A SHORTER GEOMETRY

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PREFACE

This book is intended for use in schools where it is of importance to cover the ground up to school certificate standard as rapidly as possible, without any sacrifice of thoroughness in essentials. It is assumed that the way has been prepared by some kind of course of practical geometry, for example, Durell and Tuckey, Simplified Geometry, Parts I–II.

There are no long exercises, but it is hoped that all are sufficiently comprehensive, because the examples have been selected so as to represent the various aspects of each group of properties.

In order that the book may also serve as a short revision course for matriculation or school certificate, proofs are given of all theorems demanded for elementary mathematics in such examinations; but those marked with an asterisk are not required by some examining bodies. It is suggested that the formal proofs of these theorems should not be learnt unless the particular syllabus, with which the pupil is concerned, makes it necessary.

Some alternative proofs of area and ratio theorems, with a few additional constructions, are set out in a short appendix.

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C. V. D.
SECTION I

ANGLES, PARALLELS AND CONGRUENCE

Definition.
If C is any point on the straight line AB, and if a line CD is drawn so that the angles ACD, BCD are equal, each is called a right angle.
Therefore if C is any point on the straight line AB, the angle ACB is equal to two right angles, or 180°.

The following names are important.
If two angles add up to 90° or 1 right angle, they are called complementary ; e.g. 20° and 70° are complementary angles.
If two angles add up to 180° or 2 right angles, they are called supplementary ; e.g. 20° and 160° are supplementary angles.
The opposite angles made by two lines crossing each other are called vertically opposite angles ; e.g. in Fig. 5, p. 3, x and y are vertically opposite, so are α and β.
If two angles have one arm in common and lie on opposite sides of that arm, they are called adjacent angles.

In Fig. 2, x and y are adjacent angles.
THEOREM 1

(1) If one straight line stands on another straight line, the sum of the two adjacent angles is two right angles.

(2) If at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in the same straight line.

Fig. 3

(1) Given CE meets AB at C.
To prove \( \angle ACE + \angle BCE = 180^\circ \).
\[ \angle ACE + \angle BCE = \angle ACB = 180^\circ, \text{ since } ACB \text{ is a st. line.} \]
Q.E.D.

Fig. 4

(2) Given \( \angle ACE + \angle BCE = 180^\circ \).
To prove ACB is a straight line.
Produce AC to F.
\[ \therefore \angle ACE + \angle FCE = 180^\circ, \text{ since } ACF \text{ is a st. line.} \]
But \( \angle ACE + \angle BCE = 180^\circ \), given.
\[ \therefore \angle ACE + \angle FCE = \angle ACE + \angle BCE. \]
\[ \therefore \angle FCE = \angle BCE. \]
\[ \therefore \text{CB falls along CF.} \]
\[ \text{But } ACF \text{ is a st. line; } \therefore \text{ACB is a st. line.} \]
Q.E.D.

THEOREM 2

If two straight lines intersect, the vertically opposite angles are equal.

To prove that \( x = y \) and \( \alpha = \beta \).
\[ x + \alpha = 180^\circ, \text{ adj. } \angle s \text{ on st. line.} \]
\[ \alpha + y = 180^\circ, \text{ adj. } \angle s \text{ on st. line.} \]
\[ \therefore x + \alpha = \alpha + y. \]
\[ \therefore \alpha = y. \]

Fig. 5
Similarly \( \alpha = \beta \).
Q.E.D.

THEOREM 3

If two triangles have two sides of one equal respectively to two sides of the other, and if the included angles are equal, then the triangles are congruent.

Fig. 6

Given \( AB = FQ, AC = PR, \angle BAC = \angle QPR. \)
To prove \( \triangle ABC \equiv \triangle PQR. \)
Apply the triangle ABC to the triangle PQR, so that A falls on P and the line AB along the line FQ;
Since \( AB = PQ, \therefore B \text{ falls on } Q. \)
Also since \( AB \) falls along \( PQ \) and \( \angle BAC = \angle QPR, \therefore \text{AC falls along } PR. \)
But \( AC = PR, \therefore C \text{ falls on } R. \)
\[ \therefore \text{the triangle } ABC \text{ coincides with the triangle } PQR. \]
\[ \therefore \triangle ABC \equiv \triangle PQR. \]
Q.E.D.
THEOREM 4

If one side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.

\[ \triangle ABC \]

BC is produced to D.

To prove \( \angle AOC > \angle ABC \) and \( \angle AOC > \angle BAC \).

Let F be the middle point of AG. Join BF and produce it to G, so that BF = FG. Join CG.

In the triangles AFB, CFG,

\[ AF = FC \text{ and } BF = FG, \text{ constr.} \]
\[ \angle AFB = \angle CFG, \text{ vert. opp.} \]
\[ \therefore \angle AFB \approx \angle CFG \text{ (2 sides, inc. angle)} \]
\[ \therefore \angle BAF \approx \angle GCF \]

But \( \triangle DCA \approx \) its part \( \triangle GCF \).

\[ \therefore \angle DCA \approx \angle BAF \text{ or } \angle BAC \]

Similarly, if BC is bisected and if AC is produced to E, it can be proved that \( \triangle BCE \approx \triangle ABC \).

But \( \angle ACD \approx \angle BCE, \text{ vert. opp.} \]

\[ \therefore \angle ACD \approx \angle BAC. \]

Q.E.D.

Definition.

Straight lines which lie in the same plane and which never meet, however far they are produced either way, are called parallel straight lines.

Playfair's Axiom.

Through a given point, one and only one straight line can be drawn parallel to a given straight line.

THEOREM 5

Straight lines which are parallel to the same straight line are parallel to one another.

\[ \triangle ABD \]

Given that AB and CD are each parallel to XY.

To prove that AB is parallel to CD.

If possible let AB cut CD (produced if necessary) at O.

Then through O there are two straight lines OA, OC, both of which are parallel to XY.

But this is impossible by Playfair's Axiom.

\[ \therefore \text{AB cannot cut CD and must therefore be parallel to it.} \]

Q.E.D.

A straight line which cuts two or more other straight lines is called a transversal.

The various angles formed by a transversal with two other straight lines have special names.

In Fig. 9, c and f are called alternate angles, d and e are called alternate angles.

The following pairs of angles are called corresponding: \( a, e; b, f; c, g; d, h. \)

The angles d, f are called interior angles on the same side of the cutting line; so also are c, e.
THEOREM 6*

If one straight line cuts two other straight lines such that either (1) the alternate angles are equal, or (2) the corresponding angles are equal, or (3) the interior angles on the same side of the cutting line are supplementary, then the two straight lines are parallel.

ABCD cuts PQ, RS at E, C.
(1) Given \( \angle PBC = \angle BCS \).
To prove PQ is parallel to RS.
If PQ, RS are not parallel, they will meet when produced, at H, say.
Since BCH is a triangle,
\[ \text{ext. } \angle PBC > \text{int. } \angle BCH, \]
which is contrary to hypothesis.
\[ \therefore \text{PQ cannot meet RS and is } \parallel \text{ parallel to it. } \]

(2) Given \( \angle ABQ = \angle BCS \).
To prove PQ is parallel to RS,
\[ \angle ABQ = \angle PBC, \text{ vert. opp.} \]
But \( \angle ABQ = \angle BCS \), given.
\[ \therefore \angle PBC = \angle BCS. \]
\[ \therefore \text{by (1), PQ is parallel to RS.} \]

(3) Given \( \angle QBC + \angle SCB = 180^\circ \).
To prove PQ is parallel to RS.
\[ \angle QBC + \angle PBC = 180^\circ, \text{ adj. angles, QSP a st. line.} \]
But \( \angle QBC + \angle SCB = 180^\circ \), given.
\[ \therefore \angle QBC + \angle PBC = \angle QBC + \angle SCB. \]
\[ \therefore \angle PBC = \angle SCB. \]
\[ \therefore \text{by (1), PQ is parallel to RS. } \]

Q.E.D.

FUNDAMENTAL THEOREMS

THEOREM 7*

If a straight line cuts two parallel straight lines, then (1) the alternate angles are equal; (2) the corresponding angles are equal; (3) the interior angles on the same side of the cutting line are supplementary.

\[ \angle AQR = \angle QRD. \]
To prove (1) \( \angle AQR = \angle QRD. \)
(2) \( \angle PQR = \angle QRD. \)
(3) \( \angle BQR + \angle QRD = 180^\circ. \)

(1) If \( \angle AQR \) is not equal to \( \angle QRD \), let the angle XQR be equal to \( \angle QRD \).
But these are alternate angles.
\( \therefore \text{XQ is parallel to RD.} \)
\( \therefore \text{two intersecting lines XQ, QA are both parallel to RC, which is impossible by Playfair's Axiom.} \)
\[ \therefore \angle AQR \text{ cannot be unequal to } \angle QRD. \]
\[ \therefore \angle AQR = \angle QRD. \]

(2) \( \angle PQB = \angle QRB, \text{ vert. opp.} \)
But \( \angle AQR = \angle QRD, \text{ alt. angles, by (1).} \)
\[ \therefore \angle PQB = \angle QRD. \]

(3) \( \angle BQR + \angle AQR = 180^\circ, \text{ adj. angles, BQA a st. line.} \)
But \( \angle AQR = \angle QRD, \text{ alt. angles, by (1).} \)
\[ \therefore \angle BQR + \angle QRD = 180^\circ. \]
Q.E.D.
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**THEOREM 8**

(1) If a side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.

(2) The sum of the three angles of any triangle is two right angles.

**Fig. 12.**

ABC is a triangle; BC is produced to D.

*To prove* (1) \( \angle ACD = \angle CAB + \angle ABC \).

(2) \( \angle CAB + \angle ABC + \angle ACB = 180^\circ \).

1. Let CF be drawn parallel to AB.

\[ \angle FCD = \angle ABC, \text{ corresp. angles.} \]

\[ \angle ACF = \angle CAB, \text{ alt. angles.} \]

Adding, \( \angle FCD + \angle ACF = \angle ABC + \angle ACB \).

\[ \therefore \angle ACD = \angle ABC + \angle ACB. \]

2. Add to each the angle \( \angle ACB \).

\[ \therefore \angle ACD + \angle ACB = \angle ABC + \angle CAB + \angle ACB. \]

But \( \angle ACD + \angle ACB = 180^\circ \), adj. angles, BCD a st. line.

\[ \therefore \angle ABC + \angle CAB = 180^\circ. \]

**Corollary I.** If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.

This follows from the fact that the sum of all three angles of the first triangle is equal to the sum of all three angles of the second, since each sum is two right angles.

**Corollary 2.** In a right-angled triangle (i) the right angle is the greatest angle; (ii) the sum of the remaining angles is equal to a right angle.

Since the sum of the three angles of a triangle is two right angles, if one angle is a right angle, the sum of the other two angles is a right angle, and therefore each of these two angles separately is less than a right angle.

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**ANGLES OF POLYGONS**

**THEOREM 9**

(1) All the interior angles of a convex polygon, together with four right angles, are equal to twice as many right angles as the polygon has sides.

(2) If all the sides of a convex polygon are produced in order, the sum of the exterior angles is four right angles.

Let \( n \) be the number of sides of the polygon.

*To prove* that the sum of the angles of the polygon + 4 rt. \( \angle s = 2n \) rt. \( \angle s \).

Take any point \( O \) inside the polygon and join it to each vertex.

**Fig. 13 (a).**

The polygon is now divided into \( n \) triangles.

But the sum of the angles of each triangle is 2 rt. \( \angle s \).

\[ \therefore \text{the sum of the angles of the } n \text{ triangles is } 2n \text{ rt. } \angle s. \]

But these angles make up all the angles of the polygon together with all the angles at \( O \).

Now the sum of all the angles at \( O \) is 4 rt. \( \angle s \).

\[ \therefore \text{the sum of all the angles of the polygon } + 4 \text{ rt. } \angle s = 2n \text{ rt. } \angle s. \]

(2) At each vertex, the interior \( + \) the exterior \( = 2 \) rt. \( \angle s \).

\[ \therefore \text{the sum of all the interior angles } + \text{ the sum of all the exterior angles } = 2n \text{ rt. } \angle s. \]

But the sum of all the interior angles + 4 rt. \( \angle s = 2n \) rt. \( \angle s \).

\[ \therefore \text{the sum of all the exterior } \angle s = 4 \text{ rt. } \angle s. \]

**Q.E.D.**

**Theorem 9** (1) may also be stated as follows:

The sum of the interior angles of any convex polygon of \( n \) sides is \( 2n - 4 \) right angles.
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EXERCISE I. (Numerical)

[Arrows in a diagram denote that lines are given parallel.]

Find the unknown marked angles in Figs. 14-17. Give reasons.

1. \[ \angle 40^\circ \]

2. \[ \angle 25^\circ \]

3. \[ \angle 3^\circ \]

4. \[ \angle 6^\circ \]

5. What is (i) the supplement of \( 67^\circ \), (ii) the complement of \( 72^\circ \)?

6. What angle is (i) 8 times its supplement, (ii) \( \frac{1}{3} \) times its complement?

7. OA, OB, OC are 3 lines in order such that \( \angle AOB = 54^\circ \), \( \angle BOC = 24^\circ \); OP bisects \( \angle AOC \); find \( \angle POB \).

8. A line EC meets a line AB at C. Find \( \angle ACE \), if
   (i) \( \angle ACE = 4 \angle BCE \), (ii) \( \angle ACE exceeds \angle BCE by 100^\circ \).

Find the unknown marked angles in Figs. 18-23. Give reasons.

9. \[ \angle 54^\circ \]

10. \[ \angle 3^\circ \]

11. \[ \angle 125^\circ \]

12. \[ \angle 10^\circ \]

13. \[ \angle 90^\circ \]

14. \[ \angle 105^\circ \]

15. Name pairs of parallel lines in Fig. 24? Give reasons.

16. Find the vertical angle of a triangle if
   (i) each base angle is 4 times the vertical angle,
   (ii) the vertical angle is \( \frac{2}{3} \) times each base angle.

17. In \( \triangle ABC \), the lines which bisect \( \angle B, \angle C \) meet at K. Find
   \( \angle BKC \) if (i) \( \angle BAC = 80^\circ \), \( \angle ABC = 44^\circ \), (ii) \( \angle BAC = 80^\circ \), \( \angle ABC = 2^\circ \).

18. Find the sum of the interior angles of a 20-sided polygon.

19. Prove that the sum of the angles of an 11-sided polygon is 3 times the sum of the angles of a pentagon.

20. Each angle of a polygon is \( 140^\circ \); how many sides has it?

EXERCISE II

1. In Fig. 25, ABC is a straight line and BQ bisects \( \angle CBD \); prove that
   \( \angle RBQ = \angle QBC = \angle QBD \).

2. In Fig. 25, ABC is a straight line and BQ, BR bisect \( \angle CBD \), \( \angle ABD \); prove that
   \( \angle QBR = 90^\circ \). Prove also that \( \angle ABQ + \angle CBR \) equals 3 right angles.
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3. OC bisects \( \angle AOB \); OP is any line between OB and OC; prove that \( \angle AOP = \angle POC = \frac{1}{2} \angle COP \).

4. ABCD is a quadrilateral such that AB is parallel to DC; prove that \( \angle A + \angle B + \angle C = \angle D \).

5. ABCDE is a pentagon. If AB \parallel ED, prove that \( \angle B + \angle C + \angle D = 4 \) right angles.

6. P is any point inside \( \triangle ABC \); prove that \( \angle BPC = \angle BAC + \angle PCA \).

7. D is a point on the base BC of the triangle ABC such that \( \angle DAC = \angle ABC \); prove that \( \angle ADC = \angle BAC \).

8. The diagonals of the parallelogram ABCD meet at O; prove that \( \angle AOB = \angle ADB + \angle AOC \).

9. If, in the quadrilateral ABCD, AC bisects the angle DAB and the angle DCB; prove that \( \angle ADC = \angle ABC \).

10. ABC is a triangle, right-angled at A; AD is drawn perpendicular to BC; prove that \( \angle DAC = \angle ABC \).

11. In the \( \triangle ABC \), BE and CF are perpendiculars from B, C to AC, AB; BE cuts CF at H; prove that \( \angle CHE = \angle FAD \).

12. In Fig. 26, express \( \angle x \) in terms of \( a, b, c \).

CONGRUENCE

CONGRUENCE TESTS. ISOCELES TRIANGLES

Definitions.

(i) If two sides of a triangle are equal, the triangle is called isosceles; the angle between the equal sides is often called the vertical angle, and the third side is called the base.

(ii) A polygon is called regular if all its angles are equal and all its sides are of equal length.

Congruence.

The size and shape of a triangle are fixed by any of the following sets of measurements:

(i) Two sides and the included angle. (Th. 3, p. 3.)

(ii) One side and two angles, given the situation of the side with respect to the angles. (Th. 10.)

(iii) Three sides. (Th. 12.)

Further:

(iv) If the triangle is right-angled, the hypotenuse and one other side. (Th. 13.)

But an ambiguous case arises if two sides and a not-included angle are measured. Two triangles can be drawn to fit these measurements if the given angle is opposite the shorter of the two given sides.

Fig. 28 shows the construction for drawing a triangle if two sides and a not-included angle are given. Fig. 29 shows the two different kinds of triangles that can be drawn to fit the given measurements.

\[ \text{S.A.S.} \]
**THEOREM 10**

Two triangles are congruent if two angles and a side of one are respectively equal to two angles and the corresponding side of the other.

Given either that
\[ \triangle ABC \cong \triangle PQR, \]
\[ \angle ABC = \angle PQR, \]
\[ \angle ACB = \angle PRQ, \]
or that
\[ \triangle ABC \cong \triangle PQR, \]
\[ \angle ABC = \angle QPR, \]
\[ \angle BAC = \angle QPR. \]

To prove
\[ \triangle ABC \cong \triangle PQR. \]

The sum of the three angles of any triangle is 180°.

\[ \therefore \text{ in each case, the remaining pair of angles is equal.} \]

Apply the triangle ABC to the triangle PQR so that B falls on Q and BC falls along QR.

Since BC = QR, C falls on R.

And since BC falls on QR and \( \angle ABC = \angle PQR \), \( \therefore \) BA falls along QP.

And since CB falls on RQ and \( \angle ACB = \angle PRQ \), \( \therefore \) CA falls along RP.

\[ \therefore \text{ A falls on P.} \]

\[ \therefore \text{ the triangle ABC coincides with the triangle PQR.} \]

\[ \therefore \triangle ABC \cong \triangle PQR. \]

**Q.E.D.**

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**THEOREM 11**

(1) If two sides of a triangle are equal, then the angles opposite to those sides are equal.

(2) If two angles of a triangle are equal, then the sides opposite those angles are equal.

ABC is a triangle; let the line bisecting the angle BAC meet BC at D.

(1) Given \( AB = AC \).

To prove \( \triangle ABC \cong \triangle ABD \).

In the \( \triangle ABD, ACD \),

\( AB = AC \), given.

\( AD \) is common.

\( \angle BAD = \angle CAD \), const.

\( \therefore \text{ the } \triangle s \text{ are congruent (2 sides, inc. angle).} \)

\( \therefore \triangle ABD \cong \triangle ACD \).

(2) Given \( \angle ABC = \angle ACB \).

To prove \( AC = AB \).

In the \( \triangle ABD, ACD \),

\( \angle ABD = \angle ACD \), given.

\( \angle BAD = \angle CAD \), const.

\( AD \) is common.

\( \therefore \text{ the } \triangle s \text{ are congruent (2 angles, corr. side).} \)

\( \therefore \ AB = AC. \)

**Q.E.D.**
THEOREM 12

Two triangles are congruent if the three sides of one are respectively equal to the three sides of the other.

\[ \text{Given that } AB = XY, BC = YZ, CA = ZX. \]
\[ \text{To prove } \triangle ABC \equiv \triangle XYZ. \]

Place the triangle \( \triangle ABC \) so that \( \triangle B \) falls on \( \text{Y} \) and \( \triangle \text{BC} \) along \( \text{YZ} \);
\[ \therefore \text{ since } \text{BC} = \text{YZ}, \text{C} \text{ falls on } \text{Z}. \]

Let the point \( \text{A} \) fall at a point \( \text{F} \) on the opposite side of \( \text{YZ} \) to \( \text{X} \). Join \( \text{XF} \).

Now \( \text{YF} = \text{BA} \), constr.

But \( \text{BA} = \text{XY} \), given.
\[ \therefore \text{YF} = \text{YX}. \]

But these are sides of the triangle \( \text{YFX} \).
\[ \therefore \angle \text{YXF} = \angle \text{YFX}. \]

Similarly, \( \angle \text{ZXF} = \angle \text{ZFXY} \).
\[ \therefore \text{adding in Fig. 32 (1) or subtracting in Fig. 32 (2)}, \]
\[ \angle \text{ZX} = \angle \text{YFZ}. \]

But \( \angle \text{BAC} = \angle \text{YFZ} \), constr.
\[ \therefore \angle \text{BAC} = \angle \text{YXZ}. \]

\[ \therefore \text{in the } \triangle \text{ABC, XYZ}, \]
\[ \triangle \text{ABC} = \triangle \text{XYZ}, \text{given}. \]
\[ \triangle \text{AC} = \triangle \text{AC}, \text{given}. \]
\[ \angle \text{BAC} = \angle \text{YXZ}, \text{proved}. \]
\[ \therefore \angle \text{ABC} = \angle \text{XYZ} \text{ (2 sides, inc. angle)}. \quad \text{Q.E.D.} \]

THEOREM 13

Two right-angled triangles are congruent if the hypotenuse and a side of one are respectively equal to the hypotenuse and a side of the other.

\[ \text{Given } \angle \text{ABC} = 90^\circ = \angle \text{XYZ}, \text{AC} = \text{XZ}, \text{AB} = \text{XY}. \]
\[ \text{To prove } \triangle \text{ABC} \equiv \triangle \text{XYZ}. \]

Produce \( \text{ZY} \) to \( \text{Q} \), making \( \text{YQ} = \text{BC} \), join \( \text{XQ} \).

Since \( \angle \text{ZXY} = 90^\circ \) and \( \text{ZY} \) is a straight line, \( \angle \text{QYX} = 90^\circ \).
\[ \therefore \text{in the } \triangle \text{ABC, XYZ}, \]
\[ \triangle \text{ABC} \equiv \triangle \text{XYZ}, \text{right angles}. \]
\[ \therefore \text{AC} = \text{XQ} \text{ and } \angle \text{C} = \angle \text{Q}. \]

But \( \text{AO} = \text{XZ} \), given.
\[ \therefore \text{XZ} = \text{XQ} \text{ and } \triangle \text{ZXQ} \text{ is isosceles}, \]
\[ \therefore \angle \text{Q} = \angle \text{Z}. \]

But \( \angle \text{C} = \angle \text{Q} \).
\[ \therefore \angle \text{C} = \angle \text{Z}. \]

\[ \therefore \text{in the } \triangle \text{ABC, XYZ}, \]
\[ \triangle \text{ABC} \equiv \triangle \text{XYZ} \text{ (2 angles, corr. side)}. \quad \text{Q.E.D.} \]
CONGRUENCE

13. ABCDE is a regular pentagon, prove that the line bisecting the angle BAC is perpendicular to AE.

14. In the triangle ABC, AB = AC; D is a point in AC such that AD = BD = DC; calculate $\angle BAC$.

15. ABC is an equilateral triangle; BC is produced to D so that BC = CD; prove that $\angle BAC = 90^\circ$.

EXERCISE IV

State a reason for each step in the argument.

1. Two unequal straight lines AOB, COD bisect each other; prove that AC = BD.

2. A line AP is drawn bisecting the angle BAC; PX, PY are the perpendiculars from P to AB, AC; prove that PX = PY.

3. X is the mid-point of a chord AB of a circle, centre O; prove that $\angle OXA = 90^\circ$.

4. ABC and XYZ are two triangles. By marking the data on a figure, find out which of the following sets of data make the two triangles congruent. Give reasons.

   (i) AB = ZY, AC = XZ, $\angle A = \angle Z$.
   (ii) AB = XY, $\angle A = \angle X$, $\angle B = \angle Z$.
   (iii) AC = XZ, AB = XY, $\angle C = \angle Z$.
   (iv) AB = YZ, $\angle A = \angle Z$, $\angle C = \angle X$.
   (v) AB = ZY, $\angle A = \angle Z$, $\angle B = \angle X$.
   (vi) BC = ZY, AC = XY, BG = YZ.
   (vii) $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$. (viii) AC = ZY, BC = XY, $\angle C = \angle Y$.

5. In Fig. 36, ABCD and APQR are squares. Prove that BP = DR.

6. In Fig. 37, ABC and AXY are equilateral triangles. Prove that BX = CY.
7. Two circles, centres A, B, cut at P, Q; AB cuts PQ at N. Prove that (i) \( \triangle APQ = \triangle AQB \), (ii) PN = NQ and \( \angle ANP = 90^\circ \).

8. Draw any triangle \( \triangle ABC \) and complete the parallelograms BCAP, ABSC. Prove that AP = AQ.

9. Draw two concentric circles, centre O, and draw any line PQRS cutting the larger circle at P, S and the smaller circle at Q, R. Prove that (i) \( \angle QPF = \angle ORS \), (ii) PQ = RS.

10. In the \( \triangle ABC \), AB = AC; AB is produced to D so that BD = BC; prove that \( \angle ACD = 3\angle ADC \).

11. P is a point on the line bisecting \( \angle BAC \); through P a line is drawn parallel to AC and cutting AB at Q; prove AQ = QP.

12. In \( \triangle ABC \), AB = AC; D is a point on AC produced such that BD = BA; if \( \angle CBD = 36^\circ \), prove BC = CD.

13. If in Fig. 58, AB = AC and CP = CQ, prove \( \angle SRP = 3\angle RPC \).

14. In the quadrilateral ABCD, AB = AD and \( \angle ABC = \angle ADC \); prove CD = BD.

15. ABC is an acute-angled triangle; AB < AC; the circle, centre A, radius AB cuts BC at D; prove that \( \angle ABC + \angle ADC = 180^\circ \).

16. A, B, C are three points on a circle, centre O; prove \( \angle ABC = \angle OAB + \angle OCB \), if O lies between BA and BC.

17. AD is an altitude of the equilateral triangle ABC; ADX is another equilateral triangle, prove that DX is perpendicular to AB or AC.

18. BC is the base of an isosceles triangle ABC; P, Q are points on AB, AC such that AP = PQ = QB = QC; calculate \( \angle BAC \).

19. D is the mid-point of the base BC of the triangle ABC; if AD = DB, prove \( \angle BAC = 90^\circ \).

20. In the quadrilateral ABCD, AB = CD and \( \angle ABC = \angle DCB \), prove \( \angle BAD = \angle CDA \).

21. ABC is a triangle such that \( \angle B = 90^\circ \) and \( \angle C = 90^\circ \). Prove that \( \angle ACB = 90^\circ \). (Produce CB to X so that CB = BX and show that \( \triangle ACX \) is equilateral.)

22. ABC is an equilateral triangle and Y is any point on BC; BYK is an equilateral triangle, drawn so that A and K are on opposite sides of BC. Prove that (i) \( \angle KYC = \angle KAC \); (ii) \( \angle BYK = \angle KCY \); (iii) \( \angle YAC = \angle YKC \).

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**THEOREM 14**

Inequalities.

1. If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

2. If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

---

**Fig. 30.**

1. **Given** \( AC > AB \).

To prove \( \angle ABC > \angle ACB \).

From AC cut off a part AX equal to AB. Join BX.

Since AB = AX, \( \angle ABX = \angle AXB \).

But ext. \( \angle AXB > \) int. opp. \( \angle XCB \), \( \therefore \angle ABC > \angle XCB \).

But \( \angle ABC > \angle ABX \), \( \therefore \angle ABC > \angle CXB \) or \( \angle ACB \).

2. **Given** \( \angle ABC > \angle ACB \).

To prove \( AC > AB \).

If AC is not greater than AB, it must either be equal to AB, or less than AB.

If \( AC = AB \), \( \angle ABC = \angle ACB \), which is contrary to hypothesis.

If \( AC < AB \), \( \angle ABC < \angle ACB \), which is contrary to hypothesis.

\( \therefore \) AC must be greater than AB.
THEOREM 15

Of all straight lines that can be drawn to a given straight line from an external point, the perpendicular is the shortest.

Given a fixed point $O$ and a fixed line $AB$.

