9.32 \text{ cm.} \]

EXERCISE XI. a.

1. Calculate the following angles, in degrees and minutes:
   (i) \( 5^\circ; \)
   (ii) \( 3^\circ; \)
   (iii) \( 1^\circ; \)
   (iv) \( 1^\circ; \)
   (v) \( 2^\circ; \)
   (vi) \( 0.5^\circ; \)
   (vii) \( 0.6^\circ; \)
   (viii) \( 0.6^\circ. \)

Use the Tables to check your answers.
2. Express in degrees the angles whose circular measures are:
   (i) \(\frac{\pi}{2}\); (ii) \(\frac{3\pi}{4}\); (iii) \(\frac{\pi}{6}\); (iv) \(\frac{5\pi}{6}\); (v) \(\pi\);
   (vi) \(\frac{\pi}{3}\); (vii) \(\frac{\pi}{2}\); (viii) \(\frac{\pi}{4}\); (ix) \(\frac{\pi}{8}\);
   (x) \(\frac{\pi}{12}\).

3. Express the following angles in radians in terms of \(\pi\):
   (i) \(270^\circ\); (ii) \(60^\circ\); (iii) \(150^\circ\); (iv) \(135^\circ\); (v) \(75^\circ\);
   (vi) \(30^\circ\); (vii) \(180^\circ\); (viii) \(225^\circ\); (ix) \(315^\circ\); (x) \(360^\circ\).

4. Express in radians the following angles, and compare your answers with the values given in the Tables:
   (i) \(17^\circ\); (ii) \(35^\circ\); (iii) \(68^\circ\); (iv) \(80^\circ\); (v) \(35^\circ\);
   (vi) \(54^\circ\); (vii) \(17^\circ\); (viii) \(54^\circ\); (ix) \(74^\circ\); (x) \(127^\circ\).

5. In some Tables we find \(1^\circ = 57.30'\), \(1' = 5.73'\), \(1' = 0.01\) radian, 0.01 radian = 0.06 degrees. Use these results to express in degrees to one place of decimals (i) \(1.5\) radian; (ii) \(0.435\) radian; (iii) 2.218.

6. The radius of a circle is 10 cm. Find the length of an arc which subtends at the centre an angle of (i) \(2^\circ\); (ii) \(1.5^\circ\); (iii) \(0.5^\circ\); (iv) \(37^\circ\) 30' (use Tables); (v) \(157.24^\circ\) (use Tables).

7. The radius of a circle is 4 inches. Find in radians the angle subtended at the centre by an arc of length (i) 3 in.; (ii) 5 in.; (iii) 1 ft.; (iv) 2 3/4 ft.

8. Use Tables to express \(37^\circ\) in radians, and write down the area of a sector of a circle, radius 10 cm., angle of sector \(37^\circ\).

9. Write down the values of the following, the unit being a radian:
   (i) \(\sin \frac{\pi}{3}\); (ii) \(\cos \frac{\pi}{3}\); (iii) \(\tan \frac{\pi}{4}\); (iv) \(\sin \frac{\pi}{4}\);
   (v) \(\cos \frac{\pi}{3}\); (vi) \(\sin \frac{\pi}{6}\); (vii) \(\cos 2\pi\); (viii) \(\sin 2\pi\);
   (ix) \(\tan \frac{\pi}{3}\); (x) \(\cot \frac{\pi}{4}\); (xi) \(\cos \frac{\pi}{3}\); (xii) \(\sin \frac{\pi}{4}\).

10. Simplify the following, the unit being a radian:
    (i) \(\sin (\pi - \theta)\); (ii) \(\cos \left(\frac{\pi}{2} - \theta\right)\); (iii) \(\tan \left(\frac{\pi}{2} + \theta\right)\);
    (iv) \(\cos (\pi + \theta)\); (v) \(\sin (2\pi - \theta)\); (vi) \(\tan (\pi - \theta)\);
    (vii) \(\sin \left(\frac{3\pi}{2} - \theta\right)\); (viii) \(\cos \left(\frac{3\pi}{2} + \theta\right)\); (ix) \(\cot (2\pi - \theta)\);
    (x) \(\tan (\pi + \theta)\); (xi) \(\cos \left(\frac{\pi}{2} + \theta\right)\); (xii) \(\sin \left(\frac{3\pi}{2} - \theta\right)\).

11. AB is an arc 9 in. long in a circle of radius 6 in., centre O. What angle is subtended by the chord AB, at O, at a point on the major arc AB, at a point on the minor arc AB? Give the answers in radians.

12. The arc PQ of a circle, centre O, radius 5 in., is 6 in. long. Express \(\angle POQ\) in radians; use Tables to convert this to degrees; then calculate the length of the chord PQ.

13. The area of the sector POQ of a circle, centre O, radius 10 cm., is 30 sq. cm. Express \(\angle POQ\) in radians. Calculate the length of the chord PQ.

14. A wheel of radius 20 inches is spinning on its axis at 3 radians per second. Find the speed of a point on the rim.

15. A wheel is making 20 revolutions per minute. Find in radians the angle through which a spoke turns per second.

16. The arc of a circular sector is 6 cm. long and the angle of the sector is \(\frac{1}{4}\) radian. What is the area of the sector?

17. One angle of a triangle is \(\frac{\pi}{4}\); another is \(\frac{\pi}{3}\). What is the third angle?

18. What is the complement of \(\frac{\pi}{4}\)?

19. The wheel of a carriage is 3 ft. in diameter. Through what angle (in radians) does the wheel rotate when the carriage advances 5 yards?

20. A railway line alters \(57^\circ\) in direction when passing round a circular arc of length 3 miles. What is the radius of the arc, in chains, to the nearest chain?

21. A wheel of radius \(r\) feet is rotating on its axis at \(\omega\) radians per second. What is the speed of a point on the rim?
24. In what ratio does a chord of length 8 cm. divide the circumference of a circle of diameter 10 cm.?  
25. Find from the Tables the values of (i) \( \sin 15^\circ \), (ii) \( \cos 30^\circ \), (iii) \( \tan 72^\circ \), where \( \theta = \sin 72^\circ \).  
26. Find, without using trigonometric tables, approximate values of:  
(i) \( \sin 7^\circ \);  
(ii) \( \sin 4^\circ 30' \);  
(iii) \( \sin 40^\circ \);  
(iv) \( \cos 36^\circ \);  
(v) \( \cos 84^\circ \);  
(vi) \( \cos 80^\circ \);  
(vii) \( \tan 2^\circ 30' \);  
(viii) \( \cot 83^\circ \).  
27. Use the fact that, in Fig. 264, chord \( PQ < \) arc \( PQ < PT + TQ \) to show that if \( \theta \) is acute, \( \sin \theta < \theta < \tan \theta \).  

![Diagram](image)

28. With the notation of No. 27, write down the length of the chord \( AP \), and deduce that if \( \theta \) is acute, \( \sin \theta < 2 \sin \theta < \theta \).  

29. A wheel of radius \( a \) ft. rolls, without slipping, along level ground. Initially a point \( P \) on the rim is in contact with the ground. What is the height of \( P \) above the ground when the wheel has advanced \( b \) feet?  

30. In Fig. 265, \( ABC \) is a semicircle, centre \( O \); its area is bisected by a line \( XY \) parallel to the diameter \( AB \); if \( \angle AOX = \theta \), prove that \( \frac{\pi}{2} - 2\theta = \sin 2\theta \). If \( \frac{\pi}{2} - 2\theta = \phi \), prove that \( \cos \phi = \phi \); verify from the Tables that \( \phi \approx 42^\circ 20' \), and prove that \( \angle AOX \approx 23^\circ 30' \).  

![Diagram](image)

31. In Fig. 266, \( TB \) is the tangent to the circle on \( AB \) as diameter; \( AB = 8 \text{ cm.}; \) \( TB = 1 \text{ cm.}; \) \( TP = 5 \text{ cm.} \) Show that \( \angle BOP \) is approximately one radian.  

CIRCULAR MEASURE

32. \( OT \) is a diameter of a circle, radius \( 5 \text{ cm.}; \) \( OA, AB \) are area of the circle, each \( 5 \text{ cm. long}; \) \( OA, OB \) are produced to cut the tangent at \( T \) in \( P, Q \). Calculate \( PQ \).  

33. A fly crawls from a point \( A \) on the rim of the base of a cone of semi-vertical angle \( \sin^{-1} (1) \) to the diametrically opposite point \( B \) on the rim. Show that the shortest path across the curved surface is \( \frac{3\sin 45^\circ}{\sin 22.5^\circ} \) times the distance along the rim.  

34. A string \( ABC, 1 \text{ ft. long}, \) is attached to the top of a circular cylinder of radius \( a \text{ ft.}, \) and a small heavy body is attached to \( C \), which is set swinging. (See Fig. 267.) Find an expression for the depth of \( C \) below \( A \) when the length of string in contact with the cylinder is \( s \) feet.  

35. A wheel, centre \( A \), radius \( a \text{ in.}, \) rolls, without slipping, in a vertical plane along the outer rim of a circular disc, centre \( B, \) radius \( b \); initially the spoke \( AP \) is vertical. What angle does \( AP \) make with the vertical when \( A \) has moved \( s \) inches? (See Fig. 268.)  

![Diagram](image)

36. \( C \) is a point on a given circle, centre \( O \); a circle \( AEB \) is drawn with centre \( C, \) so as to divide the area of the given circle in the ratio \( 1:2 \); if \( \angle COA = \theta \), prove that \( \sin \theta + (\pi - \theta) \cos \theta = \frac{2\pi}{3} \). (See Fig. 269.)  

![Diagram](image)
Size of a distant object.

Let $AB$ be an object whose distance $ON$ from $O=d$ feet.
Let $AN=h_1$ ft., $NB=h_2$ ft., $\angle AON=\alpha_1$, $\angle NOB=\alpha_2$.

Then

$$h_1 = d \tan \alpha_1 = d \theta_1,$$

if $\theta_1$ is small,

$$h_2 = d \tan \alpha_2 = d \theta_2,$$

if $\theta_2$ is small;

![Diagram](image)

$\therefore AB = h_1 + h_2 = d(\alpha_1 + \alpha_2)$ feet;

$\therefore$ if $\angle AOB = \theta$, where $\theta$ is small,

$$AB \approx d \cdot \theta$$

feet.

Example III. Find the diameter of the Sun, given that its distance from the Earth is 93,000,000 miles and that it subtends an angle of 31.5' at a point on the Earth.

From the Tables (or by calculation), 31.5' = 0.00915;

$\therefore$ diameter = 93,000,000 × 0.00915 miles

= 850,000 miles.

Dip of horizon.

If, from a point $T$ above the Earth, tangents are drawn in all directions to the Earth's surface, the points of contact lie on a circle $PQ$, which is called the Visible Horizon from $T$. The angle ($\angle PTH$)

![Diagram](image)

which any tangent makes with the horizontal plane through $T$ is called the Dip of the Horizon and the length of $TP$ is called the Distance of the Horizon.
Approximation for $\cos \theta$, if $\theta$ is a small angle.

Draw $\triangle CAB$, so that $CA = CB = 1$ and $\angle ACB = \theta$.

If $CN$ is perpendicular to $AB$,

$$AB = 2AN = 2AC \sin ACN = 2 \sin \frac{\theta}{2}.$$

From the cosine formula,

$$\left(2 \sin \frac{\theta}{2}\right)^2 = 1^2 + 1^2 - 2 \cos \theta;$$

$$\therefore \quad \cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} \quad \text{but sin } \frac{\theta}{2} \simeq \frac{\theta}{2};$$

$$\therefore \quad \cos \theta \approx 1 - \frac{1}{2} \theta^2.$$

From the Tables, if $\theta = \frac{1}{2}$,

$$\cos \frac{1}{2} \approx \cos 11^\circ 28' \approx 0.9801 \quad \text{and} \quad 1 - \frac{\frac{1}{2}}{2} = 1 - \frac{1}{4} = 0.62.$$

**EXERCISE XI. b.**

1. The diameter of a halfpenny is one inch. At what distance will its diameter subtend an angle of $1^\circ$?

2. What angle does the diameter of a halfpenny subtend at the eye, if held 3 feet away?

3. Find the diameter of the Moon if it subtends an angle of $31'$ at a point on the Earth at a distance of 240,000 miles.

4. What angle does the edge of a cricket screen 9 ft. high subtend at a batsman's eye 150 yards away?

5. The parallax of $\alpha$ Centauri (i.e., the angle subtended by the radius of the Earth's orbit) is $0.75'$. What is its distance?

6. The greatest angle which a diameter of the Earth subtends at a point on the Sun is $17.7'$. Taking the radius of the Earth as 3800 miles, find the distance of the Sun.
20. A vibrating pendulum is inclined at \( \theta \) to the vertical at a time \( t \) seconds after being started, where \( \theta = \frac{1}{2} \sin(6t) \). Through what angle does the pendulum swing, and what is the time of a complete oscillation?

21. A particle attached to the end of a spring is executing oscillations: it moves \( x \) inches from one extreme position in \( t \) seconds, where \( x = b \cos(\alpha t) \), \( b \) and \( \alpha \) being constants. The distance between its extreme positions is 6 inches, and the time of a complete oscillation is 0.8 sec. Find \( b, \alpha \).

22. At noon on a certain day the shadows of two vertical poles A, B, each 5 ft. high, are 3 ft. 5 in. and 3 ft. 1\( \frac{1}{2} \) in. respectively. If A is 60 miles North of B, what is the radius of the Earth according to these measurements?

23. ABCDE is a circular arc such that arc DE = arc AB = 3 in.; arc BC = arc CD = 5 in.; and the angle between the tangents at A and E is 176°. Calculate the height of C above BD. (See Fig. 278.)

24. With the data of No. 23, find the difference between the arc BD and the chord BD, given that, if \( \theta \) is small, \( \sin \theta \approx \theta - \frac{\theta^3}{6} \).

25. The angular diameter of the Sun varies by about 1° during the year. Find the ratio of the Earth's distances from the Sun at perihelion and aphelion, if the mean is about 32.

26. Prove that \( \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \).

27. If \( \alpha \) is a small angle, \( \sin(\theta + \alpha) \approx \sin \theta + \alpha \cos \theta \). Use this result to calculate \( \sin 30° \cdot 30° \), and compare your answer with the value in the Tables.

28. In Fig. 276, where CN is perpendicular to AB, show that \( CN = AC \sin \frac{\beta}{2} = \frac{1}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \), and deduce that \( \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \).

Hence, if \( \sin \theta \approx \theta + \frac{\theta^3}{6} \), where \( a, b \) are constants, show that \( a = 0, b = -\frac{1}{6} \), given that \( \cos \theta \approx 1 - \frac{1}{2} \theta^2 \), \( \theta \) being a small angle.

29. [Huygen’s Formula.] If the radius ON bisects the chord PQ, arc PQ = \( \frac{1}{2} \) chord PN – chord PQ). Prove this if \( \angle POQ \) is small, and show that the approximation even holds for \( \sin \theta \approx \theta - \frac{\theta^3}{6} \).

30. AB is the tangent at A to a circle, centre O, radius \( a \); a chord AQ subtends an angle \( \theta \) at O; from AB is cut off AP equal to AQ; PQ meets AQ produced at R. Prove that (i) \( \angle APQ = \frac{\pi}{2} - \frac{\theta}{4} \) radians, (ii) \( AR = 2a \sin \frac{\theta}{2} \cot \frac{\theta}{4} \).

Hence show that if \( \theta \) is very small, \( AR \approx 4a \).
Graph of $\sin \theta$ for values of $\theta$ from 0 to $\pi$.

From the tables we have the following values:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
<td>1.0</td>
</tr>
</tbody>
</table>

From these values we obtain the graph shown below.

**Example IV.** Solve graphically $x^2 = \frac{1}{2}x$.

With the same scale and axes as above, draw the graph of the function $\frac{1}{x}$.

These intersect where $x = 0$ and where $x = 2.475$.

The angles which satisfy the given equations are $0^\circ$ and $2.475$ radians or $0^\circ$ and $141^\circ 50'$ approx.

**EXERCISE XI. c.**

Use the graph in Fig. 279 for solving Nos. 1-15.

[Do not rule lines on the figure in the book, but use a ruler to show the position of the line when reading off a point of intersection.]

1. $\sin x = 0.6$.
2. $y = \sin 1.3^\circ$.
3. $\sin x = 0.36$.
4. $y = \sin 2.6^\circ$.
5. $\cos x = \sin \left( \frac{\pi}{2} - x \right) = 0.7$.
6. $\cos x = 0.25$.
7. $\sin x = \frac{1}{2}x$.
8. $\sin x = \frac{3}{2}x$.
9. $\sin x = \frac{x}{2}$.
10. $\sin x = \frac{1}{2}x - 1$.
11. $x + \sin x = 1$.
12. $\sin x = \frac{\pi}{2} - x$.
13. $\cos x = x$.
14. $\sin \left( x + \frac{\pi}{4} \right) = x$.
15. $\sin \left( \frac{\pi}{6} x \right) = x$.
16. $\sin \left( \frac{\pi}{4} x \right) = x$.

16. Draw the graph of $\cos x$ for values of $x$ from 0 to $\pi$, and use it to solve the equations:
   (i) $\cos x = x$;
   (ii) $\cos x = \frac{1}{2}x$;
   (iii) $\cos x = \frac{3}{2}x$;
   (iv) $\cos x = \frac{1}{2}x = 1$;
   (v) $1 + \cos x = x$.

17. How does the graph in Fig. 279 illustrate the fact $\sin^2 x = x$?

18. Find graphically a value of $x$, other than $x = 0$, such that $\tan x = x$.

19. A wire 40 cm. long is bent into a buckle composed of the arc and chord of a circle of radius 10 cm. Find graphically the angle subtended at the centre of the circle by the arc.

20. In Fig. 280, the circular portion $ACB$ of a bow is 5 ft. long and the distance $CD$ of $C$ from $AB$, where $AC = CB$, is 8 inches. Find graphically $\angle AOB$.

21. $ACB$ is an arc of a circle, centre $O$, such that the sector $AOB$ is three times the segment $ACB$. Find graphically $\angle AOB$.

22. $AB$ is a chord of a circle, centre $O$, such that the area of the segment cut off by $AB$ is one-quarter of the area of the circle. Find graphically $\angle AOB$.

23. In Fig. 281, $AS$, $AT$, $AT$ are tangents to a circle, including an angle $26^\circ$. Find $\theta$ if the arc $TS$ divides the $\triangle TAS$ into two portions of equal area.

24. $C$ is the mid-point of the arc $ACB$ of the circle, centre $O$; the centre of gravity of the sector $AOB$ is a point $G$ on $OC$, such that $GC = 2 \sin \theta$, where $\angle AOB = 26^\circ$. Find graphically the value of $\theta$ if $G$ is at the mid-point of $OC$. 

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**CIRCULAR MEASURE**

14. $\sin \left( x + \frac{\pi}{4} \right) = x$.
16. $\sin \left( \frac{\pi}{6} x \right) = x$.
25. A thread is wound round a disc, centre O, radius a in.; whose plane is vertical; initially there is a straight horizontal portion AB of length 2a in.; this is now wound into the position AP'B', where B' is at the same level as O. Find graphically \( \angle OP'P' \).

26. Sketch, with the same axes and scale the graph of
\[
y = 5 \sin \frac{2\pi}{9} (x - 3t)
\]
for \( t = 0, 1, 2, 3, 4 \), the unit of angle being a radian.
Suppose \( x \) and \( y \) are measured in feet and \( t \) represents seconds. Then the series of graphs represents the progress of a wave.

What is
(i) the height of the crest above the trough;
(ii) the distance between successive crests;
(iii) the speed of advance of the wave?

27. Repeat No. 26 for the relation \( y = a \sin \frac{2\pi}{\lambda} (x - ct) \), where \( a, \lambda, c \) are constants.

CHAPTER XII.

TRIANGLES AND POLYGONS.

Area of triangle.
It has been proved on p. 108 that the area \( \Delta \) of any triangle \( ABC \) is given by the formula
\[
\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B.
\]

By using the cosine formula, this can be expressed in terms of the lengths of the three sides:
\[
c^2 = a^2 + b^2 - 2ab \cos C ; \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} ;
\]
\[
\sin^2 C = 1 - \cos^2 C = 1 - \left( \frac{a^2 + b^2 - c^2}{2ab} \right)^2
\]
\[
= \left( \frac{1}{2ab} \right) \left( \frac{2ab}{1} - \frac{a^2 + b^2 - c^2}{2ab} \right)
\]
\[
= \frac{2ab + a^2 + b^2 - c^2}{2ab} = \frac{2ab - a^2 - b^2 + c^2}{2ab}
\]
\[
= \frac{(a+b)^2 - c^2}{2ab} = \frac{c^2 - (a-b)^2}{2ab}
\]
\[
= \frac{(a+b+c)(a+b-c)(a+b)(c-a+b)}{4ab}
\]
Put: \( a + b + c = 2s \); \( a + b - c = 2s - 2a - 2c, \) etc.;
\[ \Delta^2 = \frac{1}{16} 2a \cdot 2b \cdot \sin^2 C = \frac{2(2s - 2a)(2s - 2b)(2s - 2c)}{16} \]
\[ = s(s - a)(s - b)(s - c); \]
\[ \therefore \Delta = \sqrt{s(s - a)(s - b)(s - c)}. \]
This result was first given by Hero of Alexandria about 120 B.C.; it is sometimes called Heron's formula for the area of a triangle.

**Area of parallelogram.**

ABCD is a parallelogram, having
\[ AB = x, \ AD = y, \ \angle BAD = \theta. \]

Then area of ABCD = \( 2 \times \text{AB} \times \text{AD} \times \sin \theta = 2xy \sin \theta. \]

**Area of trapezium.**

ABCD is a trapezium, with AB and CD as parallel sides.

Let \( AB = x, \ DC = y, \) and distance between AB and DC be \( h. \)

Area of ABCD = \( \Delta ABD + \Delta BCD = \frac{1}{2} x \cdot h + \frac{1}{2} y \cdot h = \frac{1}{2} (x + y)h \]
= half sum of parallel sides \( \times \) distance between them.

**Area of quadrilateral.**

ABCD is a quadrilateral, whose diagonals cut at O;
\[ AC = x, \ BD = y \] and \( \angle AOB = \theta. \]

Let \( OA = f, \ OC = h, \ BO = p, \ OD = r, \) so that \( f + h = x \) and \( p + r = y. \)

Area of ABCD = \( \Delta AOB + \Delta BOC + \Delta COD + \Delta DOA \]
\[ = \frac{1}{2} fpy \sin \theta + \frac{1}{2} ph \sin (180^\circ - \theta) + \frac{1}{2} hpr \sin \theta \]
\[ + \frac{1}{2} rpy \sin (180^\circ - \theta). \]

**Triangles and Polygons**

But \( \sin (180^\circ - \theta) = \sin \theta; \)
\[ \therefore \text{area of } ABOC = \frac{1}{2} \sin \theta (fp + ph + hr + rf) \]
\[ - \frac{1}{2} \sin \theta (f + h)(p + r) \]
\[ = \frac{1}{2} \times xy \sin \theta. \]

Note. If we are given the 4 sides and one angle \( \theta \) of a quadrilateral, we can find the opposite angle \( \phi, \) because
\[ \theta^2 + \phi^2 - 2ab \cos \theta = x^2 = \phi^2 + a^2 - 2ab \cos \phi. \]

The area is then given by \( \frac{1}{2} ab \sin \theta + \frac{1}{2} cd \sin \phi. \)

**Area of a regular \( n \)-sided polygon.**

(i) Suppose the polygon is inscribed in a circle, centre \( O, \)
radius \( R. \)

Let \( AB \) be one of the sides.

Then \( \angle AOB = \frac{2\pi}{n}; \)
\[ \therefore \text{area of } \Delta OAB = \frac{1}{2} \sin \left( \frac{2\pi}{n} \right). \]
\[ \therefore \text{area of polygon} = \frac{nR^2}{2} \sin \left( \frac{2\pi}{n} \right). \]

(ii) Suppose the polygon circumscribes a circle, centre \( O, \)
radius \( r. \)

Let \( AB, \) one of the sides, touch the circle at \( N. \)

The \( \angle AOB = \frac{2\pi}{n}; \)
\[ \therefore \angle AON = \frac{\pi}{n}; \]
\[ \therefore AN = r \tan \left( \frac{\pi}{n} \right); \]
\[ \therefore \text{area of } \Delta OAB = ON \times AN = r^2 \tan \left( \frac{\pi}{n} \right). \]
\[ \therefore \text{area of polygon} = nr^2 \tan \left( \frac{\pi}{n} \right). \]
(iii) The area can also be expressed easily in terms of the length of a side.

Let \( AB = a \); then \( ON = \frac{a}{2} \cot \left( \frac{\pi}{n} \right) \).

\[ \therefore \Delta OAB = \frac{a^2}{4} \cot \left( \frac{\pi}{n} \right). \]

\[ \therefore \text{area of polygon} = \frac{na^2}{4} \cot \left( \frac{\pi}{n} \right). \]

Note. The reader should observe the form these results take when the number of sides, \( n \), becomes very large.

In (i) we see that the area \( \approx \frac{nR^2}{2} \times \frac{2\pi}{n} = \pi R^2. \)

In (ii) we see that the area \( \approx \frac{n^2a^2}{2 \times \frac{\pi}{n}} = \pi a^2. \)

In (iii) we see that the area \( \approx \frac{n^2a^2}{4} \times \frac{\pi}{n} = \frac{\pi a^2}{4}. \)

\[ \approx \left( \frac{\text{perimeter}}{4}\right)^2 \times \pi. \]

EXERCISE XII. a.

Calculate the areas of the following figures, Nos. 1-9.

1. Fig. 290.

2. Fig. 291.

3. Fig. 292.

4. Fig. 293.

5. Fig. 294.

6. Fig. 295.
24. A piece of wire 5 ft. long is bent into a triangle; two of the angles are 108°, 47°. Find the area.

25. A piece of ground on a sloping hill-side has an area of 2,175 square miles. On a map it is shown as an area of 1,042 miles. What angle is the hill-side inclined to the horizontal?

26. In Fig. 299, ABCD is a rectangle. Find PQ and the area of AQCP in terms of $l$, $\theta$.

27. In Fig. 300, ABCD is a parallelogram. Prove that

\[ \tan \theta = \frac{2ab \sin \alpha}{a^2 - b^2}. \]

28. In Fig. 301, $\angle ACB = 90^\circ$ and $AP = PB$. Prove that

\[ \sin \theta = \frac{2ab}{c^2}. \]

29. In Fig. 302, AD bisects $\angle BAC$. Prove that $AD = \frac{bc \sin 2\theta}{(b + c) \sin \theta}$ and use the relation $\sin 2\theta = 2 \sin \theta \cos \theta$ (Ex. XI. b, No. 28) to simplify the expression.

Pyramids.

If the base of a pyramid is a regular polygon, and if the perpendicular from the vertex to the base passes through the centre of the regular polygon, the solid is called a right pyramid.

It has already been mentioned (p. 158) that the volume of any pyramid is measured by $\frac{1}{3}$ base-area $\times$ height.

Sections of a pyramid parallel to the base are the same shape as the base, and therefore the ratio of the volumes of any two such sections equals the square of the ratio of corresponding sides or the square of the ratio of the areas of the bases.

The ratio of the distances of the sections from the vertex of the pyramid.

The volume of any frustum of a pyramid may be obtained, as on PP. 153-154, by completing the pyramid.

The slant faces of a pyramid are triangles, and their areas may therefore be obtained by using the ordinary triangle formulae.

**Exercise XII. b.**

1. A right pyramid 6 cm. high stands on a square base of side 10 cm. Calculate (i) its volume, (ii) the area of its total surface.

2. A right pyramid vertex O stands on a square base ABCD; $AB = 8$ in., $\angle AOB = 60^\circ$. What is the volume of the pyramid?

3. The base area of a pyramid is 90 sq. cm. and its height is 12 cm. Find (i) its volume, (ii) the area of a section parallel to the base and 3 cm. from the base, (iii) the volume of the frustum bounded by the base and a plane parallel to the base and 3 cm. from it.

4. A frustum of a pyramid is bounded by two rectangles 6 in. by 8 in. and 9 in. by 12 in. at a distance 5 in. apart. What is its volume?

5. If, with the data of No. 4, all the slant edges are equal, find the total area of the surface.

6. A chimney 40 ft. high tapers uniformly, the base being 12 ft. square and the top 10 ft. square; the central hollow space has a uniform circular section, 4 ft. in diameter. Find the nearest ton the weight of brickwork, if 16 cu. ft. weigh a ton.

7. Fig. 303 represents in plan a stack on a rectangular base; the ridge $EF$ is 15 ft. above the base ABCD. Find the volume of the stack.

8. Find the volume of a regular tetrahedron (a pyramid on a triangular base) if each edge is 4 inches.

9. A pyramid, vertex $O$, stands on a rectangular base ABCD; the slant edges are all equal; $\angle AOB = 37^\circ$, $AB = 6$ cm., $BC = 8$ cm. Find its volume.

10. A pyramid stands on a square horizontal base; each face makes an angle $\alpha$ with the vertical; each edge makes an angle $\beta$ with the vertical. Find a relation connecting $\alpha$ and $\beta$. 

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TRIGONOMETRY

Radius of the circumcircle of \( \triangle ABC \).
Let \( O \) be the circumcentre and \( R \) the length of the circumradius of \( \triangle ABC \), and let \( CO \) meet the circumcircle at \( P \).
Then
\[
\angle CPB = 90^\circ, \angle \text{ in semicircle.}
\]

Also
\[
\triangle BPC = \triangle BAC = A, \text{ in Fig. 304,}
\]
and
\[
\angle BPC = 180^\circ - \angle BAC = 180^\circ - A, \text{ in Fig. 305.}
\]
\[
\therefore \quad a = BC = CP \sin BPC = 2R \sin A \quad \text{(Fig. 304)}
\]
\[
= 2R \sin (180^\circ - A) \quad \text{(Fig. 305).}
\]
\[
\therefore \quad a = 2R \sin A ;
\]
\[
\therefore \quad R = \frac{a}{2 \sin A} .
\]

Similarly,
\[
R = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} .
\]

Note. Since \( A = \frac{1}{2}bc \sin A = \frac{1}{2}bc \cdot \frac{a}{2R} \)
\[
\therefore \quad A = \frac{abc}{4R} \quad \text{and} \quad R = \frac{abc}{4A}.
\]

Radius of the inscribed circle of \( \triangle ABC \).
Let \( I \) be the centre and \( r \) the length of the radius of the inscribed circle, which touches the sides at \( X, Y, Z \).
Then
\[
\Delta = \text{area of } \triangle ABC + \text{area of } \triangle ICA + \text{area of } \triangle IAB
\]
\[
= \frac{1}{2} X . \ BC + \frac{1}{2} Y . \ CA + \frac{1}{2} Z . \ AB
\]
\[
= \frac{1}{2} ra + \frac{1}{2} rb + \frac{1}{2} rc = \frac{1}{2} (a + b + c) ;
\]
put \( a + b + c = 2s \),
\[
= \frac{1}{2} r . 2s = rs ;
\]
\[
\therefore \quad r = \frac{A}{s} .
\]

TRIANGLES AND POLYGONS

Radius of an escribed circle of \( \triangle ABC \).
Let \( I_1 \) be the centre and \( r_1 \) the length of the radius of the circle escribed to \( BC \), which touches \( BC, CA, AB \) at \( X_1, Y_1, Z_1 \).