ON is the perpendicular from $O$ to $AB$, and $OP$ is any other line from $O$ to $AB$.

To prove $ON < OP$.

Since the sum of the angles of a triangle is 2 rt. angles, and since $\angle ONP = 1$ rt. angle.

\[ \therefore \angle NPO + \angle NOP = 1 \text{ rt. angle}. \]

\[ \therefore \angle NPO < \angle NOP. \]

ON < OP. Q.E.D.

THEOREM 16*

Any two sides of a triangle are together greater than the third side.

Given the triangle $ABC$.

To prove $BA + AC > BC$.

Produce $BA$ to $P$ and cut off $AX$ equal to $AC$. Join $CX$.

Since $AX = AC$ and $\angle ACX = \angle AXC$.

But $\angle BCP > \angle ACX$. 

\[ \therefore \angle BCP > \angle AXC. \]

\[ \therefore \text{ in the triangle } BXC, \angle BCP > \angle BXC. \]

\[ \therefore BX > BC. \]

But $BX = BA + AX - BA + AC$. 

\[ \therefore BA + AC > BC. \quad \text{Q.E.D.} \]

EXERCISE V

1. The bisectors of the angles $ABC$, $ACB$ of $\triangle ABC$ meet at $I$; if $AB > AC$, prove that $IB > IC$.

2. $AD$ is a median of $\triangle ABC$; if $BC < 2AD$, prove that $\angle BAC < 90^\circ$.

3. $ABC$ is an equilateral triangle; $P$ is any point on $BC$; prove $AP < BP$.

4. $AD$ is a median of $\triangle ABC$; if $AB > AC$, prove that $\angle BAD < \angle CAD$.

5. $ABC$ is an acute-angled triangle, such that $\angle ABC = 2\angle ACB$; prove $AC < 2AB$.

6. $ABCD$ is a quadrilateral; prove that $AB + BC + CD > AD$.

7. Prove that any side of a triangle is less than half its perimeter.

8. How many triangles can be drawn such that two of the sides are of lengths 4 feet, 7 feet, and such that the third side contains a whole number of feet?

9. $ABC$ is a $\triangle$; $D$ is any point on $BC$; prove that $AD < \frac{1}{2}(AB + BC + CA)$.

10. $ABCD$ is a quadrilateral; $AB < BC$; $\angle BAD < \angle BCD$; prove $AD > CD$.

11. $O$ is any point inside the triangle $ABC$; prove that (i) $\angle BOC > \angle BAC$; (ii) $BO + OC > BA + AC$.

12. $A$, $B$, $C$, are any two points on the same side of $CD$, $A'$ is the image of $A$ in $CD$ (i.e. $CD$ bisects $AA'$ at right angles); $A'B$ cuts $CD$ at $O$; if $P$ is any other point on $CD$; prove that $AP + PB > AO + OB$.

13. $AD$ is a median of $\triangle ABC$; prove $AD < \frac{1}{2}(AB + AC)$.

14. $O$ is any point inside $\triangle ABC$; prove $OA + OB + OC > \frac{1}{2}(BC + CA + AB)$.

15. Prove that the sum of the diagonals of a quadrilateral is greater than the semiperimeter and less than the perimeter of the quadrilateral.

16. $AB$ is a diameter of a circle, centre $O$; $K$ is any point on the circumference; $P$ is any point on $OB$. Prove that $PA > PK > PB$.

17. $XY$ is a diameter of a circle, $H$ is any point on the circumference; $XH$ is produced to any point $Q$. Prove that $QX > QH > QY$. 

SHORTER GEOMETRY

Definition. A quadrilateral whose opposite sides are parallel is called a parallelogram, and the lines joining opposite corners are called diagonals.

THEOREM 17

(1) The opposite sides and angles of a parallelogram are equal.
(2) Each diagonal bisects the parallelogram.

Given ABCD is a parallelogram.

To prove
(1) \( AB = CD \) and \( AD = BC \).
\( \angle DAB = \angle DCB \) and \( \angle ABC = \angle ADC \).

(2) \( AC \) and \( BD \) each bisect the parallelogram.

Join \( BD \).

In the \( \triangle ADB \), \( CBD \),
\( \angle ADB = \angle CBD \), alt. \( \angle s \), \( AD \parallel BC \).
\( \angle ABD = \angle CDB \), alt. \( \angle s \), \( AB \parallel DC \).
\( BD \) is common.

\( \therefore \) \( \triangle ADB = \triangle CDB \) (2 angles, corr. side).

\( \therefore \) \( AB = CD \), \( AD = BC \), \( \angle DAB = \angle BCD \),
and \( BD \) bisects the parallelogram.

Similarly, by joining \( AC \) it may be proved that \( \angle ABC = \angle ADC \), and that \( AC \) bisects the parallelogram. Q.E.D.

PARALLELOGRAMS

THEOREM 18

The diagonals of a parallelogram bisect one another.

The diagonals \( AC \), \( BD \) of the parallelogram \( ABCD \) intersect at \( O \).

To prove \( AO = OC \) and \( BO = OD \).

In the \( \triangle AOD \), \( COB \),
\( \angle DAO = \angle BCO \), alt. \( \angle s \), \( AD \parallel BC \).
\( \angle ADO = \angle CBO \), alt. \( \angle s \), \( AD \parallel BC \).
\( AD = BC \), opp. sides of \( \parallel \) gram.

\( \therefore \) \( \triangle AOD = \triangle COB \) (2 angles, corr. side).

\( \therefore \) \( AO = CO \) and \( BO = DO \). Q.E.D.

Definitions.

A rectangle is a parallelogram, one angle of which is a right angle.

A square is a rectangle, having two adjacent sides equal.

A rhombus is a parallelogram, having two adjacent sides equal, but none of its angles right angles.

A trapezium is a quadrilateral having one pair of opposite sides parallel.

The following facts are important:

(i) The diagonals of a rectangle are equal.
(ii) The diagonals of a rhombus cut at right angles.
(iii) The diagonals of a square are equal and cut at right angles.
THEOREM 19
A quadrilateral is a parallelogram if
either (1) its opposite angles are equal,
or (2) its opposite sides are equal,
or (3) its diagonals bisect each other.

![Diagram of parallelogram](image)

(1) Given ∠A = ∠C and ∠B = ∠D.
To prove ABCD is a parallelogram.
The angles of any quadrilateral add up to 360°.
∴ ∠A + ∠B + ∠C + ∠D = 360°.
But ∠A = ∠C and ∠B = ∠D, given.
∴ 2∠A + 2∠D = 360°,
∴ ∠A + ∠D = 180°.
∴ AD is parallel to BC.
Similarly, it may be proved that AB is parallel to DC.
∴ ABCD is a parallelogram. Q.E.D.

(2) Given AB = CD and AD = CB.
To prove ABCD is a parallelogram.
Join BD.
In the Δs ADB, CBD, AD = CB, given.
AB = CD, given.
DB is common.
∴ ΔADB ≅ ΔCBD (3 sides).
∴ ∠ADB = ∠CBD, but these are alternate angles.
∴ AD is parallel to BC.
Similarly, since ∠ABD = ∠CDB, AB is parallel to DC.
∴ ABCD is a parallelogram. Q.E.D.

PARALLELOGRAMS

(3) Given AO = OC and BO = OD.
To prove ABCD is a parallelogram.

![Diagram of parallelogram](image)

In the Δs AOB, COD, AO = OC, given.
BO = OD, given.
∠AOB = ∠COD, vert. opp.
∴ ΔAOB ≅ ΔCOD (2 sides, inc. angle).
∴ ∠BAO = ∠DCO, but these are alternate angles.
∴ AB is parallel to CD.
Similarly, it may be provided that AD is parallel to BC.
∴ ABCD is a parallelogram. Q.E.D.

Construction. Through a given point C, draw a straight line parallel to a given line AB.

![Construction diagram](image)

With C as centre, draw any arc PQ cutting AB at P; with P as centre and the same radius, draw an arc cutting AB at R.
With centre P and radius equal to CR, draw an arc cutting arc PQ at Q on the same side of AB as C. Join CQ. Then CQ is parallel to AB.
For ΔCRP ≅ ΔPQC (3 sides). ∠CRP = alt. ∠PCQ.
THEOREM 20

The straight lines which join the ends of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

Given AB is equal and parallel to CD.

To prove AC is equal and parallel to DB.

Join BC.

In the \( \Delta ABC, DCB, \)
\[ AB = DC, \text{ given.} \]
BC is common.
\[ \angle ABC = \angle DCB \text{ alt. angles, } AB \text{ being } \parallel \text{ to } CD, \]
\[ \therefore \Delta ABC = \Delta DCB \text{ (2 sides, inc. angle).} \]
\[ \therefore AC = DB \text{ and } \angle ACB = \angle DBC. \]
But these are alt. angles; \( \therefore AC \) is parallel to DB.
Q.E.D.

This theorem can also be stated as follows:

A quadrilateral which has one pair of equal and parallel sides is a parallelogram.

EXERCISE VI

1. The diagonals of the rectangle ABDC meet at O; \( \angle BOC = 44^\circ \); calculate \( \angle OAD \).
2. ABCD is a rectangle; \( \angle BAC = 32^\circ \); calculate \( \angle DBC \).
3. ABCD is a rhombus; \( \angle ABC = 50^\circ \); calculate \( \angle ACD \).
4. The diagonals of the parallelogram ABCD cut at O; any line through O cuts AB, CD at X, Y; prove XO = OY.
5. ABCD is a parallelogram; P is the mid-point of BC; DP and AB are produced to meet at Q; prove AQ = 2AB.
6. ABCD, ABXY are two parallelograms; BC and BX are different lines; prove that DCXY is a parallelogram.
7. The diagonals of a square ABCD cut at O; from AB a part AK is cut off equal to AO; prove \( \angle AOK = 2 \angle BOK \).
8. ABCD is a straight line such that \( AB = BC = CD \); BCPQ is a rhombus; prove that AQ is perpendicular to DP.
9. ABCD is a parallelogram such that the bisectors of \( \angle s DAB, \)
   ABC meet on CD; prove AB = 2BC.
10. In \( \angle ABC, \angle BAC = 90^\circ ; \) BADH, ACKE are squares outside the triangle; prove that HAK is a straight line.
11. X, Y are the mid-points of the sides AB, AC of the \( \Delta ABC \);
P is any point on a line through A parallel to BC; PX, PY are produced to meet BC at Q, R; prove QR = BC.
12. ABC is a triangle; the perpendicular bisectors of AB, AC meet at O; prove OB = OC.
13. ABC is a triangle; the lines bisecting the angles ABC, ACB meet at I; prove that the perpendiculars from I to AB, AC are equal.
14. The line joining the mid-points E, F of AB, AC is produced to G so that EF = FG; prove that BE is equal and parallel to CG.
15. In the 5-sided figure ABCDE, the angles at A, B, C, D are each 120°; prove that AB = BC = DE.
16. ABC is a triangle; lines are drawn through C parallel to the bisectors of the angles CAB, CBA to meet AB produced in D, E; prove that DE equals the perimeter of the triangle ABC.
17. ABC is an equilateral triangle; a line parallel to AC cuts BA, BC at P, Q; AC is produced to R so that BQ = CR; prove that PR bisects CQ.
18. In \( \Delta ABC, \angle BAC = 90^\circ ; \)
   ABPQ, ACRS, BQXY are squares outside ABC; prove that (i) BQ is parallel to CS; (ii) BR is perpendicular to AX.

P.S.G.
CONSTRUCTION 1

From a given point in a given straight line, draw a straight line making with the given line an angle equal to a given angle.

Given a point A on a given line AB and an angle XYZ.
To construct a line AC such that \( \angle CAB = \angle XYZ \).

With centre Y and any radius, draw an arc of a circle cutting YX, YZ at P, Q.

With centre A and the same radius, draw an arc of a circle EF, cutting AB at E.

With centre E and radius equal to PQ, describe an arc of a circle, cutting the arc EF at F.

Join AF and produce it to C.

Then AC is the required line.

Proof. Join PQ, EF.

In the \( \triangle PYQ, FAE \),

\[ \begin{align*}
YF &= AF, \text{ constr.} \\
YQ &= AE, \text{ constr.} \\
PQ &= EF, \text{ constr.} \\
\therefore \triangle PYQ &= \triangle FAE \text{ (3 sides).} \\
\therefore \angle XYZ &= \angle BAC.
\end{align*} \]

†Definition. A circle is a plane figure bounded by a curved line, all points of which are equidistant from a fixed point, called the centre of the circle. The curved line is called the circumference of the circle. The distance between the centre and any point on the circumference is called the radius.

CONSTRUCTION 2

Bisect a given angle.

Given an angle BAC.

To construct a line bisecting the angle.

With A as centre and any radius, draw an arc of a circle, cutting AB, AC at P, Q.

With centres P, Q and with any sufficient radius, the same for each, draw arcs of circles, cutting at R. Join AR.

Then AR is the required bisector.

Proof. Join PR, QR.

In the \( \triangle APR, AQR \),

\[ \begin{align*}
AP &= AQ, \text{ radii of the same circle.} \\
PR &= QR, \text{ radii of equal circles.} \\
AR &= \text{common.} \\
\therefore \triangle APR &= \triangle AQR \text{ (3 sides).} \\
\therefore \angle PAR &= \angle QAR. \\
\text{Q.E.D.}
\end{align*} \]
CONSTRUCTION 3

Draw the perpendicular bisector of a given finite straight line.

\[ \text{FIG. 51.} \]

\[ A \quad C \quad B \]

**Given a finite line AB.**

To construct the line bisecting AB at right angles.

With centres A, B and any sufficient radius, the same for each, draw arcs of circles to cut at P, Q.

Join PQ and let it cut AB at C.

Then C is the mid-point of AB, and PCQ bisects AB at right angles.

**Proof.** Join PA, PB, QA, QB.

In the \( \triangle \)s PAQ, PBQ,

\[ PA = PB, \text{ radii of equal circles.} \]
\[ QA = QB, \text{ radii of equal circles.} \]
\[ PQ \text{ is common.} \]
\[ \therefore \triangle \)PAQ = \triangle PBQ \text{ (3 sides).} \]
\[ \therefore \angle \)PAQ = \angle PBQ. \]

In the \( \triangle \)s APC, BPC,

\[ PA = PB, \text{ radii of equal circles.} \]
\[ PC \text{ is common.} \]
\[ \angle \)APC = \angle BPC, \text{ proved.} \]
\[ \therefore \angle \)APC = \angle BPC \text{ (2 sides, inc. angle).} \]
\[ \therefore AC = CB. \]

and \[ \angle \)ACP = \angle BCP. \]

These are adj. \( \angle \)s on st. line, \( \therefore \) each is a right angle.

Q.E.F.

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CONSTRUCTION 4

Draw a straight line at right angles to a given straight line from a given point in it.

\[ \text{FIG. 52.} \]

\[ A \quad P \quad B \]

**Given a point C on a line AB.**

To construct a line from C perpendicular to AB.

With centre C and any radius, draw an arc of a circle cutting AB at P, Q.

With centres P, Q and any sufficient radius, the same for each, draw arcs of circles to cut at R. Join CR.

Then CR is the required perpendicular.

**Proof.** Join PR, QR.

In the \( \triangle \)s RCP, RQO,

\[ RP = RQ, \text{ radii of equal circles.} \]
\[ CP = CQ, \text{ radii of the same circle.} \]
\[ CR \text{ is common.} \]
\[ \therefore \triangle \)RCP = \triangle RCO \text{ (3 sides).} \]
\[ \therefore \angle \)RCP = \angle RCO. \]

These are adj. \( \angle \)s on st. line, \( \therefore \) each is a right angle.

Q.E.F.
CONSTRUCTION 5

Draw a perpendicular to a given straight line of unlimited length from a given point outside it.

Given a line AB and a point C outside it.

To construct a line from C perpendicular to AB.

With C as centre and any sufficient radius, draw an arc of a circle, cutting AB at P, Q.

With P, Q as centres and any sufficient radius, the same for each, draw arcs of circles, cutting at R. Join CR and let it cut AB at X.

Then CX is perpendicular to AB.

Proof. Join CP, CQ, RP, RQ.

In the \( \triangle \text{CPR}, \triangle \text{CQR}, \)

\( 
\begin{align*}
\text{CP} &= \text{CQ}, \text{radii of the same circle}, \\
\text{RP} &= \text{RQ}, \text{radii of equal circles}, \\
\text{CR} &= \text{common}.
\end{align*}
\)

\( \therefore \triangle \text{CPR} = \triangle \text{CQR} \) (3 sides).

\( \triangle \text{PCR} = \triangle \text{QCR}. \)

In the \( \triangle \text{CPX}, \triangle \text{CQX}, \)

\( \begin{align*}
\text{CP} &= \text{CQ}, \text{radii}, \\
\text{CX} &= \text{common}.
\end{align*} \)

\( \angle \text{PCR} = \angle \text{QCR}, \) proved.

\( \therefore \triangle \text{CPX} = \triangle \text{CQX} \) (2 sides, inc. angle).

\( \therefore \angle \text{CXP} = \angle \text{CXQ}. \)

These are adj. \( \triangle \)s on st. line, \( \therefore \) each is a right angle.

Q.E.F.

EXERCISE VII

1. Draw a circle of radius 3 cm. and take points A, B, C on it such that AB = 4 cm., AC = 6 cm. Measure \( \angle \text{BAC} \): is there more than one answer?

2. Draw a triangle ABC (not isosceles); construct a point P on BC such that the perpendiculars from P to AB and AC are equal.

3. Construct a parallelogram ABCD, given AB = 7 cm., AC = 10 cm., BD = 8 cm., measure BC, CD.

4. Construct an isosceles triangle with a base of 6 cm. and a vertical angle of 70\(^{\circ}\); measure its sides.

5. Construct a rhombus ABCD, given AB = 5 cm., AC = 6 cm.; measure \( \angle \text{BAD} \).

6. Construct the rhombus ABCD, given AC = 6 cm., BD = 9 cm., measure AB.

7. Construct the rhombus ABCD, given \( \angle \text{ABC} = 60^{\circ}, \text{BD} = 7 \text{ cm.}, \) measure AC.

8. Construct a rectangle ABCD, given BD = 8 cm. and that AC makes an angle of 34\(^{\circ}\) with BD; measure AB, BC.

9. Construct a trapezium ABCD with AB, CD its parallel sides such that AB = 8, BC = 4, CD = 3, AD = 2; measure \( \angle \text{BAD} \).

10. AE is a median of the triangle ABC; given AB = 4 cm., AC = 7 cm., AE = 5 cm., construct \( \angle \text{ABC} \); measure BC.

11. AD is an altitude of the triangle ABC; given AB = 6 cm., AD = 4 cm., \( \angle \text{ABC} = 60^{\circ}, \) construct \( \angle \text{ABC} \); measure BC.

12. AD is an altitude of the triangle ABC; AD = 4 cm., \( \angle \text{ABC} = 75^{\circ}, \) \( \angle \text{ABC} = 50^{\circ}, \) construct \( \angle \text{ABC} \); measure BC.

13. Construct a parallelogram of height 4 cm., having its diagonals 5 cm., 8 cm. in length; measure one of the longer sides.

14. Construct an equilateral triangle of height 4 cm.; measure its side.

15. Draw an angle BAC of 50\(^{\circ}\); construct on AB, AC points P, Q, such that \( \angle \text{QPA} = 90^{\circ} \) and FP = 4 cm. Measure AP.

16. Draw an angle BAC of 70\(^{\circ}\); construct a point P whose distances from AB, AC are 3 cm., 4 cm. Measure AP.

17. Draw a line AB and take a point C distant 2" from AB; construct two points P, Q, each of which is 1½" from AB and 1½" from C. Measure PQ.
Equal Intercept Theorems.

THEOREM 21

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

\[\begin{align*}
\text{Given } AH &= HB \text{ and } HK \text{ is parallel to } BC. \\
\text{To prove } AK &= KC.
\end{align*}\]

Through K draw a line KX parallel to AB to cut BC at X.

Since HK is parallel to BX, given, and KX is parallel to HB constr.

\[\therefore \text{ BHXX is a parallelogram.}\]

\[\therefore \text{ KX } = \text{ HB.}\]

But AH = HB, given. \[\therefore \text{ KX } = \text{ AH.}\]

\[\therefore \text{ in the } \triangle \text{AHK, } \text{KXK,}\]

\[\angle \text{HAK } = \angle \text{XKK}, \text{ corresp. } \angle \text{s., AH } || \text{KX.}\]

\[\angle \text{HKA } = \angle \text{XCK}, \text{ corresp. } \angle \text{s., HK } || \text{XK.}\]

\[\text{AH } = \text{KX.}\]

\[\therefore \triangle \text{AHK } = \triangle \text{KXC} \text{ (2 angles, corr. side).}\]

\[\therefore \text{ AK } = \text{KC.}\]

Q.E.D.

THEOREM 22

The straight line joining the middle points of two sides of a triangle is parallel to the base and equal to half the base.

\[\begin{align*}
\text{Given } H, K \text{ are the middle points of } AB, AC. \\
\text{To prove } HK \text{ is parallel to } BC \text{ and } HK = \frac{1}{2}BC.
\end{align*}\]

Through C, draw CP parallel to BA to meet HK produced at P.

In the \(\triangle \text{s AHK, CPK,}\)

\[\angle \text{AHK } = \angle \text{CPK, alt. } \angle \text{s., HA } || \text{CP,}\]

\[\angle \text{HAK } = \angle \text{CPK, alt. } \angle \text{s., HA } || \text{CP,}\]

\[\text{AK } = \text{KC, given.}\]

\[\therefore \triangle \text{AHK } = \triangle \text{CPK} \text{ (2 angles, corr. side).}\]

\[\therefore \text{ CP } = \text{AH.}\]

But AH = BH, given.

\[\therefore \text{ CP } = \text{BH.}\]

Also CP is drawn parallel to BH.

\[\therefore \text{ the lines CP, BH are equal and parallel.}\]

\[\therefore \text{ BCPH is a parallelogram.}\]

\[\therefore \text{ HK is parallel to } BC.\]

Also HK = KP from congruent triangles.

\[\therefore \text{ HK } = \frac{1}{2}KP.\]

But HP = BC opp. sides of parallelogram.

\[\therefore \text{ HK } = \frac{1}{2}BC.\]

Q.E.D.
THEOREM 23

If there are three or more parallel straight lines, and if the intercepts made by them on any straight line cutting them are equal, then the intercepts made by them on any other straight line that cuts them are equal.

Given three parallel lines cutting a line AE at B, C, D and any other line PT at Q, R, S and that BO = CD.

To prove QR = RS.

Draw BH, CK parallel to PT to meet CR, DS at H, K.

Then BH is parallel to CK.

\[ \triangle BCH \cong \triangle CKD \] (corresponding angles equal)

\[ BC = CD \] (given)

\[ \therefore \triangle BHC \cong \triangle DCK \] (2 angles, corresponding side)

\[ \therefore BH = CK \]

But BQKH is a parallelogram since its opposite sides are parallel.

\[ \therefore BH = QR \]

And CRKS is a parallelogram since its opposite sides are parallel.

\[ \therefore CK = RS \]

\[ \therefore QR = RS \]

Q.E.D.

CONSTRUCTION 6

Divide a given straight line into any given number of equal parts.

Given a line AB.

To construct points dividing AB into any given number (say 5) equal parts.

Through A, draw any line AC.

Along AC, step out with compasses equal lengths, the number of such lengths being the required number of equal parts (in this case 5).

Let the equal lengths be AF, FG, GH, HK, KL.

Join LS, and through F, G, H, K draw lines parallel to BL, meeting AB at P, Q, R, S.

Then AP, PQ, QR, RS, SB are the required equal parts.

Proof. Since the parallel lines FP, GQ, HR, KS, LB cut off equal intercepts on AC, they cut off equal intercepts on AB.

EXERCISE VIII

1. Draw a line AB and construct a point P on AB such that
   \[ \frac{AP}{PB} = \frac{2}{3} \]

2. Draw a line AB and construct a point Q on AB produced, such that
   \[ \frac{AQ}{BQ} = \frac{3}{4} \]

3. Divide a given line in the ratio 5:3 both internally and externally.
4. In $\triangle ABC$, $\angle BAC = 90^\circ$; $D$ is the mid-point of $BC$; prove that $AD = \frac{1}{2}BC$. (From $D$, drop a perpendicular to $AC$.)

5. In Fig. 58, if $AC = CB$ and if $AP$, $BQ$, $CR$ are parallel, prove that $CR = \frac{1}{2}(AP + BQ)$.

6. In Fig. 59, if $AC = CB$, and if $AP$, $BQ$, $CR$ are parallel, prove that $CR = \frac{1}{2}(BQ - AP)$.

7. $P$, $Q$, $R$, $S$ are the mid-points of the sides $AB$, $BC$, $CD$, $DA$ of the quadrilateral $ABCD$; prove that $PQ$ is equal and parallel to $SR$.

8. In $\triangle ABC$, $\angle ABC = 90^\circ$; $BCX$ is an equilateral triangle; prove that the line from $X$ parallel to $AB$ bisects $AC$.

9. Prove that the lines joining the mid-points of opposite sides of any quadrilateral bisect each other.

10. If the diagonals of a quadrilateral are equal and cut at right angles, prove that the mid-points of the four sides are the corners of a square.

11. $ABC$ is a $\triangle$; $AX$, $AY$ are the perpendiculars from $A$ to the bisectors of the angles $ABC$, $BAC$; prove that $XY$ is parallel to $BC$.

12. In Fig. 60, if $P$, $Q$, $X$, $Y$ are the mid-points of $AB$, $AD$, $CD$, $DA$, prove that $PQ = XY$. What can you say about $PX$?

13. $AD$, $BE$ are altitudes of $\triangle ABC$ and intersect at $H$; $P$, $Q$, $R$ are the mid-points of $HA$, $AB$, $BC$; prove that $\angle PQR = 90^\circ$.

14. $ABC$ is a $\triangle$; $E$, $F$ are the mid-points of $AC$, $AB$; $BE$ cuts $CF$ at $G$; $AG$ is produced to $X$ so that $AG = GX$ and cuts $BC$ at $D$; prove that (i) $BCXD$ is a parallelogram; (ii) $DG = \frac{1}{2}GA = \frac{1}{4}DA$.

15. $ABCD$ is a parallelogram; $XY$ is any line outside it; $AP$, $BQ$, $CR$, $DS$ are perpendiculars from $A$, $B$, $C$, $D$ to $XY$; prove that $AP + CR = BQ + DS$.

REVISION PAPERS 1-5

1. 1. Fig. 61 represents a "Penagram" (i.e. the inner figure is a regular pentagon); calculate the acute angle at each corner.

2. $X$ is a point inside the triangle $ABC$ such that $\angle XAB = \angle XCA$; prove $\angle ABC + \angle BAC = 180^\circ$.

3. If in $\triangle ABC$, $AB = AC$ and $\angle BAC = 20^\circ$, and if $D$ is a point on $AC$ such that $\angle BDC = 60^\circ$, prove $AD = DS$.

4. $AB$, $DC$ are the parallel sides of the trapezium $ABCD$; if $AD = DC$, prove that $AC$ bisects $\angle BAD$.

2. 1. If, in Fig. 62, $AB$ is parallel to $EF$, calculate the angle $x$.

2. In $\triangle ABC$, $AB = AC$; $BC$ is produced to $D$ so that $CD = AB$; prove $\angle ABD = 2\angle ADB$.

3. What is the name of a quadrilateral in which:
   (i) The diagonals bisect each other?
   (ii) The diagonals bisect each other at right angles?
   (iii) The diagonals are equal and bisect each other?
   (iv) One pair of opposite sides are equal and the other pair are parallel and unequal?

4. $ABCD$ is a quadrilateral; $ADXC$, $BDFY$ are parallelograms; prove that $XY$ bisects $AB$.

3. 1. The sum of one pair of angles of a triangle is $100^\circ$, and the difference of another pair is $60^\circ$; prove that the triangle is isosceles.

2. In $\triangle ABC$, $\angle ACB = 3\angle ABC$; from $AB$ a part $AD$ is cut off equal to $AC$; prove $CD = DB$. 41
3. In \( \triangle ABC \), \( AB = AC \); from any point \( P \) on \( AB \) a line is drawn perpendicular to \( BC \) and meets \( CA \) produced in \( Q \); prove \( AP = AQ \).

4. \( ABCD \) is a square; the bisector of \( \angle BCA \) cuts \( AB \) at \( P \); \( FQ \) is the perpendicular from \( P \) to \( AC \); prove \( AQ = PB \).