Then
\[
\Delta = \text{area of } \triangle I_1 CA + \text{area of } \triangle I_1 AB - \text{area of } \triangle I_1 BC
\]
\[
= \frac{1}{2} Y_1 . \ CA + \frac{1}{2} Z_1 . \ AB - \frac{1}{2} X_1 . \ BC
\]
\[
= \frac{1}{2} r_1 b + \frac{1}{2} r_1 c - \frac{1}{2} r_1 a = \frac{1}{2} r_1 (b + c - a).
\]

But if \( a + b + c = 2s \), \( b + c - a = 2s - 2a \),
\[
\therefore \quad \Delta = \frac{1}{2} r_1 (2s - 2a) = r_1 (s - a) ;
\]
\[
\therefore \quad r_1 = \frac{\Delta}{s - a} .
\]

Similarly, if \( r_2, r_3 \) are the radii of the circles escribed to \( CA, AB, \)
\[
r_2 = \frac{\Delta}{s - b} \quad \text{and} \quad r_3 = \frac{\Delta}{s - c} .
\]

Note. (i) Since \( I_3, I_4 \) bisect the angles at \( B, C, \)
\[
BX = r \cot \frac{B}{2} \quad \text{and} \quad XC = r \cot \frac{C}{2} ;
\]
\[
\therefore \quad r \left( \cot \frac{B}{2} + \cot \frac{C}{2} \right) = BX + XC = BC = a .
\]

(ii) Since \( I_3, I_4 \) bisect the external angles at \( B, C, \)
\[
\angle I_3 BX_1 = \frac{1}{2} (180^\circ - B) = 90^\circ - \frac{B}{2} ; \quad \angle I_4 BX_1 = \frac{B}{2} .
\]
\[
\therefore \quad BX_1 = r_1 \tan \frac{B}{2} \quad \text{and} \quad X_1 C = r_1 \tan \frac{C}{2} ;
\]
\[
\therefore \quad r_1 \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right) = BX_1 + X_1 C = BC = a .
\]

These results express \( r, r_1, \text{ etc.} \), in terms of one side, and two angles of the triangle.
TRIGONOMETRY

The positions of the points of contact of the in-circle and ex-circle are obtained from the following results:

(i) In Fig. 306, \( AY = AZ = s - a \); \( BZ = BX = s - b \); \( CX = CY = s - c \).
Since the tangents from a point to a circle are equal, \( AY = AZ \),
\( BZ = BX \), \( CX = CY \);
\( \therefore AY + BX + CX = \text{semi-perimeter} = s \).
But \( BX + XC = BC = s \); \( \therefore AY = s - a \),
and similarly for the other tangents.

(ii) In Fig. 307, \( AY_1 = AZ_1 = s \); \( BZ_1 = BX_1 = s - c \); \( CX_1 = CY_1 = s - b \);
\( AY_1 + AZ_1 = AB + BZ_1 + AC + CY_1 = AB + BX_1 + AC + CX_1 = 2s \);
\( \therefore \ AY_1 = AZ_1 = s \);
\( \therefore BZ_1 = AZ_1 - AB = s - c \), and similarly for \( CY_1 \).

Note. These results give alternative forms for \( r \), \( r_1 \).

Thus
\( r = BX \tan \frac{B}{2} = (s - b) \tan \frac{B}{2} \), etc.,
\( r_1 = BX_1 \cot \frac{B}{2} = (s - c) \cot \frac{B}{2} \), etc.

EXERCISE XII. c.

1. Find the radius of a circle if a chord 3-6 in. long subtends an angle of 119° at the circumference.

2. Find the radius of a circle if a chord 5-72 in. long subtends an angle of (i) 62°, (ii) 128° at the circumference.

3. Find \( R \) and \( r \) in a triangle whose sides are 6, 7, 8 inches.

4. Find the radius of each escribed circle of the triangle whose sides are 3, 4, 5 inches.

5. In \( \triangle ABC \), \( b = c = 6 \), \( A = 50^\circ \). Find \( R \) and \( r \).

6. In \( \triangle ABC \), \( a = 3 \), \( B = 57^\circ \), \( C = 42^\circ \). Find \( R \) and \( r \).

7. Given \( R = 14 \) cm., \( a = 12 \) cm. Find \( A \).

8. Three places \( A \), \( B \), \( C \) are each 4 miles distant from a place \( O \). If the angle \( \angle BOC = 71^\circ \), find the distance of \( A \) from \( C \).

9. The cross-section of a long prism is a triangle with sides 8, 9, 11 cm. long. What is the internal diameter of the smallest cylindrical pipe through which it can be passed?

10. \( ABCD \) is a quadrilateral; \( BA = 4 \) in., \( AD = 5 \) in., \( \angle BAD = 41^\circ \);
\( \angle ABC = \angle ADC = 90^\circ \). Find \( AC \).

11. In Fig. 306, find \( BX \) and \( CX \), (i) if \( a = 5 \), \( b = 6 \), \( c = 7 \); (ii) if \( a = 14 \) cm., \( b = 20 \) cm., \( c = 30 \) cm.

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12. In Fig. 307, find \( BX \) and \( AZ \), with the measurements of No. 11.

13. The radii of three circles, centres \( P \), \( Q \), \( R \), which touch each other externally, are \( a \), \( b \), \( c \) inches. Find expressions for (i) the area, (ii) the radius of the inscribed and escribed circles of the triangle \( PQR \).

14. A bracket consists of three rods forming a triangle of sides 6, 9, 11 inches, fixed in a horizontal plane, a sphere of diameter 8 inches rests on the bracket. Find the height of the highest point of the sphere above the bracket.

15. A line is drawn through the vertex \( A \) of a triangle \( ABC \) to meet \( BC \) at \( D \). Show that the ratio of the radii of the circumscribed circles of \( \triangle ABD \), \( \triangle ACD \) is \( \frac{b}{c} \).

Prove that the radius of a circle through \( A \) and touching \( BC \) at \( C \) is \( \frac{1}{2} \sin \frac{C}{2} \).

16. Prove that for an equilateral triangle, \( r_1 = 3r \).

17. Prove that \( rr_1 = \Delta \tan \frac{C}{2} \).

18. Prove that \( \Delta = r_1 r_2 r_3 \).

19. Prove that \( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1 r_2 r_3} \).

20. Prove that \( \frac{1}{r_1} = \frac{1}{r_2} = \frac{1}{r_3} \).

21. Prove that \( 2r \sin A = 2 \sin B \sin C = \Delta \).

22. Prove that \( r_1 + r_2 + r_3 = \Delta \).

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R. 27.

1. The continuous line in Fig. 308 shows a section of some corrugated iron. The curve is formed of equal arcs of circles and the centre \( O \) of the first arc is 4 inches below the straight line \( AB \). Find the length of the curved line shown in the figure. What would be the full width of the corrugated iron covering a roof 10 ft. wide, if it were beaten out flat?
2. An ink-bottle is in the form of a cylinder with a large conical opening. When it is filled level with the bottom of the opening, it can just be turned upside down without any ink spilling. Prove that the depth of the cone is \( \frac{1}{3} \) depth of the whole bottle. (See Fig. 308.)

3. Find the other sides of a triangle in which \( a = 14.7 \) cm, \( \angle A = 72^\circ 30', \angle B = 7^\circ 42' \).

![Fig. 309.](image)

4. In a range-finder, mirrors 30 in. apart are focussed on the object whose range is being taken. At what angle will the mirrors be inclined to each other when the range of an object 600 yds. away is taken?

5. Find the area of the quadrilateral ABCD in Fig. 310.

![Fig. 310.](image)

R. 28.

1. AB and CD are two diameters at right angles of a circle, radius 10 cm.; areas are drawn as in Fig. 311, with A and B as their centres. Calculate the perimeter and area of the shaded portion.

![Fig. 311.](image)

2. A tower at a distance of 1 mile subtends an angle of 30' 36''. Find its height approximately.

3. Show that, if \( \theta \) is measured in radians and is small, \( \frac{\sin \theta}{\theta} \approx \frac{1}{\cos \theta} \) is approximately equal to \( \theta \). Test this result when \( \theta = 0.1 \), and show they agree to three figures.

R. 29.

1. Apply the approximate rule, that the area of a small segment of a circle is \( \frac{1}{2} \) base x height, to the minor segment cut off by a chord equal to the radius of the circle. Show that in this instance the rule is equivalent to taking \( \pi \) equal to \( 4 - \frac{\sqrt{3}}{2} \), and find the error per cent. to one significant figure.

2. A base-line \( AB \) is 40 chains long. A point P is observed in the same horizontal plane, and it is found that \( \angle PAB = 76^\circ 48' \), \( \angle ABP = 58^\circ 32' \).

Find the distances of P from A and from B.

3. A, B, C are three bullet-marks on a target. \( AB = 1 \) in, \( BC = 0.8 \) in, \( CA = 0.7 \) in. Find the diameter of the 'group' that they form, i.e., the diameter of the smallest circle into which they just fit.

4. Find the area of a triangle ABC in which \( a = 47.56 \) in., \( b = 20.78 \) in., \( c = 68^\circ 45' \).

5. A right pyramid, height 6', stands on a square base, side 5'. Find the total surface area of the pyramid and the inclination of the edges to the plane of the base.

R. 30.

1. A box with a square section ABCD, side 2', is rolled along the ground, turning in succession about the corners A, B, C, D. Sketch the path followed by the corner D until the side AD is again on the ground, and calculate its length. (See Fig. 313.)

![Fig. 313.](image)

2. An arc of a circle is 2 ft. long and subtends an angle of 25° 30' at the centre of the circle. Find the radius of the circle and the length of the chord of the arc.
3. A man is at a place in Lat. 51° N. and motors 100 miles due North. What latitude is he then in?

4. An octagonal tower whose sides are 8 ft. ends in a pyramid whose faces slope at 70° to the vertical. Find the height and volume of the pyramid.

5. A cone is such that its curved surface when laid out flat makes an exact semicircle of radius $a$. What is the length of the shortest distance across the curved surface of the cone between two points at opposite ends of a diameter of the base of the cone?

R. 31.

1. In Fig. 314, $AB = BC = CD = 2$ in.; semicircles are drawn as shown. Prove that the shaded area is one-third of the area of the whole circle.

2. The gradient of a railway changes from 1 in 40 to 1 in 45. Find approximately the change in the angle of slope.

3. Find the four parallels of latitude, two N. of the Equator and two S. of it, which divide the Earth's surface into 5 zones of equal area.

4. Fig. 315 is cut out in cardboard and the four congruent triangles are folded about the sides of the square to form a pyramid. Find the total surface area and the volume of the pyramid so formed.

5. In Fig. 316, $O$ is the centre of a circle, radius 3 cm., and $OC = 6$ cm. Calculate the radius of a circle which touches this circle at $A$ and passes through $C$.

R. 32.

1. In Fig. 317, $AB$ is a diameter; $AP = 8$ cm., $BQ = 4$ cm. Calculate $\angle ARQ$.

2. With the data of No. 1, calculate the length of the arc $PB$ and the area bounded by the arc $PB$ and the straight lines $AP$, $AB$.

3. In the triangle $ABC$, $AC = 25$, $BC = 20$, $\angle BAC = 40^\circ$. Calculate $\angle ACB$.

4. The discrepancy between observation and theory (Newtonian) of the rotation of the axis of Mercury's orbit is about 42 seconds of angle per century. At what distance does a halfpenny (diameter 1$\frac{1}{2}$ in.) subtend this angle?

5. Fig. 318 represents a prism on a square horizontal base $abcd$ with vertical edges $aa'$, $bb'$, $cc'$, $dd'$; the prism is cut by a plane in a section $ABCD$, which is therefore a parallelogram. Calculate $\angle ABC$, and find the area of $ABCD$. Hence deduce the angle between the planes $ABCD$ and $abcd$.

R. 33.

1. A conical funnel, vertex $O$, vertical angle $50^\circ$, is suspended from a point $A$ on the rim of the base; $G$ is a point on the axis $OC$ of the cone, such that $OG = \frac{2}{3} OC$. (See Fig. 319.)

   If the funnel rests with $G$ vertically below $A$, find the angle which $OC$ makes with the vertical.

2. If a regular pentagon and a regular decagon, sides $a$ cm., $d$ cm., respectively, are inscribed in a circle of radius $r$ cm., then $r^2 = a^2 + \frac{1}{2} d^2$. Use Tables to verify this result.

3. The ends $B$, $C$ of two rods $AB$, $AC$ are joined by a stretched elastic string $AB = 2$ ft., $AC = 3$ ft. Initially $BC$ is 1 ft. 6 in.; through what further angle must the rods be opened to cause an additional extension of 6 inches in the string?
4. What is the height of a light-house above sea-level, if its flash is visible at a distance of 12 miles?

5. In $\triangle ABC$, $\angle ABC = 33^\circ$, $\angle ACB = 65^\circ$, $BC = 5$ cm. Calculate the radius of the circle, inscribed to $BC$.

R. 34.

1. A rectangular block rests, with one edge through $A$ on the ground, across a cylinder of diameter 10 cm. also on the ground.

Find the height of $C$ above the ground and the distance of $C$ from the vertical plane through the axis of the cylinder.

2. With the data of No. 1, calculate the length of the minor arc $EF$.

3. $AB$ is a diameter and $AC$ is a chord of a circle; $E$ is the mid-point of $AC$; $AB = 3$ in., $AC = 2$ in. Calculate $\angle ABE$.

4. In Fig. 321, $AE$ is perpendicular to $BC$; $AE = h$, $\angle ABC = 90^\circ$. Express the radius of the circle, in terms of $h$, $\theta$.

5. A pyramid has a square base with 4 equal isosceles triangles for faces. Prove that if each of these faces makes an angle of $45^\circ$ with the base, the cosine of the vertical angle of each isosceles triangle will be $\frac{1}{2}$, and that the angle between a pair of triangular faces will be $120^\circ$.

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CHAPTER XIII.

IDENTITIES INVOLVING SINGLE ANGLES.

The following relations between the ratios of a single angle $A$ have been deduced in Part I. from the definitions and Pythagoras' theorem, for acute angles. The extended definitions in Chapter VII. show that they hold for angles of any magnitude.

$$\tan A = \frac{\sin A}{\cos A}; \quad \cot A = \frac{\cos A}{\sin A} \quad (1)$$

$$\sin^2 A + \cos^2 A = 1 \quad (2)$$

$$\sec^2 A = 1 + \tan^2 A \quad (3)$$

$$\csc^2 A = 1 + \cot^2 A \quad (4)$$

By means of these formulae, expressions involving the ratios of a single angle can often be simplified or transformed.

Example I. Express $\sin A \cos A$ in terms of $\tan A$ only.

$$\sin A \cos A = \frac{\sin A \cos A}{\sin^2 A + \cos^2 A} \quad (5)$$

Divide numerator and denominator by $\cos^2 A$;

$$\csc A \cos A = \frac{\sin^2 A}{\sin^2 A + \cos^2 A} = \frac{\tan A}{\tan^2 A + 1} \quad (6)$$

Note. We have used the formula (2) above to make the numerator and denominator homogeneous in $\sin A$ and $\cos A$.

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Example II. Prove that

\[
\frac{1}{\tan A + \cot A} = \frac{\sin A \cos A}{\sin A + \cos A}.
\]

Left side = \[
\frac{1}{\tan A + \cot A} = \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}} = \frac{1}{\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}} = \frac{\sin A \cos A}{\sin^2 A + \cos^2 A}
\]

= \frac{\sin A \cos A}{1} = \sin A \cos A.

Right side = \[
\frac{1}{\sin A + \cos A} = \frac{1}{\frac{\sin A}{\sin A} + \frac{\cos A}{\cos A}} = \frac{1}{\frac{\sin^2 A + \cos^2 A}{\sin A \cos A}} = \frac{\sin A \cos A}{\sin^2 A + \cos^2 A}
\]

= \frac{\sin A \cos A}{\sin A \cos A} = \sin A \cos A.

Left side = Right side.

Note. (i) It may happen that the easiest way of proving two expressions equal is to simplify each separately, as above. Never start by saying they are equal and reducing both sides together. The sign = means "is equal to"; it does not mean "is to be proved equal to."

(ii) It is usually best to write both expressions in terms of sines and cosines; sometimes, however, it is possible to express them in terms of tangents only.

EXERCISE XIII.

Write down simple expressions for the following, Nos. 1-15:

1. \(1 - \sin^2 \theta\).
2. \(1 - \cos^2 \theta\).
3. \(\sec^2 \theta - 1\).
4. \(\frac{1}{\sin^2 \theta} - 1\).
5. \(1 - \sin^2 \theta - \cos^2 \theta\).
6. \(1 + \frac{\sin^2 \theta}{\cos^2 \theta}\).
7. \(1 + \frac{\cos^2 \theta}{\sin^2 \theta}\).
8. \(\frac{1}{\cos^2 \theta} - \tan^2 \theta\).
9. \(\cot^2 \theta - 1\).
10. \(\tan \theta \cdot \cos \theta\).
11. \(5 \sin^2 \theta + 4 \cos^2 \theta\).
12. \(\sin^2 \theta - 1\).
13. \(\frac{\sec^2 \theta - 1}{\tan^2 \theta}\).
14. \(\sin^2 \theta + \sin^2 \theta \cos^2 \theta\).
15. \(1 - \cot^2 \theta + \cosec^2 \theta\).

Prove the following identities:

31. \(\cot \theta = 1 - 2 \sin^2 \theta\).
32. \(\sin \theta + \cos \theta = 1 + 2 \sin \theta \cos \theta\).
33. \(\cosec^2 \theta - \cot^2 \theta = 1\).
34. \(\cot^2 \theta + \cosec^2 \theta = \tan^2 \theta + \sec^2 \theta\).
35. \(\tan \theta + \cot \theta = \cosec \theta + \sec \theta\).
36. \(\sin \theta \cdot \cos \theta = \tan \theta \cdot \cot \theta\).
37. \(\tan \theta = -1 - 2 \cos^2 \theta\).
38. \(\sin \theta = 1 - \cos \theta\).
39. \(\cosec \theta = 1 - \cot \theta\).
40. \(\cot \theta = (\sec \theta + \cos \theta)\).
41. \(\sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A\).
42. \(\cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \sin^2 A - \sin^2 B\).
43. \(3 \cos A + 4 \sin A)^2 + (4 \cos A - 3 \sin A)^2 = 25\).
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44. \( \cot \theta = \frac{\cos \theta}{1 + \sin \theta} = 2 \tan \theta + \frac{\cos \theta}{\cos A - \cos A} \). 45. \( \sec A - \cos A \) etc.

46. \( (1 - \cos A)/(1 + \sec A) = \sin A \). tan A.

47. \( 1 - \sin \theta = 1 - \cos \theta = \cot^2 \theta \). 48. \( \sec \theta - \tan \theta = \cot \theta - \cos \theta \)

49. \( 1 + \cos \theta = 2 \sin \theta \). \( 1 + \sin \theta = \cot \theta + \cos \theta \).

50. \( \cos^2 A + \sin^2 A = 2 \sin A \cos A \).

51. \( (1 + \sec^3 \theta) + (1 + \tan^2 \theta) \sin^2 \theta = (1 + \sec^2 \theta)(1 + \tan^2 \theta)(1 + \sin^2 \theta) \).

52. \( \tan \theta - \cot \theta = 2 \cos \theta \).

53. If \( 2 \tan^2 \theta + 3 \sec^2 \theta = 8 \), find \( \sin \theta \).

54. If \( x = \sec \theta - \tan \theta \), prove that \( x^2 + 1 = 2 \sec \theta \), and express \( x \) in terms of \( \theta \).

55. If \( 16 \cos^2 \theta - 3 \sin^2 \theta = 7 \), find \( \tan \theta \).

56. If \( a \cos \theta + b \cos \theta = a \), express \( \cos \theta \) in terms of \( a, b \).

57. Find an equation connecting \( x, y \), independent of \( \theta \), if \( (i) x = 3 \cos \theta, y = 2 \sin \theta \); \( (ii) x = 4 \sin \theta, y = 5 \tan \theta \).

58. If \( x = r \cos \theta \), \( y = r \sin \theta \), \( z = r \cos \theta \), \( \phi = \cos \theta \), prove that \( x^2 + y^2 + z^2 = r^2 \).

59. Eliminate \( \theta \) from the equations:

\[
\begin{align*}
\frac{a \sin \theta + b \cos \theta = p;}{a \cos \theta - b \sin \theta = q.}
\end{align*}
\]

60. If \( (\sec A + 1)(\sec B + 1)(\sec C + 1) \) equals \( (\sec A + 1)(\sec B - 1)(\sec C - 1) \), and if all the angles are acute, prove that each expression equals \( \tan A \tan B \tan C \).

CHAPTER XIV.

THE GENERAL ANGLE.

Negative angles. The trigonometrical ratios of an angle \( \theta^\circ \) have been defined, on p. 99, for values of \( \theta \) from 0 to 360.

These definitions apply without any modification to angles of any magnitude, positive or negative.

From a fixed point \( O \) draw a line \( OX \) in a fixed sense, and regard this as defining the standard direction.

Suppose now a line rotates about \( O \) from the position \( OA \) along \( OX \) into a new position \( OP \) or \( OQ \); then the new position depends not only on the amount of rotation, but its direction.

If the direction of rotation is clockwise (Fig. 322), then we say that the angle \( XOQ \), as described, is positive; if the direction of rotation is anticlockwise (Fig. 323), then we say that the angle \( XOQ \), as described, is negative. Thus in Fig. 322, \( \angle XOQ = + \theta^\circ \), and in Fig. 323, \( \angle XOQ = - \phi^\circ \), \( \theta \) and \( \phi \) being positive numbers.

It should be noted that this is a conventional distinction between positive and negative rotations, but we are already
bound to choose the anti-clockwise direction as the positive
direction, for in previous work the OY axis has been chosen
90° ahead of the OX axis in this direction.

The amount of rotation need not be less than 1 revolution,
e.g., in Fig. 322, the rotating line OA could reach the position
OP after turning through θ° or (360° + θ°) or (720° + θ°), etc.,
or through (−360° + θ°) or (−720° + θ°), etc.

Therefore, by using the coordinate-definition of the ratios
given in Chapter VII., we see that:

\[
\sin θ° = \sin (360° + θ°) = \sin (720° + θ°) = ... \\
\sin (−360° + θ°) = \sin (−720° + θ°) = ...
\]

and similarly for the other ratios.

In fact the addition or subtraction of any multiple of 360°
does not alter the value of any ratio of an angle. Other
properties of the angle are, of course, altered.

If the angle is measured in radians and is \( \alpha \), this statement
takes the following form:

\[
\sin \alpha = \sin (\alpha + 2n\pi) ; \quad \cos \alpha = \cos (\alpha + 2n\pi) ; \quad \tan \alpha = \tan (\alpha + 2n\pi),
\]

where \( n \) is any integer, positive or negative.

Ratios of \( (−\phi°) \).

Figures 324 and 325 represent the positions of the bounding
line OQ, where \( \angle XOQ = (−\phi°) \). Draw QN perpendicular to OX.
For \( \angle MOP \) is complementary to \( \angle QON \), and therefore equal to \( \angle NQO \).

Hence in the \( \triangle MOP, \triangle QNO \),

\[
\begin{align*}
\angle M &= \angle N, \text{ right angles}, \\
\angle MOP &= \angle NQO, \text{ proved above.}
\end{align*}
\]

\( \therefore \) the \( \triangle s \) are congruent, and the lengths of \( OM \) and \( MP \) are equal to the lengths of \( QN \) and \( NO \) respectively.

\( \therefore \) the \( y \)-coordinate of \( Q \) is the \( x \)-coordinate of \( P \), in each case.

[In the figures drawn they are both negative],

and the \( x \)-coordinate of \( Q = -x \)-coordinate of \( P \), in each case.

[In Fig. 326 \( x \)-coordinate of \( Q \) is negative and \( y \)-coordinate of \( P \) is positive; in Fig. 327 \( x \)-coordinate of \( Q \) is positive and \( y \)-coordinate of \( P \) is negative].

\( \therefore \sin (90^\circ + \theta) = \cos \theta \) and \( \cos (90^\circ + \theta) = -\sin \theta \);

\( \therefore \tan (90^\circ + \theta) = -\frac{\cos \theta}{\sin \theta} = -\cot \theta \).

\textbf{Note.} The reader should draw for himself figures showing the two other cases possible, that is when the bounding line OP lies in the 1st quadrant or in the 4th quadrant. He should see that the above argument still applies and that the result is therefore true for all values of \( \theta \), since it is true for all possible positions of the line OP.

\textbf{Ratios of (90° - \( \theta \)).}

It has been proved (see pp. 24 and 40) that if \( \theta \) is acute,

\( \sin (90^\circ - \theta) = \cos \theta \); \( \cos (90^\circ - \theta) = \sin \theta \);

\( \tan (90^\circ - \theta) = \cot \theta \).

These results are true for all values of \( \theta \), and may be proved by a similar method to that used above for ratios of \( (90^\circ + \theta) \).

In figures 328, 329, \( \angle XOP = (90^\circ - \theta) \), \( \angle POQ = 90^\circ \), so that \( \angle XOQ = (90^\circ + 90^\circ) \).

Fig. 328 illustrates the case when \( 90^\circ < \theta < 180^\circ \), and Fig. 329, when \( 270^\circ < \theta < 360^\circ \). The reader should draw a

\( 180^\circ < \theta < 270^\circ \), and supply the necessary proof. Since the proof can be applied for all possible positions of OP, it is valid for all values of \( \theta \).

\textbf{Ratios of 270° ± \( \theta \).}

Using the results on p. 102 and the results of this chapter, we have

\( \sin (270^\circ - \theta) = \sin [180^\circ + (90^\circ - \theta)] \)

\( = -\sin (90^\circ - \theta) = -\cos \theta \),

\( \cos (270^\circ - \theta) = \cos [180^\circ + (90^\circ - \theta)] \)

\( = -\cos (90^\circ - \theta) = -\sin \theta \),

\( \sin (270^\circ + \theta) = \sin [360^\circ - (90^\circ - \theta)] \)

\( = -\sin (90^\circ + \theta) = -\cos \theta \),

\( \cos (270^\circ + \theta) = \cos [360^\circ - (90^\circ - \theta)] \)

\( = -\cos (90^\circ - \theta) = \sin \theta \);

\( \therefore \tan (270^\circ - \theta) = -\frac{\cos \theta}{\sin \theta} = \cot \theta \)

and \( \tan (270^\circ + \theta) = -\frac{\cos \theta}{\sin \theta} = -\cot \theta \).

\textbf{Note.} These results may also be obtained from first principles by drawing a diagram and using a method similar to that used above for ratios of \( (90^\circ + \theta) \).
Summary of results.

It is unnecessary to memorise every result obtained above separately, because any relation which is required can be written down by using the following guiding principles.

First determine the appropriate ratio, regardless of sign, in accordance with rule (i) or (ii) below, whichever applies; then determine the sign in accordance with rule (iii).

(i) The ratio of any angle, \( \theta \), is numerically equal to the same ratio of any angle whose sum with \( \theta \) or difference from \( \theta \) is an even multiple of 90°, i.e. a multiple of 180°.

(ii) The ratio of any angle, \( \theta \), is numerically equal to the co-ratio of any angle whose sum with \( \theta \) or difference from \( \theta \) is an odd multiple of 90°.

(iii) The sign to be attached to the transformed ratio is most easily obtained by drawing a rough diagram, and noting in which quadrant the original angle lies, on the assumption that \( \theta \) is acute, and then using the method of p. 102.

Radian notation. If \( \alpha \) is any angle, rules (i) and (ii) above can be stated more concisely.

(iv) Any ratio of \( 2\pi \pm \alpha \) is numerically equal to the same ratio of \( \alpha \).

(v) Any ratio of \( \frac{2n+1}{2} \pi \pm \alpha \) is numerically equal to the co-ratio of \( \alpha \).

Here \( n \) may be any integer positive or negative. The sign to be attached must be settled in accordance with rule (iii), as before.

Example 1. Express (i) \( \sec (270° - \theta) \), (ii) \( \cosec (360° - \theta) \), (iii) \( \tan (450° + \theta) \) in terms of ratios of \( \theta \).

(i) \( 270° - \theta \) is an angle whose sum with \( \theta \) is an odd multiple of 90°.

\( \therefore \) ratios of \( (270° - \theta) \) are numerically equal to co-ratios of \( \theta \).

\[ \therefore \sec (270° - \theta) = - \cosec \theta \]

(ii) Similarly, ratios of \( (360° - \theta) \) are numerically equal to ratios of \( \theta \).

Fig. 331 shows that, if \( \theta \) were acute, \( 360° - \theta \) would be in the 4th quadrant, where a cosecant is negative.

\[ \cosec (360° - \theta) = - \cosec \theta \]

(iii) Ratios of \( (450° + \theta) \) are numerically equal to co-ratios of \( \theta \).

Fig. 332 shows that, if \( \theta \) were acute, \( 450° + \theta \) would be in the 2nd quadrant, where a tangent is negative.

\[ \tan (450° + \theta) = - \cot \theta \]

EXERCISE XIV.

1. In what quadrants do the following angles lie? State the sign of the sine and cosine of each.

(a) \(-100°\); (b) \(-200°\); (c) \(400°\); (d) \(-480°\);

(e) \(-1°\); (f) \((2\pi + 1)°\); (g) \(-6°\); (h) \(-4°\).

2. Find from the Tables the sine, cosine and tangent of the following angles:

(a) \(-20°\); (b) \(-110°\); (c) \(410°\);

(d) \(90°\); (e) \(-210°\); (f) \((1 - 2\pi)°\).

3. Express the following in terms of ratios of \( \theta \):

(a) \( \cot (90° + \theta) \); (b) \( \cosec (90° + \theta) \); (c) \( \sec (270° - \theta) \);

(d) \( \cosec (270° + \theta) \); (e) \( \cot (270° - \theta) \); (f) \( \sin (450° + \theta) \);

(g) \( \cos (630° - \theta) \); (h) \( \tan (630° + \theta) \).
4. Express the following in terms of ratios of \( \theta \):
(a) \( \tan \left( \frac{\pi}{2} + \theta \right) \); (b) \( \sec \left( \frac{3\pi}{2} + \theta \right) \); (c) \( \cos \left( \frac{3\pi}{2} - \theta \right) \);
(d) \( \csc \left( \frac{3\pi}{2} - \theta \right) \); (e) \( \sin \left( \frac{3\pi}{2} - \theta \right) \); (f) \( \sin \left( \frac{3\pi}{2} + \theta \right) \);
(g) \( \tan \left( \frac{\pi}{2} - \theta \right) \); (h) \( \cos \left( \frac{\pi}{2} + \theta \right) \).

5. Draw rough sketches, and show that
(a) \( \sin 170^\circ = \cos 280^\circ \); (b) \( \tan 290^\circ = \cot 160^\circ \);
(c) \( \sin 110^\circ = \cos (-20)^\circ \); (d) \( \cot 190^\circ = \tan (-100)^\circ \).

What can you say about \( \theta \) in Nos. 6-9:
6. (i) \( \sin \theta = \sin 70^\circ \); (ii) \( \sin \theta = \sin 200^\circ \);
(iii) \( \sin \theta = - \sin 40^\circ \); (iv) \( \sin (20^\circ) = \sin 80^\circ \).
7. (i) \( \cos \theta = \cos 140^\circ \); (ii) \( \cos \theta = \cos 160^\circ \);
(iii) \( \cos \theta = - \cos 40^\circ \); (iv) \( \cos (30^\circ) = \cos 60^\circ \).
8. (i) \( \tan \theta = \tan 20^\circ \); (ii) \( \tan \theta = \tan 220^\circ \);
(iii) \( \tan \theta = - \tan 10^\circ \); (iv) \( \tan (150^\circ) = \tan 50^\circ \).
9. (i) \( \sin \theta = \cos 70^\circ \); (ii) \( \sin \theta = \cos 160^\circ \);
(iii) \( \sin \theta = - \cos 40^\circ \); (iv) \( \cos \theta = \sin 200^\circ \).

10. Draw figures as on p. 188 to illustrate the fact that
\( \sin (-\phi) = - \sin \phi \) in the following cases:
(i) \( 90^\circ < \phi < 180^\circ \); (ii) \( 270^\circ < \phi < 360^\circ \).

11. Draw figures as on p. 199 to illustrate the fact that
\( \sin (90^\circ + \theta) = \cos \theta \) in the following cases:
(i) \( 0^\circ < \theta < 90^\circ \); (ii) \( 270^\circ < \theta < 360^\circ \).

12. Draw a figure as on p. 201 to illustrate the position of the line OP when \( 270^\circ < \theta < 360^\circ \), and prove in full that in this case \( \sin (90^\circ - \theta) = \cos \theta \).

13. Draw a figure, and prove from first principles that
\( \sin (270^\circ - \theta) = - \cos \theta \) when \( 0^\circ < \theta < 90^\circ \).

14. Draw a figure, and prove from first principles that
\( \cos (270^\circ + \theta) = \sin \theta \) when \( 90^\circ < \theta < 180^\circ \).

15. An arm OA (see Fig. 330) rotates about O in a vertical plane in an anti-clockwise direction at a speed of 30° per second. Find an expression for the distance of A to the right of PQ after \( t \) seconds;

\[ P \quad 90^\circ \quad 120^\circ \quad A \]

Fig. 330.

Evaluate the result when \( t = 3, 6, 9, 12 \), and explain the results. Will these results be affected by the direction of rotation? What fact about \( \cos (-\theta) \) does this illustrate?

16. With the data of No. 15, find an expression for the height of A above O after \( t \) seconds, and evaluate the result when \( t = 3, 6, 9, 12 \). How will these results be affected if the direction of rotation is changed? What fact about \( \sin (-\theta) \) does this illustrate?

17. A wheel, centre O, radius 2 ft. (see Fig. 334), rolls along a level straight line XY. If a spoke OA is initially vertical as shown, what will be the height of A above XY when the wheel has turned through an angle \( \theta \)? Does this result depend upon the direction in which the wheel is rolled?

18. With the data of No. 17, what will be the distance of A to the right of its starting point, if the wheel rolls through an angle \( \theta \) in a clockwise direction? What will be the result of substituting \( -\theta \) for \( \theta \) in this answer and what will it signify?

19. If \( \sin \theta = a \), what is the value of
(i) \( \cos \left( \frac{3\pi}{2} - \theta \right) \); (ii) \( \sin (-\theta) \); (iii) \( \sec \left( \frac{3\pi}{2} + \theta \right) \);
(iv) \( \sin \left( \frac{3\pi}{2} - \theta \right) \); (v) \( \cos (-\theta) \);
(vi) \( \csc \left( \frac{3\pi}{2} + \theta \right) \); (vii) \( \csc \left( \frac{\pi}{2} + \theta \right) \).
20. If \( n \) is any integer, express the following as ratios of \( \theta \):

(i) \( \sin (2n\pi + \theta) \);  
(ii) \( \sin (2n - 1\pi + \theta) \);  
(iii) \( \tan (n\pi + \theta) \);

(iv) \( \cos \left( \frac{4n + 1}{2} \pi + \theta \right) \);  
(v) \( \cos \left( \frac{4n + 3}{2} \pi + \theta \right) \);

(vi) \( \sin \left( \frac{4n + 1}{2} \pi - \theta \right) \);  
(vii) \( \sin \left( \frac{4n + 3}{2} \pi - \theta \right) \).

21. If \( n \) is any integer, find simple forms for

(i) \( \sin \left( \frac{2n + 1}{2} \pi + \alpha \right) \);  
(ii) \( \cos \left( \frac{2n + 1}{2} \pi - \alpha \right) \);

(iii) \( \tan \left( \frac{2n + 1}{2} \pi + \alpha \right) \).

22. If \( n \) is any integer, prove that

\[ \sin \left( \frac{n + \frac{1}{2}}{2} \pi + \alpha \right) = (-1)^n \cos \alpha. \]

Find similar simple expressions for

(i) \( \sin \left( \frac{n + \frac{1}{2}}{2} \pi - \alpha \right) \);  
(ii) \( \cos \left( \frac{n + \frac{1}{2}}{2} \pi + \alpha \right) \);

(iii) \( \cos \left( \frac{n + \frac{1}{2}}{2} \pi - \alpha \right) \);  
(iv) \( \tan \left( \frac{n + \frac{1}{2}}{2} \pi + \alpha \right) \).

23. The length of a vibrating spring \( t \) seconds after it is set in motion is \( (10 + 2\cos (\pi t)) \) in. (i) Show that its length after 1, 3, 5 second is the same. (ii) Show that its length after \( x, (2 + x), (4 + x), \ldots, (2n + x) \) seconds is the same for all values of \( x \) and integral values of \( n \). (iii) What is its greatest length and its smallest length when it is vibrating?

24. A pendulum is set swinging, and \( t \) seconds after it is set in motion its inclination to the vertical is \( 3 \sin (720^\circ) \) degrees. (i) Show that its inclination after \( \frac{t}{2}, \frac{3}{2}, \frac{5}{2} \) seconds is the same. (ii) Show that its inclination after \( x, (x + \frac{1}{2}), (x + 1), \ldots, (x + \frac{n}{2}) \) seconds is the same for all values of \( x \) and integral values of \( n \). (iii) What is its greatest inclination to the vertical? (iv) What is the time of one complete swing backwards and forwards?

25. A particle moves in a plane so that its coordinates \( (x, y) \) after \( t \) seconds are given by \( x = 3\sin(5t^\circ), y = 2\cos(5t^\circ) \). Show that its path is a closed orbit, and find the time it takes to describe its orbit.

CHAPTER XV.

COMPOUND ANGLES.

We shall base the general proof of the addition theorems on the idea of coordinates; it is therefore necessary to obtain an expression for the distance between two points in terms of their coordinates. Those who prefer at a first reading to use the more elementary proofs (valid only for restricted values of \( A \) and \( B \)) should omit pp. 207-211. The treatment by the method of projections is reserved for Part IV, on account of its intrinsic difficulty.

Distance between two points.

Let the coordinates of two points \( P, Q \) be \( (X, Y) \) and \( (x, y) \).

To prove \( PQ^2 = (X - x)^2 + (Y - y)^2 \).

Draw \( PM, QM \) perpendicular to \( OX \), and draw \( QR \) perpendicular to \( PM \).
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Then
\[ QR = NM = OM - ON = X - x \]
and
\[ RP = MP - MR = MP - NQ = Y - y \]
\[ \therefore PQ^2 = QR^2 + RP^2 = (X - x)^2 + (Y - y)^2. \]

Note. This formula is true for all positions of P and Q.

Example I. Find PQ if P is (-8, 7) and Q is (-3, -5).

By the formula, 
\[ PQ^2 = [-8 - (-3)]^2 + [7 - (-5)]^2 \]
\[ = (-8 + 3)^2 + (7 + 5)^2 = 25 + 144 \]
\[ = 169; \]
\[ \therefore PQ = 13. \]

Note. In Fig. 336, RQ = 5 and RP = 12.

EXERCISE XV. a.

[Draw free-hand sketches to represent the data.]

1. Find PQ if P is (6, 4) and Q is (2, 1).

2. Find the distance between the following pairs of points:
   (i) (2, 3) and (5, 7);
   (ii) (-1, -4) and (5, 6);
   (iii) (4, -1) and (-1, 11);
   (iv) (-2, 12) and (5, -12).

3. Show that the following points lie on a circle, centre the origin:
   (i) (3, 4);
   (ii) (-4, 3);
   (iii) (-3, 4);
   (iv) (-4, -3);
   (v) (5, 0).

4. Show that the points
   (-9, -7); (-3, -5); (3, 1); (-3, -1)
are the corners of a parallelogram.

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5. Find the distance from the origin of
   (i) \((a \cos \theta, a \sin \theta)\);
   (ii) \((-a \cos \theta, a \sin \theta)\).

6. Find the distance between the points \((a \cos A, a \sin A)\) and
   \((a \cos B, a \sin B)\), and simplify the result.

7. Show that the distance between \((h, k)\) and
   \((h + r \cos \theta, k + r \cos \theta)\)
is constant for all values of \(h, k, \theta, r\).

To prove \(\cos (A - B) = \cos A \cos B + \sin A \sin B\).

Take perpendicular coordinate-axes \(OX, OY\), and draw a circle, centre \(O\), of unit radius.

Taking \(OX\) as initial line, describe \(\angle XOP = A\) and \(\angle XOQ = B\),
\(P\) and \(Q\) being points on the circle. Join \(PQ\).

Then in \(\triangle POQ\), \(\angle QOP\) may be either \((A - B)\), as in Fig. 337,
or \(360^\circ - (A - B)\), as in Fig. 338, or if \(A, B\) are negative or greater
than \(360^\circ\) it may be \(360^\circ \pm (A - B)\), where \(n\) is a positive or
negative integer or zero.

\[ \therefore \text{in every case, } \cos \angle QOP = \cos (A - B). \]

Since the circle is of unit radius, the coordinates of \(P\) and \(Q\)
for all values of \(A\) and \(B\) are, by definition, \((\cos A, \sin A)\) and
\((\cos B, \sin B)\).

Apply the cosine formula to \(\triangle POQ\).

\[ \therefore \cos (A - B) = \cos \angle QOP = \frac{OP^2 + OQ^2 - PQ^2}{2OP \cdot OQ}. \]

D.W.T. III.
Since this formula is true for negative angles, we may write -B for B; we then obtain
\[ \cos (A - (-B)) = \cos A \cos (-B) + \sin A \sin (-B) \]
\[ \therefore \cos (A + B) = \cos A \cos B + \sin A \sin B \] \hspace{1cm}(2)

Next write \((90° - A)\) for A in (1); we then obtain
\[ \cos [(90° - A) - B] = \cos (90° - A) \cos B + \sin (90° - A) \sin B \]
But \[ \cos [(90° - A) - B] = \cos [90° - A + B] = \sin [90° - (A + B)] = \sin (A + B) \]
\[ \therefore \sin (A + B) = \sin A \cos B + \cos A \sin B \] \hspace{1cm}(3)

Lastly, write \((90° + A)\) for A in (1); we then obtain
\[ \cos (90° + A - B) = \cos (90° + A) \cos B + \sin (90° + A) \sin B \]
\[ \therefore - \sin (A - B) = - \sin A \cos B + \cos A \sin B \]
\[ \therefore \sin (A - B) = \sin A \cos B - \cos A \sin B \] \hspace{1cm}(4)

Note. We have deduced (2), (3), (4), each from (1) direct, although it would have been shorter to deduce, say, (4) from (3) by writing -B for B in (3); because for examination purposes that is the simplest procedure. If the proof of (4) is required in an examination, (1) must be proved as above and then (4) deduced as shown. It would not be sufficient to quote (1). The formula for the distance between two points may, of course, be quoted, without proof.

Summary of Results.
\[ \sin (A + B) = \sin A \cos B + \cos A \sin B, \]
\[ \sin (A - B) = \sin A \cos B - \cos A \sin B, \]
\[ \cos (A + B) = \cos A \cos B - \sin A \sin B, \]
\[ \cos (A - B) = \cos A \cos B + \sin A \sin B. \]

These results must be committed to memory. The arrangement of signs in the expansions may be remembered by noting that for acute angles, the larger the angle, the larger the sine but the smaller the cosine. If then A, B and (A + B) are acute, \(\sin (A + B)\) is greater than \(\sin (A - B)\) but \(\cos (A + B)\) is less than \(\cos (A - B)\).
Alternative method.

The following proofs for the expansions of \( \sin (A \pm B) \) and \( \cos (A \pm B) \) are valid only for restricted values of \( A \) and \( B \), and should be omitted by those who have read the general proofs given above.

I. Obtain expansions for \( \sin (A + B) \) and \( \cos (A + B) \), given that \( A, B \) are acute and \( A + B < 90^\circ \).

\[
\begin{align*}
\text{Let } \angle EOP &= A, \angle POQ &= B, \text{ so that } \angle EQO &= A + B. \\
\text{Draw } QM, QN \text{ perpendicular to } OE, OP; \text{ draw } NH, NK \text{ perpendicular to } OE, QM.
\end{align*}
\]

Then \( \angle KQN = 90^\circ - \angle KNQ = \angle KNO = \angle NOH = A \).

\[
\begin{align*}
\sin (A + B) &= \frac{MQ}{OQ} = \frac{MK + KQ}{OQ} = \frac{HN}{OQ} + \frac{KQ}{OQ} = \sin A \cos B + \cos A \sin B.
\end{align*}
\]

But \( HN = ON \cdot \sin A \) and \( ON = QO \cdot \cos B \).

\[
\therefore \, HN = QO \cdot \cos B \cdot \sin A \quad \text{or} \quad \frac{HN}{OQ} = \sin A \cos B.
\]

Also \( KQ = QN \cdot \cos A \) and \( QN = QO \cdot \sin B \).

\[
\therefore \, KQ = QO \cdot \sin B \cdot \cos A \quad \text{or} \quad \frac{KQ}{OQ} = \cos A \sin B.
\]

\[
\therefore \, \sin (A + B) = \sin A \cos B + \cos A \sin B.
\]

Similarly, \( \cos (A + B) = \frac{OM}{OQ} = \frac{ON - MH}{OQ} = \frac{OH - NK}{OQ} = \frac{OH}{OQ} \cdot \cos A \).

But \( OH = ON \cdot \cos A = QO \cdot \cos B \cdot \cos A \).

\[
\therefore \, \frac{OH}{OQ} = \cos A \cdot \cos B.
\]

II. Obtain expansions for \( \sin (A - B) \) and \( \cos (A - B) \), given that \( A, B \) are acute and that \( A > B \).

\[
\begin{align*}
\text{Let } \angle EOP &= A, \angle POQ &= B, \text{ so that } \angle EQO &= A - B. \\
\text{Draw } QM, QN \text{ perpendicular to } OE, OP; \text{ draw } NH, NK \text{ perpendicular to } OE, QM.
\end{align*}
\]

Then \( \angle KQN = 90^\circ - \angle KNQ = \angle KNP = \angle EOP = A \).

\[
\begin{align*}
\sin (A - B) &= \frac{MQ}{OQ} = \frac{MK - KQ}{OQ} = \frac{HQ + QK}{OQ} = \frac{HN}{OQ} - \frac{KQ}{OQ} = \sin A \cos B - \cos A \sin B.
\end{align*}
\]

But \( HN = ON \cdot \sin A \) and \( ON = QO \cdot \cos B \).

\[
\therefore \, HN = QO \cdot \cos B \cdot \sin A \quad \text{or} \quad \frac{HN}{OQ} = \sin A \cos B.
\]

Also \( QK = QN \cdot \cos A \) and \( QN = QO \cdot \sin B \).

\[
\therefore \, QK = QO \cdot \sin B \cdot \cos A \quad \text{or} \quad \frac{QK}{OQ} = \cos A \sin B.
\]

\[
\therefore \, \sin (A - B) = \sin A \cos B - \cos A \sin B.
\]

Similarly, \( \cos (A - B) = \frac{OM}{OQ} = \frac{OH + HM}{OQ} = \frac{OH}{OQ} \cdot \cos A \).

But \( OH = ON \cdot \cos A = QO \cdot \cos B \cdot \cos A \).

\[
\therefore \, \frac{OH}{OQ} = \cos A \cdot \cos B.
\]
But \[ \text{OH} = \text{ON} \cos A = \text{OQ} \cos B \cdot \cos A; \]
\[ \therefore \text{OH} = \cos A \cos B. \]
\[ \therefore \text{OQ} = \sin A \sin B; \]
\[ \therefore \cos (A - B) = \cos A \cos B + \sin A \sin B. \]

Note. (i) The same construction is used in I. and II.; in each case we take a point Q on the arm bounding the compound angle and draw perpendiculars from it to the other lines through O. The method of proof is also the same in both cases.

(ii) This method can be adapted to other special cases, e.g. if A is obtuse, B acute, and \( A + B < 180^\circ \), etc.

**Exercise XV. b. (Oral.)**

Use the addition theorems to obtain alternative forms of the following:

1. \( \sin (\theta + \phi) \) 
2. \( \cos (\alpha - \beta) \) 
3. \( \cos (\theta + \phi) \) 
4. \( \sin (\gamma - \delta) \) 
5. \( \cos (2x + y) \) 
6. \( \sin (x - 3y) \) 
7. \( \sin (A + 2B) \) 
8. \( \cos (2B - 2C) \) 
9. \( \sin (\theta + 2\phi) \) 
10. \( \cos (\theta - 2\phi) \) 
11. \( \sin (\phi - 2\beta) \) 
12. \( \cos (\phi - 2\beta) \) 
13. \( \sin (\theta + \phi) \) or \( \sin (2\beta) \) 
14. \( \cos (\theta + \phi) \) or \( \cos (2\beta) \) 
15. \( \sin 2\beta \) or \( \tan 2\beta \) 
16. \( \sin \theta \cos \phi - \cos \theta \sin \phi \) 
17. \( \cos \theta \cos \phi + \sin \theta \sin \beta \) 
18. \( \sin \alpha \cos \gamma + \cos \alpha \sin \gamma \) 
19. \( \cos \alpha \cos \gamma - \sin \alpha \sin \gamma \) 
20. \( \sin 2A \cos B + \cos 2A \sin B \) 
21. \( \cos 2x \cos 2y + \sin 2x \cos 2y \) 
22. \( \sin \theta \cos \phi - \cos \theta \sin \phi \) 
23. \( \cos \gamma \sin 2x + \sin \gamma \cos 2x \) 
24. \( \cos 2\gamma \sin \delta - \sin 2\gamma \cos \delta \) 
25. \( \sin \theta \cos 40^\circ - \cos \theta \sin 40^\circ \) 
26. \( \phi \cos 70^\circ - \sin \phi \sin 70^\circ \).
Simplify the following, Nos. 1-6:
1. \( \sin (A + B) + \sin (A - B) \)
2. \( \cos (\theta + \phi) + \cos (\theta - \phi) \)
3. \( \cos (\alpha + \beta) - \cos (\alpha - \beta) \)
4. \( \sin (\gamma + \delta) - \sin (\gamma - \delta) \)
5. \( \sin (A + B) \cos (A + C) - \cos (A + B) \sin (A + C) \)
6. \( \cos (A + B) \cos (A + C) + \sin (A + B) \sin (A + C) \)

Prove the following, Nos. 7-21:
7. \( \tan A + \tan B = \frac{\sin (A + B)}{\cos A \cos B} \)
8. \( \cot A + \cot B = \frac{\sin (A + B)}{\sin A \sin B} \)
9. \( \cot \theta + \tan \phi = \frac{\cos (\theta - \phi)}{\sin \theta \sin \phi} \)
10. \( \cos (\theta + \phi) \sin \theta \sin \phi = \cos \theta \cot \phi - 1 \)
11. \( \sin \alpha + \cos \alpha, \sin \beta + \cos \beta = \sin (\alpha + \beta) + \cos (\alpha - \beta) \)
12. \( \sin \theta + \cos \theta, \cos \phi - \sin \phi = \sin (\theta - \phi) + \cos (\theta + \phi) \)
13. \( \sin \theta = \sin (\frac{\theta}{2}) \cos (\frac{\theta}{2}) = \frac{1}{2} \sin \theta \cos \theta \)
14. \( \cos \theta = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \)
15. \( \sin (\theta + 45^\circ) = \frac{1}{\sqrt{2}} \sin (\theta + \cos \theta) \). Find a similar form for \( \cos (\theta + 45^\circ) \).
16. \( \cos (A + B) = \cos A \cos B - \sin A \sin B \)
17. \( \tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \tan A + \tan B \)

What is \( \tan (A - B) \)?
18. \( \cot (A + B) = \frac{\cos (A + B) - \cot A \cot B - 1}{\sin (A + B) - \cot A + \cot B} \)

\( \cot (A - B) \)?
19. \( \sin (A + B + C) = \sin A \cos (B + C) + \cos A \sin (B + C) = \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B - \sin A \sin B \sin C \)
20. \( \cos (A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin C \sin A - \cos C \sin A \sin B \)
21. \( \sin (A - B) = \sin A \sin B = \sin (C + D) \sin (A - C) \sin B = 0 \)
22. Evaluate \( 15^\circ \) and \( 15^\circ \) in terms of \( \sqrt{2} \), \( \sqrt{3} \).
23. If \( \sin \theta = \frac{1}{2} \), and \( \phi = \frac{1}{2} \), and if \( \theta, \phi \) are acute, evaluate
   (i) \( \sin (\theta + \phi) \); (ii) \( \cos (\theta - \phi) \); (iii) \( \sin 2\theta \); (iv) \( \cos 2\phi \)
24. If \( \cos \theta = \frac{1}{2} \), and if \( \theta \) is acute, evaluate
   (i) \( \sin 2\theta \); (ii) \( \cos 2\theta \); (iii) \( \sin 2\phi \); (iv) \( \cos 2\phi \)
25. If \( A + B + C = 180^\circ \), prove that \( \sin A = \sin B \cos C + \cos B \sin C \).
26. If \( A + B + C = 90^\circ \), express \( \sin A \) in terms of ratios of \( B \) and \( C \).
27. What does the formula for \( \sin (A - B) \) become if \( 90^\circ - A \) is written for \( A \)?
28. Assuming the truth of the formula for \( \sin (A + B) \), deduce from it the formulae for \( \sin (A \pm B) \) by appropriate substitutions.
29. Use the facts in Fig. 343 that \( \Delta QPR = \Delta QPM + \Delta MPR \) and the area formula \( \frac{1}{2} \sin C \) for the area of a triangle to prove that \( \sin (A - B) = \sin A \cos B + \cos A \sin B \).

This method assumes that \( A + B < 180^\circ \).
\[ \tan (A + B) \text{ and } \tan (A - B). \]
\[ \tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}. \]

Divide numerator and denominator by \( \cos A \cos B \): \[ \sin A \cos B, \cos A \sin B \cos A \cos B, \cos A \cos B \sin A \sin B \cos A \cos B \]
\[ \therefore \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}. \tag{5} \]

By a similar method, or by writing \(-B\) for \(B\), we have \[ \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \tag{6} \]

\( \tan^{-1}x \pm \tan^{-1}y. \)

The addition theorem for tangents may be put into another form, as follows:

Let \( \tan A = x \) and \( \tan B = y \);
\[ \therefore A = \tan^{-1}x \text{ and } B = \tan^{-1}y. \]

Now, from (5), \( A + B = \tan^{-1} \left( \frac{\tan A + \tan B}{1 - \tan A \tan B} \right) \);
\[ \therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1} \left( \frac{x + y}{1 - xy} \right). \]

Similarly, \( \tan^{-1}x - \tan^{-1}y = \tan^{-1} \left( \frac{x - y}{1 + xy} \right) \).

**Double angles.**

(i) \( \sin 2A = \sin (A + A) = \sin A \cos A + \cos A \sin A \):
\[ \therefore \sin 2A = 2 \sin A \cos A. \]

(ii) \( \cos 2A = \cos (A + A) = \cos A \cos A - \sin A \sin A \):
\[ \therefore \cos 2A = \cos^2A - \sin^2A. \]

Since \( \cos^2A = 1 - \sin^2A \) and \( \sin^2A = 1 - \cos^2A \), we have also \( \cos 2A = 1 - 2 \sin^2A \).

\[ \cos 2A = \cos^2A - \sin^2A = 1 - 2 \sin^2A \]

and \( \cos 2A = \cos^2A - 1 + \cos^2A = 2 \cos^2A - 1; \)
\[ \therefore \cos 2A = \cos^2A - \sin^2A = 1 - 2 \sin^2A = 2 \cos^2A - 1. \]
It is important to notice that \( \sin \theta \) and \( \cos \theta \) can each be expressed rationally in terms of \( \tan \frac{\theta}{2} \).

Thus \[ \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}}. \]

Divide numerator and denominator by \( \cos \frac{\theta}{2} \).

Then \[ \sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}. \]

and \[ \cos \theta = \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}}. \]

\( \therefore \) dividing as before, \[ \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}}. \]

**Multiple angles.**

\[
\tan(A + B + C) = \frac{\tan A + \tan(B + C)}{1 - \tan A \tan(B + C)}
\]

\[
\tan A + \tan B + \tan C = \frac{1}{\tan A \tan B + \tan C}
\]

\[
\therefore \tan(A + B + C) = \frac{\tan A + \tan B + \tan C}{1 - \tan A \tan B \tan C}
\]

**Exercises.**

1. \( \sin \theta; (20). \)
2. \( \cos \theta; (30). \)
3. \( \tan \theta; (50). \)
4. \( 1 + \cos 2\alpha; \alpha. \)
5. \( 1 - \cos 2\beta; \beta. \)
6. \( \sin \frac{\alpha}{2}; \frac{\pi}{4}. \)
7. \( \cos \frac{\alpha}{2}; \frac{\pi}{4}. \)
8. \( \sin \theta + \cos \theta; (20). \)
9. \( \cos^2 \theta; \sin^2 \theta; (45). \)
10. \( \cos(\alpha + \beta); \alpha \beta. \)
11. \( 1 - 2 \sin^2 \theta; \frac{\theta}{2}. \)
12. \( 1 - \cos x; \frac{x}{2}. \)
13. \( \sin A - \cos A; \frac{\pi}{2}. \)
14. \( 2 \sin 3\theta; \frac{\pi}{6}. \)
15. \( \cos 2\alpha; \alpha \beta. \)
16. \( \sin 2\alpha; (\tan \alpha). \)
17. \( \sin^2 \theta; (20). \)
18. \( \cos \theta; (25). \)
19. \( \cos \alpha; (30). \)
20. \( \tan \theta; \frac{\pi}{4}. \)
21. \( 1 - \cos \theta; \frac{\theta}{2}. \)
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Write down simple alternative forms of the following:

22. 1 - cos 4\theta
23. \sin^2\alpha - \cos^2\alpha
24. 4\cot^2\theta - 1
25. 2\tan^2\theta
26. \frac{2\tan^2\theta}{1 + \tan^2\theta}
27. \frac{1 - \tan^2\theta}{2}

28. \sin 40^\circ \cos 40^\circ
29. \cos^2 50^\circ
30. 1 - \cos 50^\circ
31. 1 - \sin 40^\circ
32. 1 + \cos 10^\circ
33. 1 + \sin 60^\circ
34. \sin^2 80^\circ
35. \sin^2 50^\circ - \cos^2 50^\circ
36. (\sin 15^\circ - \cos 15^\circ)^2
37. What is \cos 3\alpha in terms of \cos \alpha? Verify when \alpha = 0.
38. What is \sin 3\alpha in terms of \sin \alpha? Verify when \alpha = 30^\circ.
39. What result is obtained from the expansion of tan(\alpha + \beta + \gamma) if \alpha + \beta + \gamma = 180^\circ?
40. What relation connects tan A, tan B, tan C if A + B + C = 90^\circ?

Example IV. Prove that \cot \frac{\theta}{2} - \tan \frac{\theta}{2} = \frac{\cos \theta}{2 \sin \frac{\theta}{2}}.

\cot \frac{\theta}{2} - \tan \frac{\theta}{2} = \frac{\cos \theta}{\sin \frac{\theta}{2}} - \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}

= \frac{\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\cos \theta}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \cot \theta;

\therefore \cot \frac{\theta}{2} - \tan \frac{\theta}{2} = \frac{\cos \theta}{2 \sin \frac{\theta}{2}}.

Or, as follows:

\tan \theta = \frac{\tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}.

\therefore \cot \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{2 \tan \frac{\theta}{2}} = \frac{(\cot \frac{\theta}{2} - \tan \frac{\theta}{2})}{2 \tan \frac{\theta}{2}}.

\therefore 2 \cot \theta = \cot \frac{\theta}{2} - \tan \frac{\theta}{2}.

EXERCISE XV. c.

1. Evaluate sin 20 and cos 20 if sin \theta = \frac{1}{2}.
2. Find the possible values of sin \frac{\theta}{2} and cos \frac{\theta}{2} if \cos \theta = \frac{7}{25}.
3. Find tan (\theta + \phi) if tan \theta = \frac{1}{3}, tan \phi = \frac{1}{2}. What is (\theta + \phi)?
4. Find tan 2A if tan A = \frac{1}{2}. What happens if tan A is put equal to 1 in the formula for tan 2A? Explain the result.
5. Find the possible values of tan \frac{\theta}{2} if tan \theta = \frac{5}{12}.
6. Find tan (A + B + C) if tan A = \frac{1}{2}, tan B = \frac{1}{3}, tan C = \frac{1}{4}. What is (A + B + C)?
7. Find sin 3\theta if sin \theta = \frac{1}{2}.
8. Find cos 3\theta if cos \theta = \frac{1}{2}.
9. Determine the signs which must be taken in the formulae

\frac{\sin \frac{\theta}{2}}{\sin \theta} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{and} \quad \cos \theta = \pm \sqrt{\frac{1 + \cos \theta}{2}}.

If \theta is (i) 100^\circ, (ii) 200^\circ, (iii) 400^\circ, (iv) 600^\circ.
10. If \theta, \phi are acute angles such that tan \theta = \frac{1}{4} and tan \phi = \frac{1}{3}, prove that \theta + 2\phi = 45^\circ.
11. Simplify (i) tan^-1(\frac{1}{4}) + tan^-1(\frac{1}{3}); (ii) 2 tan^-1(\frac{1}{4}).
12. Prove that tan^-1(\frac{1}{4}) + tan^-1(\frac{1}{3}) = 15^\circ + 180^\circ.
13. Prove that

\tan^{-1}(\theta) + \tan^{-1}(\frac{1}{\theta}) + \tan^{-1}(\frac{2}{\theta}) = 45^\circ + 180^\circ\theta.

14. Prove that cot^-1(\theta) + cot^-1(\frac{1}{\theta}) + cot^-1(\frac{2}{\theta}) = cot^-1(2).

Simplify the following:

15. sin (45^\circ - \theta) cos (45^\circ - \theta).
16. cos^3(45^\circ - \theta) - sin^3(45^\circ - \theta).
17. \sqrt{2 - 2 \cos 2\alpha}.
18. \sqrt{2 + 2 \cos 2\alpha}.
19. \sin 2\alpha.
20. \sin^2\alpha.
21. \cos^2\alpha.
22. \frac{1}{1 - \sin \alpha}.
23. tan (1 + \cot 2\alpha).
24. sin \theta cos \theta cos 2\theta.
25. sec \theta - 1.
26. sec \theta + 1.
27. \frac{1}{2} \sin \alpha \tan \alpha.
28. tan A + tan \frac{\alpha}{2}.
29. cot \theta - tan \theta.
30. cot \theta + tan \theta.
Prove the following identities:

31. \( \sin \frac{\theta}{2} + \cos \frac{\theta}{2} = 1 + \sin \theta \)

32. \( \sin \frac{\theta}{2} - \cos \frac{\theta}{2} = \pm \sqrt{1 - \sin \theta} \)

33. \( \tan A + \cot A = 2 \csc 2A \)

34. \( \tan 2\theta - 1 = \sec 2\theta \)

35. \( \cot \frac{\theta}{2} - \cot \theta = \csc \theta \)

36. \( 1 + \cot \theta \cot 2\theta = \csc \theta \cot \theta \cot \frac{\theta}{2} \)

37. \( \frac{1 + \sin A}{1 + \cos A} = \tan \frac{A}{2} \)

38. \( \frac{1 + \cos A}{1 - \sin A} = \sec \frac{A}{2} \)

39. \( \frac{\cos \theta}{\sin \theta} = \tan \frac{\theta}{2} \)

40. \( \sin 2A = \frac{2 \sin A \cos A}{1 + \cos 2A} \)

41. \( 1 + \tan 2A = \tan^{2}(45° + A) \)

42. \( 1 - \cos A - \sin A = \cot \frac{A}{2} \)

43. \( \sec (A + B) = \sec (A - B) = \cos A \cos B \)

44. \( \cos \theta + \sin \theta = \tan 2\theta + \sec 2\theta \)

45. \( \tan \left( \frac{\pi}{4} + \theta \right) + \tan \left( \frac{\pi}{4} - \theta \right) = 2 \sec 2\theta \)

The next exercise may be reserved for a revision course.