4.

1. In Fig. 63, express \( z \) in terms of \( a, b, c \).

![Fig. 63](image)

2. \( D \) is any point on the bisector of \( \angle BAC \); \( DP, DQ \) are drawn parallel to \( AB, AC \), to meet \( AC, AB \) at \( P, Q \); prove \( DP = DQ \).

3. In \( \triangle ABC \), \( AB = AC \); \( D \) is a point on \( AC \) such that \( DB = BC \); prove \( \angle BDC = \angle BAC \).

4. \( P, Q, R, S \) are points on the sides \( AB, BC, CD, DA \) of a square; if \( PR \) is perpendicular to \( QS \), prove \( PR = QS \).

5.

1. In Fig. 64, express \( z \) in terms of \( a, b, x, y \).

2. In \( \triangle ABC \), \( AD \) is perpendicular to \( BC \) and \( AP \) bisects \( \angle BAC \); if \( \angle ABC > \angle ACB \), prove \( \angle ABC = \angle ACB = \angle PAD \).

3. \( ABCD \) is a straight line such that \( AB = BC = CD \); \( BPQC \) is a parallelogram; if \( BP = 2BC \), prove \( PD \) is perpendicular to \( AQ \).

4. The sides \( AB, AC \) of \( \triangle ABC \) are produced to \( D, E \); \( AH, AK \) are lines parallel to the bisectors of \( \angle BAC, \angle CBD \) meeting \( BC \) in \( H, K \); prove \( AB + AC = BC + HK \).

### SECTION II

**AREAS**

**Definitions.**

(i) If any side of a triangle is taken as its base, the perpendicular to that side from the opposite corner is called the altitude or height.

(ii) If any side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it is called the altitude or height.

A triangle has therefore three distinct altitudes \( AD, BE, CF \) (Fig. 65).

![Fig. 65](image)

A parallelogram has two distinct altitudes \( PQ, RS \) (Fig. 66).

(iii) If two figures are of equal area, they are said to be equivalent. The symbol for "equivalent" is \( \equiv \).

If the length of a line can be expressed by a whole number or fraction in terms of a chosen unit, the length is called commensurable with that unit. By taking the unit sufficiently small, the length can then be expressed by a whole number.
Thus, if the sides of a triangle are $2\frac{1}{2}$ in., $2\frac{3}{4}$ in., $3\frac{3}{4}$ in., these lengths can be written 75, 80, 96 units, if the unit is $\frac{1}{4}$ in.

If the side of a square is of unit length, the square is said to contain 1 unit of area. The area of any figure is the number of units of area it contains. We shall assume that theorems proved for all commensurable magnitudes are true also for incommensurable magnitudes.

THEOREM 24*

The area of a rectangle is measured by the product of the measures of two adjacent sides.

\[ \text{Area} = \text{length} \times \text{width} \]

*Fig. 67.

Given a rectangle $ABCD$ in which $AB$ is $x$ units long and $AD$ is $y$ units long, where $x$ and $y$ are whole numbers.

To prove that $ABCD$ contains $xy$ units of area.

Divide $AB$ into $x$ equal parts and through each point of division draw a line parallel to $AD$.

Divide $AD$ into $y$ equal parts and through each point of division draw a line parallel to $AB$.

Then $ABCD$ is divided into a number of compartments each of which is a square of unit area. But the compartments are arranged in $y$ rows with $x$ compartments in each row; therefore there are in all $xy$ squares of unit area.

\[ \therefore \text{ABCD contains } xy \text{ units of area.} \]

Q.E.D.

THEOREM 25

The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels.

*Fig. 68.

Given $ABCD$ is a rectangle and $ABPQ$ is a parallelogram on the same base $AB$ and between the same parallels $AB$, $DP$.

To prove that the area $ABCD$ = the area $ABPQ$.

In the $\triangle$s $AQD$, $BPC$,

\[ \angle AQD = \angle BPC, \text{ corresp. } \angle s, \text{ } AQ \parallel BP. \]

\[ \angle ADQ = \angle BCP, \text{ corresp. } \angle s, \text{ AD } \parallel \text{ BC}. \]

$AD = BC$, opp. sides \| gram.

\[ \therefore \triangle AQD \cong \triangle BPC \text{ (2 angles, corr. side).} \]

From the whole figure $ABPD$, subtract in succession each of the equal triangles $AQD$, $BPC$.

\[ \therefore \text{the remaining figures } ABPQ, ADBC \text{ are equal in area.} \]

Q.E.D.

Corollary 1. The area of a parallelogram is measured by the product of its base and its altitude.

The area of $ABPQ$ = the area of $ABCD = AB \times BC$ = base $AB \times$ height $BC$.

Corollary 2. Parallelograms on the same or equal bases and of equal altitudes are equal in area.

This follows from Corollary 1. (See also p. 145.)

Note.—The proof of Theorem 25 applies, word for word, to any two parallelograms, $ABCD$ and $ABPQ$, on the same base and between the same parallels.
THEOREM 26

The area of a triangle is equal to one-half of the area of a rectangle on the same base and between the same parallels.

Given a triangle ABC and a rectangle PQBC on the same base BC, and between the same parallels BC, QA.

To prove that the area of \( \triangle ABC \) is equal to half the area of PQBC.

Complete the parallelogram ABCY.

Then area ABC = \( \frac{1}{2} \) area ABCY, diagonal bisects \( \| \)gram, but area ABCY = area QBCP, same base BC, same parallels BC, QY.

\[ \therefore \text{area ABC} = \frac{1}{2} \text{area QBCP}. \]

Corollary 1. The area of a triangle is measured by one-half the product of the measures of its base and its altitude.

The area of QBCP = BC \( \times \) CP.

But the perpendicular from A to BC equals CP, opp. sides of a rectangle.

\[ \therefore \text{area ABC} = \frac{1}{2} \text{base} \times \text{altitude}. \]

Corollary 2. Triangles on the same or equal bases and of equal altitudes are equal in area.

This follows from Corollary 1. (See also p. 145.)

Corollary 3. If triangles of the same area have the same or equal bases, their altitudes are equal.

This follows from Corollary 1.

Corollary 4. If two triangles of the same area stand on the same side of the same base or of equal bases in the same straight line, they are between the same parallels.

By Corollary 3 (or see p. 146), the altitudes AP, XQ are equal.

Also AP is parallel to XQ since each is perpendicular to BQ.

\[ \therefore \text{AP is equal and parallel to XQ.} \]

\[ \therefore \text{AX is parallel to PQ or BZ.} \]

AREAS

THEOREM 27

If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to half that of the parallelogram.

Given the triangle ABC and the parallelogram ABXY on the same base AB and between the same parallels AB, CX.

To prove \( \triangle ABC = \frac{1}{2} \| \text{gram ABXY} \).

Join BY.

The \( \triangle ABC \), ABY are on the same base and between the same parallels.

\[ \therefore \triangle ABC = \triangle ABY \text{ in area}. \]

Since the diagonal BY bisects the \( \| \)gram ABXY,

\[ \triangle ABY = \frac{1}{2} \| \text{gram ABXY}; \]

\[ \therefore \triangle ABC = \frac{1}{2} \| \text{gram ABXY}. \]

Q.E.D.

The following formula for the area of a triangle is important:

If \( a, b, c \) are the lengths of the sides of a triangle and if \( s = \frac{1}{2}(a + b + c) \), the area of the triangle \( = \sqrt{s(s-a)(s-b)(s-c)}. \)

The Area of a Trapezium.

ABCD is a trapezium with AB parallel to DC; DE is the perpendicular from D to AB.

If \( AB = a \) in., \( DC = b \) in., \( DE = h \) in., the area of the trapezium \( ABCD = \frac{1}{2}(a+b) \text{ sq. in.} \).

Join DB, draw BF perpendicular to DC, produced.

Then BF = DE = \( h \) in., opp. sides of a \( \| \)gram.

Area of \( \triangle ADB = \frac{1}{2} \text{DE} \cdot AB = \frac{1}{2} ah \text{ sq. in.} \)

Area of \( \triangle CDB = \frac{1}{2} BF \cdot DC = \frac{1}{2}bh \text{ sq. in.} \);

\[ \therefore \text{area of ABCD} = \frac{1}{2} ah + \frac{1}{2}bh = \frac{1}{2}(a + b) \text{ sq. in.} \] Q.E.D.

In words, the area of a trapezium is measured by the product of half the sum of the parallel sides and the distance between them.
CONSTRUCTION 7

(1) Reduce a quadrilateral to a triangle of equal area.
Given a quadrilateral ABCD.
To construct a triangle equal in area to it.

Join BD.
Through C, draw CK parallel to DB to meet AB produced at K.
Join DK.

Then ADK is the required triangle.

Proof. The triangles BCD, BKD are on the same base BD
and between the same parallels BD, KC.

:. area of \( \triangle ABC = \text{area of } \triangle BKD. 
\)

Add to each \( \triangle ABD. 
\)

:. area of quad. ABCD = area of \( \triangle AKD. 
\)

:. AKD is the required triangle.
Q.E.F.

In order to reduce a five-sided figure ABCDE to an equivalent
triangle, proceed exactly as in Construction 7; we then obtain
an equivalent four-sided figure AKDE. Repeat the process
and we obtain an equivalent triangle. (Cf. Fig. 79, p. 51.)

The method can clearly be applied to any polygon.

EXERCISE IX. (Numerical)

1. In Fig. 74, AD, BE, CF are altitudes of \( \triangle ABC. \)
   (i) If \( AD = 7 \text{ in.}, BC = 5 \text{ in.}, \) find area
   of \( \triangle ABC. 
\)
   (ii) If area of \( \triangle ABC = 40 \text{ sq. cm.}, \)
   and \( AC = 8 \text{ cm.}, \) find BE.
   (iii) If \( BE = 5 \text{ in.}, AB = 6 \text{ in.}, CF = 4 \text{ in.}, \)
   find AC.

2. Draw a line ABCDE so that \( AB = 6 \text{ cm.}, \)
   \( BC = 2 \text{ cm.}, CD = 4 \text{ cm.}, DE = 2 \text{ cm.} \)
   Take a point P, 5 cm. from the line AE. (i) What are
   the areas of \( \triangle PAB, \triangle PBD, \triangle PDE ? \)
   (ii) What fraction is \( \triangle PCD 
\)
   of \( \triangle PAE ? \)
   (iii) What triangle equals in area \( \triangle PBE ? 
\)

3. In Fig. 75, AP, AQ are altitudes of the \( \square \)gram. ABCD.
   (i) If AQ = 4 cm., CD = 5 cm., find area of ABCD.
   (ii) If area of ABCD = 24 sq. in., AB = 6 in., find AQ.
   (iii) If \( AB = 5 \text{ in.}, AP = 4 \text{ in.}, AD = 6 \text{ in.}, \) find AQ.

4. In Fig. 76, ABQ and DCP are parallel lines.
   (i) Find the area of ABCD, \( \triangle PBQ, \triangle BOP. \)
   (ii) Find the length of the perpendicular from B to AD,
   from Q to BP, from C to PD.
   (iii) Find the area of AQPD.

5. In quad. ABCD, BC = 8", AD = 3", and BC is parallel to AD;
   if the area of \( \triangle ABC \) is 16 sq. in., find the area of \( \triangle ABD. 
\)

6. In quad. ABCD, AB = 5", BC = 3", CD = 2", \( \angle ABC = \angle BCD = 90^\circ; \)
   find the area of ABCD.

7. The area of \( \triangle ABC \) is 36 sq. cms., \( AB = 8 \text{ cms.}, AC = 9 \text{ cms.}, \)
   D is the mid-point of BC; find the lengths of the perpendiculars
   from D to AB, AC.

8. In the parallelogram ABCD, AB = 8", BC = 6"; the perpen-
dicular from A to CD is 5"; find the perpendicular from B to AD.

9. Find the area of a rhombus whose diagonals are 5", 6".

10. Find the area of the quadrilaterals whose vertices are:
    (i) (0, 0); (3, 2); (1, 5); (0, 7).
    (ii) (1, 3); (3, 2); (5, 5); (2, 7).

11. Find in acres the areas of the fields of which the following
    field book measurements have been taken:

    | YARDS. | YARDS. |
    |--------|--------|
    | to D   | to D   |
    | 250    | 300    |
    | 200    | 220    |
    | 150    | 200    |
    | 100    | 100    |
    | 40 to E| 50 to E|
    | to C   | to C   |
    | 80 to F|
    | From A | From A |
12. Draw a triangle whose sides are 5, 6, 8 cms., and obtain its area in three different ways.

13. Construct a parallelogram of area 15 sq. cm. with sides 5 cm., 6 cm.; measure its acute angle.

14. Draw a parallelogram with sides 4 cm., 6 cm., and one angle 70°; construct a parallelogram of equal area with sides 5 cm., 7 cm.; measure its acute angle.

15. Draw a quadrilateral ABCD such that AB = 6 cm., BC = 5 cm., CD = 4 cm., \( \angle ABC = 110^\circ, \angle BCD = 95^\circ \). Reduce it to an equivalent triangle with AB as base and its vertex on BC. Find its area.

16. Given a triangle ABC and a point K on BC produced; construct a point P on BA so that area BPK = area BAC. (Note that area APK must equal area ACK.)

17. Given a triangle ABC and a point K on BC; construct the line KP which bisects the area of the triangle. (D is mid-point of BC, DP is parallel to KA; join DA, see Fig. 77.) Give proof.

**Fig. 77.**

**Fig. 78.**

18. Given a rectangle ABCD, see Fig. 78, and a point K on AB produced; construct a point M on AD so that the rectangle MAKL equals the rectangle ABCD in area. (Hint: \( \triangle BAD = \triangle KAM \).)

**EXERCISE X**

1. ABC is a \( \triangle \); a line parallel to BC cuts AB, AC at P, Q; prove \( \triangle APC = \triangle AQB \).

2. Two lines AQS, COD intersect at O; if AC is parallel to BD, prove \( \triangle AOD = \triangle BOC \).

3. P is any point on the median AD of \( \triangle ABC \); prove \( \triangle ABP = \triangle APC \).

4. ABCD is a parallelogram; P is any point on AD; prove that \( \triangle PAB + \triangle PCD = \triangle PBC \).

5. ABCD is a parallelogram; P is any point on BC; DQ is the perpendicular from D to AP; prove that the area of \( \triangle ABCD = \triangle DQ \cdot AP \).

6. ABCD is a parallelogram; P is any point on BD; prove \( \triangle PAB = \triangle PBC \).

7. ABCD is a parallelogram; a line parallel to BD cuts BC, DC at P, Q; prove \( \triangle ABP = \triangle ADQ \).

8. In Fig. 79, ABCDE is any pentagon; BP, EQ are parallel to AC, AB. Prove that \( \triangle APQ \) equals pentagon ABCDE.

9. X, Y are the mid-points of the sides AB, AC of \( \triangle ABC \); prove that \( \triangle XBY = \triangle XCY \) and deduce that XY is parallel to BC.

10. Two parallelograms ABCD, AXYZ of equal area have a common angle at A; X lies on AB; prove DX, XY are parallel.

11. The sides AB, BC of the parallelogram ABCD are produced to any points P, Q; prove \( \triangle PCD = \triangle QAD \).

12. The medians BE, CF of \( \triangle ABC \) intersect at G; prove that \( \triangle BGC = \triangle BGA = \triangle AC \).

13. With data of No. 12, prove \( \triangle BGC = \triangle AEG \).

14. In Fig. 80, the sides of \( \triangle ABC \) are equal and parallel to the sides of \( \triangle XYZ \); prove that BAXY + ACZX = BCZY.

**Fig. 80.**

**Fig. 81.**

15. ABP, AQD are equivalent triangles on opposite sides of AB; prove AB bisects PQ.

16. In Fig. 81, ABCD is divided into four parallelograms; prove P0SD = ROQB. Hence perform the construction in Ex. IX., No. 18.

17. In Fig. 81, prove \( \triangle APR + \triangle ASQ = \triangle ABD \).
THEOREM 28. (Pythagoras' Theorem)

In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

Given $\angle BAC$ is a right angle.

To prove the square on $BC = \text{the square on } BA + \text{the square on } AC$.

Let $ABHK$, $ACMN$, $BCPQ$ be the squares on $AB$, $AC$, $BC$.

Join $CH$, $AQ$. Through $A$, draw $AXY$ parallel to $BQ$, cutting $BC$, $QP$ at $X$, $Y$.

Since $\angle BAC$ and $\angle BAK$ are right angles, $KA$ and $AC$ are in the same straight line.

Again $\angle HBA = 90^\circ = \angle QBC$.

Add to each $\angle ABC$, $\therefore \angle HBC = \angle ABQ$.

In the $\triangle HBC$, $ABQ$.

$HB = AB$, sides of square.
$CB = QB$, sides of square.
$\angle HBC = \angle ABQ$, proved.
$\therefore \triangle HBC = \triangle ABQ$ (2 sides, inc. angle).

Now $\triangle HBC$ and square $HA$ are on the same base $HB$ and between the same parallels $HB$, $KA$;
$\therefore \triangle HBC = \frac{1}{2}$ square $HA$.

Also $\triangle ABQ$ and rectangle $BQYX$ are on the same base $BQ$ and between the same parallels $BQ$, $AXY$.
$\therefore \triangle ABQ = \frac{1}{2}$ rect. $BQYX$.
$\therefore$ square $HA = \text{rect. } BQYX$.

Similarly, by joining $AP$, $BM$, it can be shown that square $MA = \text{rect. } CPYX$;
$\therefore$ square $HA + \text{square } MA = \text{rect. } BQYX + \text{rect. } CPYX = \text{square } BP$.

Q.E.D.

THEOREM 29

If the square on one side of a triangle is equal to the sum of the squares on the other sides, then the angle contained by these sides is a right angle.

Given $AB^2 + BC^2 = AC^2$.

To prove $\angle ABC = 90^\circ$.

Construct a triangle $XYZ$ such that $XY = AB$, $YZ = BC$, $\angle XYZ = 90^\circ$.

Since $\angle XYZ = 90^\circ$, $XZ^2 = XY^2 + YZ^2$.

But $XY = AB$ and $YZ = BC$.
$\therefore XZ^2 = AB^2 + BC^2 = AC^2$ given.
$\therefore XZ = AC$.

$\therefore$ in the $\triangle ABC$, $XZ$.

$AB = XY$, constr.
$BC = YZ$, constr.
$AC = XZ$, proved.
$\therefore \triangle ABC = \triangle XYZ$ (3 sides).
$\therefore \angle ABC = \angle XYZ$.
But $\angle XYZ = 90^\circ$ constr.
$\therefore \angle ABC = 90^\circ$.

Q.E.D.

Note.—The proof of Th. 28 shows that the square on $AB$ equals in area the rectangle $BXYQ$ in Fig. 82.
But the area of $BXYQ$ is measured by $BX \times BQ$, that is, by $BQ \times BC$. This important fact may be stated as follows:
If $\triangle ABC$ is right-angled at $A$, and if $AX$ is an altitude, then $BA^2 = BX \cdot BC$.

Similarly, $CA^2 = CX \cdot CB$. 
EXERCISE XI. (Numerical)

1. In Fig. 82, AB = 5", CA = 12"; calculate BC, BX.
2. In Fig. 82, BC = 6", CA = 10"; calculate AB, CX.
3. In \( \triangle ABC \), \( AB = AC = 9" \), BC = 8"; calculate area of \( \triangle ABC \).
4. In \( \triangle ABC \), \( AB = AC = 13" \), BC = 16"; calculate the length of the altitude BE.
5. In \( \triangle ABC \), \( AC = 3" \), AB = 8", \( \angle ACB = 90^\circ \); find the length of the median AD.
6. AD is an altitude of \( \triangle ABC \); \( AD = 2" \), BD = 1", DC = 4"; prove \( \angle EAC = 90^\circ \).
7. Construct a line of length (i) \( \sqrt{2} \), (ii) \( \sqrt{5} \) inches.
8. Construct a square of area (i) 13 sq. in., (ii) 7 sq. in.
9. Find the distance between the points (1, 2), (5, 4).
10. Prove that the points (5, 11), (6, 10), (7, 7) lie on a circle whose centre is (2, 7); and find its radius.
11. In \( \triangle ABC \), \( \angle ABC = 90^\circ \), \( \angle ACB = 60^\circ \), AC = 8"; find AB.
12. In \( \triangle ABC \), \( \angle ABC = 90^\circ \), \( \angle ACB = 60^\circ \), AB = 5"; find BC.
13. In Fig. 84, AB = 2", BC = 4", CD = 1"; find AD.

EXERCISE XII

1. AD is an altitude of the equilateral triangle \( \triangle ABC \); prove that \( AD^2 = \frac{1}{3} BC^2 \).
2. In \( \triangle ABC \), CD is an altitude; prove \( AC^2 + BD^2 = BC^2 + AD^2 \).
3. ABN, PQN are two perpendicular lines; prove that \( PA^2 + QB^2 = PB^2 + QA^2 \).
4. The diagonals AC, BD of the quadrilateral ABCD are at right angles; prove that \( AB^2 + CD^2 = AD^2 + BC^2 \).
5. If in the quadrilateral ABCD, \( \angle ABC = \angle ADC = 90^\circ \); prove that \( AB^2 - AD^2 = CD^2 - BC^2 \).
6. P is a point inside a rectangle ABCD; prove that \( PA^2 + PC^2 = PB^2 + PD^2 \). Is this true if P is outside ABCD?
7. In \( \triangle ABC \), \( \angle BAC = 90^\circ \); H, K are the mid-points of AB, AC; prove that \( BH^2 + CH^2 = 2BC^2 \).
8. ABCD is a rhombus; prove that \( AC^2 + BD^2 = 2AB^2 + 2BC^2 \).
9. Given a square, show how to construct a square of twice the area.
10. Given two squares, show how to construct a square equal in area to (i) the sum of the two squares, (ii) the difference of the two squares.
11. In \( \triangle ABC \), \( \angle BAC = 90^\circ \); AD is an altitude; prove \( AD = \frac{AB \cdot AC}{BC} \).
12. In \( \triangle ABC \), \( \angle BAC = 90^\circ \); AD is an altitude; prove that \( AD^2 = BD \cdot DC \).
13. A pyramid of height 8" stands on a square base each edge of which is 1". Find the area of each face and the length of an edge.
14. ABCD is a rectangle; \( AB = 6" \), \( BC = 8" \); it is folded about BD so that the planes of the two parts are at right angles. Find the new distance of A from C.
15. Fig. 86 shows a square of side \( a + b \) divided up; use area formulae to prove Pythagoras' theorem \( a^2 + b^2 = c^2 \).
16. ABC is a straight line; ABXY, BCPQ are squares on the same side of AC; prove \( PX^2 + CY^2 = 3(AB^2 + BC^2) \).
SECTION III

LOCI

If we look at the tip of the seconds hand of a watch we see that it occupies a series of positions in the course of each minute: the tip of the seconds hand traces out a curve which is called its locus. If a point moves about, subject to some fixed condition, the path traced out by the point is called the locus of the point, and we may regard the locus as the aggregate of all possible positions of the point, subject to the given law. When we state that the locus of a point, which can move subject to some given condition, is a certain curve, two complementary ideas are involved:

(i) Every point on the curve satisfies the given condition.
(ii) Every point which satisfies the given condition lies on the curve.

It may happen, however, that the conditions of the problem prevent the point from describing the whole of a curve: in this case it should always be stated what part of the curve forms the locus.

Oral Examples. Describe the following loci:

1. A child in a swing.
2. The centre of the wheel of an engine running along a straight railway line.
3. A donkey tethered to a peg, if it keeps its chain taut.
4. The top of a child’s head if he slides down stairs on a tea-tray.
5. A man who walks about so that he remains at equal distances from two trees.
6. A dog is chained to a low straight rail 10 feet long by a chain 6 feet long, which can slide along the rail. What is his locus if he keeps the chain taut?

THEOREM 30

The locus of a point, which is equidistant from two given points, is the perpendicular bisector of the straight line joining the given points.

Given two fixed points $A$, $B$ and any position of a point $P$ which moves so that $PA = PB$.

To prove that $P$ lies on the perpendicular bisector of $AB$.

Bisect $AB$ at $N$. Join $PN$.

In the $\triangle ANP$, $\triangle BNP$, $AN = BN$, constr.

$AP = BP$, given. $PN$ is common.

$\therefore \triangle ANP = \triangle BNP$ (3 sides).

$\therefore \angle ANP = \angle BNP$.

These are adj. $\angle s$ on st. line, $\therefore$ each is a right angle.

$\therefore PN$ is perpendicular to $AB$ and bisects it.

$\therefore P$ lies on the perpendicular bisector of $AB$.

Conversely, any point $P$ on the perpendicular bisector of $AB$ is equidistant from $A$ and $B$.

For in the $\triangle ANP$, $\triangle BNP$, $AN = BN$ given,

$PN$ is common,

$\therefore \triangle ANP = \triangle BNP$ (rt. angle, hyp., side).

$\therefore AP = BP$.

Q.E.D.
THEOREM 31

The locus of a point which is equidistant from two given intersecting straight lines is the pair of lines which bisect the angles between the given lines.

Given two fixed lines $AOB$, $COD$ and any position of a point $P$ which moves so that the perpendiculars $PH$, $PK$ from $P$ to $AOB$, $COD$ are equal.

To prove $P$ lies on one of the two lines bisecting the angles $BOC$, $BOD$.

Suppose $P$ is situated in the angle $BOD$.

In the right-angled triangles $PHO$, $PKO$,

$PH = PK$, given.

$PO$ is the common hypotenuse.

$\therefore \angle PHO = \angle PKO$ (right angle, hypotenuse, side).

$\therefore \angle POH = \angle POK$.

$\therefore P$ lies on the line bisecting the angle $BOC$.

In the same way if $P$ is situated in either of the angles $BOC$, $COA$, $AOD$, it lies on the bisectors of these angles.

Conversely, any point $P$ on the line bisecting either of the angles between the lines $AOB$, $COD$ is equidistant from these lines.

For in the $\triangle PHO$, $PKO$, $\angle PHO = \angle PKO$ given, $\angle OMP = 90^\circ$ $\angle OKP$, $OP$ is the common hypotenuse.

$\therefore \angle PHO = \angle PKO$ (2 angles, corresponding side) and $PH = PK$.

Q.E.D.

Definition.

If the straight line $AB$ is the perpendicular bisector of the line joining two points $P$, $P'$, then $P'$ is called the image or reflection of $P$ in $AB$.

EXERCISE XIII

1. A variable point is at a given distance from a given point, what is its locus, (i) in a plane, (ii) in space?

2. A variable point is at a given distance from a given line, what is its locus, (i) in a plane, (ii) in space?

3. A variable circle passes through two fixed points, what is the locus of its centre, (i) in a given plane through the two points, (ii) in space?

4. A variable circle of given radius passes through a fixed point, what is the locus of its centre, (i) in a plane, (ii) in space?

5. $A$, $B$ are fixed points; $APB$ is a triangle of given area; what is the locus of $P$?

6. $P$ is a variable point on a given line; $O$ is a fixed point outside the line; find the locus of the mid-point of $OP$.

7. $A$, $B$ are fixed points; $PAQB$ is a variable parallelogram of given area; find the complete locus of $P$.

8. $A$, $B$ are fixed points; $ABPQ$ is a variable parallelogram; if $AP$ is of given length, find the locus of $Q$.

9. What is the locus of a point which moves inside a large triangle so that it is always two inches from some side and never more than two inches from any side.

10. $ABC$ is an equilateral triangle of side one inch; a point $P$ moves in the plane of $ABC$ so that it is always one inch away from one of the three points $A$, $B$, $C$ and never less than one inch from the other two points. What is the locus of $P$?

11. A cubical packing-case, section $ABCD$, rests with $AB$ on the floor; it is rolled, without slipping, on to its side $BC$, and then rolled over again on to $CD$, and then on to $DA$. What is the locus of $A$?

12. Repeat Ex. 11 for a case whose section is an equilateral triangle $ABC$.

13. If $P'$ is the image of $P$ in the line $AB$, prove that $AP = AP'$.

14. $A$, $B$ are two points on the same side of a line $CD$; $A'$ is the image of $A$ in $CD$; $A'B$ cuts $CD$ at $O$; prove that $(i)$ $AO$ and $OB$ make equal angles with $CD$; (ii) if $P$ is any other point on $CD$, $AP + PB > AO + OB$. 
Intersection of Loci.