EXERCISE XV. 1.

Prove the following identities:

1. \( \sin 2\theta = \frac{\sin 2\theta + \cos 2\theta}{1 + \cos 2\theta} = \frac{\sin 2\theta - \cos 2\theta}{1 - \cos 2\theta} \)

2. \( 1 + \tan 2\theta = \frac{1 + \tan 2\theta}{\sec 2\theta + \tan 2\theta} \)

3. \( (1 + \tan \theta + \sec \theta)(1 + \tan \theta - \sec \theta) = \sin 2\theta \sec^2 \theta \)

4. \( \frac{\sec \theta + \tan \theta}{\sec \theta - \tan \theta} = \tan \left( \frac{\pi}{4} + \theta \right) \)

5. \( \sec 2\theta - \tan 2\theta = \tan \left( \frac{3\pi}{4} - \theta \right) \)

6. \( \cos^2 \theta + \cos^2 \left( \frac{2\pi}{3} + \theta \right) + \cos^2 \left( \frac{2\pi}{3} - \theta \right) = \frac{3}{2} \)

7. \( \sin \frac{3\theta}{2} = \sin \frac{\theta}{2} (1 + 2 \cos \theta) \)

22. In Fig. 245, find \( \tan \angle ACB \) in terms of \( n \). Find the least integral value of \( n \) for which \( \angle ACB < 1^\circ \), if \( AB = OC = 1 \).
24. In Fig. 346, prove \( \tan \theta = 3 \tan \phi \).

\[ \text{Fig. 346.} \]

In \( \triangle ABC \), \( 2a = 3b \) and \( A = 3B \). Calculate \( B \).

26. Prove that \( \cot \theta - 2 \cot 2\theta = \tan \theta \), and deduce that
\[ \tan x + 2 \tan 2x + 4 \tan 4x + 8 \tan 8x = \cot x - 16 \cot 16x. \]

27. Show that if \( \tan \theta = -c \) there will be two distinct values \( t_1, t_2 \) of \( \tan \frac{\theta}{2} \), and that \( t_1 t_2 = -1 \).

28. In Fig. 347, prove \( \tan \theta = \frac{1 - n}{1 + n} \tan \phi \). What happens if \( n = 1 \)?

\[ \text{Fig. 347.} \]

29. Prove that \( x + \tan^{-1}(\cot 2x) = \tan^{-1}(\cot x) \).

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CHAPTER XVI.

SUMS AND PRODUCTS FORMULAE.

This chapter contains various transformations of the four fundamental formulae, tabulated on p. 211; without them, no further advance can be made.

Products as sums and differences.

\[
\begin{align*}
\sin (A + B) &= \sin A \cos B + \cos A \sin B, \\
\sin (A - B) &= \sin A \cos B - \cos A \sin B; \\
\therefore \text{adding,} \quad 2 \sin A \cos B &= \sin (A + B) + \sin (A - B), \quad \ldots \ldots (1) \\
\text{and subtracting,} \quad 2 \cos A \sin B &= \sin (A + B) - \sin (A - B). \quad \ldots \ldots (2)
\end{align*}
\]

Again, \( \cos (A + B) = \cos A \cos B - \sin A \sin B, \)
\( \cos (A - B) = \cos A \cos B + \sin A \sin B; \)
\( \therefore \text{adding,} \quad 2 \cos A \cos B &= \cos (A + B) + \cos (A - B), \quad \ldots \ldots (3) \\
\text{and subtracting the first from the second,} \quad 2 \sin A \sin B &= \cos (A - B) - \cos (A + B). \quad \ldots \ldots (4)
\]

Formulae (1)-(4) must be committed to memory: it is sometimes found easier to remember them in words, thus:

(1) \( \text{twice } \sin \cdot \cos = \sin (\text{sum}) + \sin (\text{difference}), \)
(2) \( \text{twice } \cos \cdot \sin = \sin (\text{sum}) - \sin (\text{difference}), \)
(3) \( \text{twice } \cos \cdot \cos = \cos (\text{sum}) + \cos (\text{difference}), \)
(4) \( \text{twice } \sin \cdot \sin = \cos (\text{difference}) - \cos (\text{sum}), \)

where the word "difference" is used to mean "first - second."
Until the reader knows these by heart, he should work them out whenever he needs them.

Note (i). (1) and (2) are substantially equivalent, for if $B$ and $A$ are interchanged in (1) we obtain (2), since

$$\sin (B - A) = - \sin (A - B).$$

When applying the formula, it is best to put the larger angle first.

Thus $2 \sin 20^\circ \cos 50^\circ = 2 \cos 50^\circ \sin 20^\circ = \sin 70^\circ - \sin 30^\circ$.

(ii) Formula (4) requires special notice; it is easy to remember if the reader realises that if $A$, $B$ and $(A + B)$ are acute, $\cos (A - B)$ is greater than $\cos (A + B)$.

**Example I.** Express as sums or differences:

(i) $2 \sin 50^\circ \cos 32^\circ$;

(ii) $2 \cos 70^\circ \sin 43^\circ$;

(iii) $2 \cos 61^\circ \cos 39^\circ$;

(iv) $2 \sin 61^\circ \sin 39^\circ$.

Thus

$$\begin{align*}
(i) & ~2 \sin 50^\circ \cos 32^\circ = \sin (50^\circ + 32^\circ) + \sin (50^\circ - 32^\circ) = \sin 82^\circ + \sin 18^\circ; \\
(ii) & ~2 \cos 70^\circ \sin 43^\circ = \cos (70^\circ + 43^\circ) - \sin (70^\circ - 43^\circ) = \sin 113^\circ - \sin 27^\circ. \\
(iii) & ~2 \cos 61^\circ \cos 39^\circ = \cos (61^\circ + 39^\circ) + \cos (61^\circ - 39^\circ) = \cos 100^\circ + \cos 22^\circ. \\
(iv) & ~2 \sin 61^\circ \sin 39^\circ = \cos (61^\circ - 39^\circ) - \cos (61^\circ + 39^\circ) = \cos 22^\circ - \cos 100^\circ.
\end{align*}$$

Note that this equals $\sin 67^\circ - \sin 27^\circ$.

Express the following, Nos. 1-36, as sums or differences of sines and cosines:

1. $2 \sin 35^\circ \cos 17^\circ$.
2. $2 \sin 43^\circ \cos 10^\circ$.
3. $2 \cos 48^\circ \cos 16^\circ$.
4. $2 \cos 80^\circ \cos 6^\circ$.
5. $2 \sin 61^\circ \sin 10^\circ$.
6. $2 \cos 22^\circ \sin 50^\circ$.
7. $2 \sin 28^\circ \cos 45^\circ$.
8. $2 \sin 31^\circ \cos 44^\circ$.
9. $2 \sin 35^\circ \cos 51^\circ$.
10. $2 \sin 56^\circ \cos 50^\circ$.
11. $2 \sin 50^\circ \sin 70^\circ$.
12. $2 \cos 10^\circ \cos 80^\circ$.
13. $2 \sin 100^\circ \sin 150^\circ$.
14. $2 \sin 10^\circ \cos 20^\circ$.
15. $2 \sin 2A \cos A$.
16. $2 \cos 5x \cos x$.
17. $2 \sin 3A \sin 4A$.
18. $2 \cos 3A \sin 7A$.
19. $2 \sin (A + B) \cos (A - B)$.
20. $2 \cos (\theta + \phi) \sin (\theta - \phi)$.
21. $\frac{A + B}{2} \cos \frac{A - B}{2}$.
22. $\frac{A + B}{2} \cos \frac{A - B}{2}$.
23. $\frac{A + B}{2} \sin \frac{A - B}{2}$.
24. $\frac{A + B}{2} \sin \frac{A - B}{2}$.
25. $\frac{A + B}{2} \sin \frac{A - B}{2}$.
26. $\frac{A + B}{2} \sin \frac{A - B}{2}$.
27. What do the formulae (1), (3), (4) become when $B$ is put equal to $A$?
28. Express $\cos 75^\circ \cos 45^\circ$ as a sum, and deduce the value of $\cos 75^\circ$.
29. Express $\sin 75^\circ \cos 45^\circ$ as a sum, and deduce the value of $\sin 75^\circ$.
30. Prove that $\sin (A + B) \cdot \sin (A - B) = \sin^2 A - \sin^2 B$.
31. Simplify $2 \sin (45^\circ + A) \cos (45^\circ - A)$.
32. Prove that $2 \sin 3A \cos 2A + \cos 3A \sin 2A = \sin 4A + \sin 6A$.
33. Prove that $2 \sin 3A = \frac{3A}{2} + \cos 3A\cos \frac{A}{2} = \cos A + \cos 5A$.
34. Prove that $2 \cos^2 \frac{A}{2} - 1 = \cos 2A = \cos^2 \alpha - \cos^2 \frac{A}{2}$.
35. Express $2 \cos A \cos 3A$ as the sum of two terms; hence express $4 \cos A \cos 3A \cos 5A$ as the sum of four terms.
48. Express \( \sin 60^\circ \cos 60^\circ \cos 9^\circ \) as a sum of four terms (positive or negative).

47. Prove that \( \cos 25^\circ \cos 35^\circ - \sin 20^\circ \sin 10^\circ = \frac{1}{2} (\sqrt{3} + 1) \).

48. Prove that \( \sin 56^\circ \cos 7^\circ + \sin 26^\circ \cos 80^\circ = \sin 78^\circ \cos 30^\circ \).

49. Prove that \( \sin (A + C) \sin (B + C) - \sin A \sin B = \sin C \cdot \sin (A + B + C) \).

50. Prove that \( \tan \frac{B + C}{2} = \cot \frac{B - C}{2} = \frac{\sin B + \sin C}{\sin B - \sin C} \).

**Sums and differences as products.**

We have shown that

\[
\sin (A + B) + \sin (A - B) = 2 \sin A \cos B.
\]

Now put \( A + B = x \) and \( A - B = y \);

\[
\therefore 2A = x + y \quad \text{or} \quad A = \frac{x + y}{2}.
\]

and \( 2B = x - y \) or \( B = \frac{x - y}{2} \);

\[
\therefore \sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2} \quad \text{..............(5)}
\]

Again, \( \sin (A + B) - \sin (A - B) = 2 \cos A \sin B \);

\[
\therefore \sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2} \quad \text{..............(6)}
\]

And \( \cos (A + B) + \cos (A - B) = 2 \cos A \cos B \);

\[
\therefore \cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2} \quad \text{..............(7)}
\]

And \( \cos (A + B) - \cos (A - B) = -2 \sin A \sin B \);

\[
\therefore \cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2} \quad \text{..............(8)}
\]

Formulae (5)-(8) must be committed to memory; it is sometimes found easier to remember them in words, thus:

(5) \( \sin + \sin = 2 \sin \left( \frac{1}{2} \text{ sum} \right) \cos \left( \frac{1}{2} \text{ difference} \right) \),

(6) \( \sin - \sin = -2 \cos \left( \frac{1}{2} \text{ sum} \right) \sin \left( \frac{1}{2} \text{ difference} \right) \),

(7) \( \cos + \cos = 2 \cos \left( \frac{1}{2} \text{ sum} \right) \cos \left( \frac{1}{2} \text{ difference} \right) \),

(8) \( \cos - \cos = -2 \sin \left( \frac{1}{2} \text{ sum} \right) \sin \left( \frac{1}{2} \text{ difference} \right) \).

where the word "difference" is used to mean "first - second."
Example V. Express in factors \( \sin x + \cos y \).

[We must write the given expression either as the sum of two
sines or the sum of two cosines.]

\[
\sin x + \cos y = \sin x + \sin \left( \frac{\pi}{2} - y \right) = 2 \sin \frac{1}{2} \left( x + \frac{\pi}{2} - y \right) \cos \frac{1}{2} \left( x - \frac{\pi}{2} + y \right) = 2 \sin \left( \frac{x + y}{2} \right) \cos \left( \frac{x - y}{2} \right)
\]

EXERCISE XVI. h.

Express in factors the following, Nos. 1-32:

1. \( \sin 40^\circ + \sin 20^\circ \)
2. \( \sin 40^\circ - \sin 20^\circ \)
3. \( \cos 40^\circ - \cos 20^\circ \)
4. \( \cos 40^\circ + \cos 20^\circ \)
5. \( \sin 100^\circ - \sin 50^\circ \)
6. \( \sin 100^\circ + \sin 50^\circ \)
7. \( \cos 80^\circ + \cos 50^\circ \)
8. \( \cos 80^\circ - \cos 50^\circ \)
9. \( \sin 150^\circ + \sin 140^\circ \)
10. \( \sin 150^\circ - \sin 140^\circ \)
11. \( \cos 110^\circ + \cos 130^\circ \)
12. \( \cos 110^\circ - \cos 130^\circ \)
13. \( \sin 0^\circ + \sin 2A \)
14. \( \cos \phi - \cos A \)
15. \( \sin 3\phi + \sin 3A \)
16. \( \sin 3\phi - \sin 3A \)
17. \( \cos x - \cos 2x \)
18. \( \cos x + \cos 2x \)
19. \( \cos 2e - \sin 5X \)
20. \( \cos 2e + \cos 4x \)
21. \( \cos (A + B) - \cos (A + 2B) \)
22. \( \sin (A - B) - \sin A \)
23. \( \sin (A + B + C) + \sin (A - B + C) \)
24. \( \cos (A + B) - \cos A \)
25. \( \sin (2A + 2B) - \sin (2B + 2C) \)
26. \( \cos (A + B + C) - \cos (A - B - C) \)
27. \( \sin 40^\circ + \cos 12^\circ \)
28. \( \cos 80^\circ - \sin 50^\circ \)
29. \( \cos 130^\circ + \sin 110^\circ \)
30. \( \cos 170^\circ - \sin 100^\circ \)
31. \( \cos A + \cos B \)
32. \( \cos A - \sin B \)

Prove the following identities, Nos. 33-50:

33. \( \tan A + \sin B = \frac{\tan A + B}{2} \)
34. \( \tan A - \sin B = \frac{\tan A - B}{2} \)
35. \( \sin (A + B) = \frac{\tan A + B}{2} \)
36. \( \tan (A + B) = \frac{\tan A + B}{2} \)
37. \( \sin 3x + \sin 5x = \sin 4x \)
38. \( \cos 3x + \sin 5x = \sin 5x \)
39. \( \cos 3x - \sin 5x = \cos 2x \)
40. \( \cos 3x - \sin 5x = \cos 2x \)
41. \( \cos 7x + \cos 15^\circ = \cos 72^\circ \)
42. \( \sin 7x + \sin 15^\circ = \sin 72^\circ \)
43. \( \cos 90^\circ + \cos 80^\circ = \cos 20^\circ \)
44. \( \sin 71^\circ - \cos 79^\circ = \cos 41^\circ \)
45. \( \sin 38^\circ + \cos 68^\circ = \cos 8^\circ \)
46. \( \sin 10^\circ + \cos 40^\circ = \sin 70^\circ \)
47. \( \sin \phi + \cos \phi = \tan \phi \)
48. \( \sin \phi - \cos \phi = \tan \phi \)
49. \( \cos (\phi + \theta) + \sin (\phi - \theta) = \tan \phi \)
50. \( \sin (\phi + \theta) - \sin (\phi - \theta) = \cos \theta \)
51. \( \tan (\phi + \theta) + \tan (\phi - \theta) = \frac{2 \sin 2\phi}{\cos 2\phi + \cos 2\theta} \)
52. \( \cos^2 (\phi - \theta) - \sin^2 (\phi + \theta) = \cos 2\phi - 2 \sin 2\theta \)
53. \( \cos^2 (\phi - \theta) + \sin^2 (\phi + \theta) = \cos 2\phi - 2 \sin 2\theta \)
54. \( \cos (A + B) - \cos (A - B) = 2 \sin 2B \cos 2A \)
55. \( \cos (A + B) + \cos (A - B) = 2 \cos 2B \cos 2A \)
56. \( \sin A + \sin B + \sin C = \frac{1}{2} \sin (A + B + C) \)
57. \( \sin A - \sin B - \sin C = \frac{1}{2} \sin (A - B - C) \)
58. \( \sin 2A + \sin 2B + \sin 2C = \frac{3}{2} \sin (A + B + C) \)
59. \( \sin 2A + \sin 2B + \sin 2C = \frac{3}{2} \sin (A + B + C) \)
60. \( \cos 3A + \cos 4A + \cos 5A = \cos 3A \)
61. \( \cos 3A + \cos 4A + \cos 5A = \cos 3A \)
TRIGONOMETRY

REVISION PAPERS. R. 35-50.

R. 35.

1. In Fig. 348, find PN and MN in terms of $p, f, g, \alpha$.

2. Use Tables to evaluate:
   (i) $\cos 125^\circ 20'$;
   (ii) $\sin 260^\circ$;
   (iii) $\cos 322^\circ 40'$;
   (iv) $\tan 203^\circ$;
   (v) $\sec 213^\circ$.

3. The sides of a triangle are 207, 417, 450. Prove that one angle is $60^\circ$.

4. (i) Simplify $\sin^2 \theta \cos \theta + \cos^2 \theta \sin \theta$.

5. Prove that
   (i) $\csc \theta - \cot \theta = \tan \frac{\theta}{2}$;
   (ii) $\sin^2 A + \sin^2 B = 1 - \cos (A - B) \cos (A + B)$.

R. 38.

1. If $\tan \beta = \frac{2}{3}$ and $\beta$ is an angle in the third quadrant, find the other trigonometrical ratios of $\beta$.

2. (i) Find the area of $\triangle ABC$, given $a = 10$ in., $B = 41^\circ$, $C = 67^\circ$.
   (ii) Find the area of the quadrilateral $PQRS$ in Fig. 349.

R. 37.

1. In Fig. 350, find $PQ$.

2. Fig. 351 represents a bent wire $APQB$. Find the distance of $A$ from $B$.

3. Simplify
   (i) $\sin \left(\frac{3\pi - \theta}{2}\right) \sin \left(\frac{\pi - \theta}{2}\right)$;
   (ii) $\cos (2\pi - \theta) + \sin \left(\frac{\pi}{2} + \theta\right)$;
   (iii) $\tan \left(\frac{3\pi}{2} - \theta\right), \tan (2\pi - \theta)$.

4. Prove that
   (i) $\tan^2 \theta - \sin^2 \theta = \sin^2 \theta \sec^2 \theta$;
   (ii) $\sin \theta + \sin 70^\circ = 2 \cos 29^\circ - 1$.

5. Given $\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$ and $\frac{x}{a} \sec \phi - \frac{y}{b} \tan \phi = 1$, prove that $x = a \cos \frac{\theta + \phi}{2} \sec \frac{\theta + \phi}{2}$ and find a similar expression for $y$. 

REVOLUTION PAPERS

1. Two ships sail from the same port at the same time; one travels at 14 knots on a bearing S. 71° W., the other travels due West at 12 knots. Find their distance apart 10 hours later to the nearest sea-mile.

2. Prove that
   (i) $\sin A = \frac{\cos A}{1 + \tan A}$;
   (ii) $\frac{\cos x - \cos y}{\sin x + \sin y} = \frac{\sin x - \sin y}{\cos x + \cos y}$.

3. Prove that $\tan 75^\circ + \cot 75^\circ = 4$.

What is the value of $\tan 15^\circ + \cot 15^\circ$?
R. 38.

1. If \( \cos \theta = c \), write down the value of
   (i) \( \cos (\theta - 30^\circ) \); (ii) \( \cos (\theta + 30^\circ) \); (iii) \( \cos (\theta - 180^\circ) \);
   (iv) \( \sin (\theta - 90^\circ) \); (v) \( \cos (\theta - 90^\circ) \).

2. A and B are points 200 yards apart on a straight river bank; C is a point on the far bank, and the angles CAB, CBA are 47° and 72° respectively. Find the width of the river.

3. Fig. 352 represents a pyramid whose base ABC is equilateral; PEC is also equilateral. Find the inclination of PA to the base.

4. Prove that
   (i) \( \frac{\sin x + \sin 2x}{1 + \cos x + \cos 2x} = \tan x \);
   (ii) \( \frac{\sin x - \sin 2x}{\cos 2x - \cos x} = \cos 3x \).

5. (i) If \( \tan \alpha = \frac{1}{3} \), find \( \tan 3\alpha \).
   (ii) If \( \sin \alpha = \frac{1}{3} \), find \( \sin 3\alpha \).

R. 39.

1. In Fig. 353, ABCD is a square. Calculate \( \angle DPC \).

2. Find the area of the minor segment of a circle of radius 4 in., cut off by a chord of length 6 1/4 in.

3. Without using trigonometric tables, find the approximate values of (i) \( \sin 2^\circ 30' \); (ii) \( \cos 8^\circ \).

R. 40.

1. If \( \theta \) be an angle in the 4th quadrant and \( \tan \theta = \frac{4}{3} \), find the values of \( \tan \theta \) and \( \cos \theta \).

2. Find the length of a tunnel through the earth between two places in latitude 51°, whose difference of longitude is 6°. [Radius of Earth = 3960 miles.]

3. Find two possible values of \( c \), if in a triangle \( a = 5^\circ, b = 4^\circ \), and \( B = 42^\circ \).

4. Prove that
   (i) \( \sin^2(45^\circ + x) + \sin^2(45^\circ - x) = 1 \);
   (ii) \( \sin^2 \theta \tan \theta + \cos^2 \theta \cot \theta + 2 \sin \theta \cos \theta = \tan \theta + \cot \theta \).

5. Prove that
   (i) \( \tan(45^\circ + A) - \tan(45^\circ - A) = 2 \tan 2A \);
   (ii) \( \csc \theta - 2 \cot 2\theta \cos \theta = 2 \sin \theta \).

R. 41.

1. The diameter AB of a semi-circle is of length 10 cm. Find the radius of the circle which touches AP (see Fig. 354) and touches the semicircle at B.

2. ABCD is a rectangular court; a flagstaff at A subtends angles of 35° at B and 51° at D; AB = 30 yd. Find AD.

3. If A, B, C are the angles of a triangle, prove that \( \sin (A + B) + \sin (A + C) \) is not less than \( \sin (B + C) \).

4. (i) Prove that \( \frac{1 - \sin \theta}{1 + \sin \theta} = (\sec \theta - \tan \theta)^2 \).
   (ii) Simplify \( \frac{\sin(\theta + \phi) - \sin(\theta - \phi)}{\cos(\theta + \phi) - \cos(\theta - \phi)} \).

5. If \( \cos \theta = \cos \phi \cos \psi \), prove that \( \tan \frac{\theta}{2} \tan \frac{\phi - \psi}{2} = \tan \frac{\theta + \phi}{2} \tan \frac{\theta - \phi}{2} \).
R. 42.

1. Prove that the area of a regular 12-sided figure inscribed in a circle is \( \frac{1}{6} \) the area of a square circumscribed to the circle.

2. Find (i) in radians, (ii) in minutes, the angle subtended at the eye by a target 2 ft. wide at a distance of 500 yards.

3. A rectangle ABCD is creased along one diagonal AC, and the triangle ADC is folded about AC till the two halves of the rectangle are at right angles. Find the distance of D from B in this position, given AB = 9 in., BC = 4 in.

4. Prove that
   (i) \( \sec^2 \theta \cos^2 \theta - \tan^2 \theta + \cot^2 \theta = 2 \);
   (ii) \( \sec x - \tan x = \tan \left( \frac{45^\circ - x}{2} \right) \).

5. Prove that
   (i) \( \sin (A + C) \sin B - \sin (B + C) \sin A = \sin C \sin (B - A) \);
   (ii) \( \cos A - \cos 3A = \sin A \).

R. 43.

1. A door 5 ft. wide, 8 ft. high, is opened through an angle of 155°. Find the angle between the initial and final positions of a diagonal.

2. Find the distance of Winchester (lat. 51° N., long. 1° W.) from Calais (lat. 51° N., long. 2° E.). (Radius of Earth = 3960 mi.)

![Fig. 355.](image)

3. A boy AB, 5 ft. 6 in. high, stands vertically on a hillside (see Fig. 355), and his shadow BC falls on a slope of 18° when the sun’s elevation is 49° 30’. Find the length of BC.

4. Prove that
   (i) \( \csc \theta - \csc \theta \cdot \csc \theta + 1 = 2 \sec^2 \theta \);
   (ii) \( \cot \theta - \tan \theta + \cot (\theta - \tan \theta) - 2 \cot \theta = \frac{2 \cos \theta \sin \theta}{\sin^2 \theta - \sin^2 \theta} \).

5. If \( \tan \theta + \tan \phi = p \) and \( \sec \theta + \sec \phi = q \), prove that
   \( \tan \frac{\theta + \phi}{2} = \frac{2p}{q^2 - p^2} \).

R. 44.

1. If \( \tan \alpha = -\frac{2}{3} \) and \( \alpha \) is an obtuse angle, find the values of
   (i) \( \tan \left( \frac{\pi}{2} + \alpha \right) \);
   (ii) \( \tan \left( \frac{\pi}{6} + \alpha \right) \);
   (iii) \( \sin \alpha \);
   (iv) \( \sin \left( \frac{\pi}{2} + \alpha \right) \);
   (v) \( \cos (\pi + \alpha) \).

2. Find the angles of a triangle whose sides are 20, 21, 22 inches.

3. In Fig. 356 prove \( \tan \theta = 2 \tan \phi \).

![Fig. 356.](image)

4. (i) Prove that \( \sin \theta (1 + \tan \theta) + \cos \theta (1 + \cot \theta) = \sec \theta + \csc \theta \).
   (ii) If \( \tan \theta = \alpha \), prove that \( \cos 2\theta + \alpha \sin 2\theta = 1 \).

5. Prove that
   (i) \( \sin 3x + \sin x = \tan 2x \);
   (ii) \( \cos (x + 1)A - \cos (x - 1)A = \sin^2 A \).

R. 45.

1. A cubical box, edge 6 in., is placed in a conical funnel of vertical angle 124°; the upper face of the box is horizontal and perpendicular to the axis of the funnel. Find the height of the top of the box above the vertex of the funnel.

2. In Fig. 357, the square and the circle are of equal area and have the same centre O. Calculate \( \angle POQ \).

![Fig. 357.](image)

3. In \( \triangle ABC \), \( BC = 3 \text{ in.} \), \( \angle B = 123^\circ \), \( \angle C = 31^\circ 20' \); the bisector of \( \angle BAC \) cuts BC at D. Calculate AD.
4. (i) Prove that \( \csc^2 \theta - \cot^2 \theta = 2 \csc^2 \theta - 1 \).
(ii) Simplify \( \frac{\sin 2A - \cos 2A + 1}{\sin 2A + \cos 2A + 1} \).

5. If \( \tan A \tan (A + B) = k \), prove that \( \cos (2A + B) = \frac{1 - k}{1 + k} \cos B \).

**R. 46.**

1. Find the values of
   (i) \( \sin \frac{2\pi}{3} \); (ii) \( \sin \frac{3\pi}{4} \); (iii) \( \csc \left( -\frac{\pi}{4} \right) \);
   (iv) \( \tan \left( -\frac{\pi}{3} \right) \); (v) \( \cot \frac{5\pi}{6} \); (vi) \( \cos \frac{7\pi}{6} \); (vii) \( \sec \frac{4\pi}{3} \).

2. From the top of a lighthouse 120 ft. high the light of another which is 100 ft. high can just be seen. Find their distance apart. [Radius of earth = 3960 miles.]

3. BAC is a vertical section of a window and an awning inclined to the vertical at \( 40^\circ \) (see Fig. 358). Find the length of the awning if the whole of the window is just in shadow when the sun is at an elevation of \( 70^\circ \).

4. Prove that \( (\cos A + \cos B)^2 + (\sin A + \sin B)^2 = 4 \cos^2 \frac{A - B}{2} \).

5. Prove that
   (i) \( \tan (\theta + 60^\circ) \cdot \tan (\theta - 60^\circ) = \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \),
   (ii) \( \cot A \cot B - \tan A \tan B = \frac{2(\cos 2B + \cos 2A)}{\sin 2A \sin 2B} \).

**E. 47.**

1. In Fig. 359, F, G are pegs at the same level; B, falling vertically, is drawn A up a slope EF. Find the difference in the vertical distance between A and B when the horizontal distance decreases by 1 ft. If A is still below B.

2. To inscribe a regular heptagon in a circle, diameter AB, proceed as in Fig. 360, where \( \triangle AOB \) is equilateral and \( AP = \frac{1}{2} AB \). Show that AQ is approximately equal to a side of the required heptagon.

3. Show that from a height of \( h \) ft. above the Earth's surface, the area visible is approx. \( \frac{338}{7} \) sq. miles.

4. (i) Simplify \( \frac{1 - \tan x)^2 + (1 + \tan x)^2}{(1 + \cot x)^2 + (1 - \cot x)^2} \).
   (ii) Prove that \( \frac{\sin A + \sin (A + B)}{\sin (A - B)} = \tan (A + B) \).
   \( \cos A + \cos (A + B) + \cos (A - B) \).

5. If \( \tan^2 \theta = \tan (\alpha - \beta) \cdot \tan (\alpha + \beta) \), prove that \( \tan 2\theta = \frac{2 \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} \).

**R. 48.**

1. Find the possible values of \( \cos \frac{\theta}{2} \) if \( \cos \theta = -\frac{1}{3} \), and determine the actual value:
   (i) if \( \frac{\pi}{2} < \theta < \pi \); (ii) if \( \pi < \theta < \frac{3\pi}{2} \); (iii) if \( \frac{3\pi}{2} < \theta < 3\pi \).

2. A hill-side is a plane inclined at \( 10^\circ \) to the horizontal. A straight path across it slopes upwards at \( 4^\circ \) to the horizontal. Find the bearing of the path if the lines of greatest slope bear due North.
3. Fig. 361 represents a section of a solid formed of a right cone and a hemisphere. Find its volume.

4. If \( \tan \theta = \frac{3}{4} \), find two possible values of \( \tan \frac{\theta}{2} \).

5. Prove that
   
   (i) \( \tan \theta + \sec \theta = \tan \left( 45^\circ + \frac{\theta}{2} \right) \);
   
   (ii) \( \frac{\sin (n+1)\theta + 2 \sin n\theta + \sin (n-1)\theta}{\cos (n-1)\theta - \cos (n+1)\theta} = \cot \frac{\theta}{2} \).

**R. 49.**

1. In Fig. 362, ABCD is a rectangle. Express PR in terms of \( a, \theta \).

2. ABCD is a cyclic quadrilateral; \( AB = 5, BC = 7, CD = 11 \), \( DA = 8 \). Calculate \( \angle ABC \).

3. A sector of a circle of radius 4 in., angle of sector 200\(^\circ\), is bent into the form of a circular cone. Calculate the semi-vertical angle of the cone.

4. If \( \tan \beta = \frac{n \sin \alpha \cos \alpha}{1 - n \sin^2 \alpha} \), prove that \( \tan (\alpha - \beta) = (1 - n) \tan \alpha \).

5. (i) If \( \sec \theta - \tan \theta = x \), prove that \( 4 \sec^4 \theta = x^2 + 4 \).
   
   (ii) Prove that \( [\cot (A + B) + \cot (A - B)][\sin^2 A - \sin^2 B] = \sin 2A \).

**R. 50.**

1. The error in measuring the chord instead of an arc of a circle is 0.1\%. If \( \theta \) radians is the angle subtended by the chord at the centre, use the fact that \( \sin \theta \approx \theta - \frac{\theta^3}{6} \) to find \( \theta \) approximately.

2. Find the height and volume of a regular tetrahedron, side \( a \).

3. By applying the cosine formula \( a^2 = b^2 + c^2 - 2bc \cos A \) to a triangle in which \( a, b, \) and \( A \) are given, prove that if \( c_1 \) and \( c_2 \) are two possible values of the third side, then \( c_1 + c_2 = 2b \cos A \). Show also from this formula that no such triangle can be drawn if \( b \sin A > a \).

4. Prove that
   
   (i) \( \cos A \sin (B - C) + \cos B \sin (C - A) + \cos C \sin (A - B) = 0 \);
   
   (ii) \( \sin^{-1} \frac{1}{\sqrt{3}} + \cot^{-1} 3 = 45^\circ \).

5. If \( 1 + n \sin 2\theta + (1 - n) \cos 2\theta = 1 + n \), prove that \( \tan \theta = 1 \) or \( n \).
CHAPTER XVII.

SOLUTION OF TRIANGLES : HALF-ANGLE FORMULAE.

The sine formula lends itself to logarithmic computation, but the cosine formula does not. It is therefore desirable to obtain alternative forms of the latter, to which logarithms can be applied.

Half-angle formulae.

(1.) Given three sides of a triangle, to find its angles.

We shall make use of the following results (p. 219):

\[
\sin^2 \frac{A}{2} = \frac{1}{2} (1 - \cos A) ; \quad \cos^2 \frac{A}{2} = \frac{1}{2} (1 + \cos A).
\]

From the cosine rule, \( \cos A = \frac{b^2 + c^2 - a^2}{2bc} \):

\[
\therefore 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc}.
\]

Now put \( a + b + c = 2s \); then \( a + b - c = 2s - 2c = 2(s - c) \) and \( a - b + c = 2s - 2b = 2(s - b) \);

\[
\therefore \sin^2 \frac{A}{2} = \frac{1}{2} (1 - \cos A) = \frac{2(s - c) \cdot 2(s - b) (s - b)}{4bc} = \frac{(s - b)(s - c)}{bc}.
\]

\[\therefore \sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}}. \quad \text{(See note (i) p. 246.)}\]
By symmetry, we have the following corresponding formulae:

\[ \sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}; \quad \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}; \]

\[ \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}; \quad \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}; \]

\[ \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}; \quad \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \]

Note. (i) We must give to each square root its positive value, because \( \frac{A}{2}, \frac{B}{2}, \frac{C}{2} \) must each be acute angles, since \( A, B, C \) cannot be reflex.

(ii) The above work offers an alternative method of proving Hero's formula (p. 178),

\[ \Delta = \frac{1}{2}bc \sin A = \frac{1}{2}bc \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2} \]

\[ = bc \sqrt{\frac{(s-b)(s-c)}{bc} \cdot \frac{s(s-a)}{bc}} \]

\[ = \sqrt{(s-a)(s-b)(s-c)}. \]

(iii) The alternative form for \( r \) mentioned above is sometimes useful; it may be written,

\[ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}. \]

(iv) To solve a triangle, given three sides, use the tangent formula above. Each tangent formula involves only the same four expressions, \( s, s-a, s-b, s-c \); consequently only four logarithms need be found from the Tables. If the sine or cosine formula is used, it is necessary to look up the logarithms of \( a, b, c \) also.

**Example I.** Solve the triangle \( ABC \), given that \( a = 24.76 \), \( b = 16.38 \), \( c = 15.12 \).

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<th>( a )</th>
<th>( s-a )</th>
<th>( \log )</th>
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<table>
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<th>( s-b )</th>
<th>( \log )</th>
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<table>
<thead>
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<th>( s-c )</th>
<th>( \log )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.12</td>
<td>13.01</td>
<td>1.1418</td>
</tr>
</tbody>
</table>

\[ 2s = 56.26 \quad \frac{s}{2} = 28.13 \]

\[ \log \frac{BC}{2} \quad 1.4692 \]

\[ \log \frac{B+C}{2} = \log B + \log C - \log \frac{s}{2} \]

\[ \log \frac{B-C}{2} = \log B - \log C - \log \frac{s}{2}. \]
Since \( \frac{B+C}{2} = \frac{180° - A}{2} = \frac{90° - A}{2} \tan \frac{B+C}{2} = \cot \frac{A}{2} \).

We may therefore write this formula as follows:

\[
\tan \frac{B-C}{2} = \frac{b-c}{b+c} \quad \tan \frac{B+C}{2} = \frac{b+c}{b-c} \cot \frac{A}{2}.
\]

Note. This formula was discovered by Finecke, a Danish mathematician, in 1583. It should be committed to memory, but the reader will probably find it easier to remember it in its symmetrical ratio-form, rather than in the final form mentioned.

By symmetry, we have the following corresponding formulae:

\[
\begin{align*}
\tan \frac{B-C}{2} &= \frac{c-a}{c+a} \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \\
\tan \frac{B+C}{2} &= \frac{a+b}{a-b} \quad \tan \frac{A+B}{2} = \frac{c+a}{c-a}.
\end{align*}
\]

Example II. Solve the triangle ABC, given that \( a = 24°76', b = 16°38', c = 36°26' \).

\[
\begin{align*}
\tan \frac{A-B}{2} &= \frac{a-b}{a+b} \\
\tan \frac{A+B}{2} &= \frac{a+b}{a-b}.
\end{align*}
\]

Now \( a-b = 24°76' - 16°38' = 8°38', \)
\( a+b = 24°76' + 16°38' = 41°14', \)
\( A+B = \frac{1}{2}(180° - C) = \frac{1}{2}(180° - 36°26') = \frac{1}{2} \text{ of } 143°34' = 71°47'; \)
\( \tan \frac{A-B}{2} = 8°38 \tan 71°47'; \)
\( \tan \frac{A+B}{2} = \frac{8°38}{41°14}; \)
\( A-B = 31°45'; \)
\( A+B = 71°47'; \)
\( \therefore \text{ adding, } A = 103°32', \text{ and subtracting, } B = 40°2'. \)
17. Find $\Delta$ and $R$ if $a=8:02$, $b=7:14$, $c=9:88$.
18. With the data of No. 17, find $r$.
19. Find $\tan \frac{A}{2}$ if $a:b:c = 4:5:6$.
20. What does the formula for $\tan \frac{A}{2}$ become if $a=b=c$? Interpret the result.
21. Find $R$ if $b=148$, $c=188$, $A=74^\circ 30'$.
22. With the data of No. 21, find $r$.
23. Find the angles of the trapezium in Fig. 364.

![Fig. 364](image)

25. If $a=4$, $b=3$, $c=6$, prove $C=2A$.
26. If $\angle ACB=90^\circ$, prove $\sin \frac{B}{2} = \frac{c-a}{2c}$.
27. If $A=60^\circ$ and $b=2c$, find $B$ and $C$.
28. If the sides of a triangle are $m^2+m+1$, $2m+1$, $m^2-1$, where $m > 1$, prove that the greatest angle is $120^\circ$.

Problems on heights and distances.

The practical surveyor obtains data for his calculations by observations of points on the ground. The principal instruments used are a chain for measuring the length of a base-line and a theodolite for measuring angles either in a vertical or in a horizontal plane. Angles in an oblique plane are not measured, and in this connection the reader is reminded that all compass bearings refer to angles in a horizontal plane (see p. 73).

For the practical details to be observed in making a survey the reader is referred to technical books on surveying; the simplest and most interesting problems that arise are concerned with the determination of the heights of a distant point. Some of the cases that may arise are given below, the measurements being denoted by letters because the methods to be used in numerical computation have already been illustrated in full. It is suggested that the reader should first attempt to obtain the solution from the data for himself, and only consult the solution given if necessary.

Example III.

$AB$ is a horizontal base line 1 ft. long in the same vertical plane as a point $P$. The angles of elevation of $P$ from $A$, $B$ are $\theta$, $\phi$ respectively. Find the height $PM$ of $P$ above $AB$.

![Fig. 365](image)

\[ \angle APB = \phi - \theta; \]
\[ \therefore \text{in } \triangle PAB, \quad \frac{PB}{\sin \theta} = \frac{l}{\sin (\phi - \theta)}; \]
\[ \therefore \quad PB = \frac{l \sin \theta}{\sin (\phi - \theta)}; \]
\[ \therefore \quad PM = PB \sin \phi = \frac{l \sin \theta \sin \phi}{\sin (\phi - \theta)} \text{ feet.} \]

Note. This result could also be obtained in the equivalent form $\frac{l}{\cot \theta - \cot \phi}$ from the right-angled triangles $APM$, $BPM$, but this form is not so well adapted for use with logarithms. In similar examples it is shorter and more accurate to work with the sine formula, rather than with two right-angled triangles. (Cf. Exercises XVII. b., Nos. 1-3.)
Example IV. \( AB \) is a base line 1 ft. long in the same vertical plane as \( P \). The angles of elevation of \( P \) from \( A \) and \( B \) are \( \theta \) and \( \phi \); the angle of elevation of \( B \) from \( A \) is \( \psi \). Find the height \( PM \) of \( P \) above \( A \). (See Fig. 366.)

Produce \( PB \) to meet \( AM \) at \( Q \).

\[
\angle PBA = \angle BQA + \angle BAQ = 180^\circ - \phi + \psi,
\angle APB = \angle PBM = \angle PAQ = \phi - \theta;
\]

\[
\therefore \text{from } \triangle PAB, \quad \frac{PA}{\sin (180^\circ - \phi + \psi)} = \frac{PB}{\sin (\phi - \theta)};
\]

\[
\therefore \quad PA = \frac{l \sin (\phi - \psi)}{\sin (\phi - \theta)};
\]

\[
\therefore \quad PM = PA \sin \theta = \frac{l \sin (\phi - \psi) \sin \theta}{\sin (\phi - \theta)} \text{ feet.}
\]

Note, as a check, that if \( \psi = 0 \), this result reduces to that of Example III.

Example V. \( AB \) is a horizontal base line 1 ft. long, not in the same vertical plane as \( P \). The angle of elevation of \( P \) from \( A \) is \( \theta \); the difference of the bearings of \( B \) and \( P \) from \( A \) is \( \alpha \), and the difference of the bearings of \( A \) and \( P \) from \( B \) is \( \beta \). Find the height \( PM \) of \( P \) above \( A \). (See Fig. 367.)

Note that \( \alpha \) and \( \beta \) are angles in a horizontal plane, being obtained by use of a theodolite.

\[
\angle AMB = 180^\circ - \alpha - \beta;
\]

\[
\therefore \text{from } \triangle AMB, \quad \frac{AM}{\sin \beta} = \frac{l}{\sin (180^\circ - \alpha - \beta)}.
\]

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\[
\therefore AM = \frac{l \sin \beta}{\sin (\alpha + \beta)};
\]

\[
\therefore PM = AM \tan \theta = \frac{l \sin \beta \tan \theta}{\sin (\alpha + \beta)} \text{ feet.}
\]

Note. (i) In practical work this result would be checked by

observing in addition the angle of elevation \( \phi \) of \( P \) from \( B \); a similar calculation would then give

\[
PM = \frac{l \sin \alpha \tan \phi}{\sin (\alpha + \beta)}.
\]

(ii) The distance of \( P \) from \( A \) in the preceding Examples III–V is \( PM \cos \theta \), but this distance, as it appears on a map, is the horizontal projection \( AM \), which is \( PM \cot \theta \).

Exercise XVII. b.

1. From the top of a house 42 ft. high the elevation of a spire is 14° 18' and from the bottom of the house it is 23° 19'. Find the height of the spire.

2. From a point \( C \) a wireless mast \( AB \), 300 ft. high, is visible on a hill; the angles of elevation of \( A \) and \( B \) from \( C \) are 7° 35' and 10° 10'. Find the height of \( A \) above \( C \).

3. \( AB \) is a horizontal line 1500 yd. long. An aeroplane crosses the vertical plane through \( AB \) between \( A \) and \( B \); its elevation from \( A \) at this moment is observed to be 42° and from \( B \) 61° 30'. Find the height of the aeroplane above \( AB \).

4. \( A, B \) are two points on a level plain, 3 miles apart. \( P, Q \) are conspicuous landmarks in line with \( B \) on the same level, and angles are observed as in Fig. 368. Find the distance \( PQ \).
5. The distance of a lighthouse L from a point A is 15 miles, and its bearing is N. 62° E. What is the distance and bearing of L from a ship which has travelled 10 miles from A on a course N. 33° E.?

6. B is 200 ft. above the horizontal plane APQ (Fig. 369). BA is vertical; Q is due East of B and P is S. 41° E. of B. Find PQ.

7. In Fig. 370 A, C, D are three points on the same level and AB is vertical. Find AB.

8. A whaler, travelling N.W. at 10 knots, observed a stationary iceberg N.E. and the elevation of its summit was 2° 30'. Six minutes later the iceberg was on a bearing S. 78° E. Calculate the height of the summit above the observer in feet. (1 sea mile = 6960 ft.)

9. An aeroplane is observed to be due North of A at an elevation 38° 30', and at the same moment it bears N. 42° W. from a point B, 3000 yards east of A on the same level as A. Find the height of the aeroplane above A.

10. A, B in Fig. 371 represent successive positions of an aeroplane flying horizontally. AM and BN are vertical and M, X, N are in a horizontal plane. If the aeroplane travels from A to B in 2 minutes, find its speed in miles per hour.

11. From a stationary balloon at an altitude of 4000 ft. two places P and Q at sea-level are observed. P bears due West and Q bears N. 14° W. The angles of depression of P and Q are 33° 10' and 27° 30' respectively. Find the distance PQ.

12. From a point A, S.E. of a tower CD, the elevation of the top D is 32°; an observer walks 300 ft. from A in a direction S. 80° W. to B, and then finds that the tower bears N. 37° E. If A, B, C lie in a horizontal plane, find the height of the tower.

13. A lighthouse L is observed from a ship S to bear N. 42° 10' W. after steaming 6 miles due N. and then 6 miles N. 38° E., the bearing of L from S is N. 65° 12' W. How far is S from L at the second observation?

14. From A, a peak P bears due North and its elevation is 15° 10'; from B, two miles West of A and level with A, the elevation of P is 14° 50'. Find in feet the height of P above A.

15. From the top of a cliff 200 ft. high two ships are observed, one due South with angular depression 15°, the other S. 42° W. with angular depression 11° 30'. Find the bearing and distance of the second ship from the first.

16. From a point S, 475 ft. above sea-level, the angles of depression of two buoys A, B on the sea are 10° 32' and 14° 54'; also \( \angle ASB = 60° 13' \). Find the distance of A from B.

17. X, Y are two points 300 yd. apart in the same horizontal plane at the foot N of a tower PN. The elevation of P from X is 7° 38'; also \( \angle NXY = 64° 20' \) and \( \angle NYX = 57° 35' \). Find the height of the tower in feet.

18. From a train running S.E. at 25 m.p.h. an aeroplane is observed to be due North at elevation 30°; two minutes later it bears N. 60° E. at elevation 11°. The aeroplane is flying due East at a constant height. Find its height and speed.

19. H, L, N, R represent Hereford, Ledbury, Newent, Ross; the given distances are in miles. Neglecting differences of level, find the distance of Hereford from Newent. (See Fig. 372.)

20. In Fig. 373, A, B are observation points not on the same level. M, A, N are points in the same horizontal plane, and PB, BM are vertical. From observations with a theodolite, \( \angle NMA = 41° 35' \), \( \angle NAM = 65° 14' \). If \( AB = 3000 \) ft., find the height of P above A.
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The following exercise may be reserved for a second reading:

EXERCISE XVII. e.

1. AP, BQ are two vertical poles, 10 ft., 16 ft. high, at a distance 12 ft. apart; their tops P, Q are joined by a taut rope; the shadow of AP on the ground, which is level, is 7 ft. long and the bearing of the Sun is S. 10° W.; A is due East of B. Find the bearing of the shadow of the rope on the ground.

2. AB, CD are two chimneys of equal height, standing on level ground; a person walks 100 ft. from O in a direction perpendicular to AC, and finds that the elevations of B, D are 45°, 60°. Find AB, AC. (See Fig. 374.)

3. A rod AB, hinged at A, rests with B on a level table CD; AB is rotated through an angle β in a vertical plane. Find the new height of B above CD. [See Fig. 375.]

4. When the lamina in Fig. 376 is suspended from O it hangs so that G is vertically below O. Find an expression for the angle θ which OE makes with the vertical.

5. ABCD is a parallelogram; AB = 8 cm., BC = 5 cm., ∠ABC = 115°; the parallelogram rotates about AC as axis; through what distance does B move per revolution?

6. AB is a tower seen from points D, E on a hill-side in the same vertical plane as AB. (See Fig. 377); ED = DC = OA = 400 yd., AF = 80 yd., FG = 40 yd. Find θ and the height of the tower.

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7. ABCD is a tetrahedron such that AB = AC = DB = DC = 4 in., BC = 3 in., AD = 5 in. Find (i) the inclination of AD to the plane ABC; (ii) the angle between the planes OAB, DAC.

8. Fig. 378 represents a pair of shear logs OA, OB and a back log OC used for unloading ships. A, B, C are in a horizontal plane; OC is at right angles; OA = 41 ft., AB = 18 ft., OC = 60 ft., CN = 25 ft. Find the inclination of (i) the plane OAB, (ii) the log OA to the horizontal.

9. The roofs of the buildings around a rectangular courtyard are inclined at 38° to the horizontal. Find the inclination to the horizontal of the line of intersection of the roofs on two adjacent sides.

10. Prove that the area of a triangle ABC can be expressed in the form

\[ \frac{1}{2} \sin A (\cos (B - C) + \cos A). \]

Hence find the angles B, C of the triangle ABC, given that its area is 8 sq. in., BC = 5 in., \( \angle BAC = 57°. \)

11. From a ship steaming due North at 5 knots, a lighthouse is observed to bear N. α° E.; one hour later it bears S. β° E. Find the distance of the lighthouse from the ship's course.

12. An aeroplane flying horizontally at uniform speed is seen due East at an elevation θ; ten minutes later it bears due North at an elevation φ; show that it bears N.W. after another

\[ \frac{10 \cot \phi}{\cot β - \cot φ} \] minutes.

13. From a point C due N. of a tower AB, the angle of elevation of B is α, and from a point D due E. ofAB the angle of elevation of B is β; CD = l ft. and C, D lie in the horizontal plane through A; prove that \( AB = l \sqrt{(\cot α + \cot β)}. \)

14. The angular elevation of the summits of two spires which appear in line is α, and the angular depressions of their images in still water as viewed from a point c ft. above the water level are β, γ. Show that the horizontal distance between the spires is

\[ \frac{2c \cos α \sin (β - γ)}{\sin (β - α) \sin (γ - α)} \] feet.

15. The roofs of the buildings along two adjacent sides of a rectangular court make angles θ, φ with the horizontal and the line of intersection of the roofs makes an angle α with the horizontal. Prove that \( \cot α = \cot θ + \cot φ. \)
CHAPTER XVIII.

EQUATIONS AND ELIMINATION.

The various methods that can be used in solving trigonometrical equations are best illustrated by examples. There is no general rule.

It must be remembered that although the number of solutions of an algebraic equation is finite, and the number of values of, say, \( \sin \theta \) in a trigonometrical equation is finite, yet there is no limit to the number of possible values of \( \theta \). The general solution of a trigonometrical equation is considered later in Part IV.; in this chapter we shall be concerned only with a finite number of solutions, and for the most part solutions contained between 0 and 360°.

Note. The angles throughout this chapter must be understood to be in degrees; the sign \( ^{°} \) has been omitted for convenience.

Example 1. Solve the equation \( 2 \sin^2 \theta + \cos \theta = 1 \), obtaining all possible solutions from 0 to 360°.

[Use the relation \( \sin^2 \theta = 1 - \cos^2 \theta \) to obtain an equation involving one unknown ratio only.]

Then \( 2(1 - \cos^2 \theta) + \cos \theta = 1 \);

\( \therefore 2 - 2 \cos^2 \theta + \cos \theta - 1 = 0 \);

\( \therefore 2 \cos^2 \theta - \cos \theta - 1 = 0 \);

\( \therefore (2 \cos \theta + 1)(\cos \theta - 1) = 0 \);

\( \therefore \cos \theta = 1 \) or \( \cos \theta = -\frac{1}{2} \).

EXERCISE XVIII. a.

Solve the following equations, giving all solutions from 0 to 360°:

1. \( \sin \theta = 0.42 \).
2. \( \cos \theta = \frac{1}{3} \).
3. \( \cos \theta = -\frac{1}{3} \).
4. \( \tan \theta = 4 \).
5. \( \csc \theta = 3 \).
6. \( \cot \theta = \frac{1}{3} \).
7. \( \sec \theta = 6 \).
8. \( 5 \sin \theta + 1 = 0 \).
9. \( 3 \cos \theta + 2 = 0 \).
10. \( 4 \sin^2 \theta = 3 \).
11. \( 4 \cos^2 \theta = 3 \).
12. \( \tan^2 \theta = 2 \).
13. \( 4 \cos \theta = \sec \theta \).
14. \( \tan \theta - 2 \sin \theta = 0 \).
15. \( 3 \sin \theta = 2 \cos \theta \).
16. \( 2 \sin^2 \theta = 2 + 3 \cos \theta \).
17. \( 5 \cos^2 \theta + 2 \sin \theta = 2 \).
18. \( 1 + \cos \theta = \sin^2 \theta \).
19. \( 2 \cos^2 \theta = 1 + \sin \theta \).
20. \( 3 \sec \theta = 5 \tan \theta = 5 \).
21. \( \tan \theta + 2 \cos \theta = 3 \).
22. \( 2 \sin \theta = 5 \tan \theta = 5 \).
23. \( 6 \tan \theta = 5 \cos \theta \).
24. \( 3 \tan \theta \) = 2 \tan \theta.
25. \( \sin^2 \theta + 2 \sin \theta \cos \theta + 3 \cos^2 \theta \).
26. \( 2 \sin^3 \theta = 3 \sin \theta \cos \theta + 2 \cos^3 \theta \).
27. \( 7 \sin \theta \cos \theta + \cos^2 \theta = 2 \).
28. \( 2 \cos 2\theta = 1 - 4 \cos \theta \).

The solutions of the equations in the next exercise depend on the formulae obtained in Chapters XV, XVI.
Example II. Solve the equation \(6 \sin \theta = \sec \theta\), obtaining solutions between 0 and 360°.

\[6 \sin \theta = \sec \theta\]
\[\therefore 6 \sin \theta = \frac{1}{\cos \theta}\]
\[\therefore 6 \sin \theta \cos \theta = 1\]
\[\therefore 3 \sin 2\theta = 1\]
\[\therefore \sin 2\theta = \frac{1}{3}\]

Now \(\frac{1}{3} \approx \sin 19^\circ 28'\) from the Tables.
Hence, from Fig. 330, we can say that possible values of 2θ are
\[2\theta = 19^\circ 28', 169^\circ 28', 339^\circ 28', 529^\circ 28', \ldots\]

\[\therefore \theta = 9^\circ 44'\] or \(80^\circ 16'\) or \(189^\circ 44'\) or \(269^\circ 16'\).

Note that to obtain values of \(\theta\) between 0 and 360°, it is necessary to consider values of 2θ between 0 and 720°.

Example III. Solve
\[\sin 5\theta - \sin 3\theta + \sin \theta = 0\]
for values of \(\theta\) from 0 to 180°.

[Here we notice that if we use the fact that
\[\sin 5\theta + \sin \theta = 2 \sin 3\theta \cos 2\theta\]
we shall obtain \(\sin 3\theta\) as a common factor of the left-hand side of the equation.]

Then
\[2 \sin 3\theta \cos 2\theta - \sin 3\theta = 0\]
\[\therefore \sin 3\theta (2 \cos 2\theta - 1) = 0\]
\[\therefore \text{either } \sin 3\theta = 0 \text{ or } 2 \cos 2\theta = 1\]

Exercise XVIII. Solve the following equations, obtaining all solutions from 0 to 360°:

1. \[4 \cos \theta = \csc \theta\]
2. \[3 \sin \theta = \sec \theta\]
3. \[\sin 4\theta + \sin 2\theta = 0\]
4. \[\cos 4\theta - \cos 2\theta = 0\]
5. \[2 \sin (\theta + 20^\circ) = 1\]
6. \[\sin (\theta + 10^\circ) = 1\]
7. \[\sin \theta + \sin (\theta + 20^\circ) = 1\]
8. \[4 \cos \theta \cos (\theta + 20^\circ) = 1\]
9. \[\sin \theta \cos (\theta - 60^\circ) = 0.3\]
10. \[\cos \theta - \cos (\theta + 45^\circ) = 0.4\]
11. \[\cos \theta + \cos 2\theta + \cos 3\theta = 0\]
12. \[\sin 7\theta - \sin \theta = \sin 3\theta\]
13. \[\sin \theta + \sin 2\theta + \sin 3\theta = 0\]
14. \[\sin 9\theta = \sin 5\theta - \sin 2\theta\]
15. \[\cos 3\theta + \cos 5\theta + \cos 7\theta = 0\]
16. \[\cos 2\theta - \cos \theta + 1 = 0\]
17. \[\cos 5\theta - \cos 2\theta + \cos \theta = 1\]
18. \[3 \sin 2\theta - 5 \sin \theta = 0\]
19. \[\cos \theta = \csc 2\theta\]
20. \[\cos 3\theta = \cos 2\theta \cos \theta\]
21. \[\sin (\theta + 30^\circ) = 2 \sin (\theta - 30^\circ)\]
22. \[\tan \theta + \tan (\theta + 45^\circ) = 1\]
23. \[2 \tan \theta + \tan (\theta + 45^\circ) = 1\]
24. \[4 \sin (\theta + 60^\circ) \sin (\theta - 60^\circ) = \sin \theta\]

Example IV. Find solutions from \(-90^\circ\) to \(+90^\circ\) of

(i) \[3\theta = \sin 2\theta\]
(ii) \[2\theta = \cos 3\theta\]

(i) If \(3\theta = \sin 2\theta\), then
\[3\theta = 2\theta + 180^\circ - 2\theta\]
or \(3\theta\) may differ from \(2\theta\) or from \((180^\circ - 2\theta)\) by any multiple of \(360^\circ\).
Possible solutions are therefore

\[ 36° = 2\theta, \ (360° + 2\theta), \ (-360° + 2\theta), \ \ldots \ \text{etc.}, \]

and

\[ 30° = (180° - 2\theta), \ (540° - 2\theta), \ (-180° - 2\theta), \ \ldots \ \text{etc.} \]

These reduce to equations \( \theta = 0, \ 360° \ \text{or} \ -360°, \ \text{etc.}, \)

and

\[ 54° = 180°, \ 540°, \ -180°, \ \text{etc.}; \]

\( \therefore \) solutions required are \( \theta = 0 \) or \( 36° \ \text{or} \ -36°. \)

(ii) If \( \sin 2\theta = \cos 3\theta, \) then by the property of complementary angles,

\[ \sin 2\theta = \sin (90° - 3\theta). \]

We then have the following possibilities:

\[ 2\theta = 90° - 3\theta, \ 450° - 3\theta, \ -270° - 3\theta, \ \text{etc.}, \]

and

\[ 2\theta = 180° - (90° - 3\theta), \ 540° - (90° - 3\theta), \ -180° - (90° - 3\theta), \ \text{etc.}. \]

Hence

\[ 5\theta = 90° \ \text{or} \ 450° \ \text{or} \ -270°, \ \text{etc.}, \]

or

\[ \theta = 90° \ \text{or} \ 450° \ \text{or} \ -270°, \ \text{etc.}; \]

\( \therefore \) the solutions required are \( 18°, \ 90°, \ -54°, \ -90°. \)

\( \sin 18° \) and \( \sin 54°. \)

Example IV. (ii) shows that \( \theta = \pm 90°, \ \theta = 18°, \ \theta = -54° \) are four particular solutions of \( \sin 2\theta = \cos 3\theta. \)

This equation may be solved as follows:

\[ 2 \sin \theta \cos \theta = 4 \cos \theta - 3 \cos \theta; \]

\[ \therefore \ \cos \theta = 0 \ \text{or} \ 2 \sin \theta = 4 \cos \theta - 3, \]

\( \cos \theta = 0 \) corresponds to \( \theta = \pm 90°. \)

From the other condition, we have

\[ 2 \sin \theta = 4 (1 - \sin^2 \theta) - 3; \]

\[ \therefore \ 4 \sin^2 \theta + 2 \sin \theta - 1 = 0; \]

\( \therefore \) \( \sin \theta = \frac{-1 + \sqrt{5}}{4} \) or \( \frac{-1 - \sqrt{5}}{4} \).

Now \( \theta = 18° \) does not satisfy \( \cos \theta = 0, \) nor does it satisfy

\( \sin \theta = \frac{-1 - \sqrt{5}}{4}, \) since this is negative;

\[ \therefore \ \sin 18° = \frac{-1 + \sqrt{5}}{4}. \]

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Again, \( \theta = -54° \) does not satisfy \( \cos \theta = 0, \) nor does it satisfy \( \sin \theta = \frac{-1 + \sqrt{5}}{4}, \) since this is positive;

\[ \therefore \ \sin (-54°) = \frac{-1 - \sqrt{5}}{4}. \]

But \( \sin (-54°) = - \sin 54°; \quad \therefore \ \sin 54° = \frac{1 + \sqrt{5}}{4}. \)

Note. \( \cos 36° = \sin 54° = \frac{1 + \sqrt{5}}{4}; \ \cos 72° = \sin 18° = \frac{-1 + \sqrt{5}}{4}. \)

The other ratios may be obtained from the fundamental relations connecting \( \sin, \ \cos, \ \tan, \ \text{etc.} \)

\[ \text{EXERCISE XVIII.} \]

Solve the following equations, Nos. 1-10, finding values of \( \theta \) from 0 to 180°:

1. \( \sin \theta = \sin 20°. \)
2. \( \sin \theta = \cos 20°. \)
3. \( \sin (\theta + 10°) = \sin (30° + 40°). \)
4. \( \sin (\theta - 10°) = \cos 30°. \)
5. \( \tan 3\theta = \tan 70°. \)
6. \( \tan 3\theta = \cot \theta. \)
7. \( \cos 59° = \cos 29°. \)
8. \( \sin 20° \ \cos \theta = \sin 30° \ \cos 25°. \)
9. \( \sin 4\theta \ \cos \theta = \sin 30° \ \cos 29°. \)
10. \( \tan (\theta + 50°) = \cot \theta. \)

11. Find \( \cos 36° \) by solving \( \sin 39° = \sin 29°; \) hence show that \( \sin 36° = \frac{1}{2} (10 - 2\sqrt{5}). \)

12. Use the fact that \( \sin a \pm \cos a = \sqrt{2} \sin (a \pm 45°) \) to solve \( \sin 29° + \cos 29° = \sin \theta - \cos \theta \) between 0 and 90°.