If the position of a point is given by two distinct conditions, it may be possible to trace the two corresponding loci, and so fix its position from the intersection of these lines or curves.

EXERCISE XIV

1. ABC is an equilateral triangle of side three inches, what is the locus of all points two inches from A? What is the locus of all points one inch from BC? Construct the position (or positions of) a point P two inches from A and one inch from the line BC.

2. With the data of Ex. 1, find a point on AC which is 2.8 inches from B.

3. With the data of Ex. 1, find a point on BC whose distance from AB is two inches.

4. A is a point whose distance from a line BC is 7 cm.; a circle centre A radius 5 cm. is drawn. Construct a point Q on the circle such that if QR makes an angle of 45° with BC, and cuts BC at R, then QR = 4 cm.

5. ABC is an equilateral triangle of side 6 cm. Construct a point P such that its perpendicular distances from AB, AC are 2 cm., 3 cm. respectively.

6. ABC is a triangle such that AB = 5 cm., BC = 6 cm., CA = 7 cm. Construct a point on AC which is equidistant from AB and BC.

7. With the data of Ex. 6, construct a point P such that PA = PB and PC = 6 cm.

8. With the data of Ex. 6, construct a point P whose distance from BC is 3 cm., and such that it is equidistant from A, C.

9. With the data of Ex. 6, construct a point P such that it is 2 cm. from AB, and is equidistant from AC and BC.

10. With the data of Ex. 6, construct the centre of a circle, if the centre lies on AC and the circle passes through B and C.

11. Draw a triangle of any shape, and construct the position of a point equidistant from the three corners.

12. Draw a triangle of any shape, and construct the position (or positions of) a point equidistant from the three sides, produced if necessary.

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6.

1. P is a point on the side AB of ∆ABC such that AP = PC = CB. If CP bisects ∠ACB, calculate ∠BAC.

2. ABCD is a parallelogram and O is the mid-point of AB; if OC = 2AD, prove ∠DOC = 90°.

3. The vertices of a triangle are (2, 0), (0, 5), (3, 7), the unit on each axis being one inch. Calculate the area of the triangle.

4. ABCD is a quadrilateral; if AB = BC = CD and ∠ABD = ∠BCD = 90°, prove that AD = CD.

7.

1. What is the length of the diagonal of a box whose sides are 3", 4", 12"?

2. ABC is an equilateral triangle; K is a point on BC such that ∠CAK = 3∠KAB; a line KL is drawn perpendicular to BC to meet AB at L; prove that AL = LK.

3. The side BC of the parallelogram ABCD is produced to any point K; prove ∆ABK = quad. ACKD.

4. In ∆ABC, ∠BAC = 90°; H, K are the mid-points of AB, AC; prove that BH² + HK² + KC² = BC².

8.

1. In Fig. 89, the triangle ABC is inscribed in a rectangle; find its area and the distance of A from the mid-point of BC.

2. A, B are fixed points; X is a variable point such that ∠AXB is obtuse; the perpendicular bisectors of AX, BX cut AB at Y, Z; prove that the perimeter of ∆XYZ is constant.

3. ABC is a ∆; a line XY parallel to BC cuts AB, AC at X, Y, and is produced to Z so that XZ = BC; prove ∆BXZ = ∆AYZ.

4. The sides of a triangle are 8 cm., 9 cm., 12 cm. Is it obtuse angled?

D.S.G.
SECTION IV

CIRCLES

Definitions. The straight line joining any two points on the circumference of a circle is called a chord. A chord which passes through the centre of the circle is called a diameter. Any portion of the circumference is called an arc.

THEOREM 32

(1) The straight line which joins the centre of a circle to the middle point of a chord (which is not a diameter) is perpendicular to the chord.

(2) The line drawn from the centre of a circle perpendicular to a chord bisects the chord.

(1) Given a circle, centre O, and a chord AB, whose mid-point is N.
To prove \( \angle ONA \) is a right angle.
Join OA, OB.
In the \( \triangle ONA \), ONB,
\[ OA = OB \text{, radii.} \]
\[ AN = BN \text{, given.} \]
\[ ON \text{ is common.} \]
\( \therefore \) \( \triangle ONA \equiv \triangle ONB \) (3 sides).
\( \therefore \angle ONA = \angle ONB. \)

These are adj. \( \angle s \) on st. line, \( \therefore \) each is a right angle.

(2) Given that ON is the perpendicular from the centre O of a circle to a chord AB.
To prove that N is the mid-point of AB.
In the right-angled triangles ONA, ONB,
\[ OA = OB \text{, radii.} \]
\[ ON \text{ is common.} \]
\( \therefore \triangle ONA \equiv \triangle ONB \) (rt. angle, hyp., side). \( \therefore AN = NB \) . Q.E.D.

Corollary. The perpendicular bisector of a chord of a circle passes through the centre of the circle.
THEOREM 33

In equal circles or in the same circle:
(1) Equal chords are equidistant from the centres.
(2) Chords which are equidistant from the centres are equal.

(1) Given two equal circles $ABC$, $CDY$, centres $P$, $Q$, and two equal chords $AB$, $CD$.
To prove that the perpendiculars $PH$, $QK$ from $P$, $Q$ to $AB$, $CD$ are equal.
Join $PA$, $QC$.
Since $PH$, $QK$ are the perpendiculars from the centres to the chords $AB$, $CD$, $H$ and $K$ are the mid-points of $AB$ and $CD$.
\[ AH = \frac{1}{2} AB \text{ and } CK = \frac{1}{2} CD. \]
But $AB = CD$, given.
\[ AH = CK. \]
\[ \therefore \text{ in the right-angled triangles } PAH, QCK, \text{ the hypotenuse } PA = \text{the hypotenuse } QC, \text{ radii of equal circles}. \]
\[ AH = CK, \text{ proved}. \]
\[ \therefore \Delta PAH = \Delta QCK \text{ (rt. angle, hyp., side)}. \]
\[ \therefore PH = QK. \quad Q.E.D. \]

CHORDS OF A CIRCLE

(2) Given that the perpendiculars $PH$, $QK$ from $P$, $Q$ to the chords $AB$, $CD$ are equal.
To prove that $AB = CD$.
In the right-angled triangles $PAH$, $QCK$, the hypotenuse $PA = \text{the hypotenuse } QC$, radii of equal circles.
\[ PH = QK, \text{ given}. \]
\[ \therefore \Delta PAH = \Delta QCK \text{ (rt. angle, hyp., side)}. \]
\[ \therefore AH = CK. \]
But the perpendiculars $PH$, $QK$ bisect $AB$, $CD$.
\[ \therefore AB = 2AH \text{ and } CD = 2CK. \]
\[ \therefore AB = CD. \quad Q.E.D. \]
The proof is unaltered if the chords are in the same circle.

Definition.
If $ABC$ is any triangle, the circle which passes through $A$, $B$, $C$ is called the circum-circle of the triangle, and the centre $O$ of this circle is called the circum-centre. The radius of the circle is called the circum-radius, and the process of constructing the circle is described as circumscribing a circle to a given triangle.

Constructions. The construction in Theorem 34 shows:
(i) How to draw a circle through 3 points.
(ii) How to draw the circumcircle of a triangle.
(iii) How to find the centre of a circle, an arc of which is given.
THEOREM 34*

There is one circle and only one circle which passes through three given points not in the same straight line.

Given three points A, B, C not in a straight line.

To prove that one and only one circle can be drawn to pass through A, B, C.

Draw the perpendicular bisector of AB and the perpendicular bisector of BC.

Since AB and BC are different straight lines, these perpendicular bisectors are not parallel and must therefore intersect. Call the point of intersection O.

Now the locus of all points equidistant from A and B is the perpendicular bisector of AB, and the locus of all points equidistant from B and C is the perpendicular bisector of BC.

\[ \therefore \text{O is equidistant from } A, B, C, \text{ and is the only point which is equidistant from them.} \]

The circle, centre O, radius OA, passes through A, B, C; and there is no other circle which passes through A, B, C.

Corollary. Two circles cannot intersect in more than two points.

EXERCISE XV

1. AB is a chord of a circle of radius 10 cm.; AB = 8 cm.; find the distance of the centre of the circle from AB.

2. A chord of length 10 cm. is at a distance of 12 cm. from the centre of the circle; find the radius.

3. A chord of a circle of radius 7 cm. is at a distance of 4 cm. from the centre; find its length.

4. ABC is a \( \triangle \) inscribed in a circle; \( AB = AC = 13' \), BC = 10'; calculate the radius of the circle.

5. In a circle of radius 5 cm., there are two parallel chords of lengths 4 cm., 6 cm.; find the distance between them.

6. Two parallel chords AB, CD of a circle are 3" apart; \( AB = 4' \), \( CD = 10' \); calculate the radius of the circle.

7. The perpendicular bisector of a chord AB cuts AB at C and the circle at D; \( AB = 6' \), CD = 1"; calculate the radius of the circle.

8. ABC is a straight line, such that \( AB = 1'' \), BC = 4"; PQ, is the chord of the circle on AC as diameter, perpendicular to AC; find PQ.

9. P is a point on the diameter AB of a circle; \( AP = 2'' \), PB = 8"; find the length of the shortest chord which passes through P.

10. Two circles, centres A, B, intersect at X, Y; prove that AB bisects XY at right angles.

11. Two circles, centres A, B, intersect at C, D; PCQ is a line parallel to AB cutting the circles at P, Q; prove PQ = 2AB.

12. A line PQRS cuts two concentric circles at P, S and Q, R; prove PQ = RS.

13. ABC is a triangle inscribed in a circle; if \( \angle BAC = 90' \), prove that the midpoint of BC is the centre of the circle.

14. In Fig. 96, if PQ is parallel to RS, prove PQ = RS.

15. APB, QPD are intersecting chords of a circle, centre O; if OP bisects \( \angle APD \), prove \( AB = CD \).
16. The diagonals of the quadrilateral ABCD meet at O; circles are drawn through A, O, B; B, O, C; C, O, D; D, O, A; prove that their four centres are the corners of a parallelogram.

17. In Fig. 97, A, C, B are the centres of three unequal circles; if \( AC = GB \), prove \( PQ = RS \).

Definitions.

The area bounded by two radii of a circle and the arc they cut off is called a sector of the circle.

The area bounded by a chord of a circle and the arc it cuts off is called a segment of the circle. A segment greater than a semicircle is called a major segment; if less, a minor segment.

Any number of points are said to be concyclic, if a circle can be drawn to pass through all of them.

If the vertices of a quadrilateral lie on a circle, it is called a cyclic quadrilateral.

If \( ABC \) is any arc of a circle (see Fig. 101) and if \( B \) is any point on the remaining part of the circumference, the angle \( ABC \) is said to stand on the arc \( ABC \).

N.B.—The arc on which an angle stands is the part of the circumference intercepted between the arms of the angle.

The following example is suggested for oral work:

Draw on the blackboard a large circle and mark (say) eight points \( A, B, C, D, E, F, G, H \) on its circumference.

Join every pair of points by a straight line, and consider such questions as:

(i) On what arcs do the angles \( BCE, HAB \), etc. stand?
(ii) What angles stand on the arc \( EGA \)?
(iii) What angles stand on the same arc as the angle \( HAD \)?
(iv) What angles stand on the chord \( AC \)?

\[ \text{ANGLE PROPERTIES OF A CIRCLE} \]

\[ \text{THEOREM 35} \]

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

\[ \text{Fig. 98 (1).} \]

\[ \text{Fig. 98 (3).} \]

\[ \text{Fig. 98 (3).} \]

\( \text{Given } AB \text{ is an arc of a circle, centre } O; \ P \text{ is any point on the remaining part of the circumference.} \)

\( \text{To prove } \angle AOB = 2 \angle APB. \)

Join PO, and produce it to any point N.

Since \( OA = OP \), radii, \( \angle OAP = \angle OPB. \)

But ext. \( \angle NOA = \text{int. } \angle OAP + \text{int. } \angle OPA. \)

\( \therefore \angle NOA = 2 \angle OPA. \)

Similarly \( \angle NOB = 2 \angle OPB. \)

\( \therefore \text{adding in Fig. 98 (1) and subtracting in Fig. 98 (2), we have } \angle AOB = 2 \angle APB. \)

Fig. 98 (3) shows the case where the angle \( AOB \) is reflex, i.e. greater than \( 180^\circ \); the proof for Fig. 98 (3) is the same as for Fig. 98 (1).
THEOREM 36

(1) Angles in the same segment of a circle are equal.
(2) The angle in a semicircle is a right angle.

(1) Given two angles APB, AQB in the same segment of a circle.

To prove \( \angle APB = \angle AQB \).
Let O be the centre. Join OA, OB.
Then \( \angle AOB = 2 \angle APB \). \( \angle \) at centre = twice \( \angle \) at Oec.
and \( \angle AOB = 2 \angle AQB \).
\[ \therefore \angle APB = \angle AQB. \]
Q.E.D.

(2) Given AB a diameter of a circle, centre O, and P a point on the circumference.

To prove \( \angle APB = 90^\circ \).

\[ \angle AOB = 2 \angle APB. \] \( \angle \) at centre = twice \( \angle \) at Oec.
But \( \angle AOB = 180^\circ \), since AOB is a straight line ;
\[ \therefore \angle APB = 90^\circ. \]
Q.E.D.

THEOREM 37

(1) The opposite angles of a cyclic quadrilateral are supplementary.
(2) If a side of a cyclic quadrilateral is produced, the exterior angle is equal to the interior opposite angle.

(1) Given ABCD is a cyclic quadrilateral.

To prove \( \angle ABC + \angle ADC = 180^\circ \).
Let O be the centre of the circle. Join OA, OC.

Let the arc ADC subtend angle \( x^\circ \) at the centre, and let the arc ABC subtend angle \( y^\circ \) at the centre.
\[ \therefore x^\circ + y^\circ = 360^\circ. \]
Now \( x^\circ = 2 \angle ABC \). \( \angle \) at centre = twice \( \angle \) at Oec.
and \( y^\circ = 2 \angle ADC \).
\[ \therefore 2 \angle ABC + 2 \angle ADC = 360^\circ. \]
\[ \therefore \angle ABC + \angle ADC = 180^\circ. \]
Q.E.D.

(2) Given the side AD of the cyclic quadrilateral ABCD is produced to P.

To prove \( \angle DPC = \angle ABC. \)

Now \( \angle ADC + \angle DPC = 180^\circ \), adj. \( \angle \)s, AO a st. line, and \( \angle ADC + \angle ABC = 180^\circ \), opp. \( \angle \)s cyclic quad.
\[ \therefore \angle DPC = \angle ABC. \]
Q.E.D.
EXERCISE XVI. (Numerical)

1. ABC is a $\triangle$ inscribed in a circle, centre $O$; $\angle AOC = 130^\circ$, $\angle BOC = 150^\circ$, find $\angle ACB$.

2. AB, CD are perpendicular chords of a circle; $\angle BAC = 35^\circ$, find $\angle ABD$.

3. ABCD is a quadrilateral such that $AB = AC = AD$; if $\angle BAD = 140^\circ$, find $\angle BCD$.

4. ABCD is a quadrilateral inscribed in a circle; AB is a diameter; $\angle AOC = 127^\circ$, find $\angle BAC$.

5. Two circles APQB, AQSB intersect at A, B : PAQ, RBS are straight lines; if $\angle QPR = 80^\circ$, $\angle PRS = 70^\circ$, find $\angle PQS$, $\angle QSR$.

6. P, Q, R are points of a circle, centre $O$; $\angle POQ = 54^\circ$, $\angle ORQ = 36^\circ$; P, Q, R are on opposite sides of OQ; find $\angle QPR$ and $\angle QQR$.

7. The diagonals of the cyclic quadrilateral ABCD meet at O; $\angle BAC = 42^\circ$, $\angle BOC = 114^\circ$, $\angle ADB = 33^\circ$; find $\angle BCD$.

8. ABCD is a cyclic quadrilateral, EABF is a straight line; $\angle EAD = 55^\circ$, $\angle FBC = 74^\circ$, $\angle BDC = 60^\circ$; find angle between AC, BD.

9. Two chords AB, DC of a circle, centre $O$, are produced to meet at $E$; $\angle AOB = 100^\circ$, $\angle EBC = 72^\circ$, $\angle ECB = 84^\circ$; find $\angle COD$.

10. A chord QR and a diameter AB, when produced, meet at P; $\angle QPA = 28^\circ$, $\angle QAR = 42^\circ$, find $\angle QRA$.

EXERCISE XVII

1. AB, XY are parallel chords of a circle; XY cuts BX at O; prove OX = OY.

2. Two circles BAPR, BAsQ cut at A, B; PAQ, RAS are straight lines; prove $\angle PBR = \angle QBS$.

3. AB is a chord of a circle, centre $O$; P is any point on the minor arc AB; prove $\angle AOB + 2 \angle APB = 360^\circ$.

4. ABCD is a cyclic quadrilateral; if AC bisects the angles at $A$ and $C$, prove $\angle ABC = 90^\circ$.

5. Two lines OAB, OCD cut a circle at A, B and C, D; prove $\angle OAD = \angle OCB$.

6. AB is a diameter of a circle APQR; prove $\angle APQ + \angle QRB = 270^\circ$.

7. ABCDEF is a hexagon inscribed in a circle; prove that $\angle FAB + \angle BCD + \angle DEF = 360^\circ$.

8. Two circles ABPR, ABQS cut at A, B; PBQ, RAS are straight lines; prove PR is parallel to QS.

9. AP, AQ are diameters of the circles APB, AQB; prove that PBQ is a straight line.

10. OA is a radius of a circle, centre $O$; AP is any chord; prove that the circle on OA as diameter bisects AP.

11. If in Fig. 103, O is the centre of the circle, prove that $x + y = z$.

12. ABC is a $\triangle$ inscribed in a circle, centre $O$; D is the midpoint of BC; prove $\angle BOD = \angle BAC$.

Fig. 103.

Fig. 104.

13. OA, OB, OC are three equal lines; if $\angle AOB = 90^\circ$, prove $\angle ACB = 45^\circ$ or $135^\circ$.

14. Two lines OAB, OCD cut a circle at A, B and C, D; if OB = OD, prove OC = CA.

15. In Fig. 104, AB = AC and C is the centre of the circle; prove that DE is parallel to the line bisecting $\angle ABC$.

16. ABCD is a rectangle; any circle through A cuts AB, AC, AD at X, Y, Z; prove that ABCD is equiangular triangles.

17. ABC is a triangle inscribed in a circle; $AB = AC$; BC is produced to D; AD cuts the circle at E; prove $\angle ACE = \angle ADB$.

18. Two given circles ABP, ABQ intersect at A, B; a variable line PAQ meets them at P, Q; prove $\angle PBQ$ is of constant size.

Any three points lie either on a straight line or a circle. In general, if any four points are taken, it is impossible to draw either a straight line or a circle to pass through all of them. The next theorems give two tests for determining whether four points are concyclic. These tests are the converses of Theorems 36, 37.
THEOREM 38

If the line joining two points subtends equal angles at two other points on the same side of it, then the four points lie on a circle.

*Fig. 105 (1).*

*Fig. 105 (2).*

Given that \( \angle APB = \angle AQB \), where \( P, Q \) lie on the same side of \( AB \).

To prove that \( A, P, Q, B \) lie on a circle.

One of the angles \( APB, ABQ \) must be the greater; suppose it is \( \angle APB \), so that \( BQ \) lies in the angle \( ABP \).

Draw the circle through \( A, B, P \) and suppose, if possible, that it does not pass through \( Q \).

Since \( BQ \) lies in the angle \( APB \), the circle must cut \( BQ \) or \( BQ \) produced, at \( X \) say.

Then \( \angle AXB = \angle APB \), same segment.

But \( \angle AQB = \angle APB \), given.

\[ \therefore \angle AXB = \angle AQB. \]

But one of these is the exterior angle and the other is the interior opposite angle of the triangle \( AQX \).

\[ \therefore \text{they cannot be equal unless } X \text{ coincides with } Q. \]

\[ \therefore \text{the circle through } A, B, P \text{ must pass through } Q. \]

Q.E.D.

TESTS FOR CONCYCLIC POINTS

THEOREM 39

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

*Fig. 106 (1).*

*Fig. 106 (2).*

Given that in the quadrilateral \( ABCD \), \( \angle ABC + \angle ADC = 180^\circ \).

To prove that \( A, B, C, D \) lie on a circle.

Draw the circle through \( B, C, D \) and suppose, if possible, that it does not pass through \( A \).

Since \( CA \) lies in the angle \( BCD \), the circle must cut either \( CA \) or \( CA \) produced, at \( X \) say.

Then \( \angle ABC + \angle ADC = 180^\circ \), given.

But \( \angle BXC + \angle XDC = 180^\circ \), opp. \( \angle \) a cyclic quad.

\[ \therefore \angle ABD + \angle ADC = \angle BXC + \angle XDC. \]

But one of these is only a part of the other, therefore they cannot be equal unless \( X \) coincides with \( A \).

\[ \therefore \text{the circle through } B, C, D \text{ must pass through } A. \]

Q.E.D.
THEOREM 40

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

Given a triangle ABC, right-angled at A.
To prove the circle on BC as diameter passes through A.
Bisect BC at O; through O draw OX parallel to CA to cut AB at X; join OA.
Since BO = OC and OX is parallel to CA.
\[
\therefore \text{OX bisects BA.}
\]
Since \( \angle BAC = 90^\circ \) and OX is parallel to CA.
\[
\therefore \text{OX is perpendicular to BA.}
\]
\[
\therefore \text{OA = OB.}
\]
But OB = OC, \( \because \) OA = OB = OC.
\[
\therefore \text{the circle on BC as diameter passes through A.}
\]
Q.E.D.

Note.—This theorem is really a special case of Theorem 38, making use of Theorem 36 (2).
THEOREM 41*

In equal circles (or in the same circle), if two arcs subtend equal angles at the centres or at the circumferences, they are equal.

\[ \triangle AHB = \triangle CKD \]

Given two equal circles, ABP, CQD, centres H, K.

(1) Given that \( \angle AHB = \angle CKD \).

To prove that arc AB = arc CD.

Apply the circle AB to the circle CD so that the centre H falls on the centre K and HA along KD.

Since the circles are equal, A falls on C and the circumferences coincide.

Since \( \angle AHB = \angle CKD \), HB falls on KD, and B falls on D.

\[ \because \text{the arcs } AB, CD \text{ coincide.} \]

\[ \because \text{arc } AB = \text{arc } CD. \]

(2) Given that \( \angle APB = \angle CQD \).

To prove that arc AB = arc CD.

Now \( \angle AHB = 2 \angle APB \), \( \angle \) at centre = twice \( \angle \) at centre, and \( \angle CKD = 2 \angle CQD \).

But \( \angle APB = \angle CQD \), given.

\[ \therefore \text{arc } AB = \text{arc } CD. \]

Corollary. In equal circles (or in the same circle), equal chords cut off equal arcs.

For, if chord AB = chord CD, \( \triangle AHB = \triangle CKD \).

THEOREM 42*

In equal circles (or in the same circle), if two arcs are equal, they subtend equal angles at the centres and at the circumferences.

\[ \triangle AHB = \triangle CKD \]

Given two equal circles ABP, CDQ, centres H, K, and two equal arcs AB, CD.

To prove (1) \( \angle AHB = \angle CKD \).

(2) \( \angle APB = \angle CQD \).

(1) Apply the circle AB to the circle CD so that the centre H falls on the centre K and HA along KD.

Since the circles are equal, A falls on C and the circumferences coincide.

But arc AB = arc CD; \( \therefore \) B falls on D and HB on KD.

\[ \because \text{arc } AB = \text{arc } CD. \]

(2) Now \( \angle APB = \frac{1}{2} \angle AHB \), \( \angle \) at centre = \( \frac{1}{2} \angle \) at centre, \( \angle CQD = \frac{1}{2} \angle CKD \).

But \( \angle AHB = \angle CKD \), just proved.

\[ \therefore \angle APB = \angle CQD \]

Q.E.D.

Corollary. In equal circles (or in the same circle), the chords of equal arcs are equal.

For \( \angle AHB = \angle CKD \), \( \therefore \triangle AHB = \triangle CKD \).
MENSURATION

1. For a circle of radius \( r \) inches,
   (i) the length of the circumference = \( 2\pi r \) in.
   (ii) the area of the circle = \( \pi r^2 \) sq. in.
   (iii) the length of an arc which subtends \( \theta \) at the centre
       of the circle, = \( \frac{\theta}{360} \times 2\pi r \) in.
   (iv) the area of a sector of a circle of angle \( \theta \) = \( \frac{\theta}{360} \times \pi r^2 \)
       sq. in.

2. For a sphere of radius \( r \) inches,
   (i) the area of surface of sphere = \( 4\pi r^2 \) sq. in.
   (ii) the volume of the sphere = \( \frac{4}{3} \pi r^3 \) cub. in.
   (iii) the area of the surface intercepted between two parallel planes at distance \( d \) inches apart = \( 2\pi rd \)
       sq. in.

3. For a circular cylinder, radius \( r \) inches, height \( h \) inches,
   (i) the area of the curved surface = \( 2\pi rh \) sq. in.
   (ii) the volume of the cylinder = \( \pi r^2 h \) cub. in.

4. For a circular cone, radius of base \( r \) inches, height \( h \) inches
   length of slant edge \( l \) inches,
   (i) \( P = r^2 + h^2 \).
   (ii) area of the curved surface = \( \pi rl \) sq. in.
   (iii) volume of cone = \( \frac{1}{3} \pi r^2 h \) cub. in.

5. (i) The volume of any cylinder = area of base \( \times \) height.
    (ii) The volume of any pyramid = \( \frac{1}{3} \) area of base \( \times \) height.
        \( \pi = \frac{22}{7} \) approx. or 3.1416 approx.

EXERCISE XIX

1. Find (i) the circumference, (ii) the area of a circle of radius
   (i) 4", (ii) 100 yards.

2. The circumference of a circle is 5 inches; what is its radius
   correct to \( \frac{1}{10} \) inch ?

3. The area of a circle is 4 sq. cm.; what is its radius correct to
   \( \frac{1}{10} \) cm. ?

4. An arc of a circle of radius 3 inches subtends an angle of 40°
   at the centre; what is its length correct to \( \frac{1}{10} \) inch ?

5. The angle of a sector of a circle is 108°, and its radius is 2-5
   cm.; what is its area ?

6. A square ABCD is inscribed in a circle of radius 4 inches
   what is the area of the minor segment cut off by AB.

7. AB is an arc of a circle, centre O; AO = 5 cm. and arc AB
   = 5 cm.; find \( \angle AOB \), correct to nearest minute.

8. A piece of flexible wire is in the form of an arc of a circle of
   radius 4-8 cm. and subtends an angle of 240° at the centre of
   the circle; it is bent into a complete circle; what is the radius ?

9. Find (i) the volume, (ii) the total surface of a closed cylinder,
   height 8", radius 5".

10. How many cylindrical glasses 2" in diameter can be filled to
    a depth of 3" from a cylindrical jug of diameter 5" and height 12" ?

11. Find (i) the volume, (ii) the area of the curved surface of a
    circular cone, radius of base 5", height 12".

12. Find (i) the volume, (ii) the total area of the surface of a
    pyramid, whose base is a square of side 6" and whose height is 4".

13. Find (i) the volume, (ii) the area of the surface of a sphere
    of diameter 6 cm.

14. Draw a circle of radius 5 cm. and place in it a chord AB of
    length 4 cm.; find the area of the major segment AB, making
    any measurements you like.

15. A sector of a circle of radius 5 cm. and angle 60° is bent to
    form the surface of a cone; find the radius of its base.

16. The curved surface of a circular cone, height 3", radius of
    base 4" is folded out flat. What is the angle of the sector so
    obtained ?
EXERCISE XX
Arks, Angles, and Chords

1. ABCD is a square and AEF is an equilateral triangle inscribed in the same circle; calculate the angles of \( \triangle ECD \).

2. AB is a side of a regular hexagon and AC of a regular octagon inscribed in the same circle; calculate the angles of \( \triangle ABC \).

3. ABCD is a quadrilateral inscribed in a circle; AC cuts BD at O; DA, CB when produced meet at E; AB, DC when produced meet at F; if \( \angle AEB = 65^\circ \), \( \angle BFC = 35^\circ \), \( \angle DOC = 50^\circ \), prove are BC = twice are AB.