The equation \( a \cos \theta + b \sin \theta = c. \)

Example V. Find an acute angle such that

\[ 8 \cos \theta - \sin \theta = 4. \]

First method. Use the relations

\[ \cos \theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \quad \text{and} \quad \sin \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \]

(see p. 220),

and put \( \frac{\theta}{2} = t. \)
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Then
\[ \frac{8(1 - \phi)}{1 + \phi^2} = \frac{2t}{1 + \phi^2} = 4. \]
\[ 8 - 8\phi^2 - 2t = 4 + 4\phi^2; \]
\[ \therefore 12\phi^2 + 2t - 4 = 0 \text{ or } 6\phi^2 + t - 2 = 0; \]
\[ \therefore (2t - 1)(3\phi + 2) = 0; \]
\[ \therefore t = \frac{1}{2} \text{ or } -\frac{2}{3}. \]

Since an acute angle solution is required, we must take
\[ \phi = \frac{1}{2}, \] which gives \[ \phi = 26° 34'; \]
\[ \therefore \phi = 53° 8'. \]

Second method. Construct an acute angle \( \alpha \), so that \( \tan \alpha = \frac{1}{2}. \)
From Fig. 382, (not drawn to scale),
\[ \cos \alpha = \frac{8}{\sqrt{65}}, \quad \sin \alpha = \frac{1}{\sqrt{65}}. \]

The given equation may now be written
\[ \frac{8}{\sqrt{65}} \cos \phi - \frac{1}{\sqrt{65}} \sin \phi = \frac{4}{3}; \]
\[ \therefore \cos \alpha \cos \phi - \sin \alpha \sin \phi = \frac{4}{3}; \]
\[ \therefore \cos (\phi + \alpha) = 0.4962; \]
\[ \therefore \phi + \alpha = 60° 15' \text{ or } 360° - 60° 15', \text{ etc.; } \]
but
\[ \alpha = \tan^{-1} \left( \frac{1}{2} \right) = 7° 7'; \]
\[ \therefore \text{taking the acute-angled value of } \phi, \]
\[ \phi + 7° 7' = 65° 15', \]
\[ \therefore \phi = 53° 8'. \]

Note. The following method of solution should never be used:
\[ 8 \cos \phi = 4 + \sin \phi. \]

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Square each side;
\[ \therefore 64 \cos^2 \theta = (4 + \sin \theta)^2; \]
\[ \therefore 64 - 64 \sin^2 \theta = 16 + 8 \sin \theta + \sin^2 \theta. \]

This is a quadratic in \( \sin \theta \), and can therefore be solved and
values of \( \theta \) obtained. But when each side of the equation was
squared, a step was taken which is not reversible. For the
equation \( 64 \cos^2 \theta = (4 + \sin \theta)^2 \) is equivalent to
\[ 8 \cos \theta = \pm (4 + \sin \theta). \]

Consequently some of the solutions obtained will be roots of
\[ 8 \cos \theta = -(4 + \sin \theta). \]

EXERCISE XVIII. d.

Solve the following equations, Nos. 1-8, for values of \( \theta \) from
0 to 180°:
1. \( 3 \sin \theta + 4 \cos \theta = 5. \)
2. \( \cos \theta + 3 \sin \theta = 2. \)
3. \( 3 \sin \theta + 10 \cos \theta = 11. \)
4. \( 6 \cos \theta - 3 \sin \theta = 4 - 0. \)
5. \( 10 \tan \theta - 2 \sec \theta = 5. \)
6. \( 4 \cot \theta + \cosec \theta = 3. \)
7. \( \sin 2\theta = 1 + \cos 2\theta. \)
8. \( 3 \sin 2\theta = 1 + 2 \cos 2\theta. \)
9. Find the angles of an isosceles triangle in which the perimeter
is four times the height.
10. Show that if \( a^2 + b^2 < c^2 \), the equation \( a \cos \theta + b \sin \theta = c \) has
no real roots.

EXERCISE XVIII. e.

Solve the following equations, giving all solutions from 0 to
360°, unless otherwise stated:
1. \( \sin 3\theta = \sin \theta. \)
2. \( 2 \cos^2 \theta + 1 = 3 \cos \theta. \)
3. \( 4 \tan \theta = 9 \cot \theta. \)
4. \( 4 \cosec 30° = \sec 20°. \)
5. \( \tan 30° = \cot 50° \) (between 0° and 90°).
6. \( \cos 30° + \sin 30° = 0 \) (between 0° and 90°).
7. \( 1 - \sin \theta = \cos^2 \theta. \)
8. \( \sec^2 \theta = 1 + \tan \theta. \)
9. \( \cos^2 \theta + 3 \sin \theta = 2. \)
10. \( \sin 4\theta = 3 \sin^2 2\theta. \)
11. \( 2 \sin^2 \theta + \sin^2 2\theta = 2. \)
12. \( 16 \tan^2 \theta = 15 \sin \theta. \)
13. \( 3 \sin 2\theta = 5 \tan \theta. \)
14. \( \cos \theta \cot \theta = 1 + \sin \theta. \)
15. \( \tan 2\theta = 3 \tan \theta \).  
16. \( 3 \sin \theta = 2 \sin (90^\circ - \theta) \).

17. \( \sin \theta = 2 \cos (45^\circ + \theta) \).  
18. \( 4 \sin \theta \sin 3\theta = 1 \).

19. \( 2 \cos 2\theta = \cos \theta - \sin \theta \).  
20. \( 3 \theta = \cos \theta - \sin \theta \).

21. \( \sin 3\theta + \cos 3\theta = -\sin 2\theta - \cos 2\theta \).  
22. \( \sin \theta = 2 \cos \theta \).

23. \( \sin \theta + \cos \theta = 0.6. \)  
24. \( \sin \theta - \cos \theta = 0.6 \).

25. \( \tan^{-1} \left( \frac{2}{3} \right) - \tan^{-1} \left( \frac{1}{5} \right) = \tan^{-1} \left( \frac{1}{2} \right) \).

26. \( \tan \phi = \frac{90^\circ}{\theta} \); \( \theta + 4 \sin \phi = 4 \).

27. \( \tan^{-1} (2x) + \tan^{-1} (3x) = \tan^{-1} (1) \).

28. \( \tan (45^\circ + \theta) = 3 \tan (45^\circ - \theta) \).

29. \( \cos 3\theta = \sin (\theta + 90^\circ) \).

30. \( \tan 2\theta + \cot \theta = 8 \cos^2 \theta \).

Elimination. If we have two independent equations, each containing a certain variable, it is possible to deduce an equation from which that variable is excluded; this process is known as elimination.

Example VI. Eliminate \( \theta \) from the equations

\[ x \sin \theta + y \cos \theta = a; \quad x \cos \theta - y \sin \theta = b. \]

We have

\[ (x \sin \theta + y \cos \theta)^2 + (x \cos \theta - y \sin \theta)^2 = a^2 + b^2; \]

\[ \therefore \quad x^2 (\sin^2 \theta + \cos^2 \theta) + y^2 (\cos^2 \theta + \sin^2 \theta) = a^2 + b^2; \]

\[ \therefore \quad x^2 + y^2 = a^2 + b^2. \]

Example VII. Eliminate \( \theta \) from the equations

\[ \cos \theta = \sin \theta = a; \quad \sec \theta = \cos \theta = b. \]

\[ \frac{1}{\sin \theta} - \sin \theta = a; \quad \therefore \quad 1 - \sin^2 \theta = a \sin \theta; \]

\[ \therefore \quad \cos^2 \theta = a \sin \theta; \]

similarly,

\[ \sin^2 \theta = b \cos \theta; \]

\[ \therefore \quad \sin^2 \theta \cos \theta = ab \sin \theta \cos \theta \quad \text{or} \quad \sin \theta \cos \theta = -ab; \]

\[ \therefore \quad \cos^2 \theta = a \sin \theta \cos \theta = a^2 b; \]

and

\[ \sin^2 \theta = b \sin \theta \cos \theta = a^2 b; \]

\[ \therefore \quad \cos^2 \theta = (-a^2 b)^{\frac{1}{2}} \text{ and } \sin^2 \theta = (a^2 b)^{\frac{1}{2}}; \]

\[ \therefore \quad a^3 b^3 + a^3 b^3 = 1. \]
CHAPTER XIX.

MISCELLANEOUS IDENTITIES.

This chapter is entirely concerned with formal work in the manipulation of trigonometrical expressions. Facility in this work can only be acquired by constant practice and revision at frequent intervals. For this reason a large number of examples is provided, far more than should be taken at a first reading.

Example I. Prove that
\[
\sin 2A + \sin 2B + \sin 2C - \sin(2A + B + C)
= 4 \sin(A + B) \sin(B + C) \sin(C + A).
\]
\[
\{\sin 2A + \sin 2B \} + \{\sin 2C - \sin(2A + B + C)\}
= 2 \sin(A + B) \cos(A - B) - 2 \sin(A + B) \cos(A + B + 2C)
= 2 \sin(A + B) \{\cos(A - B) - \cos(A + B + 2C)\}
= 2 \sin(A + B) \{2 \sin(A + B) \sin(B + C)\}
= 4 \sin(A + B) \sin(B + C) \sin(C + A).
\]

Example II. Find the maximum value of
\[
\cos(\theta + 60^\circ) \cdot \cos(\theta - 20^\circ),
\]
and a value of \(\theta\) for which it is a maximum.
\[
\cos(\theta + 60^\circ) \cdot \cos(\theta - 20^\circ) = \frac{1}{2} \{\cos(2\theta + 40^\circ) + \cos 80^\circ\},
\]
but \(\cos 80^\circ\) is constant and the maximum value of \(\cos(2\theta + 40^\circ)\) is 1, a value which it will have when \(2\theta + 40^\circ = 0\).
\[
\therefore \text{the maximum value required is}
\]
\[
\frac{1}{2}(1 + \cos 80^\circ) = \frac{1}{2}(1 + 0.1736)
= 0.587,
\]
and a value of \(\theta\) which will give this maximum is \(\theta = -20^\circ\).

Note. The minimum value is \(\frac{1}{2}(-1 + \cos 80^\circ)\).

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EXERCISE XIX. (a).

Prove the following identities, Nos. 1-16:
1. \(\cos(A + B) \cdot \cos(A - B) = \cos^2 A - \sin^2 B\).
2. \(2(\sin^2 A - \sin^2 B) = \tan(A + B)\).
3. \(\tan(\alpha + \beta) = \frac{2 \cos \alpha \cos \beta}{\sin \alpha + \sin \beta}\) where
\[
\tan \alpha = \cos(\alpha + \beta) \text{ and } \tan \beta = \cos(\alpha - \beta).
\]
4. \(\cot 10^\circ \cdot \tan 10^\circ = 2 \tan 10^\circ\).
5. \(\cot 70^\circ \cdot \tan 20^\circ = 2 \cot 40^\circ\).
6. \(\cot(\theta + 15^\circ) \cdot \tan(\theta - 15^\circ) = \frac{4 \cos 2\theta}{1 + 2 \sin 2\theta}\).
7. \(\sin(\alpha + 1) \theta = \sin(\alpha - 1) \theta = \tan \frac{\theta}{2}\).
8. \(\sin A \sin(B - C) + \sin B \sin(C - A) + \sin C \sin(A - B) = 0\).
9. \(\sin A + \sin B + \sin(A + B) = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{A + B}{2}\).
10. \(\cos A + \cos B + \cos C + \cos(A + B + C) = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \cos \frac{A + B + C}{2}\).
11. \(\cos \theta + \cos \left(\frac{\theta + 2\pi}{3}\right) + \cos \left(\theta + \frac{4\pi}{3}\right) = 0\).
12. \(\sin \theta + \sin \left(\frac{\theta + 2\pi}{3}\right) + \sin \left(\theta + \frac{4\pi}{3}\right) = 0\).
13. \(\sin(A + B - C) + \sin(B + C - A) + \sin(C + A - B) = \sin(A + B + C) = 4 \sin A \sin B \sin C\).
14. \(\cos(A + B + C) + \cos(A + B - C) + \cos(B + C - A) + \cos(C + A - B) = 4 \cos A \cos B \cos C\).
15. \(\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{4}\).
16. \(\tan A \cot B - \cot A \tan B = \frac{2(\cos 2B - \cos 2A)}{\sin 2A \sin 2B}\).
17. Find the maximum value of \(2 \cos(\theta + 10^\circ) \cos(\theta - 20^\circ)\), and a value of \(\theta\) which makes it a maximum.
18. Find the maximum value of \(\sin(\theta + 15^\circ) \cos(\theta - 25^\circ)\).
19. Find the minimum value of \(\sin(2x - 10^\circ) \cos(2x + 46^\circ)\).
20. A stone is thrown up a hill, which is a plane sloping at 10° to
the horizontal. If the velocity with which it is thrown is V ft. per
sec., the distance up the hill at which it will hit the ground is
\[ \frac{V^2}{16} \cos^2 10° \cos \theta' \sin (\theta' - 10°) \],
where \( \theta' \) is the angle of projection. Find \( \theta' \), if it is to hit the ground
as far as possible up the hill.

21. Find the maximum value of \( \cos \theta + \cos (\theta + 30°) \), and a value
of \( \theta \) which makes it a maximum.

22. Find the minimum value of \( \cos \theta - \sin (\theta - 10°) \).

23. In Fig. 383, OP is 10 in. and \( \theta \) varies.

Find the maximum value of
(i) OM + ON;
(ii) FN + PM;
(iii) PM + ON.

Prove the following identities:

24. \[ 8 \sin \left( \frac{\theta + \pi}{4} \right) \sin \left( \frac{\phi + \pi}{4} \right) \sin \left( \frac{\theta - \pi}{4} \right) \sin \left( \frac{\phi - \pi}{4} \right) = \cos 2(\theta + \phi) + \cos 2(\theta - \phi) \].

25. \( \tan (A - B) + \tan (B - C) + \tan (C - A) = \tan (A - B) \cdot \tan (B - C) \cdot \tan (C - A) \).

26. \[ \tan \left( \frac{\theta + \pi}{3} \right) + \tan \left( \frac{\theta - \pi}{3} \right) + \tan \theta \tan \left( \frac{\theta + \pi}{3} \right) + \tan \theta \tan \left( \frac{\theta - \pi}{3} \right) = -3 \].

27. \[ \sin \frac{\pi}{6} \sin \frac{2\pi}{9} \sin \frac{3\pi}{9} \sin \frac{4\pi}{9} \sin \frac{5\pi}{9} = \frac{3}{16} \].

28. \( \sin \alpha \sin \left( \frac{\pi}{3} - \alpha \right) \sin \left( \frac{\pi}{3} + \alpha \right) = \frac{\alpha}{\sin 3\alpha} \).

29. \( \cos \alpha \cos \left( \frac{\pi}{3} - \alpha \right) \cos \left( \frac{\pi}{3} + \alpha \right) = \frac{1}{4} \cos 3\alpha \).

30. \[ \tan \theta + \tan \left( \frac{\pi}{4} + \theta \right) + \tan \left( \frac{\pi}{4} + \theta \right) = 3 \tan 3\theta \].
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\[
\begin{align*}
\therefore \cos A + \cos B + \cos C &= 1 \\
&= 2 \sin \frac{C}{2} \cos \frac{A - B}{2} - 2 \sin \frac{C}{2} \\
&= 2 \sin \frac{C}{2} \left( \cos \frac{A - B}{2} - \cos \frac{C}{2} \right) \\
&= 2 \sin \frac{C}{2} \left( \cos \frac{A - B}{2} - \cos \frac{A + B}{2} \right) \\
&= 2 \sin \frac{C}{2} \cdot 2 \sin \frac{A}{2} \sin \frac{B}{2} \\
\therefore \cos A + \cos B + \cos C &= 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.
\end{align*}
\]

Note. (i) Having obtained \( \sin \frac{C}{2} \) as a factor, we then express the remainder of the expression in terms of \( A, B \).

(ii) It is often easier to start with the product term, here \( 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \), and express it as a sum.

Example V. If \( \alpha + \beta + \gamma = 90^\circ \), prove that

\[
tan \alpha \tan \beta + tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1.
\]

Since

\[
tan (\beta + \gamma) = tan (90^\circ - \alpha) = cot \alpha;
\]

\[
tan \beta + tan \gamma = \frac{1}{1 - tan \beta \tan \gamma} = tan \alpha;
\]

\[
tan \alpha (tan \beta + tan \gamma) = 1 - tan \beta tan \gamma;
\]

\[
tan \alpha tan \beta \tan \gamma + tan \gamma tan \alpha = 1.
\]

Alternative method.

\[
tan (\alpha + \beta + \gamma) = tan \alpha + tan \beta + tan \gamma - tan \alpha tan \beta tan \gamma;
\]

but, since \( \alpha + \beta + \gamma = 90^\circ \), \( \tan (\alpha + \beta + \gamma) \) is infinite;

\[
\therefore \tan \alpha \tan \beta + tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1.
\]

MISCELLANEOUS IDENTITIES

EXERCISE XIX. b.

1. Given that \( A + B + C = 180^\circ \), simplify the following:

(i) \( \sin (A + B) \); (ii) \( \cos (B + C) \); (iii) \( \sin (A + B + C) \);

(iv) \( \tan (A + C) \); (v) \( \sin \frac{A + B}{2} \); (vi) \( \cos \frac{A + C}{2} \);

(vii) \( \tan \frac{B + C}{2} \); (viii) \( \sin \left( \frac{2A + 2C}{2} \right) \); (ix) \( \cos (2A + 2B) \);

(x) \( \cos \left( \frac{2A + 2B + C}{2} \right) \); (xi) \( \sin \left( \frac{3A}{2} + \frac{3B}{2} \right) \); (xii) \( \cot \left( \frac{B + C}{2} \right) \).

2. Given that \( A + B + C = 180^\circ \), express the following in terms of two letters:

(i) \( \sin B \); (ii) \( \cos C \); (iii) \( \tan A \);

(iv) \( \cos 2A \); (v) \( \sin 2C \); (vi) \( \tan 2B \);

(vii) \( \tan \frac{C}{2} \); (viii) \( \cos \frac{B}{2} \); (ix) \( \sin \frac{A}{2} \).

Prove the identities in Nos. 3-20, given that \( A + B + C = 180^\circ \):

3. \( \cos A + \cos B + \cos C = \sin A \sin B \sin C \).

4. \( \sin A - \sin B \cos C = \cos A \sin B \cos C \).

5. \( \sin (A + B) + \sin (A + C) = \sin B + \sin C \).

6. \( \sin A + \sin B \cos C = 1 - \cos 2C \).

7. \( \sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C \).

8. \( \cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C \).

9. \( \sin 4A + \sin 4B + \sin 4C = 4 \sin 2A \sin 2B \sin 2C \).

10. \( \cos 2A + \cos 2B + \cos 2C = 4 \cos A \cos B \cos C + 1 = 0 \).

11. \( \sin 4A - \sin 4B + \sin 4C = 4 \cos 2A \sin 2B \cos 2C \).

12. \( \cos 4A - \cos 4B + \cos 4C = 1 + 2 \cos 2A \cos 2B \sin 2C \).

13. \( \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \).

14. \( \sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \).

15. \( \cos A + \cos B + \cos C = 1 + 4 \cos \frac{A + B + C}{2} \).

16. \( \cos A \sin B + \cos B \sin C + \sin A \cos C \sin A + \sin C \sin A \sin B = 1 + \cos A \cos B \cos C \).

17. \( \sin A \cos (B - C) + \sin B \cos (C - A) + \sin C \cos (A - B) = 4 \sin A \sin B \sin C \).

18. \( \tan A + \tan B + \tan C = \tan A \tan B \tan C \).

19. \( \cot A + \cot B + \cot C = \cot A \cot B \cot C \).

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20. \( \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = -\cot \frac{A}{2} - \cot \frac{B}{2} - \cot \frac{C}{2} \).

Prove the identities in Nos. 21-26, given that \( \alpha + \beta + \gamma = 180^\circ \).

21. \( \cot \alpha + \cot \beta + \cot \gamma = -\cot \alpha \cot \beta \cot \gamma \).

22. \( \sin 2a + \sin 2\beta + \sin 2\gamma = 4 \cos \alpha \cos \beta \cos \gamma \).

23. \( \sin 2\beta + \sin 2\gamma - \sin 2a = 4 \cos \alpha \sin \beta \sin \gamma \).

24. \( \cos 2\alpha + \cos 2\beta - \cos 2\gamma = 4 \cos \alpha \cos \beta \sin \gamma - 1 \).

25. \( \sin \alpha + \sin \beta + \sin \gamma = \frac{1}{4} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2} \sin \frac{\alpha + \beta}{2} \).

26. \( \cos \alpha + \cos \beta + \cos \gamma = 4 \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2} \cos \frac{\alpha + \beta}{2} \).

27. If \( \theta + \phi + \psi = 0 \), prove that \( \sin \theta + \sin \phi + \sin \psi = -4 \sin \frac{\theta}{2} \sin \frac{\phi}{2} \sin \frac{\psi}{2} \).

28. If \( \theta + \phi + \psi = 360^\circ \), prove that \( \sin 2\theta + \sin 2\phi + \sin 2\psi = 4 \sin \theta \sin \phi \sin \psi \).

29. If \( \theta + \phi + \psi = 360^\circ \), express \( 1 + \cos 2\theta + \cos 2\phi + \cos 2\psi \) in simple factors.

30. If \( \theta + \phi + \psi = 270^\circ \), prove that \( \cos 2\theta + \cos 2\phi + \cos 2\psi = 1 - 4 \sin \theta \sin \phi \sin \psi \).

If \( \alpha + \beta + \gamma = 360^\circ \), prove the identities, Nos. 31-34:

31. \( \sin \alpha + \sin \beta + \sin \gamma + \sin \delta = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \delta}{2} \).

32. \( \cos \alpha + \cos \beta + \cos \gamma + \cos \delta = 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \delta}{2} = 0 \).

33. \( \sin \alpha - \sin \beta - \sin \gamma - \sin \delta = 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \delta}{2} = 0 \).

34. \( \tan \alpha + \tan \beta + \tan \gamma + \tan \delta = \tan \alpha \tan \beta \tan \gamma \tan \delta \).

If \( A + B + C = 180^\circ \), prove the following:

35. \( \sin (B + 2C) + \sin (C + 2A) + \sin (A + 2B) = 4 \sin \frac{B - C}{2} \sin \frac{C - A}{2} \sin \frac{A - B}{2} \).

36. \( \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{B + C}{4} \cos \frac{C + A}{4} \cos \frac{A + B}{4} \).
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6. \( \sin A \sin (B - C) = \cos C - \cos B \).
7. \( \sin B + \sin C = 1 + \cos A \cos (B - C) \).
8. \( \sin (X - Y) \sin (X + Y - 2Z) + \sin (Y - Z) \sin (Y + Z - 2X) + \sin (Z - X) \sin (Z + X - 2Y) = 0 \).
9. \( \cos 2A \cos (B - C) + \cos 2B \cos (C - A) + \cos 2C \cos (A - B) = 0 \).
10. \( \sin^2 A \cos (B - C) + \sin^2 B \cos (C - A) + \sin^2 C \cos (A - B) = 0 \).
11. \( \cos^2 \theta + \cos^2 (\theta + 60^\circ) + \cos^2 (\theta + 120^\circ) = 1 \).
12. \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \sin^2 (\alpha + \beta + \gamma) = 2 + 2 \cos (\beta + \gamma) \cos (\gamma + \alpha) \cos (\alpha + \beta) \).
13. \( \cot B - \cot C = \cosec A \cot (C - B) - \cosec B \cot (B - C) \).
14. \( \frac{\sin A - \sin B + \sin C}{2} \tan \frac{B}{2} = (\sin A + \sin B - \sin C) \tan \frac{C}{2} \).
15. \( \frac{1 - \cos A + \cos B + \cos C}{2} \tan \frac{B}{2} = (1 - \cos A - \cos B - \cos C) \tan \frac{A}{2} \).
16. \( \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \).
17. \( \cos \frac{A}{2} - \cos \frac{B}{2} + \cos \frac{C}{2} = 1 + \tan \frac{A}{4} \).
18. If \( u + v + x + y = 180^\circ \), prove that \( \cos u \cos v \cos x \cos y = \sin u \sin v \sin x \sin y \).
19. If \( u + v + x + y = 360^\circ \), prove that \( \cos u + \cos v + \cos x + \cos y = 4 \cos \frac{u + v}{2} \cos \frac{v + x}{2} \cos \frac{x + y}{2} \).
20. \( \sin^2 A + \sin^2 B + \sin^2 C = 3 \cos A \cos B \cos C + \cos \frac{A + 90^\circ}{2} \cos \frac{B + 90^\circ}{2} \cos \frac{C + 90^\circ}{2} \).
21. \( \sin^2 A \cos A + \sin^2 B \cos B + \sin^2 C \cos C = \sin A \sin B \sin C + \frac{1}{2} \sin 2A \sin 2B \sin 2C \).
22. \( \cot A + \cot B + \cot C = \frac{1}{\cot A} \cot B \cot C + \cot A \cot B \cot C \).
23. \( \tan A + \tan B + \tan C + \tan A \tan B \tan C = \frac{\sin 90^\circ}{\cos 90^\circ \cos 90^\circ} \).
24. If \( \theta = \alpha + \beta + \gamma \), prove that \( \cos (\theta - \alpha) \cos (\theta - \beta) \cos (\theta - \gamma) = \cos \theta \cos \alpha \cos \beta \cos \gamma + \sin \alpha \sin \beta \sin \gamma \).

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Fig. 284.

To prove that:
\[ r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \] and 
\[ r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \]

With the usual notation (see p. 185),
\[ BX = r \cot \frac{B}{2} \quad CX = r \cot \frac{C}{2} \]
\[ B + C \cot \frac{B + C}{2} = B + C = g \]

but \[ \cot \frac{B + C}{2} = \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}} \]
\[ \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}} = \frac{B + C}{\sin \frac{B}{2} \sin \frac{C}{2}} \]
\[ \frac{\cos A}{\sin \frac{B}{2} \sin \frac{C}{2}} \]

\[ \frac{\cos A}{2} \]
\[ r = \frac{A}{\sin \frac{A}{2}} = a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2}; \]

\[ r = 4R \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \]

Similarly, \( B \tan_1 = r \tan \frac{B}{2}, \) \( C \tan_1 = r \tan \frac{C}{2}. \)

\[ r_1 \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right) - B \tan_1 + C \tan_1 = c; \]

but \( \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{\sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \)

\[ = \frac{\sin \frac{B}{2} \cos \frac{C}{2} + \sin \frac{C}{2} \cos \frac{B}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} = \frac{B + C}{\cos \frac{B}{2} \cos \frac{C}{2}}. \]

Thus \( r_1 = 4R \sin \left( \frac{180^\circ - B}{2} \right) \sin \left( \frac{180^\circ - C}{2} \right) \sin \left( -\frac{A}{2} \right) \)

\[ = -4R \cos \frac{B}{2} \cos \frac{C}{2} \sin \frac{A}{2}. \]

Example VII. In any triangle, prove that

\[ \frac{b^2 - c^2}{a^2} = \frac{\sin (B - C)}{\sin (B + C)} \]

Since \( a = 2R \sin A, \) etc.,

\[ \frac{b^2 - c^2}{a^2} = \frac{4R^2 \sin^2 B - 4R^2 \sin^2 C}{4R^2 \sin^2 A} \]

\[ = \frac{\sin^2 B - \sin^2 C}{\sin^2 A} \]

\[ = \frac{\sin (B + C) \cdot \sin (B - C)}{\sin^2 (B + C)} \]

\[ = \frac{\sin (B - C)}{\sin (B + C)} \]
Example VIII. Prove that \( r_1 + r_2 + r_3 - r = 4R \).

First method.

\[
\begin{align*}
  r_1 + r_2 &= \Delta \frac{1}{s-a} + \frac{1}{s-b} = \Delta \frac{(s-b) + s-a}{(s-a)(s-b)} = \Delta \frac{c}{s-a} \\
  r_3 - r &= \Delta \frac{1}{s-c} = \Delta \frac{(s-c) + s-b}{(s-b)(s-c)} \\
  \therefore \quad r_1 + r_2 + r_3 - r &= \Delta \frac{(s-a) + (s-b) + (s-c)}{(s-a)(s-b)(s-c)} \\
  \therefore \quad r_1 + r_2 + r_3 - r &= 4R.
\end{align*}
\]

Second method.

\[
\begin{align*}
  r_1 + r_2 &= 4R \sin A \frac{1}{2} \cos \frac{B}{2} \cos \frac{C}{2} + 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \\
  &= 4R \cos \frac{A}{2} \left( \sin \frac{B}{2} \cos \frac{A}{2} + \cos \frac{B}{2} \sin \frac{A}{2} \right) \\
  &= 4R \cos \frac{A}{2} \sin \frac{A+B}{2} \\
  &= 4R \cos \frac{C}{2} \\
  r_3 - r &= 4R \sin A \frac{1}{2} \cos B \sin C - 4R \sin A \frac{1}{2} \sin B \sin C \\
  &= 4R \sin \frac{1}{2} \cos \frac{1}{2} \sin \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{A}{2} \cos \frac{B}{2} \\
  &= 4R \sin \frac{A+B}{2} \cos \frac{C}{2} \\
  &= 4R \sin \frac{C}{2} \cos \frac{A+B}{2} \\
  \therefore \quad r_1 + r_2 + r_3 - r &= 4R \left( \cos \frac{C}{2} + \sin \frac{C}{2} \right) = 4R.
\end{align*}
\]

Note. A summary of trigonometrical formulae will be found at the beginning of the book.
TRIGONOMETRY

33. \((a + b)^2 \sin^2 \frac{C}{2} + (a - b)^2 \cos^2 \frac{C}{2} = c^2\).

34. \(a^2 (\cos B - C + \cos A) = 4\Delta \sin A\).

35. \((b - c)^2 = a^2 - 4\Delta \tan \frac{A}{2}\).

36. \(r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2\).

37. \(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{a^2 + b^2 + c^2}{4\Delta}\).

38. \(n \cot A + b \cot B + c \cot C = 2(r + r_1)\).

39. \(2r - r_1 = 2R \cos A\).

40. \(\left(\frac{s}{a} - 1\right)\left(\frac{s}{b} - 1\right)\left(\frac{s}{c} - 1\right) = \frac{r^2}{4R^2}\).

41. \(\cos A + \cos B + \cos C = 1 + \frac{r}{R}\).

42. \(a \cot B - b \cot C = \frac{aR}{s}(\cos B - \cos C)\).

43. \(r_1 \cos A + r_2 \cos B + r_3 \cos C = \frac{a^2}{2R}\).

44. \(\Delta = r_1 r_2 \tan \frac{C}{2} = r_2 r_3 \cot \frac{A}{2} = \frac{cr_1}{r_1 + r_2} = r_3 \cot \frac{B}{2} = r_2 \cot \frac{C}{2}\).

45. \(\sin A \sin B \sin C = r_1 r_2 r_3 = r_1(r_2 + r_3) = r_2(r_3 + r_1) = r_3(r_1 + r_2)\).

46. \(2\Delta = ab \sin C = \frac{ab}{2} \sin A + \frac{ab}{2} \sin B + \frac{ab}{2} \sin 2C\).

47. \(R = \frac{1}{2}(a + b + c) \sec \frac{A}{2} = \frac{a + b + c}{2} \sec \frac{A}{2} \geq \frac{a + b + c}{2} \sec \frac{A}{2}\).

48. \(a^2 \cos (B - C) + b^2 \cos (C - A) + c^2 \cos (A - B) = 3abc\).

49. \(r_1 - r_2 (r_2 - r)(r_3 - r) = 4R \cdot r^2\).

50. \(r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 + b^2 + c^2\).