4. ABCD is a quadrilateral inscribed in a circle; \( \angle AOB = 25^\circ \), \( \angle DBC = 65^\circ \); prove are AB + are CD = are BC + are AD.

5. AB, DC are parallel chords of a circle; prove are AD = are BC.

6. ABCD is a cyclic quadrilateral; if AB = CD, prove \( \angle ABC = \angle BCD \).

7. ABP, ABQ are two equal circles; PBQ is a straight line; prove AP = AQ.

8. AB, BO, CD are equal chords of a circle, centre O; prove that AC cuts BD at an angle equal to \( \angle AOB \).

9. ABCD is a square and APQ an equilateral triangle inscribed in the same circle, P being between B and C; prove are BP = \( \frac{1}{2} \) are PC.

10. On a clock-face, prove that the line joining 4 and 7 is perpendicular to the line joining 6 and 12.

11. If AB and CD are equal arcs of a circle, prove that the chords BC, AD are either equal or parallel.

12. ABCD is a rectangle inscribed in a circle; DP is a chord equal to DC; prove PB = AD.

13. ABCDEF is a hexagon inscribed in a circle; if \( \angle ABC = \angle DEF \), prove AF is parallel to CD.

14. In \( \triangle ABC \), AB = AC; BC is produced to D; prove that the circles ABD, ACD are equal.

15. AB, AC are equal chords of a circle; BC is produced to D so that CD = CA; DA cuts the circle at E; prove that BE bisects \( \angle ABC \).

16. Two fixed circles cut at A, B; P is a variable point on one; PA, PB when produced cut the other at Q, R; prove QR is of constant length.

TANGENT PROPERTIES

The Tangent to a Circle.

Definitions.

If a straight line cuts a circle at two distinct points it is called a secant.

If a straight line has one point, and only one point, in common with a circle, however far it is produced, the straight line is called a tangent to the circle, and the common point is called the point of contact.

meeting and cutting  meeting and touching

The words "touching" and "meeting" must not be confused. If a line meets a circle, it may when produced meet it at a second distinct point, and if it does so the line is a secant: on the other hand it may only have one point in common with the circle, however far either way the line is produced, and, if this is so, the line is a tangent.

If two circles touch the same circle at the same point they are said to touch each other at that point. If the circles lie on opposite sides of the line they are said to touch externally, p. 88, Fig. 115 (1); if they lie on the same side of the line they are said to touch internally, p. 88, Fig. 115 (2).

If a straight line touches each of two circles, it is called a common tangent to the two circles, an exterior common tangent if the circles lie on the same side of it, p. 90, Fig. 118, an interior common tangent if the circles lie on opposite sides of it, p. 91, Fig. 119.
THEOREM 43

(1) The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

(2) A tangent to a circle is perpendicular to the radius drawn through the point of contact.

(1) Given that $O$ is the centre and $OA$ a radius of a circle, and that $BAC$ is a line perpendicular to $OA$.

To prove that $BAC$ touches the circle at $A$.

Let $P$ be any point on $BC$; join $OP$.

\[
\angle OAP = 90^\circ \text{ given, } \therefore \text{ each other angle of } \triangle OAP \text{ is less than } 90^\circ.
\]

\[
\therefore \angle OPA < \angle OAP, \quad \therefore OA < OP.
\]

But $OA$ is a radius, $\therefore P$ lies outside the circle.

Similarly every point on $BC$ except $A$ lies outside the circle.

$\therefore$ $BC$ touches the circle at $A$. Q.E.D.

(2) Given that the line $BAC$ touches the circle, centre $O$, at $A$.

To prove that $\angle OAC = 90^\circ$.

If $OA$ is not perpendicular to $BC$, draw $OP$ perpendicular to $BC$.

Since $\angle OPA = 90^\circ$, each other angle of $\triangle OAP$ is less than $90^\circ$.

\[
\therefore \angle OAP < \angle OPA, \quad \therefore OP < OA.
\]

But $OA$ is a radius, $\therefore P$ lies inside the circle.

$\therefore$ $AP$ if produced must cut the circle again, which is impossible since $AP$ is a tangent.

$\therefore$ $OA$ must be perpendicular to $BC$. Q.E.D.

Corollary 1. At every point of a circle, one and only one tangent can be drawn to the circle.

The line through the point perpendicular to the radius is a tangent; $\therefore$ there is one tangent. Further there cannot be more than one tangent, because it must be perpendicular to the radius through the point.

Corollary 2. The perpendicular to a tangent at its point of contact passes through the centre of the circle.

For the radius through the point of contact is this perpendicular.

THEOREM 44

If a straight line touches a circle and, from the point of contact, a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments of the circle.

Given that the line $BAC$ touches the circle at $A$ and that $AD$ is any chord.

To prove (i) $\angle DAO = \text{angle in alternate segment } APD$.

(ii) $\angle DAB = \text{angle in alternate segment } AQD$.

(i) Draw the diameter $AP$ through $A$.

Since $AP$ passes through the centre of the circle and since $AC$ is a tangent $\angle PAC = 90^\circ$.

Since $AP$ is a diameter, $\angle ADP = 90^\circ$.

But the sum of the angles of $\triangle APD$ is $180^\circ$.

\[
\angle PAD + \angle APD = \angle PAC = \angle PAD + \angle DAO.
\]

\[
\therefore \angle APD = \angle DAC.
\]

\[
\therefore \angle DAO = \text{angle in alternate segment } APD.
\]

(ii) Take any point $Q$ on the minor arc $AD$.

$APDQ$ is a cyclic quadrilateral, $\therefore \angle AQD + \angle APD = 180^\circ$.

Also $BAC$ is a straight line, $\therefore \angle BAD + \angle DAC = 180^\circ$.

\[
\therefore \angle BAD + \angle DAC = \angle AQD + \angle APD.
\]

But $\angle DAC = \angle APD$, proved,

\[
\therefore \angle BAD = \angle AQD.
\]

\[
\therefore \angle BAD = \text{angle in alternate segment } AQD. \quad \text{Q.E.D.}
\]
EXERCISE XXI

1. A line TBO cuts a circle ABC at B, C; TA is a tangent; if \( \angle TAC = 118^\circ \), \( \angle ATC = 28^\circ \), find \( \angle ABC \).

2. ABC is a minor arc of a circle; the tangents at A, C meet at T; if \( \angle ATC = 54^\circ \), find \( \angle ABC \).

3. AOC, BOD are chords of a circle ABCD; the tangent at A meets DB produced at T; if \( \angle ATD = 24^\circ \), \( \angle COD = 82^\circ \), \( \angle TBC = 146^\circ \), find \( \angle BAC \). Find also the angle between BD and the tangent at C.

4. The sides BC, CA, AB of a \( \triangle \) touch a circle at X, Y, Z; \( \angle ABC = 64^\circ \), \( \angle ACB = 52^\circ \); find \( \angle XYZ \), \( \angle XZY \).

5. TBP, TCQ are tangents to the circle ABC; \( \angle PBA = 146^\circ \), \( \angle QCA = 128^\circ \); find \( \angle BAC \) and \( \angle BTC \).

6. A chord AB of a circle is produced to T; TC is a tangent from T to the circle; prove \( \angle TBC = \angle ACT \).

7. Two circles APB, AQB intersect at A, B; AP, AQ are the tangents at A, prove \( \angle ABP = \angle ABQ \).

8. DA is the tangent at A to the circle ABC; if DB is parallel to AC, prove \( \angle ADB = \angle ABC \).

9. BC, AD are parallel chords of the circle ABCD; the tangent at A cuts CB produced at P; PO cuts the circle at Q; prove \( \angle PAQ = \angle BPQ \).

10. Two circles ACB, ADB intersect at A, B; CA, DB are tangents to circles ADB, ACB at A, B; prove \( \angle ADB = \angle ABC \).

11. CA, CB are equal chords of a circle; the tangent ADE at A meets BC produced at D; prove \( \angle BDE = \frac{1}{2} \angle CAD \).

12. Two circles intersect at A, B; the tangents at B meet the circles at P, Q; if \( \angle PBQ \) is acute, prove \( \angle PAQ = 2 \angle PBQ \). What happens if \( \angle PBQ \) is obtuse?

13. ABC is a \( \triangle \) inscribed in a circle; the tangent at C meets AB produced at T; the bisector of \( \angle ABC \) cuts AB at D; prove \( \angle TO = TD \).

14. AOB is a diameter of a circle, centre O; the tangent at B meets any chord AP at T; prove \( \angle ATB = \angle OPB \).

15. AB is a diameter of a circle ABC; TC is the tangent from a point T on AB produced; TO is drawn perpendicular to TA and meets AC produced at D; prove \( \angle TC = TD \).

16. The diameter AB of a circle, centre O, is produced to T so that CB = BT; TP is a tangent to the circle; prove \( TP = PA \).

THEOREM 45

If two tangents are drawn to a circle from an external point.

1. The tangents are equal.

2. The tangents subtend equal angles at the centre.

3. The line joining the centre to the external point bisects the angle between the tangents.

Given TP, TQ are the tangents from T to a circle, centre O.

To prove

(1) \( TP = TQ \).

(2) \( \angle TOP = \angle TOQ \).

(3) \( \angle OTP = \angle OTQ \).

Since TP, TQ are tangents at T, Q, the angles TPO, TQO are right angles.

\( \therefore \) in the right-angled triangles TPO, TQO

\( OP = OQ \), radii.

\( OT \) is the common hypotenuse.

\( \therefore \) \( \angle TOP = \angle TOQ \) (rt. angle, hyp., side).

\( \therefore \) \( TP = TQ \),

and \( \angle TOP = \angle TOQ \),

and \( \angle OTP = \angle OTQ \). Q.E.D.
THEOREM 46

If two circles touch one another, the line joining their centres (produced if necessary) passes through the point of contact. *Given two circles, centres A, B, touching each other at P.*

*To prove* AB (produced if necessary) passes through P.

Since the circles touch each other at P, they have a common tangent XPY at P.

Since XP touches each circle at P, the angles XPA, XPB are right angles.

∴ A and B each lie on the line through P perpendicular to PX.

∴ A, B, P lie on a straight line.

Q.E.D.

*Note.*—If two circles touch each other externally (Fig. 115 (1)), the distance between their centres equals the sum of the radii.

If two circles touch each other internally (Fig. 115 (2)), the distance between their centres equals the difference of the radii.

CONSTRUCTION OF TANGENTS

(1) Construct a tangent to a circle at a given point on the circumference.

(2) Construct the tangents to a circle from a given point outside it.

(1) Given a point A on the circumference of a circle.

To construct the tangent at A to the circle.

Construct the centre O of the circle. Join AO. Through A, construct a line AT perpendicular to AO.

Then AT is the required tangent.

*Proof.* The tangent is perp. to the radius through the point of contact. But AO is a radius and $\angle OAT = 90^\circ$,

∴ AT is the tangent at A.

(2) Given a point T outside a circle.

To construct the tangents from T to the circle.

Construct the centre O of the circle. Join OT and bisect it at F. With centre F and radius FT, describe a circle and let it cut the given circle at P, Q. Join TP, TQ.

Then TP, TQ are the required tangents.

*Proof.* Since TF = FO, the circle, centre F, radius FT, passes through O, and TO is a diameter.

∴ $\angle TPO = 90^\circ = \angle TQO$. $\angle$ in semicircle.

But OP, OQ are radii of the given circle.

∴ TP, TQ are tangents to the given circle. Q.E.F.
CONSTRUCTION 9

(1) Draw the direct (or exterior) common tangents to two circles.

(2) Draw the transverse (or interior) common tangents to two non-intersecting circles.

---

CONSTRUCTION OF TANGENTS

(2) Given. Two non-intersecting circles, centres A, B.

To construct the transverse common tangents.

Let \(a, b\) be the radii of the circles, centres A, B.

![Diagram of circles and tangents]

With A as centre and \(a + b\) as radius, describe a circle and construct the tangents BP, BP’ to it from B. Join AP, AP’ and produce them to meet the circle, radius \(a\), in Q, Q’. Through Q, Q’ draw lines QR, Q’R’ parallel to PB, P’B.

Then QR, Q’R’ are the required common tangents.

Proof. Through B draw BR, BR’ parallel to AQ, AQ’ to meet QR, Q’R’ at R, R’.

By construction, PBRQ is a parallelogram.

\[\therefore BR = PQ = AQ - AP = a - (a - b) = b.\]

\[\therefore R\] lies on the circle, centre B, radius \(b\).

Also, since BP is a tangent, \(\angle BPA = 90^\circ\).

\[\therefore \angle QRA = 90^\circ\] and \(\angle BRQ = 90^\circ\), by parallels.

\[\therefore QR\] is a tangent at Q and R to the two circles.

Similarly, it may be proved that Q’R’ is also a common tangent.

---

EXERCISE XXII

1. Three circles, centres A, B, C, touch each other externally; AB = 4, BC = 6, CA = 7; find their radii.

2. In \(\triangle ABC\), AB = 4, BC = 7, CA = 5; two circles with B, C as centres touch each other externally; a circle with A as centre touches the others internally; find their radii.

3. Two circles of radii 3 cm, 12 cm, touch each other externally; find the length of their common tangent.
4. Fig. 120 is formed of three circular arcs of radii 6.7 cm., 2.2 cm., 3.1 cm.; X, Y, Z are the centres of the circles; find the lengths of the sides of \( \triangle XYZ \).

5. In Fig. 121, AB is a quadrant touching AD at A and the quadrant BC at B; \( \angle ADC = 90^\circ \), \( AB = 12 \) cm., \( BC = 9 \) cm.; find the radius of the circle.

6. C is a point on AB such that \( AC = 5 \) cm., \( CB = 3 \) cm.; find the radius of the circle which touches AB at C and also touches the circle on AB as diameter.

7. A, B are the centres of two circles of radii 5 cm., 3 cm.; \( AB = 12 \) cm.; BC is a radius perpendicular to BA; find the radius of a circle which touches the larger circle and touches the smaller circle at C. (Two answers.)

8. AB, BC are two equal quadrants touching at B; their radii are 12 cm.; find the radius of the circle which touches AB, BC, AC.

9. In \( \triangle ABC \), \( AB = 4 \) cm., \( BC = 6 \) cm., \( CA = 7 \) cm.; a circle touches BC, CA, AB at X, Y, Z; find BX and AY.

10. In \( \triangle ABC \), \( AB = 5 \) cm., \( BC = 7 \) cm., \( CA = 9 \) cm.; a circle touches CA produced, CB produced, AB at Q, P, R; find AQ, BR.

11. The distance between the centres of two circles of radii 11 cm., 5 cm. is 20 cm.; find the lengths of their exterior and interior common tangents.

12. A circle touches the sides of \( \triangle ABC \) at X, Y, Z; if Y, Z are the mid-points of AB, AC, prove that X is the mid-point of BC.

13. Two circles touch each other at A; any line through A cuts the circles at P, Q; prove that the tangents at P, Q are parallel.

14. ABCD is a quadrilateral circumscribing a circle; prove that \( AB + CD = BC + AD \).

15. ABCD is a parallelogram; if the circles on AB and CD as diameters touch each other, prove that ABCD is a rhombus.

16. Two circles touch externally at A; PQ is their common tangent; prove that the tangent at A bisects PQ and that \( \angle PAQ = 90^\circ \).

17. ABCDEF is a hexagon circumscribing a circle; prove that \( AB + CD + EF = BC + DE + FA \).

18. ABCD is a quadrilateral circumscribing a circle, centre O; prove \( \angle AOB + \angle COD = 180^\circ \).

19. Two circles touch internally at A; a chord PQ of one touches the other at R; prove \( \angle PAR = \angle QAR \).

20. Two circles touch internally at A; any line PQRS cuts one at P, Q and the other at R, S; prove \( \angle PAQ = \angle RAS \).

21. Two circles touch at A; any line PAQ cuts one circle at P, the other at Q; prove that the tangent at P is perpendicular to the diameter through Q.

22. Two circles touch externally at A; a tangent to one of them at P cuts the other circle at Q, R; prove \( \angle PAQ + \angle PAR = 180^\circ \).

23. OA, OB are two radii of a circle, such that \( \angle AOB = 60^\circ \); a circle touches OA, OB and the arc AB; prove that its radius = \( \frac{1}{2} \) OA.

Inscribed and Circumscribed Regular Polygons.

If a regular polygon of \( n \) sides is inscribed in a circle or circumscribed about a circle, each side subtends an angle of \( \frac{360}{n} \) degrees at the centre of the circle.

For the values \( n = 3, 4, 6, 8, \) these angles are respectively \( 120^\circ, 90^\circ, 60^\circ, 45^\circ \), and angles of these magnitudes can be constructed by ruler and compass without using a protractor.

Fig. 123 represents a regular octagon inscribed in a circle and circumscribed about a circle.

The reader should construct inscribed and circumscribed figures of 3, 4 and 6 sides.

The simplest way of inscribing a regular hexagon in a circle is to make use of the fact that the length of each side is equal to the radius. To circumscribe a regular hexagon about a circle, draw tangents at the corners of the inscribed regular hexagon.
CONSTRUCTION OF CIRCLES

CONSTRUCTION 10

1. Construct the inscribed circle of a given triangle.
2. Construct an escribed circle of a given triangle.

Given a triangle ABC.

To construct (1) the circle inscribed in \( \triangle ABC \);

2. the circle which touches AB produced, AC produced and BC.

1. Construct the lines BI, CI, bisecting the angles ABC, ACB and intersecting at I.

Draw IX perpendicular to BC.

With I as centre and IX as radius, describe a circle.

This circle touches BC, CA, AB.

Proof. Since I lies on the bisector of \( \angle ABC \).

I is equidistant from the lines BA, BC.

Since I lies on the bisector of \( \angle ACB \).

I is equidistant from the line CB, CA.

\( \therefore \) I is equidistant from AB, BC, CA.

\( \therefore \) the circle, centre I, radius IX, touches AB, BC, CA.

2. Produce AB, AC to H, K. Construct the lines BI', CI', bisecting the angles HBC, KCB and intersecting at I'.

Draw I'X', perpendicular to BC.

With I' as centre and I'X' as radius, describe a circle.

This circle touches AB produced, AC produced and BC.

Proof. Since I' lies on the bisector of \( \angle HBC \).

I' is equidistant from BH and BC.

Since I' lies on the bisector of \( \angle KCB \).

I' is equidistant from CK and CB.

\( \therefore \) I' is equidistant from HB, BC, CK.

\( \therefore \) the circle, centre I', radius I'X', touches HB, BC, CK.

Q.E.F.

EXERCISE XXIII

1. Draw a line AB 3 cm. long, and construct a circle of radius 5 cm. to pass through A and B.

2. Draw two lines AOB and COD intersecting at an angle of 80°, make AO = 3 cm., OB = 4 cm., CO = 5 cm., OD = 2 cm., construct a circle to pass through A, B, C. Does it pass through D?
3. Construct two circles of radii 4 cm., 5 cm., such that their common chord is of length 6 cm. Measure the distance between their centres.

4. Inscribe an equilateral triangle in a given circle.

5. Draw a circle radius 4 cm. and take a point 6 cm. from the centre. Construct the tangents from this point to the circle and measure their lengths.

6. On a line of length 5 cm., construct a segment of a circle containing an angle of 70°; measure its radius.

7. On a line of length 2 inches, construct a segment of a circle containing an angle of 140°; measure its radius.

8. Given a point B on a given line ABC and a point D outside the line, construct a circle to pass through D and to touch AC at B.

9. Draw a line AB and take a point C at a distance of 3 cm. from the line AB; construct a circle of radius 4 cm. to pass through C and touch AB.

10. Draw two lines AB, AC making an angle of 65° with each other; construct a circle of radius 3 cm. to touch AB and AC.

11. Draw a circle of radius 3 cm. and take a point A at a distance of 4 cm. from its centre; construct a circle to touch the first circle and to pass through A, and to have a radius of 2 cm. Is there more than one such circle?

12. Draw a line AB of length 6 cm.; with A, B as centres and radii 3 cm., 2 cm. respectively, describe circles. Construct a circle to touch each of these circles and have a radius of 5 cm. Give all possible solutions. (The contacts may be internal or external.)

13. Draw a circle of radius 4·5 cm. and draw a diameter AB; construct a circle of radius 1·5 cm. to touch the circle and AB.

14. Draw a circle of radius 5 cm.; construct two circles of radii 1·5 cm., 3·5 cm. touching each externally and touching the first circle internally.

15. Draw a triangle whose sides are of lengths 2, 3, 4 cm., and construct the four circles which touch the sides of this triangle, and measure their radii.

16. In a circle of radius 3 cm., inscribe a triangle whose angles are 40°, 65°, 75°; measure its longest side.

17. Circumscribe about a circle of radius 2 cm. a triangle whose angles are 60°, 55°, 75°; measure its longest side.

18. In the ∆ABC, ∠ABC = 54°, ∠BAC = 78°; the bisector of ∠BCA cuts AB at X; prove that CA = CX.

19. In Fig. 126, prove that QR is parallel to ST.

20. AB is a quadrant of a circle, AC is any chord; BN is the perpendicular from B to AC; prove that BN = NC.

21. AB is a chord of a circle; AT is the tangent at A; AC is a chord bisecting ∠BAT; prove that AC = CB.

22. In ∆ABC, ∠ACB = 60°, AC = 2CB; CD is an altitude; prove by using the figure of Pythagoras' theorem or otherwise that AD = 4DB.

23. In Fig. 127, O is the centre of the circle; PQ and PT are equally inclined to TO; prove ∠QOT = 2∠POT.

24. AOB is a chord of a circle ABC; T is a point on the tangent at A; the tangent at B meets TO produced at P; ∠ATO = 35°, ∠BOT = 115°; find ∠BPT.

25. In ∆ABC, AB = AC; the circle on AB as diameter cuts BC at P; prove SP = PC.
14.

1. ABCD is a parallelogram; AB < BC; the line bisecting ∠ABC cuts AD at P; prove that BC = CD + DP.

2. In Fig. 128, O is the centre and TQ bisects ∠OTP; prove ∠TQP = 45°.

3. The radii of two circles are 2 cm., 5 cm., and the distance between their centres is 9 cm.; calculate the lengths of the internal and external common tangents.

4. A, B, C are three points on a circle, A being on the major arc BC; the tangents at B, C intersect at T; a line is drawn through T parallel to the tangent at A, and cuts AB, AC produced at P, Q; prove that TP = TQ.

15.

1. A circle is drawn touching the sides BC, CA, AB of the triangle ABC at X, Y, Z. If ∠ABC = 36°, ∠ACB = 66°, calculate ∠YXZ.

2. ABCD is a square inscribed in a circle; P is any point on the minor arc AB; prove ∠APB = 3∠BPC.

3. A, B, C are points on a circle, centre O; BO, CO are produced to meet AC, AB at P, Q; prove ∠BPC + ∠BQC = 3∠BAC.

4. In Fig. 129, AB is a diameter; ∠HPQ = ∠KQP = 90°; prove AH = BK.

SECTION V

AREAS OF RECTANGLES

Definition. A rectangle contained by two lines, AB and CD, is a rectangle having two adjacent sides equal to AB and CD respectively. The area of the rectangle is denoted by AB . CD.

Areas of rectangles can be used to illustrate various identities:

(i) Fig. 130 illustrates \[ k(a + b + c) = ka + kb + kc. \]

(ii) Fig. 131 illustrates \[ (a + b)^2 = a^2 + 2ab + b^2. \]

(iii) Fig. 132 illustrates \[ (a - b)^2 = a^2 - 2ab + b^2. \]

(iv) Fig. 133 illustrates \[ (a + b)(a - b) = a^2 - b^2. \]

These identities correspond to the geometrical theorems, Th. 47-49.
Definition.
If \( AB \) and \( CD \) are any two straight lines, and if \( AH, BK \) are the perpendiculars from \( A, B \) to \( CD \), then
\( HK \) is called the projection of \( AB \) on \( CD \).

Thus, in Fig. 82, p. 52,
\( QY \) is the projection of \( BA \) on \( Q \),
\( XC \) is the projection of \( AC \) on \( BC \).

Or, in Fig. 138, p. 103,
\( AN \) is the projection of \( AC \) on \( AB \),
\( BN \) is the projection of \( BC \) on \( AB \).

Note.—The use of the word "projection" shortens the enunciation of Theorems 50, 51. The geometrical proofs of Theorems 47-49, given below, are necessary if algebraic methods of proof for Theorems 50, 51, 53, 54 are not allowed; otherwise they may be omitted.

**THEOREM 47**

If there are two straight lines, and if the second is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the first line and the several parts of the second line.

**Given** a line \( AB \) and a second line \( AD \) divided into (say) three parts \( AP, PQ, QD \).

**To prove**
\[ AB \cdot AD = AB \cdot AP + AB \cdot PQ + AB \cdot QD. \]

**Draw** \( PX, QY \) parallel to \( AB \) to meet \( BC \) at \( X, Y \).

\[ PX = QY = AB, \text{ opp. sides } [\text{gram}] ; \]
\[ \therefore \text{rect. } PQYX = PX \cdot QY = AB \cdot PQ. \]
\[ \text{rect. } QDQY = QY \cdot QD = AB \cdot QD. \]

**But** \( ABCD = APXB + PQYX + QDCY. \)

\[ \therefore AB \cdot AD = AB \cdot AP + AB \cdot PQ + AB \cdot QD. \]

**THEOREM 48**

(i) If a straight line is divided into two parts, then the square on the whole line is equal to the sum of the squares on the two parts plus twice the rectangle contained by the two parts.

(ii) The square on the difference of two lines is equal to the sum of the squares on the two lines minus twice the rectangle contained by the lines.

(i) **Given** a line \( AB \) divided at \( P. \)

**To prove** sq. on \( AB = \)
\( \text{sq. on } AP + \text{sq. on } PB + 2 \text{rect. } AP \cdot PB. \)

**Draw** the squares \( ABCD, APHK \) on the same side of \( AB \), then \( AK \) lies along \( AD \).

**Produce** \( PH, KH \) to meet \( CD, CB \) at \( Q, G. \)

**Figures** \( PBGH, HGCQ, HKQD \) are \([\text{gram}], \text{ with one angle a right angle, and therefore are rectangles.} \)

Also \( AB = AD, AP = AK \), sides of sq. \( \therefore PB = KD. \)
\( \therefore HG = PB = KD = HQ, \text{ opp. sides of } [\text{gram}] . \)
\( \therefore HGCQ \text{ is a square and equals sq. on } PB. \)

Also rect. \( PBGH = BG \cdot PB = AP \cdot PB \), since \( BG = AK = AP \), and rect. \( KHQD = KH = KD = AP \cdot PB \), since \( KH = AP. \)
\( \therefore \text{sq. on } AB = \text{square } ABCD \)
\[ = APHK + PBGH + HKQD + HGCQ \]
\[ = \text{sq. on } AP + 2 \text{rect. } AP \cdot PB + \text{sq. on } PB. \]

(ii) **Given** a line \( AP \) produced to \( B. \)

**To prove** sq. on difference of \( AB \) and \( PB = \)
\( \text{sq. on } AB + \text{sq. on } PB - 2 \text{rect. } AB \cdot PB. \)

**Use** the same construction and figure as for (i).

\( \text{Sq. on } (AB - PB) = \text{sq. on } AP = \text{square } APHK \)
\[ = ABCD + PBQC + HKQD \]
\[ = ABCD + PBQC + KGCD + HGCQ \]
\[ = \text{sq. on } AB - 2 \text{rect. } AB \cdot PB + \text{sq. on } PB. \]
THEOREM 49

The rectangle contained by the sum and difference of two straight lines is equal to the difference of the squares on the two lines.

Given a line $AB$ divided at $P$.

To prove rectangle $(AB + AP)(AB - AP) = \text{sq. on } AB - \text{sq. on } AP$.

Draw the squares $ABCD$, $APHK$ on the same side of $AB$, then $AK$ lies along $AD$.

Produce $DC$ to $S$, so that $CS = AP$.

Complete rectangle $KDS$, then $KH$ lies along $KT$ since $\angle AKH = 90^\circ$, $\angle$ of sq.

Let $KT$ cut $BC$ at $G$, then $PBGH$ is a $\parallel$gram with one angle $90^\circ$ and is therefore a rectangle.

$KD = AD - AK = AB - AP$, sides of sq.

$DS = DC + CS = AB + AP$.

$\therefore$ rect. $(AB + AP)(AB - AP) = DS \cdot KD = \text{rect. KDS}$.

Since $CS = AP = PH$ and $CG = DK = PB$, rect. $GOST = \text{rect. BPHG}$.

$\therefore$ rect. $KDS = KDC + GOST = KDGA + PHGB = ABCD - APKH = \text{sq. on } AB - \text{sq. on } AP$.

THEOREM 50

In an obtuse-angled triangle, the square on the side opposite the obtuse angle is equal to the sum of the squares on the sides containing it plus twice the rectangle contained by one of those sides and the projection on it of the other.

Given $\angle BAC$ is obtuse and $CN$ is the perpendicular from $C$ to $BA$ produced.

To prove $BC^2 = BA^2 + AC^2 + 2BA \cdot AN$.

(Put in a small letter for each length that comes in the answer and also for the altitude.)

Let $BC = a$ units, $BA = c$ units, $AC = b$ units, $AN = x$ units, $CN = h$ units.

It is required to prove that $a^2 = c^2 + b^2 + 2cx$.

Since $\angle BNC = 90^\circ$, $a^2 = (c + x)^2 + h^2$.

$\therefore$ $a^2 = c^2 + 2cx + x^2 + h^2$.

Since $\angle ANC = 90^\circ$, $b^2 = x^2 + h^2$.

$\therefore$ $a^2 = c^2 + 2cx + b^2$.

or $BC^2 = BA^2 + AC^2 + 2BA \cdot AN$.

Q.E.D.

Alternatively, arrange the proof as follows:

Since $\angle BNC = 90^\circ$, $BC^2 = BN^2 + NC^2$

$= BA^2 + 2BA \cdot AN + AN^2 + NC^2$.

because $BN$ is the sum of $BA$ and $AN$,

$= BA^2 + 2BA \cdot AN + AC^2$. 

Q.E.D.
THEOREM 51

In any triangle, the square on the side opposite an acute angle is equal to the sum of the squares on the sides containing it minus twice the rectangle contained by one of those sides and the projection on it of the other.

\[ \text{Fig. 139 (i).} \quad \text{Fig. 139 (ii).} \]

Given \( \angle BAC \) is acute and \( CN \) is the perpendicular from \( C \) to \( AB \) or \( AB \) produced.

To prove \( BC^2 = BA^2 + AC^2 - 2AB \cdot AN \).

(Put in a small letter for each length that comes in the answer and also for the height.)

Let \( BO = a \) units, \( BA = c \) units, \( AC = b \) units, \( AN = x \) units, \( CN = h \) units. It is required to prove that \( a^2 = c^2 + b^2 - 2cx \).

In Fig. 139 (i), \( BN = c - x \); in Fig. 139 (ii), \( BN = x - c \).

Since \( \angle CNB = 90^\circ \), \( a^2 = (c - x)^2 + h^2 \) in Fig. 139 (i),

or \( a^2 = (x - c)^2 + h^2 \) in Fig. 139 (ii);

\[ \therefore \text{in each case, } a^2 = c^2 - 2cx + x^2 + h^2. \]

Since \( \angle ANC = 90^\circ \), \( b^2 = x^2 + h^2. \)

\[ \therefore a^2 = c^2 - 2cx + b^2, \]

or \( BC^2 = BA^2 + AC^2 - 2AB \cdot AN. \)

Q.E.D.

Alternatively, arrange the proof as follows:

Since \( \angle BNC = 90^\circ \), \( BC^2 = BN^2 + NC^2 \)

\[ = AB^2 - 2AB \cdot AN + AN^2 + NC^2, \]

because, in each figure, \( BN \) is the difference of \( AB \) and \( AN \),

\[ = AB^2 - 2AB \cdot AN + AC^2. \]

Definition.

The line joining a vertex of a triangle to the mid-point of the opposite side is called a median.

THEOREM 52. (Apollonius' Theorem)

In any triangle, the sum of the squares on two sides is equal to twice the square on half the base plus twice the square on the median which bisects the base.

\[ \text{Fig. 140.} \]

Given \( D \) is the mid-point of \( BC \).

To prove \( AB^2 + AC^2 = 2AD^2 + 2BD^2. \)

Draw \( AN \) perpendicular to \( BC \).

Of the angles \( ADB, ADC \), suppose \( \angle ADC \) is acute, so that \( \angle ADB \) is obtuse.

From the triangle \( ADB \), since \( \angle ADB \) is obtuse,

\[ AB^2 = AD^2 + DB^2 + 2BD \cdot DN. \]

From the triangle \( ADC \), since \( \angle ADC \) is acute,

\[ AC^2 = AD^2 + DC^2 - 2DC \cdot DN. \]

But \( BD = DC \), given;

\[ \therefore BD \cdot DN = DC \cdot DN \text{ and } BD^2 = DC^2. \]

Adding, \( AB^2 + AC^2 = 2AD^2 + 2BD^2. \)

Q.E.D.
EXERCISE XXIV

1. Find by calculation which of the following triangles are obtuse-angled, their sides being as follows: (i) 4, 5, 7; (ii) 7, 8, 11; (iii) 8, 9, 12; (iv) 15, 18, 22.

2. Each of the sides of an acute-angled triangle is an exact number of inches; two of them are 12", 15"; what is the greatest length of the third side?

3. In \( \triangle ABC \), \( BC = 6 \), \( CA = 3 \), \( AB = 4 \); \( CN \) is an altitude; calculate \( AN \) and \( CN \).

4. In \( \triangle ABC \), \( BC = 8 \), \( CA = 9 \), \( AB = 10 \); \( CN \) is an altitude; calculate \( AN \) and \( CN \).

5. In \( \triangle ABC \), \( BC = 7 \), \( CA = 13 \), \( AB = 10 \); \( CN \) is an altitude; calculate \( AN \), \( BN \), \( CN \).

6. Find the area of the triangle whose sides are 9", 10", 11".

7. \( \triangle ABC \) is a parallelogram; \( AB = 5" \), \( AD = 3" \); the projection of \( AC \) on \( AB \) is 6"; calculate \( AC \).

8. In \( \triangle ABC \), \( AC = 8 \text{ cm} \), \( BC = 6 \text{ cm} \), \( \angle ACB = 120^\circ \); calculate \( AB \).

9. In \( \triangle ABC \), \( AB = 8 \text{ cm} \), \( AC = 7 \text{ cm} \), \( BC = 3 \text{ cm} \); prove \( \angle ABC = 60^\circ \).

10. The sides of a triangle are 23, 27, 36; is it obtuse-angled?

11. In \( \triangle ABC \), \( AB = 9\)", \( AC = 11" \), \( \angle BAC > 90^\circ \); prove \( BC > 14" \).

12. In \( \triangle ABC \), \( AB = 14" \), \( BC = 10" \), \( CA = 6" \); prove \( \angle ACB = 120^\circ \).

13. The sides of a \( \triangle \) are 4, 7, 9; calculate the length of the shortest median.

14. Find the lengths of the medians of a triangle whose sides are 6, 8, 9 cm.

15. The sides of a parallelogram are 5 cm, 7 cm, and one diagonal is 8 cm; find the length of the other.

16. \( AD \) is a median of the \( \triangle ABC \); \( AB = 6 \), \( AC = 8 \), \( AD = 5 \); calculate \( BC \).

17. In \( \triangle ABC \), \( AB = 4 \), \( BC = 5 \), \( CA = 8 \); \( BC \) is produced to \( D \) so that \( CD = 5 \); calculate \( AD \).

18. \( \triangle ABC \) is an equilateral triangle; \( BC \) is produced to \( D \) so that \( BC = CD \); prove \( AD^2 = 3AB^2 \).

19. In \( \triangle ABC \), \( AB = AC \); \( CD \) is an altitude; prove that \( BC^2 = 2AB \cdot BD \).

20. \( CF \), \( BE \) are altitudes of the triangle \( ABC \); prove that \( AF \cdot AB = AE \cdot AC \).

EXTENSION OF PYTHAGORAS' THEOREM

21. \( \square ABCD \) is a parallelogram; prove that \( AC^2 + BD^2 = 2AB^2 + 2BC^2 \).

22. \( \square ABCD \) is a rectangle; \( P \) is any point in the same or any other plane; prove that \( PA^2 + PC^2 = PB^2 + PD^2 \).

23. In \( \triangle ABC \), \( AB = AC \); \( AB \) is produced to \( D \) so that \( AB = BD \); prove \( CD^2 = AB^2 + 2BC^2 \).

24. In \( \triangle ABC \), \( \angle ACB = 90^\circ \); \( AB \) is trisected at \( P, Q \); prove that \( PC^2 + QP^2 + PQ^2 = \frac{5}{9} AB^2 \).

25. The base \( BC \) of \( \triangle ABC \) is trisected at \( X, Y \); prove that \( AX^2 +AY^2 +4XY^2 = AB^2 + AC^2 \).

26. \( AD, BE, CF \) are the medians of \( \triangle ABC \); prove that \( 4(AD^2 + BE^2 + CF^2) = 3(AB^2 + BC^2 + CA^2) \).

27. \( \square ABCD \) is a quadrilateral; \( X, Y \) are the mid-points of \( AC, BD \); prove that \( AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4XY^2 \).

28. \( \square ABC \) is a triangle; \( ABPQ \), \( AOXY \) are squares outside \( ABC \); prove that \( BC^2 + QY^2 = AP^2 + AX^2 \).

29. \( ABCD \) is a tetrahedron; \( \angle BAC = \angle CAD = \angle DAB = 90^\circ \); prove that \( BCD \) is an acute-angled \( \triangle \).

Segments of a Chord.

If \( AB \) is any chord of a circle, and if \( X \) is any point either on \( AB \) (Fig. 141), or \( AB \) produced (Fig. 142), then \( AX \) and \( BX \) are called the segments of the chord formed by the point of division \( X \).

The rectangle contained by the segments, \( AX \) and \( BX \), of the chord is a rectangle whose length is \( AX \) and breadth \( BX \), so that its area is measured by \( AX \cdot BX \).
THEOREM 53. (First Proof)

If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

**Given** two chords $AB$, $CD$ of a circle, intersecting at $X$ within the circle.

**To prove** $AX \cdot XB = CX \cdot XD$.

Let $O$ be the centre of the circle, draw the perpendicular $ON$ from $O$ to $AB$; join $OA$, $OX$.

Since $ON$ is perpendicular to $AB$, $AN = NB$.

\[ AX \cdot XB = (AN + NX)(NB - NX) = (AN + NX)(AN - NX) = AN^2 - NX^2 = (AN^2 + NO^2) - (NX^2 + NO^2) = OA^2 - OX^2, \text{ Pythagoras.} \]

Similarly

\[ CX \cdot XD = OC^2 - OX^2. \]

But $OA = OC$, radii.

\[ AX \cdot XB = CX \cdot XD. \]

**Q.E.D.**

**Corollary.** If $X$ is any point inside a circle with centre $O$ and radius $r$, then the rectangle contained by the segments of any chord drawn through $X$ equals $r^2 - OX^2$.

---

THEOREM 54. (First Proof)

If from a point without a circle a secant and a tangent to the circle are drawn, the rectangle contained by the whole secant and the segment of it without the circle is equal to the square on the tangent.

**Given** a secant $XBA$ and a tangent $XT$ touching the circle at $T$.

**To prove** $XA \cdot XB = XT^2$.

Let $O$ be the centre of the circle, draw the perpendicular $ON$ from $O$ to $AB$; join $OA$, $OX$, $OT$.

Since $ON$ is perpendicular to $AB$, $AN = NB$.

\[ AX \cdot XB = (XN + NA)(XN - NB) = (XN + NA)(XN - NA) = XN^2 - NA^2 = (XN^2 + NO^2) - (NA^2 + NO^2) = XO^2 - AO^2, \text{ Pythagoras,} \]

\[ = XO^2 - OT^2, OA = OT \text{ radii,} \]

\[ = XT^2, \text{ Pythagoras, since } \angle OTX = 90^\circ. \text{ Q.E.D.} \]

**Corollary 1.** If two chords of a circle meet when produced at a point outside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Each is equal to the square on the tangent from that point to the circle.

**D.S.G.**
Corollary 2. If $X$ is any point outside a circle with centre $O$ and radius $r$, the rectangle contained by the segments of any chord drawn through $X$ equals $OX^2 - r^2$.

It should be noted that Theorem 53 and Corollary 1 of Theorem 54 are in reality two cases of the same theorem.

The converses of Theorems 53, 54 are important and may easily be proved by a reductio ad absurdum method. We may state them as follows:

I. If two straight lines $AB$ and $CD$ cut each other either both internally or both externally so that $AX \cdot XB = XD \cdot CX$, then the four points $A, B, C, D$ lie on a circle.

II. If from any point $X$ on a line $AB$ produced another straight line is drawn and a point $T$ taken on it so that $XT^2 = XA \cdot XB$, then the line $XT$ is a tangent to the circle which passes through $A, B, T$.
CONSTRUCTION 12

(i) Construct a square equal in area to a given rectangle.
(ii) Construct a square equal in area to a given polygon.
(i) Given a rectangle ABCD.

To construct a square of equal area.
Produce AB to E, making BE = BC.
On AE as diameter, describe a semicircle.
Produce CB to meet the semicircle at P.
On BP describe a square.
This is the required square.

Proof. \( BP^2 = AB \cdot BE \), but \( BE = BC \);
\[ \therefore \quad BP^2 = AB \cdot BC = \text{area of ABCD}. \quad \text{Q.E.F.} \]

(ii) Given any polygon.

To construct a square of equal area.

\[ \text{Fig. 148.} \]

By the method of Constr. 7, reduce the polygon to any equivalent triangle XYZ (see p. 48).
Draw the altitude XK and bisect YZ at Q.
Use (i) to construct a square of area equal to a rectangle whose sides are equal to YQ and XK.
This is the square required.

Proof. Area of polygon = area of \( \triangle XYZ \).
\[ = \frac{1}{2} YZ \cdot XK \]
\[ = YQ \cdot XK = \text{square}. \quad \text{Q.E.F.} \]

EXERCISE XXV

1. In Fig. 143, (i) if \( AX = 6, XB = 2, CX = 3 \), find XD.
   (ii) if \( AB = 11, BZ = 3, CX = 4 \), find CD;
   (iii) if \( AX = 12, AB = 15, CX = XD \), find CD.

2. In Fig. 143, (i) if the radius is 7 cm. and \( BX = 5 \), find \( \triangle AXB \);
   (ii) if \( OA = 6 \), \( AX = 5 \), \( CX = 4 \), find \( BX \).

3. From a point P on a circle PN is drawn perpendicular to a diameter \( AB \); \( AN = 4 \), \( NB = 16 \); find \( PN \).

4. In \( \triangle ABC \), \( \angle BAC = 90^\circ \), \( AB = 4 \), \( AC = 3 \); \( AD \) is an altitude; find \( BD \).

5. In \( \triangle ABC \), \( AB = 9, AC = 12 \), \( F \) is the mid-point of \( AC \); the circle through \( B, F, C \) cuts \( AB \) at \( E \); find \( BE \).

6. In Fig. 150, (i) if \( AB = 2, BX = 6, DX = 3 \); find CD;
   (ii) if \( AB = 2, BX = 3, DX = 2 \); find TX;
   (iii) if \( TX = 5, DX = 2 \); find CD.

7. If in Fig. 144 the circle is of radius 5 cm. and the distance of X from the centre is 9 cm., find \( XA \cdot XB \).

8. ABC is a triangle inscribed in a circle; \( AB = 10 \), \( BC = 16 \); \( AD \) is drawn perpendicular to \( BC \) and is produced to meet the circle at \( E \); find \( DE \) and the radius of the circle.

9. The roadway ABC of a bridge is a circular arc resting on supports at A, B at the same level; the highest point C of the roadway is 4 feet above AB, and AB is 8 yards. Find the radius of the arc.
10. Draw a rectangle of sides 5 cm., 8 cm., and construct a square of equal area; measure its side.

11. Construct a square equal in area to an equilateral triangle of side 3 inches; measure its side.

12. Given a line CD and two points A, B on the same side of CD. If a circle is drawn through A and B to touch CD, construct the two possible positions of the points of contact. (Start by joining AB and producing it to cut CD at X.)

13. Construct a rectangle of perimeter 20 cm. and area 20 sq. cm.; measure its sides. Show that if the perimeter of a rectangle is given its area is greatest when the figure is a square.

14. In the triangle XBC, XB = 6 cm., XC = 4 cm., \angle BXC = 90°; a circle is drawn to touch BX at B and to pass through C. Find the radius of the circle.

15. AXB, CXD are two perpendicular chords of a circle whose centre is O; AX = 3", CX = 5", XD = 6"; find OX and the radius of the circle.

16. If, in Fig. 150, XC = 2XT, prove that CD = 2DX.

17. Two circles intersect at A, B; P is any point on AB produced; prove that the tangents from P to the two circles are equal.

18. Any two circles being given, a third circle is drawn cutting one of them at A, B and the other at C, D; the lines AB and CD are produced to meet at X; prove that the tangents from X to the two first circles are equal.

19. Prove that the common chord of two intersecting circles bisects their common tangents.

20. In \triangle ABC, AB = AC and \angle BAC = 36°; the bisector of \angle ABC meets AC at P; prove that \angle ACB = \angle BAC = \angle BPA.

21. The altitudes BE, CF of \triangle ABC intersect at H; prove that (i) BH = CE = CH, HF; (ii) AF = AB = AE, AC; (iii) CE, CA = CH, CF.

22. In \triangle ABC, \angle BAC = 90°, AB = 2AC. If AD is an altitude, prove that BD = 4DC.

23. PQ is a chord of a circle, centre O; the tangents at P, Q meet at X; OX cuts PQ at N; prove that ON . OX = OP².

24. Two circles intersect at A, B; X is a point such that the tangents from X to the circles are equal; prove that X must lie on AB produced.

SECTION VI
RATIO AND SIMILAR FIGURES

RATIO

If the lengths of two straight lines are 4 cm. and 6 cm., we say that the first is \(\frac{4}{6}\) or \(\frac{2}{3}\) of the second, and that \(\frac{2}{3}\) is the ratio of the lengths of the two lines; this is often written 2 : 3. In general, if two quantities contain respectively \(a\) units and \(b\) units of the same kind, we say that their ratio is \(\frac{a}{b}\). The ratio is therefore a comparison of their magnitudes. The quantities must of course be of the same kind, it would be meaningless to compare 5 yards with 10 shillings.

If two quantities have a common measure, we can express their ratio as the ratio of two integers, e.g., if the lengths of two lines are 2-56 in., 1-12 in., their ratio is \(\frac{256}{112} = \frac{16}{7}\) or 16:7. Here a common measure is 1\(\frac{1}{2}\) inch. But we frequently meet lines whose lengths have no common measure; if the side of a square is 1 inch, the diagonal is \(\sqrt{2}\) inches (Pythagoras), and these two lengths have no common measure and are called incommensurable. The ratio of two such lengths cannot be expressed as the ratio of two integers, although we can find two integers whose ratio differs from this ratio by an amount as small as we please. Formal proofs of theorems involving the ratios of incommensurable quantities are very difficult, and we shall assume that, if a theorem has been proved for all commensurable ratios, it also remains true if the ratios are incommensurable.
If four quantities $a$, $b$, $c$, $d$ are such that the ratio of $a$ to $b$ equals the ratio of $c$ to $d$, then $a$, $b$, $c$, $d$ are said to be in proportion.

If $a$, $b$, $c$, $d$ are in proportion, we have $\frac{a}{b} = \frac{c}{d}$, and $d$ is called the fourth proportional to $a$, $b$, $c$.

If three quantities $a$, $b$, $c$ are such that $\frac{a}{b} = \frac{b}{c}$, they are said to be in continued proportion. Further, $c$ is called the third proportional to $a$, $b$, and $b$ is called the mean proportional between $a$, $c$.

![Diagram 151](image)

If $P$ is any point on a straight line $AB$ or on $AB$ produced, $AP$ and $PB$ are called segments of the line, and the line $AB$ is said to be divided at $P$ in the ratio $\frac{AP}{PB}$.

If $P$ lies between $A$ and $B$, the line $AB$ is said to be divided internally in the ratio $\frac{AP}{PB}$.

If $P$ lies on $AB$ produced or $BA$ produced, the line $AB$ is said to be divided externally in the ratio $\frac{AP}{PB}$.

It is important to notice that in all cases, Fig. 151 and Fig. 152, $AB$ is the whole line and does not appear in the ratio $\frac{AP}{PB}$; the "segments" of $AB$ are $AP$ and $PB$ whether $P$ lies on $AB$ or on $AB$ produced. This aspect may also be emphasised, if considered advisable, by a discussion on directed lengths and the interpretation of positive and negative ratios.

**EXERCISE XXVI**

1. If $\frac{a}{b} = \frac{c}{d}$, prove that:
   
   (i) $\frac{b}{a} = \frac{d}{c}$;  
   (ii) $ad = bc$;  
   (iii) $\frac{a+b}{b} = \frac{c+d}{d}$;
   (iv) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$;  
   (v) $\frac{b+d}{a+c} = \frac{b}{a}$.

2. If $\frac{a}{b} = \frac{c}{d}$, fill up the blank spaces in the following:
   
   (i) $\frac{a}{a+b} = \frac{c}{c+d}$;  
   (ii) $\frac{a-b}{a} = \frac{b-c}{b}$;  
   (iii) $\frac{a+c}{b+d} = \frac{a}{b}$.

3. Are the following in proportion (i) $3$, $5$, $8$, $12$; (ii) $8$ inches, $12$ degrees, $12$ inches?

4. Find the fourth proportional to (i) $2$, $3$, $4$; (ii) $ab$, $bc$, $cd$.

5. Find the third proportional to (i) $1$, $\frac{1}{2}$; (ii) $x$, $xy$.

6. Find a mean proportional between (i) $4$, $25$; (ii) $a^2b$, $b^2a$.

7. A line $AB$, $8\text{"}$ long, is divided internally at $P$ in the ratio $2:3$; find $AP$.

8. A line $AB$, $8\text{"}$ long, is divided externally at $Q$ in the ratio $7:3$; find $BQ$.

9. $AB$ is divided internally at $C$ in the ratio $5:6$. Is $C$ nearer to $A$ or $B$?

10. A line $AB$, $6\text{"}$ long, is divided internally at $P$ in the ratio $2:1$, and externally at $Q$ in the ratio $5:2$; find the ratios in which $PQ$ is divided by $A$ and $B$.

11. $ABCDE$ is a straight line such that $AB : BC : CD : DE = 1 : 3 : 2 : 5$. Find the ratios (i) $\frac{AB}{AC}$; (ii) $\frac{AC}{EC}$; (iii) $\frac{ED}{AD}$.

12. $ABCD$, $AXYZ$ are two straight lines such that $AB : BC : CD : DE = AX : XY : YZ$. Fill up the blank spaces in the following:
   
   (i) $\frac{AB}{AC} = \frac{AX}{AY}$;  
   (ii) $\frac{BC}{BD} = \frac{AZ}{AC}$;  
   (iii) $\frac{XZ}{AY} = \frac{AC}{AD}$.
THEOREM 55

A straight line drawn parallel to one side of a triangle divides the sides proportionally.

Given a line parallel to BC cuts AB, AC (produced if necessary) at H, K.

To prove

\[ \frac{AH}{HB} = \frac{AK}{KC} \]

Express \( \frac{AH}{HB} \) as a fraction \( \frac{x}{y} \), where x and y are integers.

This assumes that AH and HB are commensurable.

Divide AH into x equal parts and HB into y equal parts; then each part of AH is equal to each part of HB.

Through each point of division draw a line parallel to BC.

These lines divide AK into x equal parts and KC into y equal parts, and each part of AK is equal to each part of KC.

\[ \therefore \frac{AK}{KC} = \frac{x}{y} \]

\[ \therefore \frac{AH}{HB} = \frac{AK}{KC} \]

Q.E.D.

Corollary. If HK is parallel to BC, \( \frac{AH}{AB} = \frac{AK}{AC} \) and \( \frac{HB}{KC} = \frac{KB}{KC} \)

These may be proved in exactly the same way.

Note.—It is not possible to express \( \frac{AH}{HB} \) as the ratio of two integers, unless AH and HB are commensurable.

If AH and HB are incommensurable, the theorem is still true, but the difficulty of its proof makes it unsuitable for an elementary course.

THEOREM 56

If two sides of a triangle are divided in the same ratio, the straight line joining the points of section is parallel to the third side.

Given

\[ \frac{AH}{AB} = \frac{AK}{AC} \]

To prove HK is parallel to BC.

Draw HL parallel to BC to cut AC at L.

\[ \frac{AH}{AB} + 1 = \frac{AK}{AC} + 1, \quad \therefore \frac{AH}{HB} = \frac{AK}{KB} \]

\[ \therefore \frac{AB}{AC} = \frac{KC}{LC} \]

But \( \frac{AB}{AC} = \frac{KC}{LC} \), since HL is || BC;

\[ \therefore \frac{KB}{KC} = \frac{LC}{KC} \]

\[ \therefore \frac{KB}{KC} = \frac{LC}{KC} \]

\[ \therefore \frac{AH}{AB} = \frac{AK}{AC} \]

But HL is || BC, \( \therefore \) HK is || BC.

Q.E.D.

Corollary 1. If \( \frac{AH}{AB} = \frac{AK}{AC} \), HK is parallel to BC.

2. If \( \frac{AB}{AC} = \frac{KB}{KC} \), HK is parallel to BC.

Note.—Alternative methods of proof of Theorems 55, 56 are given in the Appendix.
THEOREM 57

The bisector (internal or external) of an angle of a triangle divides the opposite side (internally or externally) in the ratio of the sides containing the angle bisected.

![Diagram](image)

Given AD bisects \( \angle BAC \), internally in Fig. (1), externally in Fig. (2), and cuts BC or AC produced at D.

To prove \( \frac{BD}{BA} = \frac{DC}{AC} \).

Through \( C \) draw \( CP \) parallel to \( DA \) to cut \( BA \) or \( BA \) produced at \( P \).

Take any point \( E \) on \( BA \) in Fig. (1), and on \( BA \) produced in Fig. (2).

Then \( \angle EAD = \angle ACP \), corresp. \( \angle s \), \( AD \parallel PC \).

\( \angle DAC = \angle ACP \), alt. \( \angle s \), \( AD \parallel PC \).

But \( \angle EAD = \angle DAC \) given, \( \therefore \angle ACP = \angle ACP \);

\( \therefore \, AP = AC \).

Since AD is parallel to PC, \( \frac{BD}{BA} = \frac{DC}{AC} \).

But \( AP = AC \), \( \therefore \frac{BD}{BA} = \frac{DC}{AC} \). Q.E.D.

Corollary. If the base \( BC \) is divided internally or externally at \( D \) in the ratio \( AB : AC \), then \( AD \) bisects the angle \( BAC \) internally or externally.

This is proved by the same argument as that used to prove Theorem 57, in a reversed order.

CONSTRUCTION 13

Divide a given finite straight line in a given ratio (i) internally, (ii) externally.

![Diagram](image)

Given two lines \( p, q \) and a finite line \( AB \).

To construct (i) a point \( X \) in \( AB \) such that \( \frac{AX}{XB} = \frac{p}{q} \);

(ii) a point \( Y \) in \( AB \) produced such that \( \frac{AY}{BY} = \frac{p}{q} \).

(i) Draw any line \( AC \) and cut off successively \( AH = p \), \( HK = q \). Join \( KB \). Through \( H \) draw a line parallel to \( KB \) to cut \( AB \) at \( X \). Then \( \frac{AX}{XB} = \frac{AH}{HK} = \frac{p}{q} \) by parallels. Q.E.F.

(ii) Draw any line \( AC \); cut off \( AH = p \), and from \( HA \) cut off \( HK = q \). Join \( KB \). Through \( H \) draw a line parallel to \( KB \) to cut \( AB \) produced at \( Y \).

Then \( \frac{AY}{BY} = \frac{AH}{HK} = \frac{p}{q} \) by parallels. Q.E.F.
CONSTRUCTION 14

Construct a fourth proportional to three given lines.

\[ \frac{a}{b} = \frac{c}{d} \]

Given three lines of lengths \(a, b, c\) units.

To construct a line of length \(d\) units, such that \( \frac{a}{b} = \frac{c}{d} \).

Draw any two lines \(OX, OY\).

From \(OX\) cut off parts \(OP, OQ\) such that \(OP = a, OQ = b\).

From \(OY\) cut off a part \(OR\) such that \(OR = c\).