51. If \(A = 2B\), prove that \(a^2 = b^2 + bc\).

52. If \(r_1 = 2r_2\), prove that \(b + c = 3a\).

53. If \(r_2 + r_3 = 3r_1\), prove that \(a + b = 2c\).

54. If \(r_1 = r_2\), prove that \(A = 90^\circ\).

55. If \(2 \cos B = \sin A \sec C\), prove that \(\Delta ABC\) is isosceles.

56. If \(R = r_1\), prove that \(\cos A = \cos B + \cos C\).

57. If \(\cot A + \cot C = 2 \cot B\), prove that \(a^2 + c^2 = 2b^2\).

58. If \(a + c = 2b\), prove that \(\cot \frac{C}{2} = 3 \tan \frac{A}{2}\).
CB bearing 43° W. of N., at 5 m.p.h. Find the difference of time of arrival at B of the heads of the two columns if both start together.

5. If \( \alpha + \beta + \gamma = 90^\circ \), prove that
\[
2(1 + \sin \alpha)(1 + \sin \beta)(1 + \sin \gamma) = (\cos \alpha + \cos \beta + \cos \gamma)^2.
\]

R. 53.

1. Find the angles of a triangle whose sides are 14-5, 2, 17-6, 191-4 yards.

2. The vertical angle of an isosceles triangle is 29°, and the radius of its inscribed circle is \( r \); express the area of the triangle in terms of \( r, \theta \).

3. A wireless mast is stayed by five wire ropes, placed symmetrically, and each inclined at an angle \( a^\circ \) to the vertical. Find an expression for the angle between two adjacent wire ropes.

4. Find all the acute angles \( \theta \), such that
(i) \( \cos^2 2\theta = \frac{1}{2} \);
(ii) \( \sin 2\theta = \frac{1}{2} \).

5. Prove that in any triangle
(i) \( \cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \sin A \sin B \cos C \);
(ii) \( r_1 = \frac{1}{2} \tan A \) \( r_2 = \frac{1}{2} \tan B \).

R. 54.

1. If \( b = 15-87 \), \( c = 13-25 \), and \( A = 61° 24' \), find B and C.

2. The base of a pyramid is a regular hexagon, and each face is an isosceles triangle of vertical angle \( 90^\circ \). Find an expression for the angle between two consecutive faces.

3. Find an acute angle \( \theta \), such that
\( 3 \sin \theta + 2 \cos \theta = 2 \cdot 5 \).

4. Eliminate \( \theta \) from the equations
\( a \sin 2\theta = b \sin \theta \), \( c \cos 2\theta = d \cos \theta \)
given that \( \sin \theta \) is not zero.

5. Prove that in any triangle
(i) \( \cos \frac{A - B}{2} = \frac{a + b}{c} \sin \frac{C}{2} \);
(ii) \( r_1 + r_2 = a \cot \frac{A}{2} \).

R. 55.

1. A man walking due North along a straight level road sees a church on a bearing N. 7° E. A mile further on he sees it on a bearing N. 72° E. How far is the church from the nearest point of the road?
R. 57.

1. Find the least length of a line on the Moon which subtends an angle of 1° at the eye of an observer on the Earth, taking the distance as 240,000 miles.

2. AB is a diameter of a circle, centre O; BPQ is a tangent; prove that
\[ \frac{2AB}{PQ} = \cot \beta + \cos (\alpha + \beta) \csc \beta. \]

3. Solve for values of \( \theta \) from 0 to 180°
\[ \cot \theta + \cot (\theta + 45°) = 3. \]

4. In Fig. 301, prove that
(i) \( t = \frac{2u \sin (\theta - \alpha)}{\cos \alpha} \)
(ii) \( OP = \frac{ut}{\sqrt{p}} \sec \alpha \left[ \sin (2\theta - \alpha) - \sin \alpha \right] \).

5. (i) In any triangle ABC, prove that
\[ (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0. \]
(ii) If \( (1 - e) \tan \frac{\theta}{2} = (1 + e) \tan \frac{\phi}{2} \), prove that
\[ (1 + e \cos \theta)(1 - e \cos \phi) = 1 - e^2. \]

R. 58.

1. Fig. 322 represents 5 circles, each of radius 2 in., arranged symmetrically in contact. Find the diameter of the least circle enclosing them all.

2. In \( \triangle ABC \) (i) if \( \tan A = p \cot B \), prove that
\[ \cos C = \frac{1}{p+1} \cos (A - B). \]
(ii) If \( a + c = 2b \), prove that \( r = \frac{ac}{b^2} \).

3. Solve \( \tan^{-1} \left( \frac{1-x}{1+x} \right) = \frac{1}{2} \tan^{-1} x. \)

4. A and B are two search-lights; A is 1000 yd. due N. of B and at the same level as B. A is pointed in the direction S. 80° E. at elevation 30°; B is pointed in the direction N. 65° E. at elevation 28°. Show that it is possible for both lights to bear on the same aeroplane, and find its height.

5. If \( \tan A \tan B + \tan B \tan C + \tan C \tan A = 1 \), prove that
\[ \tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C. \]

R. 59.

1. In Fig. 303, ABCD is a rectangle. Find \( \angle APO \).

2. Find all acute angles \( x \) and \( y \), such that
\[ \sin 3x = \cos 4y, \quad \tan 4x = \cot 3y. \]
3. In Fig. 394, if AP bisects the area of the semicircle APB, prove that
\[ 2\alpha + \sin 2\alpha = \frac{\pi}{2}. \]

4. Prove that
(i) \( \sin^2 x + \sin^2 y = 1 - \cos(x + y) \cdot \cos(x - y) \)
(ii) in any triangle ABC,
\[ \frac{\sin^2 A}{2} + \frac{\sin^2 B}{2} + \frac{\sin^2 C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \]

5. If \( x = r \sin(\theta - \alpha) \) and \( y = r \sin(\theta + \alpha) \), prove that
\[ x^2 - 2xy \cos 2\alpha + y^2 = r^2 \sin^2 2\alpha. \]

R. 90.

1. Fig. 395 represents a square inscribed in a rhombus ABCD; \( AB = l \) in., \( \angle ABC = \alpha \). Find the area of the square in terms of \( l, \alpha \).

2. If \( n \) is any integer, prove that the next odd integer is
\[ n + 1 + \sin^n \left( \frac{n\pi}{2} \right). \]

3. Two circles of radii 13, 20 in. have a common chord of length 24 in.; their planes make an angle of 60° with each other. Calculate the radius of the sphere on which both circles lie.

4. Find a root between 0 and 180° of the equation
\[ 4 \sin 2\alpha + \sin 3\alpha = 0. \]

5. (i) Prove that
\[ \cos \alpha \cdot \sin(\beta - \gamma) + \cos \beta \cdot \sin(\gamma - \alpha) + \cos \gamma \cdot \sin(\alpha - \beta) \]
equall to \[ \cos(\alpha + \beta + \gamma) + \sin(\beta - \gamma) \cdot \sin(\gamma - \alpha) \cdot \sin(\alpha - \beta) \]
(ii) If \( x \sin \theta + y \cos \theta = \frac{1}{2} \sin 2\theta \) and \( x \cos \theta - y \sin \theta = \cos 2\theta \), prove that
\[ x^2 + y^2 = 1. \]

**NOTE ON TABLES**

Usually the "Difference Columns" give average differences, calculated over intervals of 1°; but, if these differences are changing rapidly, the error introduced by taking the average over so large an interval becomes serious, and it is necessary to use a smaller interval. Accordingly, where necessary, the Tables give the average Difference for 1', calculated over 12' intervals.

**Example.** Find \( \tan 75° 56' \) and \( \tan 75° 58' \).

\[
\begin{array}{ll}
\text{Add} & \text{Diff. for 2'} \\
\tan 75° 54' &= 3.9812 & 98 \quad & \text{is 49} \\
\therefore \tan 75° 56' &= 3.9910 & \therefore \text{Add. for 2'} &= 49 \times 2 - 98 \\
\tan 75° 54' &= 3.9812 & \text{or} \quad \tan 76° 0' &= 4.0108 \\
\text{Add} & \text{Diff. for 4'} & 196 & \text{Subtract Diff. for 2'} & 98 \\
\therefore \tan 75° 58' &= 4.0008 & \therefore \tan 75° 58' &= 4.0010 \\
\end{array}
\]

**Example.** Find \( \cosec 4° 32' \).

\[
\begin{array}{ll}
\cosec 4° 30' &= 12.75 & \text{Diff. for 1'}, \text{interval } 25 \text{ to } 35, \\
\text{Subtract} & \text{Diff. for 2'} & 10 & \text{is 5} \\
\therefore \cosec 4° 32' &= 12.65 & \therefore \text{Diff. for 2'} &= 5 \times 2 - 10. \\
\end{array}
\]

D.W. &
<p>|   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 |
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**Use Interpolation**

**Differences**

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**Where the integer changes, the numbers are halved.**

**Where the integer changes, the numbers are halved.**
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| 2° | +1.66 | 1.545 | 1.515 | 1.486 | 1.458 | 1.431 | 1.406 | 1.382 | 1.361 | 1.342 | 1.324 | 1.308 | 1.293 | 1.279 | 1.267 | 1.256 | 1.246 | 1.238 |
| 4° | +1.40 | 1.346 | 1.321 | 1.298 | 1.276 | 1.255 | 1.236 | 1.219 | 1.203 | 1.188 | 1.174 | 1.161 | 1.149 | 1.138 | 1.129 | 1.122 | 1.115 | 1.109 |
| 5° | +1.26 | 1.298 | 1.276 | 1.257 | 1.239 | 1.222 | 1.207 | 1.193 | 1.181 | 1.170 | 1.160 | 1.151 | 1.143 | 1.137 | 1.132 | 1.127 | 1.123 | 1.119 |

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Where the integer changes, the numbers are halved.
### LOG. SECANTS

#### Use Interpretation

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</tr>
</tbody>
</table>

#### Where the integer changes, the numbers are italicised.
| 0° | 1° | 2° | 3° | 4° | 5° | 6° | 7° | 8° | 9° | 10° | 11° | 12° | 13° | 14° | 15° | 16° | 17° | 18° | 19° | 20° | 21° | 22° | 23° | 24° | 25° | 26° | 27° | 28° | 29° | 30° | 31° | 32° | 33° | 34° |
|-----|----|----|----|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0.0000 | 0.0170 | 0.0340 | 0.0510 | 0.0680 | 0.0849 | 0.1019 | 0.1189 | 0.1359 | 0.1529 | 0.1699 | 0.1869 | 0.2039 | 0.2209 | 0.2379 | 0.2549 | 0.2719 | 0.2889 | 0.3059 | 0.3229 | 0.3399 | 0.3569 | 0.3739 | 0.3909 | 0.4079 | 0.4249 | 0.4419 | 0.4589 | 0.4759 | 0.4929 | 0.5099 | 0.5269 | 0.5439 | 0.5609 | 0.5779 | 0.5949 | 0.6119 | 0.6289 | 0.6459 | 0.6629 | 0.6799 | 0.6969 | 0.7139 | 0.7309 | 0.7479 | 0.7649 | 0.7819 | 0.7989 | 0.8159 | 0.8329 | 0.8499 | 0.8669 | 0.8839 | 0.9009 | 0.9179 | 0.9349 | 0.9519 | 0.9689 | 0.9859 |}

Use interpolation.

D.W.T. **
<table>
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<th>NATURAL TANGENTS</th>
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<tr>
<td><strong>$\theta'$</strong></td>
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<tr>
<td>------------------</td>
</tr>
<tr>
<td><strong>$'$</strong></td>
</tr>
<tr>
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</tr>
<tr>
<td>1°</td>
</tr>
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<tr>
<td>19°</td>
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<tr>
<td>20°</td>
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Where the integer changes, the numbers are Realised.
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<th>12°</th>
<th>18°</th>
<th>24°</th>
<th>30°</th>
<th>36°</th>
<th>42°</th>
<th>48°</th>
<th>54°</th>
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</thead>
<tbody>
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<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>
| 1° | 22.69| 57.30| 236.5| 191.0| 143.2| 116.4| 95.8| 81.8| 71.6| 62.6|}

<table>
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<td>13°</td>
</tr>
<tr>
<td>14°</td>
</tr>
<tr>
<td>15°</td>
</tr>
</tbody>
</table>

**Notes:**
- The numbers in the table represent natural cosecants.
- Use interpolation when the integers change, as the numbers are interpolated.

---

**Subtract Table:**
- The values in the subtract table are subtracted from the natural cosecants.
- The table is used for calculations involving subtraction of natural cosecants.

---

**Use Interpolation:**
- When the integer changes, the numbers are interpolated to obtain accurate results.
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<th>3°</th>
<th>4°</th>
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<th>6°</th>
<th>7°</th>
<th>8°</th>
<th>9°</th>
<th>10°</th>
<th>11°</th>
<th>12°</th>
<th>13°</th>
<th>14°</th>
<th>15°</th>
<th>16°</th>
<th>17°</th>
<th>18°</th>
<th>19°</th>
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Where the integer changes, the numbers are tabulated.
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Differences

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ANSWERS

Note. Four-figure tables have been used for working out the Answers given below. Angles are given to the nearest minute, but in many cases there are variations of one or two minutes, depending on the use made of the Tables.

PART I.

EXERCISE I. a. (p. 5.)
1. 270°; 45°; 60°.
2. 70°; 156°; 120° 40'; 17° 10'; 42° 35'.
3. 50°; 65° 5°; 29° 10'; 87°; 31° 50'.
4. 90°; 65°; 120°; 115° 30'.
5. 70°; 179°; 200°; 310°.
6. N. 50° E.; S. 30° W.; N. 2° W.; S. 70° E.
7. S. 10° W.; N. 14° W.; 105°; 130°.
8. Z. of debr. 18° 45'.
9. 15° 27'.
10. 28° 22' 19''; 25° 42' 51'';
11. 25-590; 108-280.
12. 15'.
13. 30'.
14. 84° 35'; 11° 59'.
15. N. 25° W.

EXERCISE I. b. (p. 7.)
1. 1.87, 1.40 in.; 0-47, 0.47.
2. 5.72 cm., 1.71 in.; 0.57, 0.57.
3. 3 cm., 1.6 in.; 6 cm., 10 in.; 0.6, 0.6.
4. 1.82 cm. 5 in.; 3.75 cm., 7.5 in.; 16, 16.
5. 72 ft.
6. 860,000 mi.; 220,000 mi.
7. 5 ft.
8. 10 x 6 cm.; 6 x 3 in., yes.
9. No.
10. 4.8 in.

EXERCISE I. c. (p. 13.)
1. 0.3649, 0.8391, 1, 1.018, 1.7921, 3.7921.
2. 14° 2', 41° 59', 58°, 66° 30'.
3. 1.8807, 0.4346, 0.2320, 2.0248.
4. 23°, 71°, 15° 24', 71° 36', 88°, 14° 14', 55° 14', 41° 44'.

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5. 38° 40', 50° 19'; 62° 50', 36° 52'.
6. 29° 41', 10° 18'; 42° 11', 47° 59', 35° 29'.
7. 3° 70', 6° 58', 0° 54', 20° 53'.
8. 9° 45', 196° 83° 9', 6° 15'.
9. √3, 1/√3; 17231, 0-5774. 10. 1126 cm., 1-1 cm., 1-80 in.
16. 13° 07', 17. 9° 5° 58', E., 8° 39° 26', E. 18. 9° 09'.
19. 21° 44'; 20. 48° 49'.
21. 62° 36', 26° 34° 90', 38° 40', 51° 20' 90°. 22. 7-08 ft.
23. 67° 23', 122° 37'. 24. 15-4 cm. 25. 21-7 sq. in.
26. 48'. 27. 2-36 cm. 28. 30° 55'.
29. 12° 55'. 30. 2-92 ft. 31. 10° 27'.
32. 8° 37'.

EXERCISE I. d. (p. 18.)

1. 0-43, 0-45, 0-57, 0-819, 0-906, 0-423; 1-0, 0-1.
2. 0-996, 0-009, 0-8994, 0-9897, 0-4006, 0-0009, 0-0096.
3. 9-793, 0-8829, 0-592, 0-0176, 0-3971, 0-3655, 0-3963, 0-3958.
4. sin δ, cos φ = 0-5, cos δ, sin φ = 0-9; sin δ, cos φ = 0-5, cos δ, sin φ = 0-9.
5. sin δ, cos φ = 0-5, cos δ, sin φ = 0-9; sin δ, cos φ = 0-5, cos δ, sin φ = 0-9.
6. tan Z, tan X, tan Y; tan B, tan C; tan B, tan C; tan B, tan C.
7. x = 0-54, y = 0-91; a = 2-65, b = 1-11; p = 9-11, q = 2-52; c = 84-8.
8. 2-77, 0-740, 9-4, 3-0, 5-43, 1-65, 1-83, 1-35.
9. 0-8, 0-4, 0-98, 0-4. 24. 3-7. 2. 23/3. 0-8, 0-16, 1-4.
10. 9° 27 ft. 11. 1120, 1800 yd. 12. 2353 ft.
13. 1-60 sq. in. 14. 42°, 42° 30', 68° 13', 74° 39'.
15. 0-0683. 16. 6-41 cm. 17. 2-99 cm. 18. 5-30, 6-25 cm.

EXERCISE II. b. (p. 31.)

2. 25°, 74° 50', 20° 42', 26° 48', 26° 44', 26° 4°, 13° 34', 18° 39', 74° 44' (457).
3. 56°, 38°, 29° 24', 79° 36', 79° 42', 79° 49', 70° 37', 40° 30', 30° 29'.
4. 31° 41', 76° 56', 32° 15', 68° 35', 39', 17° 37'.
5. 67° 52', 58° 8', 25° 37', 67° 25'; a, φ = 53° 8', β, θ = 53° 8'; 10° 16', 73° 44'.
6. 14° 29', 5° 44'. 9. 53° 8', 73° 44', 60° 33° 8'.
7. 36° 52', 143° 8'.
10. 25° 37', 18° 26'; 5° 44', 5° 43'; 5° 14', 15° 44'; 34°, 34'; along slope.
11. 10° 25', 15° 44'; 15° 44', 18° 26'.
12. 9° 35'; 13. 33° 33'; 14. 46° 29'. 15. 18°, 0° 39 in.
16. 18° 29'. 17. 24° 2'. 18. 36° 20'.
19. 24° 37'. 20. 31° 48', 106° 48'.
21. 112° 51'. 22. 27° 35'.
23. 27° 35'. 24. 27° 35'. 25. 60° 42'.
26. 30° 19'. 27. 51° 24', 51° 50'. 28. 24° 37'. 29. 11° 6 cm.
30. 40° 22', 88° 51'.
31. 3-69. 32. 2-14, 3-19, 4-37.

EXERCISE II. c. (p. 35.)

1. 2410 mi. 2. 15,000 mi. 3. 52-9, 140 yd. 4. 5-47, 11-0 ft.
5. 46° 9 yd. 6. 2-8, 0-01 mi. 7. 67° 8'. 8. N. 30° 34° E.
9. 113° 35'. 10. 1-31 mi. 11. 158, 1744-190 ft. 10-9 sec.
12. 7-95 ft. 13. 7-24, 2-76 in. 14. 04° 14'. 15. 41° 24'.
16. AN AC = 17. 88° 50'. 18. 6490 yd.
20. 69. 21. x = y + 90, 30. 22. 15° 22'.
33. 25. 35, 33. 35. 33. 35. 33. 35. 33. 35. 33. 35. 33. 35. 33. 35. 33. 35.
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EXERCISE IV. a. (p. 53.)
1. \( \sqrt{2} \). 2. \( \frac{2}{\sqrt{3}} \). 3. 1. 4. 2. 5. \( \frac{1}{\sqrt{3}} \).
6. \( \sqrt{2} \). 7. 2. 8. \( \sqrt{3} \). 9. \( \sqrt{3} \). 10. \( \frac{1}{\sqrt{3}} \).
15. 60°. 16. 3. \( \sqrt{3} \) = 6.20 in. 17. 27.7 sq. cm.
21. \( \sqrt{3} - 1, \frac{1}{2}(\sqrt{3} - 1), 15^\circ \). 22. \( \sqrt{3} + 1 \).
23. \( \frac{2}{\sqrt{2}} \).
24. \( \sin \theta, \cos \theta, \tan \theta; 1, 0, \infty; 0, 1, 0 \).

EXERCISE IV. b. (p. 56.)
1. 1-1918. 2. 2-2024. 3. 0-3640. 4. 0-4226. 5. 2-3559.
6. 1-0376. 7. 1-1918. 8. 0-2073. 9. 0-75, 1-25. 10. \( \frac{1}{2}, 2 \frac{1}{2} \).
11. 1-\( \sqrt{7} \), 11. 12. 2. 13. \( \frac{1}{2} \). 14. \( \frac{1}{4}, 2 \frac{1}{4} \).
15. 1, sin\( \theta \), cot \( \theta \). 20. \( \sqrt{(p^2-1)}, \frac{1}{p}\sqrt{(p^2-1)} \). 22. cot \( \theta \).
23. (ii), (v), (vii), (x), (xi). 24. 0, \( \infty \). 25. 0, 0.
26. 1. 27. 1. 28. 9, 98, 45°.
29. \( \frac{2}{\sqrt{2}} \). 30. \( \frac{2}{\sqrt{2}} \). 31. \( \frac{2}{\sqrt{2}} \). 32. \( \frac{2}{\sqrt{2}} \).

EXERCISE IV. c. (p. 58.)
1. \( z \cos \theta \). 2. \( x \sec \theta \). 3. \( y \sec \phi \). 4. \( y \cos \phi \).
5. tan\( \frac{1}{2}(\phi) \). 6. \( \sin \frac{1}{2}(\phi) \). 7. \( z \sin \theta \). 8. \( x \sec \phi \).
9. tan\( \frac{1}{2}(\theta) \). 10. \( \cos \frac{1}{2}(\theta) \). 11. \( x \cot \theta \). 12. \( y \sec \phi \).
13. PQ cosec R. 14. GF cot E. 15. YZ cosec X. 16. cos\( \frac{1}{2}(QR) \).
17. tan\( \frac{1}{2}(YZ) \). 18. PQ cot R. 19. XZ cos X. 20. EF sin E.
21. 60°. 22. 328 ft. 23. 172°. 24. 25° 40'.
25. 38° 56'. 26. 61° 3'. 27. 22° 1', 38° 41'. 28. 4° 39'.
29. 59° 29'. 30. 32° 37'. 31. 35° 44'. 32. 25° 55'.
33. 4° 55'. 34. 7° 28' geom. 35. 51° 45'.
36. 8° 5', 5° 25', 25 ft. 37. 34°. 38. 6-20, 8-14 cm.
39. 341 yd. 40. 20-7 mi.

REVISION PAPERS. R. I-6. (p. 48.)
R. 1. 2. 21-2 cm. 3. 242 sq. in. 4. 2-34, 81 ft. 5. 10° 10'.
6. 2. 168 ft. 3. 65° 23', 65° 23', 49° 14'.
7. 117 ft./sec. 8. 10° 22'.
8. 1. 15° 31'. 2. 302 yd. 3. 82°.
4. 21625, 2-152, 0-0007, 1. 5. 4-88.
9. 1. 5° 13'. 2. 3-19. 3. 35° 47', 8° 24'.
10. 4' 47'. 11. 5° 30' yd. 12. 300 yd. 13. 5° 55' yd. 14. 5° 25' yd.
R. 2. 1. 5° 35', 14-0 cm. 2. 2° 24'. 3. 9° 36', 9° 28'.
4. 1500 yd. 5. 285 cm. 6. 90°.
7. 2. 56° 26' or 123° 34'. 3. 10-5, 9-74 cm.
8. 60-7(5) cm. 10. 19° 27'.
9. 3-6. 4-15 in. 5. 9-74 cm.
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EXERCISE VI. a. (p. 70.)

1. 22° 1'.  2. 13° 21'.  3. 67° 23'.  4. 17° 55'.  5. 27° 56'.  6. 36° 52'.  7. 75° 56'.  8. 45°.
17. 60° 6'.  18. 60°.  19. 81° 12'.  20. 40° 30'.
21. 10° 06', 11° 57'.  22. 28° 28'.  23. N. 69° 2', E. or W.
24. 10° 41', 16° 28'.  25. 9° 60 ft.  26. 16° 16 ft.
27. 29° 34'.  28. 87° 43', 56° 19'.  29. 141 ft.
30. 1920 ft.
31. OA = AC = 2, OC = √6, 52° 15', 52° 15', 75° 30'; ∆OAC = ∆OBC; √3, √4/5 = 77° 10'.  32. 2. 65 in., 43° 11'.
33. 37° 23', 31° 43'.

EXERCISE VI. b. (p. 74.)

1. 11° 50', 18° 59', 49° 11'.  2. 81° 59'.  2. 18° 26'.
4. 28° 23'.  5. 63° 58'.  6. 25° 59'.  7. 65° 23'.
8. 30° 50'.  9. 54° 8', 28° 2'.  10. 7'.  11. 27° 51'.
12. 80° 33'.  13. 40° 49'.  14. 10° 2'.  15. N. 53° 5'E. or W.
16. S. 67° 4'E.  17. 35° 10', 109° 26'.  18. 15° 35' or 143° 1'.
19. 26° 34'.
20. 93° 11'.

EXERCISE VI. a. (p. 80.)

1. 0.0008.  2. 1.0002.  3. 0.0175.  4. 57.30.  5. 57.29.  6. 0.0175.  7. 0.0087.  8. 114.6.
9. 1.000.  10. 1.000.  11. 0.0087.  12. 114.6.
13. x > 89.  14. x > 89.4.  15. x > 87.7.  16. x < 1.
17. x < 96.  18. x < 2.8.  19. x < 0.6.  20. x < 0.6.
21. x > 89.4.  22. x > 89.  23. x > 89.4.  24. x < 0.6.
25. x = 0; z; 1; 0; z = 90; 1, 0, 0.  25. cot θ in θ; cot θ = 0;
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R. 15. 1. \( \frac{\sqrt{(b^2 - 1)}}{b} \) \( \sqrt{(b^2 - 1)} \). 2. 32° 23'. 3. 1-45 ft.
4. \( \sqrt{(p^2 + q^2)} \) 68° 12'. 5. 14° 44'.
R. 16. 1. \( \frac{2m}{2m - m^2} \). 2. 70-4, 20-7 ml.; 16° 24'.
3. 30° 56'. 4. 30'. 5. 5-7 in., 25° 6'.
R. 17. 1. 30°, 30°, 60°. 2. 3-30 ft.
3. 7° 36'. 4. 15\( \frac{1}{2} \) cm 13\( \frac{1}{2} \).
R. 18. 1. \( x^2 + y^2 = 25 \). 2. 20-7 per cent.
3. 114 ft. 4. 6-16 cm.

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PART II

EXERCISE VII. a. (p. 98.)
1. (0-4, 0-5); (-0-7, 0-3); 0-4, 0-5, 0-7, 0-3 in.
2. (-0-3, -0-6); (0-6, -0-3); 0-3, 0-6, 0-3, 0-6 in.
3. No; one of four positions. 4. G, H.
5. C, D; E, F.
6. x is --, y is +; t is +, y is --; x is +, y is +; x is --, y is --.

EXERCISE VII. b. (p. 103.)
1. -0-8, -0-6, -\( \frac{1}{2} \). 2. 0-8, 0-6, -\( \frac{1}{2} \).
3. -0-8, 0-6, -\( \frac{1}{2} \). 4. 0-6, -0-8, -\( \frac{1}{2} \).
5. -0-6, 0-8, -\( \frac{1}{2} \). 6. -0-8, -0-6, -\( \frac{1}{2} \).
7. 0-8, 0-6, \( \frac{1}{2} \). 8. -0-6, 0-8, \( \frac{1}{2} \).
9. -\( \sqrt{21} \), -0-4, \( \sqrt{21} \).
10. 0-006; -0-423; -0-766; 0-829; -0-819; 0-574; -0-085; 0-839.
11. 90° < \( \theta \) < 180°. 12. 270° < \( \theta \) < 360°.
13. 90° < \( \theta \) < 180°. 14. 180° < \( \theta \) < 270°.
15. 270° < \( \theta \) < 360°. 16. 360° < \( \theta \) < 360°.
23. sin 10°. 24. -cos 70°. 25. tan 35°.
26. sin 85°. 27. tan 50°. 28. -tan 35°.
29. 113° 30'. 30. 246° 25'. 31. 311° 24'. 32. 15° 25', 313° 25'.
33. 50° 55', 210° 56'. 34. 30° 3420'. 35. -1731. 36. -0-0063.
41. 0-8328. 42. -0-0545. 43. 0-7378. 44. 0-0237.

EXERCISE VII. c. (p. 105.)
1. 0-89, -0-89, -0-89, 0-89; 0-89, -0-89, -0-89.
2. 197-5, 252-5; 197-5, 342-5.
3. 53, 127; 57, 323.
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4. 180 < x < 360; 90 < x < 180.
5. 235 < x < 360; 230 < x < 360.
6. 0 < x < 60; 30 < x < 90; 113 < x < 246.
7. 35, 225.
10, 60°, 300°.
11. 20°, 160°.
12. 56°, 238°.
13. 230°, 310°.
14. 117°, 243°.
15. 156°, 338°.
16. 292.18.
17. 1.0557.
18. 3.721.
19. 1.0394.
20. 0.00875.
21. 0.0154.
22. 1.01557.
23. 0.03891.
24. 63° 26', 243° 25'.
25. 33° 35', 166° 25'.
26. 114° 38', 245° 22'.
27. 204° 37', 333° 23'.
28. 66° 29', 116° 34', 243° 25', 296° 34'.
29. 54° 44', 125° 16', 234° 4', 305° 16'.
30. 143° 8', 126° 55', 429° 8'.
31. cot θ = 106; sec θ = 105; cosec θ = 104.
32. sin θ = 103; cos θ = 102.
33. sin A; cos A.

EXERCISE VII. d. (p. 109.)

1. 2.5, 2.5, -2.5, -2.5 ft. 2. 3.83 ft., 13 sec.
3. 5 cos (100°) feet; 4.33, -4.33, -4.33, 4.33 feet.
4. 6.64, 29.36 sec.; 11.30, 24.64 sec.
5. 7.05, -11.4, 11.4, -7.05 ft.
6. 57 a.m., 6.37 p.m.; 11.22 a.m., 11.52 p.m.
7. 2.5, -2.5, -5, -2.5, 2.5, 5 ft.; 10 ft.; 12 sec.
8. 15° 4 cos θ, 4 sin θ ml.; 16° 8', 137; 11° 2', 1.37; 19° 0', 11° 0', 11° 2', -1.37; 19° 0', 11° 0', -1.37.
9. Yes; 0. 10. Yes; 23° 10'.
12. sin α + sin β + sin γ = 1.355, 0.565; 1.662, -0.123; -1.456, -0.387.
13. 90°, 345°, 90°, 180°.
14. cot θ = -10 (cot B = cot C); 73° 35', 49° 42', 100° 25'.

EXERCISE VIII. (p. 115.)

1. 0, 0.913, -0.913; 0.7844, -0.9000, 1.0000; 1.1003, 0.0385, 0.0529, 0.0825, 0.2141, 0.2919, 4.4277, 0.0465, 0.0642, 2.0096, 0.0395, 0.0395.
2. 1.4805; 1.6409; 1.6000; 1.4925; 1.0106; 1.0106.
3. 50° or 127°; 28° 36' or 311° 24'; 71° 36' or 231° 36'; 51° 30' or 231° 30'; 28° 48' or 231° 12'; 57° 42' or 174° 18'; 20° 44' or 109° 16'; 38° 28' or 218° 28'; 67° 33' or 292° 27'; 70° 29' or 205° 28'; 15° 39' or 164° 21'; 75° 28' or 284° 32'.
4. 1.22.
5. 0.294.
6. 0.833.
7. 0.519.
8. 2.54. 9. 1.75. 10. 8.79. 11. 0.935.
12. 0.424. 13. 1.68. 14. 1.11. 15. 1.39.

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16. 38° 24'. 17. 56° 37'. 18. 65° 30'. 19. 18° 28'.
20. 54° 39'. 21. 27° 35'. 22. 22° 12'. 23. 59° 4'.
24. 66° 5'. 25. 76° 19'. 26. 1° 55'. 27. 57° 1.
28. 88° 7'. 29. 27° 15'. 30. 91° 6', 51° 54'. 31. 59° 7'.
32. 36° 14'. 33. 5.40. 34. 91° 8'. 35. 24° 16'.

EXERCISE IX. a. (p. 117.)

1. None. 2. None. 3. One. 4. One.
5. None. 6. Any number. 7. Two. 8. One.
13. None. 14. Two. 15. A + B + C = 180°. 16. 3 + c > a, etc.

EXERCISE IX. b. (p. 122.)

1. 9.40. 2. 23.3. 3. 6.40. 4. 7.917.
5. 6.96. 6. 7.23. 7. 35° 43'. 8. 47° 29'.
9. 41° 48'. 10. 41° 23'. 11. 61° 6'. 12. 29° 39'.
23. 10 > b > -7.88; b = 7.88 or b > 10; b < -7.88.
24. B.
25. c > 10; No. 26. 68° 5' or 113° 25'. 27. None.
28. 38° 45' or 5° 15'. 29. None. 30. 90°.
31. 104° 27'. 32. None. 33. 19° 47'.
34. 17° 39' or 31° 27'. 35. 63° 58', 71° 8'.
36. 77° 27', 51° 18', 6:54 or 102° 33', 26° 12', 3:53.
37. 117° 55', 241° 5. 2:3. 38. 15° 50', 36° 30', 2:90.
40. 35° 50', 4:02, 6:92.
41. 50° 11', 60° 19', 7:09.
42. 40° 47', 43° 13', 6:79.
43. 28° 3', 61° 57', 10:36.
44. 31° 30', 19° 271.
45. 87° 53', 52° 47', 1010 or 13° 27', 127° 13', 235.

EXERCISE IX. c. (p. 128.)

[Note. The answers in this and the following Exercises are given to a higher degree of accuracy than will always be attained if the (four-figure) table of squares is employed.]

1. 1.19.
2. 1.197.
3. 9.345.
4. 2.977.
5. 40° 34'.
6. 43° 32'.
7. 109° 28'.
8. 129°.
10. 33° 48', 44° 4', 7:03.
11. 20° 42', 143° 54', 4:61.
TRIGONOMETRY

13. 90° 50', 56° 15', 29° 55'.
15. 90° 54', 52° 25', 29° 41'.
17. 18° 35', 58° 45', 5° 20'.
19. 38° 11', 47° 59', 5° 50'.
23. 74° 13', 58° 35', 47° 14'.
25. 52° 26', 75° 8', 52° 26'.

EXERCISE IX. d. (p. 129.)
1. 49° 12', 180° 128.
2. 28° 22', 54° 52', 38° 46'.
3. 78° 39', 54° 53' or 101° 21', 31° 18', 59-3.
5. 118° 11', 4-06', 3-33.
6. 107° 44', 32° 4', 40° 12'.
7. 23° 44', 32° 1', 2-89.
8. 10° 30', 15° 12', 40-2.
9. 53°, 89° 40', 23° 4 or 127°, 15° 40', 6-32.
10. 17° 65', 99° 7', 17°.
11. 25° 56', 30° 31', 117° 33'.
13. 68° 15', 60° 15', 103 or 8° 45', 119° 45', 266.
15. 59° 30', 61° 37-5.

EXERCISE IX. e. (p. 129.)
1. 84, 66 mi. 2. 1010 yd. 3. 158 mi. 4. 8-67(5), 8-67 mi.
5. 34 mi. 6. 111 ft. 7. 0-45(5) mi. 8. 0-29 yd.
13. 2490 yd. 14. 5° 8'.
15. 4470 yd. 16. 127 sea mi.
17. 3650 yd., 333 32'. 18. 10200 yd. 20. 6620 ft.
21. 19 m.p.h. 22. 590 yd. 23. 5-17 ft.
24. 24° 9'; 15-5 ft.

EXERCISE IX. f. (p. 132.)
1. 47-6 ft. 2. 238 ft.
3. 40-3, 2-67 ft. 4. 62° 43'.
5. 18° 12', 41° 24'. 6. 7-11 in.
7. 6-9361. 8. 3-34. 9. 0.
10. 18-8, 2-1 in; 0-7, 19-3 in. 11. 16-4 in, 58° 9'.
12. 22-4 in. 13. 49-1 in. 14. 18-6 in.
15. 21-6(5) ft. 17. 6-32 ft. 18. 11-9 ft. 19. 67° 9'.
20. 1 - 2 in. 15. 34-8 in. 22. 1-68. 24. 57° 58' or 35° 20'.
21. 26° 20'. 22. 5° 9'.

REVIEW PAPERS. R. 19-28. (p. 137.)

EXERCISE X. a. (p. 146.)
1. 3-14. 2. 3-12. 3. 3-17.
4. 4, 2, 2. 5. 1, 1. 6. 7.
7. 46 cm, 154 sq. cm.; 14-8 cm, 17-3(5) sq. cm.
8. 3-61, 4-48(5) in. 9. 38-6.
10. 56 sq. cm. 11. 77 in. 12. 6-73 cm.
13. 7-69 sq. cm. 14. 57-71'. 15. 0-35 in.
16. 12-6 in. 18. 1-8 ft. 19. 31-1 ft.
20. 7-75 cm. 21. 1-5(5), 1-9 sq. in. 22. 0-20 cm. 23. 37 sq. ft.
27. 1075 sq. ft., 22(5) ft. 28. 25 cm. 29. 13-3 in. 30. 0-9 in. 31. 23-4 ft.
32. 192 cm., 12-5 (9). 33. 0-20 sq. cm. 34. 3-77 ft.; 6000 ft.
35. 29-2 in. 37. 84° 33'.

EXERCISE X. b. (p. 152.)
1. 1° 30'; 10 min. 2. 829, 8290 mi. 3. 21600.
4. 42 mi. 5. 4 hr, 42 min. 6. 12-4 ft.
7. 30, 800 std.; 4, 860 mi. 8. 70° 32'. 9. 46° 11'.
TRIGONOMETRY

EXERCISE XI. c. (p. 156.)

1. 524 cu. cm., 55-8 sq. cm. 2. 410-5 cu. ft., 234 sq. ft.
3. 16-4 cu. in. 4. 277°.
5. 25° 41'. 6. 14-1 cu. in., 28-3 sq. in.
7. 197,000,000 sq. mi.; 35,200,000 sq. mi. 8. 14-9 in.
9. 6-77 in. 10. 2d in. 11. 8,150,000 sq. mi.
12. 134 cu. ft., 151 sq. ft. 13. 25-7 cu. in.
14. 102° 38'; 151 sq. in.; 402 cu. in.; 204 cu. in.; 198 cu. in.
15. 203 cu. cm.; 28° 4'; 126 sq. cm. 16. 83° 8'.
17. 0-70 cm. 18. y sin 2θ tan (45° - θ) = -y sin 2θ cos 2θ.
19. θ = 360 sin δ. 20. 1-4 in. 21. 34,700 sq. ft.
22. 4010. 23. 433 sq. cm. 24. 36° 52'.

EXERCISE XI. a. (p. 163.)

1. 57° 18', 171° 52', 28° 30', 63° 18-8', 143° 14', 45° 50', 4° 0-6', 14'.
2. 90°, 130°, 36°, 150°, 15°, 07° 30', 270°, 315°, 80°, 106°.
3. 37° 7', 30° 3', 30', 30° 3', 30° 3', 30° 3', 30° 3', 30° 3', 30° 3'.
4. 0-2967, 0-3687, 1-0122, 1-5010, 0-0102, 0-0157, 0-3124, 0-8133, 5. 1-2988, 2-2294.
6. 75-8, 24-9, 126-8. 7. 20, 13-4, 5-5, 6-56, 27-5 cm.
8. 0-75, 1-25, 3, 0-60. 9. 0-6458, 32-3 sq. cm.
10. 1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1.
11. sin δ, sin θ, - cot δ, - cos δ, - sin δ, - tan δ, - cos δ, - sin δ, - cot δ.
12. sin δ, - sin δ, - cos δ.
15. 1-2, 11-3 cm. 16. 5 ft. sec. 17. 2π.
18. π.
19. 10. 20. 24-2 ft. sec. 21. 1-5, 2-5 cm.; 4 sq. cm.
22. 299. 23. 4-29; 1. 24. 0-8415, 1-0768, 1-4019.
25. 0-122, 0-0785, 0-0116, 0-000174(3), 0-105, 0-0475, 0-0436(5), 0-0873.
26. 0-829 sq. ft. 27. 26° 47'. 28. l cos θ - sin θ tan θ = l sec θ cos θ.
29. tan θ (cos θ - sin θ tan θ) = β tan θ (cos θ - sin θ).

EXERCISE XII. a. (p. 180.)

1. 6-95. 2. 0-92. 3. 0-60 or 2-78. 4. 0-82.
9. 2-68. 10. 2-75. 11. 0-81. 12. 8-35.
13. 0-74. 14. 0-98. 15. 0-52.
19. 127°. 20. 117°. 21. 86°. 22. 132°. 23. 0-68.
27. 2n, λ, v.

EXERCISE XII. b. (p. 183.)

1. 512 cu. cm., 576 sq. cm. 2. 30-5 cu. in.
3. 320 cu. cm., 45 sq. cm., 185 cu. cm. 4. 380 cu. in.
5. 341 sq. in. 6. 272 lon. 7. 685 cu. ft.
8. 7-54 cu. in. 9. 128 cu. cm. 10. tan β = √2; tan a.
TRIGONOMETRY

EXERCISE XII. e. (p. 186.)
1. 1-96 in. 2. 3-63, 3-63 in. 3. 3-57, 1-63 in. 4. 2, 3, 6 in. 5. 3-31, 1-62. 6. 1-32, 0-674. 7. 20°2′2″. 8. 7-36 mi. 9. 11-2 cm. 10. 5-03(5). 11. 3, 2 in.; 12, 2 cm. 12. 2, 9 in.; 2, 32 cm. 13. \(\sqrt{a^2 + b^2 + c^2}\) in.; \(\sqrt{\frac{bc(a+b+c)}{a}}\) in.; etc. 14. 7-42 in.

REVISION PAPERS. R. 27-34. (p. 187.)
R. 27. 1. 25-7 in., 10-7 ft. 2. 5-10 ft. 3. 2-06(5), 15-2. 4. 3-46″. 5. 50-4.
R. 28. 1. 44-4 cm., 114 sq. cm. 2. 47 ft. 3. 45-8 cm., 38-6 sq. cm. 4. 25-3 cm., 10-1 in. 5. 4-96 sq. in.; 69-9 sq. in. 6. 4-86 sq. in.; 69-9 sq. in.
R. 29. 1. 2-03 per cent. 2. 48-6, 55-4 cm. 3. 1-91 in. 4. 469 sq. in. 5. 90 sq. in.; 96-9 sq. in.
R. 30. 1. 57-6 ft. 2. 4-49, 1-93 ft. 3. 52-7′ N. 4. 3-5(5) ft., 362 cu. ft. 5. a \(\sqrt{2}\).
R. 31. 1. 9-5(6)″. 2. 30-3′, 11-3′. 3. 75-7 sq. cm., 38-3(5) cu. cm. 4. 73-3 sq. cm., 28-3 sq. cm. 5. 6-64 cm.
R. 32. 1. 20°32′. 2. 6-43 cm., 28-1 sq. cm. 3. 13°28′ or 86°32′. 4. 136 yd. 5. 90°14′, 18-3 sq. in., 29′12″.
R. 33. 1. 25°27′. 2. 15°1′. 3. 96 ft. 4. 5-36 cm. 5. 5-36 cm
R. 34. 1. 25-5 cm., 4-9 cm. 2. 11-8 cm. 3. 17°43′.

ANSWERS

PART III.

EXERCISE XIII. (p. 194.)
1. \(\cos^2\theta\). 2. \(\sin^2\theta\). 3. \(\tan^2\theta\). 4. \(\cot^2\theta\). 5. 0. 6. \(\sec^2\theta\). 7. \(\csc^2\theta\). 8. 1. 9. -1. 10. \(\sin \theta\).
11. 4 + \sin^2\theta. 12. -1. 13. 1. 14. \(\sin \theta\). 15. 2.
16. \(\sin \theta \pm \theta\). 17. \(\cos \theta \pm \theta\). 18. \(\cos \theta \pm \theta\).
19. \(\cos \theta \pm \theta\). 20. \(\sin \theta \pm \theta\). 21. \(\sin \theta \cos \theta\). 22. 1. 23. \(\sin \theta \cos \theta\). 24. 2.
25. 0. 26. 1. 27. tan A tan B. 28. 1.
29. \(\tan \theta\). 30. 2 \(\tan \theta\). 31. \(\pm \sqrt{2}\) 32. -2 \(\tan \theta\). 33. ±1-5. 34. ± \(\sqrt{\frac{1}{a+b}}\).
35. 4\(a^2 + b^2 = 36\); 25\(a^2 + b^2 = 400\). 36. \(a^2 + b^2 = 16\).

EXERCISE XIV. (p. 203.)
1. 1, -1, +1, +1; 1, +1, -1, +1; 3, -1, -1, +1, +1, +1; +1, +1, +1, +1; +1, +1, +1.
2. 0-3420, 0-9277, -0-3640; 0-3640, -0-9277, -0-3420, 2-7475; 0-7660, 0-6428, 1-1918; 0-5, -0-8600, -0-5774; 0-5, -0-8600, -0-5774; 0-8415, 0-5405, 1-3577.
3. \(\tan \theta\), \(\sec \theta\), \(\cosec \theta\), \(\sec \theta\), \(\cosec \theta\), \(\tan \theta\), \(\sin \theta\), \(\cos \theta\), \(\cot \theta\).
4. \(\cot \theta\), \(\cosec \theta\), \(\sin \theta\), \(\sec \theta\), \(\cos \theta\), \(\cot \theta\), \(\sin \theta\).
5. \(\sin \theta\), \(\cos \theta\), \(\tan \theta\), \(\cot \theta\), \(\sec \theta\), \(\cosec \theta\).
6. D.W.T. III.
TRIGONOMETRY

6. 360° + 70 or 360° + 110; 360° + 300 or 360° - 20; 360° - 40 or
360° - 140; 180° + 30 or 180° + 50.
7. 360° + 80; 360° + 160; 360° - 140; 120° ± 20.
8. 180° ± 20; 180° ± 140; 180° ± 10; 360° ± 10.
9. 360° + 20 or 360° + 160; 360° - 10 or 360° - 170; 360° - 60 or
360° - 130; 360° ± 110.
10. cos (30°) in.; 0, -10, 0, 10; no; cos (30° - φ) = cos φ.
11. sin (30°) in.; 10, -10, 0; sign changed; sin (30° - φ) = sin φ.
12. (1 - cos φ) ft.; no.
13. (α - sin φ) ft.; -α (φ - sin φ).
14. -a, -a, a, ±√(1 - a²), ±√(1 - a²), ±√(1 - a²), ±√(1 - a²).
15. sin φ, -sin φ, tan φ, -sin φ, sin φ, cos φ, -cos φ.
16. -sin φ, -cos φ, tan φ.
17. (-1) cos φ, (-1) sin φ, (-1) cos φ, -cot φ.
18. 12, 8 in.; 36. 7 sec.
19. a, a, -a, -a, a, ±√(1 - a²), ±√(1 - a²), ±√(1 - a²), ±√(1 - a²).
20. sin φ, -sin φ, tan φ, -sin φ, sin φ, cos φ, -cos φ.
21. -sin φ, -cos φ, tan φ.
22. (-1) cos φ, (-1) sin φ, (-1) cos φ, -cot φ.
23. 12, 8 in.; 36. 7 sec.
24. 3°; 1 sec.

EXERCISE XV. a. (p. 208.)
1. 5. 2. 5, 11-7, 13, 25. 3. a, a.
6. a√(2(1 - cos A cos B - sin A sin B)).
7. r.

EXERCISE XV. b. (p. 216.)
1. 2 sin A cos B. 2. 2 cos φ cos φ. 3. -2 sin a sin β.
4. 2 cos α + sin A. 5. sin (B - C). 6. cos (B - C).
15. 1 tan A - tan B. 17. tan A + tan B. 18. cot A cot B + 1
32. 5.

EXERCISE XV. c. (p. 221.)
13. 1 + tan² x. 15. 1 + tan x. 16. 2 tan x.
20. 2 tan θ. 21. tan θ. 22. 1 - tan² θ.

EXERCISE XV. d. (p. 222.)
1. ±1. 1. -1. 2. ±1. ±1. 3. ±1. 4. 5. 4 - 5.
6. 1, (180° + 45°). 7. 4. 8. 4 - 5.
9. 9 + 1, 9 - 1, 9, -9. 11. (180° + 45°) tan⁻¹ (1).
15. 1/2 cos 2θ. 16. sin 2θ. 17. ± sin θ. 18. ± sin θ.
19. tan θ. 20. cot θ. 21. cos 2θ. 22. cot² A.
27. cos² θ. 28. sec A. 29. cos 2θ. 30. tan θ.

EXERCISE XV. e. (p. 224.)
17. a. 18. ±a - b. 21. sin⁴ θ + 4 sin² θ cos² θ + (sin² θ + 3 cos² θ).
22. ±. 23. ±1. 25. 37° 46'.

EXERCISE XVI. a. (p. 229.)
1. sin 52° + sin 18°. 2. sin 80° - sin 8°. 3. cos 56° + cos 36°.
4. cos 12° - cos 60°. 5. cos 58° + cos 74°. 6. sin 35° - sin 7°.
7. cos 41° - cos 61°. 8. sin 34° + sin 24°. 9. sin 72° + sin 28°.
24. sin 3x - sin x. 25. cos 2A - cos 6A. 26. cos 9A + cos 3A.
27. sin 10θ + sin 4A. 28. sin 4A. 29. sin(2A + 2B).
32. ±(sin 3θ + 3θ + sin (θ - φ)). 33. ±(sin A + sin B).
40. ±(cos 30° + cos 30°). 41. ±(sin 2A).
44. cos A + cos 2A. 45. cos A + cos 3A + cos 3A + cos A.
46. sin 9θ + sin 7θ + sin 3θ + sin θ.
TRIGONOMETRY

EXERCISE XVI. b. (p. 222.)

1. \(2 \sin 33^\circ \cos 15^\circ\).  2. \(2 \cos 35^\circ \cos 12^\circ\).  3. \(-2 \sin 33^\circ \sin 13^\circ\).

4. \(2 \cos 33^\circ \cos 12^\circ\).  5. \(2 \cos 75^\circ \sin 25^\circ\).  6. \(2 \sin 62^\circ \cos 23^\circ\).

7. \(2 \cos 63^\circ \cos 7^\circ\).  8. \(2 \sin 30^\circ \sin 6^\circ = -2 \sin 33^\circ \sin 5^\circ\).

9. \(-2 \sin 55^\circ \sin 6^\circ\).  11. \(2 \cos 120^\circ \cos 10^\circ = -2 \cos 10^\circ\).

12. \(2 \cos 80^\circ \sin 16^\circ\).  13. \(2 \sin 4A \cos 2A\).  14. \(-2 \sin 2A \cos A\).

16. \(2 \cos 2x \cos 2x\).  16. \(2 \cos 2x \cos 2x\).  17. \(2 \sin 2x \cos 2x\).

18. \(2 \sin 5\theta \cos 3\theta\).  19. \(-2 \sin 2\cos 3\theta\).  20. \(2 \cos 2\cos 3\theta\).

21. \(2 \cos \frac{5A + 3B}{2} = \frac{A - B}{2} \cos \frac{A + B}{2}\).

22. \(2 \sin A \cos (B - C)\).  23. \(-2 \sin \frac{2A + B}{2} \sin \frac{B - A}{2}\).

24. \(2 \cos (A + B + C) \cos (A - B - C)\).  25. \(-2 \sin A \sin (B + C)\).

26. \(2 \cos 19^\circ \).  27. \(-2 \cos 80^\circ \sin 20^\circ\).  28. \(-2 \sin 55^\circ \cos 15^\circ\).

29. \(-2 \cos 50^\circ \sin 30^\circ = -2 \cos 30^\circ\).

30. \(2 \cos (\frac{A - B + 45^\circ}{2}) \sin (\frac{A + B - 45^\circ}{2})\).

31. \(2 \cos (\frac{A + B + 45^\circ}{2}) \cos (\frac{A - B - 45^\circ}{2})\).

REVISION PAPERS. B. 35-50. (p. 234.)

R. 55.  1. \(a \cos a + b \cos a = b \sin a - a \sin a\).  2. \(\sin a \cos a\).

R. 56.  2. \(31^\circ\), 16.3.  37.  2. \(9\), 46 (6).  3. \(-\frac{1}{2} \cos \theta\).

R. 57.  3. \(\tan \theta = \frac{y}{x}\).  5. \(62^\circ \), 29.  5. \(\frac{y}{x}\).

R. 58.  2. \(100\) yd.  5. \(\frac{b}{c}\).  7. \(10\) in.  8. \(2\), 64 (6).  9. \(3, 9809\).

R. 59.  1. \(122^\circ\).  2. \(3, 875\).  3. \(1, 63\).  4. \(10\) sq. in.  5. \(0, 0459\).

R. 60.  2. \(3, 875\).  3. \(1, 63\).  4. \(10\) sq. in.  5. \(0, 0459\).

R. 61.  2. \(1, 62\).  3. \(1, 62\).  4. \(1, 62\).  5. \(1, 62\).

R. 64.  2. \(55^\circ\), 25.  3. \(64^\circ\), 61.  4. \(1, 62\), \(2\).

R. 65.  2. \(8,20(8)\).  3. \(2, 97\).  4. \(1, 62\).

R. 66.  2. \(2, 97\).  3. \(2, 97\).  4. \(1, 62\).

R. 67.  2. \(1, 62\).  3. \(1, 62\).  4. \(1, 62\).

R. 68.  1. \(\sin \theta\).  2. \(1, 62\).  3. \(1, 62\).

R. 59.  1. \(0, 0459\).

1. \(\tan \theta = \frac{b}{c}\) cos a.

4. \(\tan \theta = \frac{a - b}{c}\) sin a.

7. \(1, 62\), \(2\).

10. \(\sin a + \sin a\).

EXERCISE XVII. b. (p. 253.)

1. \(2, 855\) ft.  2. \(2, 855\) ft.

5. \(7, 80\) ft.  6. \(9, 80\) ft.

12. \(1, 20\) ft.  13. \(1, 20\) ft.

15. \(2, 50\) yd.  16. \(2, 50\) yd.

18. \(1, 20\) ft.  19. \(1, 20\) ft.

EXERCISE XVII. c. (p. 256.)

1. \(69^\circ\), \(51^\circ\).  2. \(122^\circ\), \(22^\circ\), \(28^\circ\).

3. \(\sin a + \sin b\) mi.  4. \(\sin a + \sin b\) mi.

7. \(60^\circ\), \(97^\circ\).  8. \(60^\circ\), \(97^\circ\).

10. \(\sin a + \sin b\) mi.

EXERCISE XVIII. a. (p. 259.)

1. \(24^\circ\), \(59^\circ\).  2. \(24^\circ\), \(59^\circ\).

3. \(95^\circ\), \(22^\circ\), \(28^\circ\).

4. \(24^\circ\), \(28^\circ\), \(34^\circ\).

5. \(28^\circ\), \(28^\circ\), \(34^\circ\).

6. \(28^\circ\), \(28^\circ\), \(34^\circ\).

7. \(28^\circ\), \(28^\circ\), \(34^\circ\).

8. \(28^\circ\), \(28^\circ\), \(34^\circ\).

9. \(28^\circ\), \(28^\circ\), \(34^\circ\).

10. \(\sin a + \sin b\) mi.
TRIGONOMETRY

7. 80° 24', 279° 36'. 8. 191° 29', 348° 28'.
11. 30°, 150°, 310°, 330°.
12. 54° 44', 135° 16', 234° 44', 305° 16'.
13. 60°, 120°, 240°, 300°.
14. 0°, 60°, 180°, 300°, 360°.
15. 33° 41', 213° 41'.
16. 90°, 270°.
17. 90°, 216° 52', 323° 8'.
18. 90°, 180°, 270°.
19. 30°, 150°, 270°.
20. 63° 26', 161° 34', 243° 26', 341° 34'.
21. 48°, 63° 26', 223° 26', 243° 26', 223° 26', 243° 26'.
22. 28° 34', 63° 26', 206° 34', 243° 26'.
23. 48° 11', 311° 49'.
24. 0°, 180°, 360°.
25. 45°, 189° 26', 225°, 288° 26'.
26. 63° 26', 133° 26', 243° 26', 333° 26'.
27. 8° 29', 73° 23', 188° 29', 233° 23'.
28. 60°, 300°.

EXERCISE XVIII. b. (p. 261.)

[Note. a stands for any integer or zero.]

1. 15°, 75°, 155°, 235°.
2. 20° 54', 69° 3', 200° 54', 249° 6'.
3. 90°, 270°, 600°.
4. 60°.
5. 10°, 130°.
6. 35° 20', 153° 20', 272° 20'.
7. 20° 31', 130° 20', 100° 31'.
8. 48° 31', 111° 57', 225° 3', 291° 57'.
9. 22° 17', 127° 43', 202° 17', 307° 43'.
10. 9°, 126°.
11. 120°, 240°, 45° + 90°a.
12. 60°a, 15° + 90°a, 75° + 90°a.
13. 120°, 240°, 90°a.
14. 360°a, 90° + 120°a, 100° + 120°a.
15. 60°, 120°, 240°, 300°, 18° + 36°a.
16. 60°, 90°, 270°, 300°.
17. 90, 270, 120°a.
18. 33° 34', 326° 29', 180°a.
19. 90°, 270°, 45° + 90°a.
20. 90° + 90°a.
21. 69°, 240°.
22. 71° 34', 251° 34', 180°a.
23. 26° 26', 243° 26', 180°a.
24. 90°, 228° 35', 311° 25'.

EXERCISE XVIII. c. (p. 263.)

[Note. a stands for any integer or zero.]

1. 0°, 60°, 180°.
2. 30°, 150°.
3. 32° 30', 122° 30', 165°.
4. 20°, 110°, 146°.
5. 45°a.
6. 45° + 32° 30'.
7. 120°, 360°a.
8. 0°, 180°, 45° + 22° 30'.
9. 0°, 45°, 135°, 180°.
10. 30°, 110°.
11. \(\sqrt{1 + \sqrt{5}}\).
12. 60°.

ANSWERS

EXERCISE XVIII. a. (p. 265.)

1. 30° 55'.
2. 20° 48', 122° 20'.
3. 160° 16', 36° 82'.
4. 100° 2'.
5. 36° 22'.
6. 64° 40'.
7. 45°, 90°.
8. 24° 54', 98° 45'.
9. 53° 8', 53° 8', 73° 44'.

EXERCISE XVIII. b. (p. 261.)

[Note. a stands for any integer or zero.]

1. 180°a, 90°a + 45°.
2. 0, 60°, 300°, 360°.
3. 50° 19', 123° 41', 236° 19', 303° 41'.
4. 72a + 15°.
5. 11° 15', 33° 45', 59° 15', 78° 45'.
6. 32° 45', 45°, 78° 45'.
7. 0, 90°, 180°, 360°.
8. 0, 45°, 180°, 225°, 360°.
9. 22° 27', 137° 33'.
10. 90°a, 90°a + 18° 51'.
11. 90°, 270°, 180°a + 45°.
12. 180°, 90°, 180°.
13. 180°a, 24° 6', 153° 54', 204° 6', 333° 54'.
14. 30°, 150°, 270°.
15. 180°a, 180°a + 30°.
16. 22° 25', 203° 25'.
17. 30° 21', 210° 21'.
18. 180°a + 18', 180°a + 54°.
19. 45°, 114° 18', 223°, 333° 42'.
20. 15°, 75°, 90°, 195°, 225°, 270°.
21. 270°, 72°a + 36°.
22. 45°, 225°.
23. 109° 54', 340° 6'.
24. 70° 6', 190° 54'.
25. 2, 3.
26. 0, 28° 4', 90°, 61° 56'.
27. 1, 1.
28. 15°, 75°, 195°, 255°.
29. 145°, 325°, 90°a + 17° 30'.
30. 90°, 270°, 90°a + 7° 30', 90°a + 37° 30'.

EXERCISE XVIII. d. (p. 265.)

1. \(x^2 + (y - 1)^2 = 4\).
2. \(x^2 + ay^2 = ax^2\).
3. \(2x^2 + 4axy = 4a^2y\).
4. \(y(x^2 + y^2) = 2abx + c(x^2 - ax)\).
5. \((a^2 + b^2)c = c(a^2 + b^2)\).
6. \((xy + 1)\tan(a - b) = p - q\).
7. \(b^2 - a^2 = 1 + (p^2 - 1)q\).
8. \(x + e = 1\).
9. \(a + b = p + q\).
10. \(a + b = p + q\).
11. \(a^2 + b^2 = 4a\).
12. \(a^2 - ab = 2\), unless \(sin \phi = 0\).
13. \(x^2 + y^2 = 2a^2 + 3b^2 + 9c^2\).

EXERCISE XIX. a. (p. 269.)

17. 1-985; 165°.
18. 0-021.
19. 0-914(5).
20. 50°.
22. -1-332.
23. 17-3, 10, 19-3 in.
TRIGONOMETRY

EXERCISE XIX. b. (p. 273.)

1. \( \sin C, - \cos A, 0, - \tan B, \cos \frac{C}{2}, \sin \frac{B}{2}, \cot \frac{A}{2}, - \sin 2B, \cos 2C, \cos C, - \cos \frac{C}{2}, - \cot 2A. \)

2. \( \sin (A + C), - \cos (A + B), - \tan (B + C), \cos (2B + 2C), - \sin (2A + 2B), - \tan (2A + 2C), \cot \frac{A + B}{2}, \sin \frac{A + C}{2}, \cos \frac{B + C}{2}. \)

29. \( 4 \cos \theta \cos \phi \cos \psi. \)

REVISION PAPERS. R. 51-60. (p. 283.)

R. 51. 1. 832 ft. 2. 988 sq. in. 3. 45°, 225°.

R. 52. 1. 16-6 in. 2. 88 min.

R. 53. 1. 45° 57', 62° 44', 71° 19'. 2. \( r^2 \tan \theta (1 + \cos \theta). \)

3. \( 2 \sin^{-1}(\sin a \cdot \sin 30°). \)

4. 15°, 45°, 75°; 5°, 25°, 65°, 85°.

R. 54. 1. 67° 55', 50° 41'. 2. \( 2 \sin^{-1}(\sin 60° \cdot \sec a \cdot \frac{a}{2}). \)

3. 10° 15'.

4. \( 2abc + a + b + c. \)

R. 55. 1. 225 yd. 2. \( r \cos \theta - \phi \cdot \sec \frac{\theta + \phi}{2}. \)

3. 16-4.

R. 56. 1. 31°.

R. 57. 1. 69-8 ml. 2. 21° 19', 147° 22'.

R. 58. 1. 10-8 in. 3. \( \pm \sqrt{3}. \)

4. 2740 ft.

R. 59. 1. 145° 35'. 2. \( x = y = 123° \) or \( 44° \); \( x = 64°, y = 18°. \)

R. 60. 1. \( \frac{1}{2} \tan^2 a(1 - \sin a) = \frac{1}{2} \sin^2 a / (1 + \sin a). \)

3. 20-3 in. 4. 83° 13'.