Join \(PR\).

Through \(Q\) draw a line \(QS\) parallel to \(PR\) to meet \(OY\) at \(S\).

Then \(QS\) is the required fourth proportional.

\[ \text{Proof}. \quad \text{Since} \quad PR \parallel QS, \quad \frac{OP}{OR} = \frac{a}{c}, \quad \frac{OQ}{OS} = \frac{b}{QS}. \quad \text{Q.E.F.} \]

EXERCISE XXVII. (Numerical)

1. In Fig. 153, \(AB = 10.5\) cm, \(AC = 7\) cm, \(BH = 4.5\) cm; calculate \(KC\).
2. In Fig. 153, \(AB = 19.5\) cm, \(AC = 13\) cm, \(AH = 12\) cm; calculate \(KC\).
3. If in Fig. 153, \(AH = 4\frac{1}{2}\) in, \(HB = 2\frac{2}{3}\) in, what is the largest unit of length that can be chosen if \(AH\) and \(HB\) each contain an integral number of these units?
4. In Fig. 153 (2), \(AB = 5, AH = 7, CK = 5\), calculate \(AC\).
5. In Fig. 153 (3), \(AH = 2, AC = 2\frac{1}{2}, BH = 5\), calculate \(CK\).
6. \(ACE, BDF\) are two straight lines cut by three parallel lines \(AB, CD, EF\); \(AC = 2\), \(CE = 3\), \(BF = 4\); calculate \(BD\).

RATIO CONSTRUCTIONS 123

7. Construct and measure a fourth proportional to lines of length \(4, 5, 6\) cm.
8. Construct and measure a third proportional to lines of length \(5, 6\) cm.
9. Draw a line \(AB\) and divide it internally in the ratio \(2:3\).
10. Draw a line \(AB\) and divide it externally (i) in the ratio \(5:3\), (ii) in the ratio \(3:5\).
11. Construct a line of length \(\frac{a}{b}\) cms.
12. Draw a line \(AB\) and divide it in the ratio \(2:7\).
13. In \(\triangle ABC, AB = 6\) cm, \(BC = 8\) cm, \(CA = 4\) cm; the internal and external bisectors of \(\angle BAC\) cut \(BC\) at \(P, Q\); find \(BP\) and \(BQ\).
14. In \(\triangle ABC, AB = 4\), \(BC = 3\), \(CA = 5\); the bisector of \(\angle ACB\) cuts \(AB\) at \(D\); find \(CD\).
15. In \(\triangle ABC, AB = 12, BC = 15, CA = 8\); \(P\) is a point on \(BC\) such that \(BP = 9\); prove \(AP\) bisects \(\angle BAC\); if the external bisector of \(\angle BAC\) cuts \(BC\) produced at \(Q\), and if \(D\) is the mid-point of \(BC\), prove that \(DP, DQ = DC^2\).
16. The internal and external bisectors of \(\angle BAC\) meet \(BC\) and \(BC\) produced at \(P, Q\); \(BP = 5, PC = 3\); find \(CQ\).
17. \(ABCD\) is a rectangular sheet of paper; \(AB = 4\), \(BC = 3\); the edge \(BC\) is folded along \(BD\) and the corner is then cut off along the crease; find the area of the remainder.
18. In \(\triangle ABC, AB = 6, AC = 4\); the bisector of \(\angle BAC\) meets the median \(BE\) at \(O\); the area of \(\triangle ABC\) is 8 sq. in. ; what is the area of \(\triangle AOB\)?

EXERCISE XXVIII

1. Three parallel lines \(AX, BY, CZ\) cut two lines \(ABC, XYZ\); prove that \(\frac{AX}{BY} = \frac{BC}{YZ}\).
2. If two triangles have equal heights, prove that the ratio of their areas equals the ratio of their bases.
3. \(ABC\) is a triangle; \(P, Q\) are points on \(AB, AC\) such that \(AP = \frac{1}{2}AB\) and \(CQ = \frac{1}{3}CA\); prove that the line through \(C\) parallel to \(PQ\) bisects \(AB\).
4. The diagonals of the quad. \(ABCD\) intersect at \(O\); if \(AB\) is parallel to \(DC\), prove \(\frac{AO}{BO} = \frac{AC}{BD}\).
5. O is any point inside the $\triangle ABC$; a line XY parallel to AB cuts OA, OB at X, Y; XZ is drawn parallel to BC to cut OC at Z; prove XZ is parallel to AC.

6. ABCD is a quadrilateral; P is any point on AB; lines PX, PY are drawn parallel to AC, AD to cut BC, BD at X, Y; prove XY is parallel to CD.

7. D is the foot of the perpendicular from A to the bisector of $\angle ABC$; a line from D parallel to BC cuts AC at X; prove AX = XC.

8. In Fig. 159, prove $\triangle ABC \sim \triangle ABD \sim OD'$.

9. I is the in-centre of $\triangle ABC$; prove that $\triangle IBC : \triangle ICA : \triangle IAB = BC : CA : AB$.

10. In Fig. 159, prove $\triangle ACD \sim \triangle ABC$.

11. Two circles APQ, AXY touch at A; APX, AQY are straight lines; prove $\frac{PX}{QY}$.

12. ABC is a $\triangle$; three parallel lines AP, BQ, CR meet BC, CA, AB (produced if necessary) at P, Q, R; prove that $\frac{BP}{CQ} \cdot \frac{CQ}{AR} = 1$.

13. ABC is a triangle; a line cuts BC produced, CA, AB at P, Q, R; CX is drawn parallel to PQ, meeting AB at X; prove (i) $\frac{BP}{BR} \cdot \frac{CQ}{QA} \cdot \frac{AR}{AR} = 1$.

(This is known as Menelaus' Theorem.)

14. The internal and external bisectors of $\angle BAC$ cut BC and BC produced at P, Q; prove $\frac{PC}{CQ} = \frac{1}{2}$.

15. AX is a median of $\triangle ABC$; the bisectors of $\angle s AXB, AXC$ meet AB, AC at H, K; prove HK is parallel to BC.

16. The bisector of $\angle BAC$ cuts BC at D; circles with B, C as centres are drawn through D and cut BA, CA at H, K; prove HK is parallel to BC.

17. H is any point inside the $\triangle ABC$; the bisectors of $\angle s BHC, CHA, AHB$ cut BC, CA, AB at X, Y, Z; prove $\frac{BX}{CY} \cdot \frac{AZ}{XZ} \cdot \frac{YA}{ZA} \cdot \frac{ZB}{XB} = 1$.

18. In $\triangle ABC$, $\angle BAC = 90^\circ$ and AD is an altitude; the bisector of $\angle ABC$ meets AD, AC at L, K; prove $\frac{AL}{CK} = \frac{LD}{KA}$.

19. ABCD is a quadrilateral; if the bisectors of $\angle s DAB, DCB$ meet on DB, prove that the bisectors of $\angle s ABC, ADC$ meet on AC.

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**SIMILAR TRIANGLES**

**THEOREM 58**

If two triangles are equiangular, their corresponding sides are proportional.

![Fig. 160.](image)

Given the triangles $ABC, XYZ$ are equiangular, having $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.

To prove

$$\frac{AB}{AC} = \frac{BC}{BD} = \frac{CD}{CE}.$$

From AB, AC cut off AH, AK equal to XY, XZ. Join HK.

In the $\triangle AHB, XHY$, $\angle A = \angle X$, constr.

In the $\triangle AKC, XHZ$, $\angle A = \angle X$, constr.

$\triangle AHK \sim \triangle XYZ$, given; $\therefore \angle AHK = \angle XYZ$.

But $\angle XYZ = \angle ABC$, given; $\therefore \angle AHK = \angle ABC$.

But these are corresponding angles, $\therefore HK$ is parallel to BC;

$$\frac{AB}{AC} = \frac{XH}{KA}.$$

But $AH = XY$ and $AK = XZ$.

Similarly it can be proved that $\frac{AC}{BC} = \frac{XZ}{YZ}$. Q.E.D.

**Definition.**

If two polygons are equiangular, and if their corresponding sides are proportional, they are said to be similar.

Theorem 58 proves that equiangular triangles are necessarily similar.

D.S.G.
THEOREM 59

If the three sides of one triangle are proportional to the three sides of the other, then the triangles are equiangular.

\[ \triangle ABC, XYZ \text{ are such that } \frac{AB}{BC} = \frac{CA}{XY} = \frac{BA}{YZ} = \frac{CA}{ZX}. \]

To prove \( \angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z \).

On the side of \( YZ \) opposite to \( X \), draw \( YP \) and \( ZP \) so that \( \angle YPZ = \angle ABC \) and \( \angle ZPY = \angle ACB \).

Since the \( \triangle \)s \( ABC, \) \( XYZ \) are equiangular, by construction

\[ \begin{align*}
AB &= BC, \\
YP &= YZ, \\
& \text{given;}
\end{align*} \]

But

\[ \begin{align*}
AB &= BC, \\
XY &= YZ, \\
\therefore & \quad \angle YP = \angle XY; \\
\therefore & \quad \angle YP = \angle XY; \\
\therefore & \quad \angle ZP = \angle XZ;
\end{align*} \]

Similarly

\[ \angle YP = \angle XY; \]

\( \therefore \) in the \( \triangle \)s \( XYZ, \) \( \angle PYZ \)

\[ \begin{align*}
XY &= PY, & \text{proved.} \\
XZ &= PZ, & \text{proved.} \\
YZ & \text{is common;} \\
\therefore & \quad \angle XYZ = \angle PYZ \text{ (3 sides);} \\
\therefore & \quad \angle XYZ = \angle LYZ, & \text{proved.}
\end{align*} \]

But \( \angle LYZ = \angle ABC \text{ and } \angle LZY = \angle ACB, \text{ constr.} \)

\[ \therefore \angle XYZ = \angle ABC \text{ and } \angle XYZ = \angle ACB. \]

\( \therefore \) also \( \angle YXZ = \angle BAC. \) Q.E.D.

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SIMILAR TRIANGLES

THEOREM 60

If two triangles have an angle of one equal to an angle of the other, and the sides about these equal angles proportional, the triangles are equiangular.

\[ \begin{align*}
\triangle ABC, XYZ, & \angle BAC = \angle XYZ \text{ and } \frac{AB}{AC} = \frac{XY}{XZ}.
\end{align*} \]

To prove \( \angle ABC = \angle XYZ \) and \( \angle ACB = \angle XZY. \)

From \( AB, AC \), cut off \( AH, AK \) equal to \( XY, XZ \). Join \( HK \).

In the \( \triangle \)s \( \triangle AHK, XYZ \),

\[ \begin{align*}
AH &= XY, & \text{constr.} \\
AK &= XZ, & \text{constr.} \\
\angle HAK &= \angle XYZ, & \text{given;} \\
\therefore & \quad \angle AHK = \angle XYZ \text{ (2 sides, inc. angle);} \\
\therefore & \quad \angle AHK = \angle XYZ \text{ and } \angle AKH = \angle XZY.
\end{align*} \]

Now \( AB = AC \) \( \frac{XY}{XZ} \) \( \text{and } XY = AH, XZ = AK; \)

\[ \begin{align*}
\therefore & \quad \angle ABC = \angle ACB. \\
\therefore & \quad AH = AK. \\
\therefore & \quad HK \text{ is parallel to } BO.
\end{align*} \]

\[ \therefore \angle AHK = \angle ABC \text{ and } \angle AKH = \angle ACB, \text{ corrres. } \angle BAC \]

But \( \angle AHK = \angle XYZ \) and \( \angle AKH = \angle XZY, \text{ proved.} \)

\[ \therefore \angle ABC = \angle XYZ \text{ and } \angle ACB = \angle XZY. \] Q.E.D.
EXERCISE XXIX. (Numerical)

1. Show that the triangle whose sides are 5-1", 6-8", 3-5" is right-angled.

2. How far in front of a pinhole camera must a man 6' high stand in order that a full-length photograph may be taken on a film 21" high, 24" from the pinhole?

3. Two triangles are equiangular; the sides of one are 5', 8', 9'; the shortest side of the other is 4 cm.; find its other sides.

4. The bases of two equiangular triangles are 4", 6"; the height of the first is 5"; find the area of the second.

5. In \( \triangle ABC \), \( AB = 8'' \), \( BC = 6'' \), \( CA = 5'' \); a line \( XY \) parallel to \( BC \) cuts \( AB \) at \( X \), \( Y \); \( AX = 2'' \); find \( XY \), \( CY \).

6. In quadrilateral \( ABCD \), \( AB \) is parallel to \( DC \) and \( AB = 8'' \), \( AD = 3'' \), \( DC = 5'' \); \( AD \), \( BC \) are produced to meet at \( P \); find \( PD \).

7. A line parallel to \( BC \) meets \( AB \), \( AC \) at \( X \), \( Y \); \( BC = 8'' \), \( XY = 5'' \); the lines \( BC \), \( XY \) are 2" apart. Find the area of \( \triangle AXY \).

8. In Fig. 162,
   (i) if \( AO = 3'' \), \( OB = 2'' \), \( AB = 4'' \), \( DG = 1'' \); find \( CO \), \( DO \);
   (ii) if \( AO = 5'' \), \( BO = 4'' \), \( AC = 7'' \), find \( BD \);
   (iii) if \( PA = 9'' \), \( PB = 8'' \), \( AB = 4'' \), \( PC = 3'' \); find \( PD \), \( CD \);
   (iv) if \( PA = 9'' \), \( PB = 8'' \), \( AC = 6'' \), \( PC = 4'' \); find \( BD \), \( PD \).

9. The diameter of the base of a cone is 9" and its height is 15"; find the diameter of a section parallel to the base and 3" from it.

10. \( \triangle AOB \) is a straight line; \( AC \), \( XY \), \( BD \) are the perpendiculars from \( A \), \( X \), \( B \) to a line \( CD \); \( AC = 10' \), \( BD = 16' \), \( AX = 12' \), \( XB = 6' \); find \( XY \).

11. \( A \), \( B \) are points on the same side of a line \( OX \) and at distances \( 1', 5' \) from it; \( Q \) and \( R \) divide \( AB \) internally and externally in the ratio 5:3; find the distances of \( Q \) and \( R \) from \( OX \).

12. A sphere of 5" radius is placed inside a conical funnel whose slant side is 12" and whose greatest diameter is 14"; find the distance of the vertex from the centre of the sphere.

SIMILAR TRIANGLES

EXERCISE XXX

1. \( \triangle AOB, \triangle COD \) are two intersecting chords of a circle; fill up the blank spaces in (i) \( \frac{OA}{AC} = \frac{BD}{EC} \); (ii) \( \frac{OA}{BD} = \frac{OC}{EC} \).

2. Two straight lines \( OAB, OCD \) cut a circle at \( A, B, C, D \); fill up the blank spaces in (i) \( \frac{AC}{OA} = \frac{BD}{OB} \); (ii) \( \frac{OA}{OC} = \frac{OB}{OC} \).

3. \( \triangle ABC \) is a \( \triangle \) inscribed in a circle; the bisector of \( \angle BAC \) cuts \( BC \) at \( Q \) and the circle at \( P \); prove \( \frac{AQ}{AP} = \frac{AB}{AC} \) and complete the equation \( \frac{BQ}{PC} = \frac{AB}{AC} \).

4. In \( \triangle ABC \), \( \angle BAC = 90^\circ \); \( AD \) is an altitude; prove that \( \frac{DC}{AC} = \frac{BD}{AD} \) and complete the equation \( \frac{CD}{AC} = \frac{BD}{AD} \).

5. The medians \( BY, CZ \) of \( \triangle ABC \) meet at \( G \); prove that \( \frac{BY}{GY} = \frac{1}{2} \).

6. \( BE, CF \) are altitudes of \( \triangle ABC \); prove that \( \frac{EF}{BC} = \frac{AF}{AG} \).

7. Prove that the common tangents of two non-intersecting circles divide (internally and externally) the line joining the centres in the ratio of the radii.

8. \( M \) is the mid-point of \( AB \); \( AXB, MYB \) are equilateral triangles on opposite sides of \( AB \); \( XY \) cuts \( AB \) at \( Z \); prove \( AZ = 2ZB \).

9. \( AB \) is a diameter of a circle \( \triangle ABP \); \( PT \) is the perpendicular from \( P \) to the tangent at \( A \); prove \( \frac{AP}{PA} = \frac{PT}{TB} \).

10. \( ABCD \) is a parallelogram; any line through \( C \) cuts \( AB \) produced, \( AD \) produced at \( X, Y \); prove \( \frac{AX}{AB} = \frac{DY}{AY} \).

11. \( AB, DC \) are the parallel sides of the trapezium \( ABCD \); any line parallel to \( AB \) cuts \( CA, CB \) at \( H, K \); \( DH, DK \) cut \( AB \) at \( X, Y \); prove \( AB = XY \).

12. \( BC, YZ \) are the bases of two similar triangles \( \triangle ABC \), \( \triangle XYZ \); \( AX, CQ \) are medians; prove \( \angle BAP = \angle YXQ \).
THEOREM 53. (Second Proof)

If two chords of a circle (produced if necessary) cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

(1) Given two chords AB, CD intersecting at O.
To prove \( OA \cdot OB = OC \cdot OD \).

Join BC, AD.

In the \( \triangle AOD, \triangle BOC \).
\[ \angle OAD = \angle OCB, \text{ in the same segment, Fig. 164 (1) and Fig. 164 (2)}. \]
\[ \angle AOD = \angle COB, \text{ vert. opp. in Fig. 164 (1), same } \angle \text{ in Fig. 164 (2)}. \]
\[ \therefore \text{ the third } \angle ODA = \text{ the third } \angle OBC. \]
\[ \therefore \text{ triangles are equiangular.} \]
\[ OA \cdot OD = OC \cdot OB. \]
\[ \therefore OA \cdot OB = OC \cdot OD. \]
Q.E.D.

THEOREM 54. (Second Proof)

If from any point outside a circle, a secant and a tangent are drawn, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.

Given a chord AB meeting the tangent at T in O.
To prove \( OA \cdot OB = OT^2 \).

Join AT, BT.

In the \( \triangle AOT, \triangle TBO \).
\[ \angle TAO = \angle TBO, \text{ alt. segment}. \]
\[ \angle AOT = \angle TBO, \text{ same angle}. \]
\[ \therefore \text{ the third } \angle AOT = \text{ the third } \angle TBO. \]
\[ \therefore \text{ the triangles are equiangular.} \]
\[ OA \cdot OT = OB \cdot OT^2 \]
\[ \therefore OA \cdot OB = OT^2. \]
Q.E.D.

Note.—This may also be deduced from Theorem 53 by taking the limiting case when D coincides with C in Fig. 164 (2).

The converse properties are as follows:
(i) If two lines AOB, COD are such that \( AO \cdot OB = CO \cdot OD \), then A, B, C, D lie on a circle.
(ii) If two lines OBA, ODC are such that \( OA \cdot OB = OC \cdot OD \), then A, B, C, D lie on a circle.
(iii) If two lines OBA, OT are such that \( OA \cdot OB = OT^2 \), then the circle through A, B, T touches OT at T.
These are proved easily by a \textit{reductio ad absurdum} method.
THEOREM 61.

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.

Given $\triangle ABC = 90^\circ$, and that $AD$ is perpendicular to $BC$.

To prove $\triangle s$ $ABC$, $DBA$, $DAC$ are similar.

In $\triangle s$ $ABC$, $DBA$,

$\angle BAC = \angle BDA$, right angles.

$\angle ABC = \angle DBA$, same angle.

$\therefore$ the third angles $ACB$, $DAB$ are equal, and the triangles are

equiangular.

In the same way it may be proved that $\triangle s$ $ABC$, $DAC$ are

equiangular.

$\therefore$ the three triangles $DBA$, $ABC$, $DAC$ are equiangular and therefore similar.

Q.E.D.

Corollary 1. The square on the perpendicular is equal to the rectangle contained by the segments of the base.

$AD^2 = BD \cdot DC$.

For, since $\triangle s$ $DBA$, $DAC$ are similar,

$DA \cdot DC$.

$DB = DA'$.

$DA^2 = DB \cdot DC$.

Mean Proportional.

If $a$, $b$, $c$ are three quantities such that $a:b = b:c$ or $\frac{a}{b} = \frac{b}{c}$

or $ac = b^2$, then $b$ is called the mean proportional between

$a$ and $c$.

CONSTRUCTION 15

Construct a mean proportional to two given lines.

Given two lines of lengths $a$, $b$ units.

To construct a line of length $x$ units such that $\frac{a}{x} = \frac{x}{b}$ or $x^2 = ab$.

METHOD I. Take a point $O$ on a line and cut off from the line on the same side of $O$, parts $OA$, $OB$ of lengths $a$, $b$ units.

On $OB$ as diameter, describe a circle.

Draw $AQ$ perpendicular to $OB$ to meet the circle at $Q$.

Join $OQ$. Then $OQ$ is the required mean proportional.

Proof. $\angle OQB = 90^\circ$ ; angle in semicircle.

$\therefore OQ$ is a tangent to the circle on $OQ$ as diameter.

But $\angle QAB = 90^\circ$, $\therefore$ circle on $QB$ as diameter passes through $A$

$\therefore OQ^2 = OA \cdot OB$, tangent property of circle.

$\therefore OQ^2 = a \cdot b$ or $\frac{a}{OQ} = \frac{OQ}{b}$. 
CONSTRUCTION 15

Construct a mean proportional to two given lines.

Given two lines of lengths \(a\), \(b\) units.

To construct a line of length \(x\) units such that \(\frac{a}{x} = \frac{x}{b}\) or \(x^2 = ab\).

\[ \frac{a}{x} = \frac{x}{b} \]

\[ \text{FIG. 168.} \]

METHOD II. Take a point \(O\) on a line and cut off from the line on opposite sides of \(O\), parts \(OA\), \(OB\) of lengths \(a\), \(b\) units.

On \(AB\) as diameter, describe a circle.

Draw \(OP\) perpendicular to \(AB\) to cut the circle at \(P\). Then \(OP\) is the required mean proportional.

**Proof.** Produce \(PO\) to meet the circle at \(Q\).

\(PQ\) is a chord perpendicular to the diameter \(AB\).

\[ \therefore \text{PO} = \text{OQ} \]

But \(PO \cdot OQ = AO \cdot OB\), intersecting chords of a circle.

\[ \therefore OP^2 = a \cdot b, \]

or \(\frac{a}{OP} = \frac{OP}{b} \).

Q.E.F.

**Note.**—This should be compared with Constr. 12, p. 112.

In practical constructions Method I is often preferable to Method II.

These constructions may also be proved by quoting the Corollaries of Theorem 61.

## Exercise XXXI. (Numerical)

**Note.**—For additional examples and riders on the rectangle properties of a circle, see pp. 113, 114.

1. Find a mean proportional between (i) 3 and 48; (ii) 12x, 3xy^2.

2. From a point \(P\) on a circle, \(PN\) is drawn perpendicular to a diameter \(AB\); \(AN = x\), \(NE = y\); find \(PN\).

3. In \(\triangle ABC\), \(\angle BAC = 90°\); \(AD\) is an altitude; \(AB = 5\), \(AC = 12\); find \(BD\).

4. Construct a mean proportional between 5 and 8; measure it.

5. Draw a rectangle of sides 4 cm., 7 cm., and construct a square of equal area; measure its side.

6. Construct a square equal in area to an equilateral triangle of side 5 cm.; measure its side.

7. In \(\triangle ABC\), \(AB = 8\), \(AC = 12\); a circle through \(B\), \(C\) cuts \(AB\), \(AC\) at \(P\), \(Q\); \(BP = 5\); find \(CQ\).

8. The diagonals of a cyclic quadrilateral \(ABCD\) meet at \(O\); \(AC = 9\), \(BD = 15\), \(OA = 4\); find \(OB\).

9. In Fig. 169.

(i) If \(AB = 9\), \(BO = 3\), find \(OT\).

(ii) If \(OB = 6\), \(OT = 12\), find \(AB\).

(iii) If \(OA = 3\), \(AB = 2\), \(AT = 4\), find \(BT\).

(iv) If \(AB = 8\), \(AT = 6\), \(BT = 5\), find \(OT\).

10. \(ABC\) is a triangle inscribed in a circle; \(AB = AC = 10\), \(BC = 12\); \(AD\) is drawn perpendicular to \(BC\) and is produced to meet the circle in \(E\); find \(DE\) and the radius of the circle.

11. In \(\triangle ABC\), \(\angle ABC = 90°\), \(AB = 3\), \(BC = 4\); find the radius of the circle which passes through \(A\) and touches \(BC\) at \(C\).

12. Construct a square equal in area to a quadrilateral \(ABCD\) given \(AB = BC = 4\), \(CD = 6\), \(DA = 7\), \(AC = 10\) cm.; measure its side.

13. Draw a line \(AB\); construct a point \(P\) on \(AB\) such that \(AP^2 = \frac{3}{2}AB^2\).

14. \(ABC\) is a \(\triangle\) inscribed in a circle; the tangent at \(C\) meets \(AB\) produced in \(D\); \(BC = p\), \(CA = q\), \(AB = r\), \(BD = x\), \(CD = y\); find \(x, y\) in terms of \(p, q, r\).
EXERCISE XXXII

1. The diagonals of a cyclic quadrilateral $ABCD$ intersect at $O$; prove $AD \cdot OC = BC \cdot OD$.

2. Two lines $OAB$, $OCD$ cut a circle at $A$, $B$, $C$, $D$; prove $OA \cdot BC = OC \cdot AD$.

3. Two chords $AB$, $CD$ of a circle intersect at $O$; if $D$ is the midpoint of arc $AB$, prove $CA \cdot CB = CO \cdot CD$.

4. In $\triangle ABC$, $AB = AC$; $D$ is a point on $AC$ such that $BD = BC$; prove $BC^2 = AC \cdot CD$.

5. $ABCD$ is a cyclic quadrilateral; $P$ is a point on $BD$ such that $\angle PAD = \angle BAC$; prove that (i) $BC \cdot AD = AC \cdot DP$; (ii) $AB \cdot CD = AC \cdot BD$; (iii) $BC \cdot AB + AD \cdot CD = AC \cdot BD$.

6. $AB$ is a diameter of a circle, centre $O$; $AP$, $PQ$ are equal chords; prove $AP \cdot PB = AQ \cdot PO$.

7. $AD$ is an altitude of $\triangle ABC$; prove that the radius of the circle $ACB$ equals $\frac{AB \cdot AC}{2AD}$. (Draw diameter through $A$.)

8. A line $PQ$ is divided at $T$ so that $PR^2 = PQ \cdot QT$; $TQR$ is a $\triangle$ such that $TQ = TR = PR$; prove $PT = PQ$.

9. $PQR$ is a $\triangle$ inscribed in a circle; the tangent at $P$ meets $QR$ produced at $T$; prove $\frac{TQ}{PQ^2} = \frac{PR}{TR}$.

10. In $\triangle ABC$, $\angle BAC = 90^\circ$; $E$ is a point on $BC$ such that $AE = AB$; prove $BE \cdot BC = 2AE^2$.

11. $AD$ is an altitude of $\triangle ABC$; if $AB \cdot BC = AC^2$ and if $AB = CD$, prove $\angle BAC = 90^\circ$.

12. The tangent at a point $C$ on a circle is parallel to a chord $DE$ and cuts two other chords $DE, PE$ at $A, B$; prove $\frac{AC}{AD} = \frac{CB}{BE}$.

13. $AB$ is a diameter of a circle, centre $O$; the tangents at $A, B$ meet any other tangent at $H, K$; prove $AH \cdot BK = AO^2$.

14. $ABC$ is a $\triangle$ inscribed in a circle; a line through $B$ parallel to $AC$ cuts the tangent at $A$ in $P$; a line through $C$ parallel to $AB$ cuts $AP$ in $Q$; prove $\frac{AP}{AQ} = \frac{AC}{AB}$.

[For additional riders, see p. 114.]

AREAS OF SIMILAR FIGURES

THEOREM 62

The ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding sides.

Given the triangles $ABC$, $XYZ$ are similar.

To prove \[\frac{\triangle ABC}{\triangle XYZ} = \frac{BC^2}{YZ^2} \]

Draw the altitudes $AH$, $XK$.

In the $\triangle$ $AHB$, $XKY$,

\[\frac{\triangle AHB}{\triangle XKY} \text{, given.} \]

\[\frac{\triangle AHB}{\triangle XKY} \text{, given.} \]

\[\frac{\triangle AHB}{\triangle XKY}, \text{ rt. as constr.} \]

\[\text{the third } \angle BAH = \text{the third } \angle XKY.\]

\[\therefore \frac{\triangle AHB}{\triangle XHY}, \text{ the } \triangle \text{ as constr.} \]

But $\frac{AB}{BC}$, since $\triangle ABO$, $XYZ$ are similar.

\[\frac{AX}{XY} = \frac{BY}{YZ} \]

But $\triangle ABC = \frac{1}{2} \triangle ABC \cdot BC$ and $\triangle XYZ = \frac{1}{2} \triangle XYZ \cdot YZ$.

\[\frac{\triangle ABC}{\triangle XYZ} = \frac{AH}{XK}, \frac{BC}{YZ}\]

But $\frac{\triangle AHB}{\triangle XKY} \text{, proved.} \]

\[\frac{\triangle ABC}{\triangle BCA} = \frac{BC}{YZ}\]

Q.E.D.
Areas and Volumes.

If two polygons are similar, it can be proved that they can be divided up into the same number of similar triangles.

Hence it follows that the ratio of the areas of two similar polygons is equal to the ratio of the squares on corresponding sides.

The following facts are also of importance:

(i) The ratio of the areas of the surfaces of similar solids equals the ratio of the squares of their linear dimensions.

(ii) The ratio of the volumes of similar solids equals the ratio of the cubes of their linear dimensions.

EXERCISE XXXIII

1. A screen, 6′ high (not necessarily rectangular), requires 27 sq. ft. of material for covering; how much is needed for a screen of the same shape, 4′ high?

2. On a map whose scale is 6′ to the mile, a plot of ground is represented by a triangle of area 2½ sq. inches; what is the area (in acres) of the plot?

3. The sides of a triangle are 6 cm., 9 cm., 12 cm.; how many triangles whose sides are 2 cm., 3 cm., 4 cm. can be cut out of it? How would you cut it up?

4. ABC, XYZ are similar triangles; AD, XK are altitudes; AB = 15, BC = 14, CA = 13, AD = 12, XY = 5; find XK and the ratio of the areas of △s ABC, XYZ.

5. A triangle ABC is divided by a line HK parallel to BC into two parts AHK, HKCB of areas 9 sq. cm., 16 sq. cm.; BC = 7 cm.; find HK.

6. ABC is a △ such that AB = AC = 2BC; D is a point on AC such that △ BDC = △ BAC; a line through D parallel to BC cuts AB in E; find the ratio of the areas △ ABC : △ BCD : △ BED : △ EDA.

7. If it costs $3 to gild a sphere of radius 3 ft., what will it cost to gild a sphere of radius 4 ft.?

8. Two hot-water cans are the same shape; the smaller is 9′ high and holds a quart; the larger is 15′ high; how much will it hold?

9. A metal sphere, radius 3′, weighs 8 lb.; find the weight of a sphere of the same metal 1′ in radius.

10. A cylindrical tin 5′ high holds 1 lb. of tobacco; how much will a tin of the same shape 8′ high hold?

11. Two models of the same statue are made of the same material; one is 3′ high and weighs 8 oz.; the other weighs 4 lb.; what is its height?

12. A lodger pays 8 pence for a scuttle of coal, the scuttle being 20′ deep; what would he pay if the scuttle was the same shape and 2½ feet deep?

13. A tap can fill half of a spherical vessel, radius 1¼ feet, in 2 minutes; how long will two similar taps take to fill one-quarter of a spherical vessel of radius 4 feet?

14. The sides of a △ABC are trisected as in the figure; prove that the area of PQRSXY = ²₃ △ ABC.

15. If in the △s ABC, XYZ, △BAC = △ YXZ, prove that △ABC : △ YXZ : △ ABC.

16. Two lines OAB, OCD meet a circle at A, B, C, D; prove that △ OAD : △ OBD. What result is obtained by making B coincide with A?

17. In △ ABC, △BAC = 90° and AD is an altitude; prove △ABD = △BDC.

18. ABCD is a parallelogram; P, Q are the mid-points of CB, CD; prove △ APQ = ½ parallelogram ABCD.

19. In △ ABC, △BAC = 90° and AD is an altitude; DE is the perpendicular from D to AB; prove BE = BA² / BC².

20. In △ ABC, △BAC = 90°; BCX, CAY, ABZ are similar triangles with X, Y, Z corresponding points; prove △ CAY + △ ABZ = △ BCX.
CONSTRUCTION 16

To construct a pentagon similar to a given pentagon and such that corresponding sides are in a given ratio.

Given a pentagon OABCQ and a ratio XY : XZ.

To construct a pentagon O'A'B'C'D' such that

\[
\frac{OA'}{A'B'} = \frac{XY}{XY'} \quad \text{and} \quad \frac{OA'}{AB} = \frac{XZ}{XZ'}
\]

Join OB, OC, 

Draw any line OQ and cut off parts OP, OP equal to XY, XZ.

Join PA.

Through P' draw P'A' parallel to PA to meet OA at A'.
Through A' draw A'B' parallel to AB to meet OB at B'.
Through B' draw B'C' parallel to BC to meet OC at C'.
Through C' draw C'D' parallel to CD to meet OD at D'.

Then O'A'B'C'D' is the required pentagon.

Proof. Since A'B' is parallel to AB, \(\triangle OAB' = \triangle OAB\).

Similarly \(\triangle O'B'C' = \triangle OBC\), and so on.

\[
\frac{OA}{AB} = \frac{BC}{CD} = \frac{DO}{DO'}
\]

EXERCISE XXXIV

1. Given a triangle ABC, construct a point P on BC such that the lengths of the perpendiculars from P to AB and AC are in the ratio 2 : 3.

2. ABC is an equilateral triangle of side 5 cm., construct a point P inside it such that the perpendiculars from P to BC, CA, AB are in the ratio 1 : 2 : 3. Measure AP.

3. Draw any triangle ABC, use the method indicated in Fig. 173 to construct a triangle XYZ similar to triangle ABC and such that XY = 2AB.

4. Given a quadrilateral ABCD, construct a similar quadrilateral each side of which is \(\frac{1}{2}\) of the corresponding side of ABCD.

5. Construct an equilateral triangle such that the length of the line joining one vertex to a point of trisection of the opposite side is 2"; measure its side.

6. Construct a square ABCD, given that the length of the line joining A to the mid-point of BC is 3"; measure its side.

[Fig. 174, where AF = FB, \(\angle AFG = 90^\circ\), AP = AG.]

7. Given a triangle ABC, construct a line PQ parallel to BC such that it bisects \(\triangle ABC\). Prove your method.

D.S.G.
8. Given a quadrilateral ABCD, construct a similar quadrilateral with its area $\frac{3}{4}$ of the area of ABCD.

9. Given a triangle ABC, construct an equilateral triangle of equal area.

![Diagram of triangle ABC and equilateral triangle]

10. Construct a circle to pass through two given points A, B and touch a given line CD. Prove your method.

[See Fig. 176, where $\angle OAQ = 90^\circ$, $OQ = OG = OP$.]

11. Construct a triangle ABC, given $\angle BAC = 48^\circ$, $\angle BCA = 73^\circ$, and the median BE = 5 cm.; measure AC.

12. Draw a triangle of sides 5, 6, 7 cms. and construct a square of equal area; measure its side. Check your result from the formula $\sqrt{(s-a)(s-b)(s-c)}$.

13. Given two equilateral triangles, construct an equilateral triangle whose area is the sum of their areas.

![Diagram of two equilateral triangles and constructed equilateral triangle]

14. Construct a circle to pass through two given points A, B and touch a given line CD.

Use the method indicated in Fig. 176 and obtain two solutions.

15. Construct a circle to pass through two given points A, B and to touch a given circle.

---

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16. A line AB, 8 cm. long, is divided internally and externally in the ratio 3:1 at P, Q respectively; find PQ : AB.

2. A chord AB of a circle ABT is produced to O; OT is a tangent; OA = 6", OT = 4", AT = 3"; find BT.

3. Fig. 177 represents parts of two circles which touch at a point A on CB produced; the lines CB and QP when produced intersect at right angles at the centre of the larger circle. Given PQ = 3", BC = 5", calculate the radius of each circle.

4. ABC is a triangle inscribed in a circle; AD is an altitude; AP is a diameter; prove $\angle ABD = \angle ACD$ and complete the equation $AB - AP$.

---

17. In Fig. 178, AB, CD, EF are parallel; AD = 7", DF = 3", CE = 4"; find BG. If EF = 2", AB = 3", find CD.

2. ABC is a triangle inscribed in a circle; the tangents at B, C meet at T; a line through T parallel to the tangent at A meets AB, AC produced at D, E; prove DT = TE.

3. AB, DC are parallel sides of the trapezium ABCD; AG cuts DB at O; the line through O parallel to AB cuts AD, BC at P, Q; prove PO = OQ.

4. Two circles intersect at A, B; the tangents at A meet the circles at C, D; prove $BC = \frac{BA}{BD}$.
18. Two triangles are equiangular; the sides of one are 3 cm., 5 cm., 7 cm.; the perimeter of the other is 2½ feet; find its sides.

2. AB is a diameter of a circle APB; the tangent at A meets BP at Q; prove that the tangent at P bisects AQ.

3. ABCD is a parallelogram; a line through A cuts BD, BC, CD at E, F, G; prove $AE \parallel AG$.

4. Two lines OAB, OCG meet a circle at A, B, C, D; prove that $OA \cdot AD = OB \cdot CD$.

19. In Fig. 179, if $\angle ADC = \angle BEA = \angle CFB$, prove that the triangles ABC, XYZ are equiangular.

2. A light is placed 4' in front of a circular hole 3' in diameter in a partition; find the diameter of the illuminated part of a wall 6' behind the partition and parallel to it.

3. ABC is a triangle inscribed in a circle; $AB = AC$; AP is a chord cutting BC at Q; prove $AP \perp AQ = AB'$.

4. XAY is a diameter of a circle, centre A; Z is the middle point of $AY$. If a circle is drawn on $XZ$ as diameter, prove that the length of the tangent to this circle from any point $P$ on the outer circle is equal to $PX$.

20. Show that the triangle whose vertices are (2, 1), (5, 1), (4, 2) is similar to the triangle whose vertices are (1, 1), (7, 1), (6, 3).

2. In $\triangle ABC$, $AB = AC$ and $\angle BAC = 120^\circ$; the perpendicular bisector of $AB$ cuts $BC$ at $X$; prove $BC = 3BX$.

3. AOB, COD are two perpendicular chords of a circle; prove that $AC + \text{are BD equals half the circumference}$.

4. Two lines OAB, OCG cut a circle at A, B, C, D; H, K are points on OB, OD such that $OH = OC, OK = OA$; prove that HK is parallel to BD.

APPENDIX

ALTERNATIVE PROOFS AND ADDITIONAL CONSTRUCTIONS

Parallelograms on equal bases and between the same parallels are equal in area.

Given ABCD, XYZW are parallelograms on equal bases AB, XY and between the same parallels AD, XZ.

To prove area ABCD = area XYZW. Join AW, BZ.

X is equal and parallel to WZ, opp. sides = gram.

But $AB = XY$, given.

.: $AB$ is equal and parallel to $XZ$.

.: $ABZW$ is a parallelogram.

.: area $ABCD$ = area $ABZW$, $\text{grams on same base AB and between the same parallels AB, DCWZ}$.

Similarly area $WZBA = \text{area WXYZ}$, base WZ, parallels WZ, XZ.

.: area $ABCD = \text{area WXYZ}$. Q.E.D.

Triangles on equal bases (or the same base) and between the same parallels are equal in area.

Given ABC, XYZ are triangles on equal bases AB, XY and between the same parallels ABXY, Oz.

To prove area ABC = area XYZ.

Complete the parallelograms ABCD, XYZW.

$\triangle ABC$ is half $\text{gram ABCD}$ and $\triangle XYZ$ is half $\text{gram XYZW}$, since a diagonal bisects the $\text{gram}$.

But area $ABCD = \text{area XYZW}$, $\text{grams on equal bases and between the same parallels}$. .: $\triangle ABC = \triangle XYZ$. Q.E.D.
Triangles of the same area on the same base and on the same side of it are between the same parallels.

Given $ABC$, $ABD$ are triangles of the same area and on the same side of the same base $AB$.

To prove $CD$ is parallel to $AB$.

Draw $CQ$ parallel to $AB$ to meet $AD$, produced if necessary, at $Q$; join $BQ$.

Area $ABC = \text{area } ABQ$, $\triangle$s on same base and between same parallels.

But area $ABC = \text{area } ABD$, given.

$\therefore$ area $ABQ = \text{area } ABD$,

which is impossible unless $Q$ coincides with $D$.

But $CQ$ is drawn parallel to $AB$,

$\therefore CD$ is parallel to $AB$. Q.E.D.

Corollary. Triangles of the same area on equal bases in the same straight line and on the same side of it are between the same parallels.

If the triangles are $ABC$, $XYZ$, see Fig. 181, draw $CQ$ parallel to $AY$ to meet $XZ$, or $XZ$ produced, at $Q$; join $QY$ and proceed as before.

If two triangles have equal altitudes, the ratio of their areas is equal to the ratio of their bases.

Given $BC$, $YZ$ are the bases of two triangles $ABC$, $XYZ$ of equal altitude.

To prove $\triangle ABC = \frac{BC}{\triangle XYZ} \cdot \frac{YZ}{YZ}$.

Express $\frac{BC}{YZ}$ as a fraction $\frac{p}{q}$ where $p$ and $q$ are integers.

(This assumes that $BC$ and $YZ$ are commensurable.)

Divide $BC$ into $p$ equal parts and $YZ$ into $q$ equal parts; then each part of $BC$ is equal to each part of $YZ$.

Join $A$ to each point of division of $BC$, and join $X$ to each point of division of $YZ$.

Then $\triangle ABC$ is divided into $p$ equal triangles, since their bases are equal and they have the same height.

For the same reason, $\triangle XYZ$ is divided into $q$ equal triangles.

But each of the $p$ triangles which make up $\triangle ABC$ is equal to each of the $q$ triangles which make up $\triangle XYZ$, since their bases and altitudes are equal.

$\therefore \triangle ABC = \frac{p}{q}$,

$\triangle ABC = \frac{BC}{BC}$,

$\triangle XYZ = \frac{YZ}{YZ}$.

Q.E.D.
(1) If a straight line is drawn parallel to one side of a triangle, it divides the other sides (produced if necessary) proportionally.

(2) If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

(1) Given a line parallel to BC cuts AB, AC (produced if necessary) at H, K.

To prove \( \frac{AH}{HB} = \frac{AK}{KC} \).

Join BH, CH.

The triangles KHA, KHB have a common altitude from K to AB.

\[
\triangle KHA \sim \triangle KHB
\]

The triangles HKA, HKC have a common altitude from H to AC.

\[
\triangle HKA \sim \triangle HKC
\]

But \( \triangle KHB, \triangle KHC \) are equal in area, being on the same base HK and between the same parallels HK, BC.

\[
\frac{AH}{HB} = \frac{AK}{KC}.
\]

Q.E.D.

(2) Given a line HK cutting AB, AC at H, K such that

\[
\frac{AH}{HB} = \frac{AK}{KC}.
\]

To prove HK is parallel to BC.

The triangles KHA, KHB have a common altitude from K to AB.

\[
\triangle KHA \sim \triangle KHB
\]

The triangles HKA, HKC have a common altitude from H to AC.

\[
\triangle HKA \sim \triangle HKC
\]

But

\[
\frac{AH}{HB} = \frac{AK}{KC} \text{ given.}
\]

\[
\frac{\triangle KHA}{\triangle HKB} = \frac{\triangle HKA}{\triangle HKC}.
\]

\[
\triangle KHB = \triangle HKC.
\]

But these triangles are on the same base HK and on the same side of it.

\[
\therefore \text{HK is parallel to BC.}
\]

Q.E.D.
CONSTRUCTION 17

To divide a quadrilateral ABCD into five equal parts by lines through the vertex A.

Join AC; through D draw DP parallel to AC to meet BC produced at P. Divide BP into five equal parts, BQ₁, Q₂Q₃, Q₄Q₅, Q₅P. Through the points of division which lie in BC produced, here Q₂ and Q₃, draw Q₂Q₃. Q₃Q₄ parallel to PD to meet CD in R₃, R₄.

Then AQ₁, AQ₂, AR₃, AR₄ are the required lines.

By Constr. 7, p. 48, quad. ABCD = ½ ABP.

But the areas of \( \triangle BQ₁A \), \( \triangle Q₂AQ₃ \), \( \triangle Q₄AQ₅ \), \( \triangle Q₅AP \) are equal, since their bases are equal; therefore each is \( \frac{1}{5} \triangle ABP = \frac{2}{5} \) quad. ABCD.

Further, \( \triangle ACQ₃ = \triangle ACR₃ \), \( \triangle ACQ₄ = \triangle ACR₄ \), \( \triangle ACP = \triangle ACD \), being on the same base and between the same parallels.

\( \therefore \) quad. AQ₂CR₃ = \( \triangle AQ₂Q₃ \) + \( \triangle ACR₃ \) = \( \triangle AQ₂Q₃ \) = \( \frac{1}{5} \) quad. ABCD.

Similarly quad. AQ₂CR₄ = \( \triangle AQ₂Q₄ \) = \( \frac{1}{5} \) quad. ABCD, and quad. AQ₂CD = \( \frac{1}{5} \) quad. ABCD; therefore AQ₁, AQ₂, AR₃, AR₄ are the required lines.

Note.—The same method may be used to divide any polygon into any number of equal parts by lines through a vertex.

CONSTRUCTION 18

To divide a given line AB at a point X so that

\[ AB : BX = AX^2. \]

Draw BC perpendicular to AB and equal to \( \frac{1}{2} AB. \)

Join CA. From CA cut off CP equal to CB.

From AB cut off AX equal to AP.

Let \( AB = b \) units and \( AX = x \) units.

Then \( CP = CB = \frac{1}{2} b \) units and \( AP = AX = x \) units.

\( \therefore \) \[ b^3 = AB^3 = AC^3 = CE^3 = PC^3 \]

\[ = (AC - PC)(AC + PC) = AP^2 \cdot (AP^2 + 2PC) \]

\[ = x(2 + b) - x^3 + bx. \]

\[ \therefore b^2 - bx = x^2. \]

\[ \therefore b(b - x) = x^2. \]

\[ \therefore AB : BX = AX^2. \]

\[ \therefore X \] is the required point of division.

Definition. A line AB is said to be divided in medial section at X, if \( AB : BX = AX^2. \)

With the same notation as before, we have

\[ AC^3 = b^3 + \frac{1}{2} b^3 = \frac{5b^3}{4} \]

\[ \therefore AC = \frac{b}{5}. \]

\[ \therefore AX = AP = AC - PC = \frac{1}{2} b\sqrt{5} - \frac{1}{2} b = \frac{1}{4} b(\sqrt{5} - 1). \]

\[ \therefore AX = \frac{1}{4}(\sqrt{5} - 1). \]
CONSTRUCTION 19

To construct a triangle ABC such that \( \angle B = \angle C = 2 \angle A \), given the length of AB.

With centre A, radius AB, describe a circle. Divide AB at P, so that \( AB \cdot BP = AP^2 \), and place a chord BC in the circle, so that \( BC = AP \). Join AC.

Then ABC is the required triangle.

\[ AB, BP = AP^2. \quad \text{But} \quad AP = BC. \quad \therefore \quad BA, BP = BC^2. \]

\[ \therefore \quad \triangle BCP, BAC \text{ are equiangular.} \]

But \( AB = AC. \quad \therefore \quad CB = CP. \)

And \( CB = AP. \quad \therefore \quad CP = AP. \quad \therefore \quad \triangle PAC = \triangle PCA. \)

\[ \therefore \quad \angle BCA = \angle BCP + \angle PCA = 2 \angle PAC = 2 \angle BAC; \quad \text{and} \quad \angle CBA = \angle BCA = 2 \angle BAC. \]

CONSTRUCTION 20

To inscribe a regular pentagon in a given circle.

Make the same construction as in Constr. 19; also draw CH perpendicular to AB and produce it to meet the circle at D.

Then CD is a side of a regular pentagon inscribed in the circle, centre A, radius AB.

Since \( \angle ABC = \angle ACB = 2 \angle BAC, \quad \angle BAC = 36^\circ. \)

\[ \therefore \quad \angle DAC = 2 \angle BAC = 72^\circ = \frac{1}{2} \text{ of } 360^\circ. \]

\[ \therefore \quad DC \text{ is a side of a regular pentagon inscribed in circle } CBD. \]
REVISION PAPERS 1-5. (pp. 41, 42.)

1. 1. 36°.
2. 2. 1. 108°.
4. 1. x = 540° - a - b - c.
5. 1. z = 180° - a - b - x - y.

EXERCISE IX. (p. 48.)

1. 17-5 sq. in.; 10 cm.; 4-8 in. 2. 15, 15, 5 sq. cm.; \( \frac{2}{3} \); \( \Delta \) PAC.
3. 20 sq. cm.; 4 in.; 4-8 in.
4. 20, 17-5, 7-5 sq. in.; \( \frac{3}{4} \), \( \frac{4}{8} \), \( \frac{1}{2} \) in.; 45 sq. in.
5. 6-75 sq. in.
6. 10-5 sq. in. 7. 4-5, 4 cm. 8. 4-8 in.
9. 15 sq. in.
10. 10; 11. 11. 3-3, 6-4 ac. 12. 15-9 sq. cm.

EXERCISE XI. (p. 54.)

1. 13, \( \frac{5}{3} \) in. 2. 8, 3-6 in. 3. 32-25 sq. in. 4. \( \frac{9}{10} \) in.
5. 4-77 in. 6. 9-47. 7. 5 in.
8. 8-9 in. 9. 13 in. 10. 3-5 in.
11. 9-16 in. 12. 2-6 ft. 13. 6-24 in.

EXERCISE XII. (p. 55.)

13. Each face, 60 sq. in.; 11-7 in. 14. 7-34 in.

REVISION PAPERS 6-10. (pp. 61, 62.)

6. 1. 36°. 3. 3-9 sq. in. 7. 1. 13 in.
8. 1. 12; 8-66. 9. 1. 9 in. 10. 1. 2 in.

EXERCISE XV. (p. 67.)

1. 9-16 cm. 2. 13 cm. 3. 11-5 cm.
4. \( \frac{7}{10} \) in. 5. 8-58, 0-58 cm. 6. 5-36 in.
7. 5 in. 8. 4 in. 9. 8 in.

EXERCISE XVI. (p. 72.)

1. 40°. 2. 85°. 3. 110°. 4. 37°. 5. 100°; 110°.
6. 54°; 99°. 7. 105°. 8. 72°. 9. 124°. 10. 38°.

EXERCISE XIX. (p. 81.)

1. 23-1 in.; 50-3 sq. in.; 628 yd., 31420 sq. yd. 2. 0-8 in.
3. 1-1 cm. 4. 2-1 in. 5. 5-89 sq. cm. 6. 4-57 sq. in.
7. 57° 18'. 8. 3-2 cm. 9. 628 cu. in.; 408 sq. in.
10. 25. 11. 314 cu. in.; 264 sq. in.
12. 48 cu. in.; 96 sq. in. 13. 68-4 cu. cm.; 78-5 sq. cm.
14. 77-4 sq. cm. 15. \( \frac{5}{3} \) em. 16. 288°.

EXERCISE XX. (p. 82.)

1. 30°, 45°, 105° or 15°, 30°, 135°.
2. \( \frac{3}{8} \), \( \frac{2}{3} \), \( \frac{5}{10} \), \( \frac{22}{10} \), 30°, 127° \( \frac{1}{2} \).

EXERCISE XXI. (p. 86.)

1. 62°. 2. 117°. 3. 26°, 8°. 4. 88°, 64°. 5. 94°, 8°.

EXERCISE XXII. (p. 91.)

1. 2-5, 1-5, 4-5 in. 2. 8, 4, 3 in. 3. 12 cm.
4. 5-3, 3-6, 4-8 cm. 5. 10-5, 1-5 in. 6. 1\( \frac{1}{2} \) in.
7. 32, 8 cm. 8. 3 cm. 9. 1-5, 2-5 in.
10. 9-5, 2-5 in. 11. 19-1, 15 cm.

EXERCISE XXIII. (p. 94.)

3. 6-65 cm. 5. 4-47 cm. 6. 2-66 cm. 7. 1-56 in.
15. 0-64, 1-16, 1-93, 5-80 cm. 16. 5-80 cm. 17. 8-13 cm.

REVISION PAPERS 11-15. (pp. 97, 98.)

11. 2. 3\( \frac{1}{2} \) cm. 13. 3. 15°. 14. 3. 5-66, 8-486 cm.
15. 1. 51°.

EXERCISE XXIV. (p. 106.)

1. (i), (ii), (iv). 2. 2-19 in. 3. 1\( \frac{1}{2} \); 2-67. 4. 5-85; 6-84.
5. 11; 1; 6-63. 6. 42-45 sq. in. 7. 6-35 in. 8. 12-2 cm.
10. Yes. 13. 3-5. 14. 5-45, 6-52, 7-97 cm.
15. 9-17 cm. 16. 10. 17. 12-7.

EXERCISE XXV. (p. 113.)

1. 4; 10; 12. 2. 24 sq. cm.; 4 cm. 3. 8 in. 4. 3-2 in.
5. 1. 6. 13; 6; 7\( \frac{1}{2} \). 7. 56 sq. cm. 8. 10\( \frac{1}{2} \), 8\( \frac{1}{2} \) cm.
9. 20 ft. 10. 6-325 cm. 11. 1-97 in. 13. 7-24, 2-76 cm.
14. 6-6 cm. 15. 3-54, 6-62 in.

EXERCISE XXVI. (p. 117.)

4. 6; \( \frac{\sqrt{5}}{\alpha} \). 5. \( \frac{3}{12} \); \( \sqrt{2} \). 6. 10; abs. 7. 3-2 in.
8. 6 in. 10. 2; 5; 1: 2. 11. 1-6 in.
EXERCISE XXVII. (p. 122.)
1. 3 cm. 2. 5 cm. 3. $\frac{3}{4}$ in. 4. $12\frac{1}{2}$. 5. $4\frac{1}{2}$.
6. 1-6 in. 7. 7-5 cm. 8. 7-2 cm. 13. 3, 15 cm.
14. 3-35 in. 16. 12. 17. $\frac{9}{2}$ sq. in. 18. 3 sq. in.

EXERCISE XXIX. (p. 128.)
2. 6 ft. 8 in. 3. 6-4, 7-2 cm. 4. 22$\frac{1}{2}$ sq. in.
5. 1-5, 3$\frac{1}{3}$ in. 6. 5 in. 7. 8$\frac{1}{2}$ sq. in.
8. (i) $\frac{1}{4}$ in.; (ii) $1\frac{1}{2}$ in.; (iii) $2\frac{3}{4}$, 1$\frac{1}{2}$ in.; (iv) $5\frac{1}{2}$, $3\frac{1}{2}$ in.
9. 7-2 in. 10. 14. 11. $3\frac{1}{2}$, 11 in. 12. 8$\frac{1}{2}$ in.

EXERCISE XXXI. (p. 135.)
1. 12 ; 0 ey. 2. 6 in. 3. $\frac{2}{3}$ in. 4. 6, 325.
5. 5-29 cm. 6. 3-29 cm. 7. 10. 8. 2 or 10.
9. (i) 6 ; (ii) 18 ; (iii) 2, 31 ; (iv) $21\frac{1}{2}$.
10. $4\frac{1}{2}$, $6\frac{1}{2}$ in.
11. $4\frac{1}{2}$ in. 12. 5, 00 cm. 14. $p^2r^2 \ , \ \frac{1}{pq} \ , \ \frac{1}{qr} \ , \ \frac{1}{pr}$

EXERCISE XXXIII. (p. 138.)
1. 1-2 sq. ft. 2. 40 ac. 4. 4 : 9. 5. 4-2 cm.
6. 16 : 4 : 3 : 9. 7. $5\frac{1}{2}$. 8. $4\frac{1}{2}$ qts. 9. 512 lb.
10. 1-024 lb. 11. 6 in. 12. 2s. 3d. 13. $9\frac{3}{4}$ min.

EXERCISE XXXIV. (p. 141.)
2. 3-63 cm. 5. 5-27 in. 6. 2-68 in. 11. 5-78 cm. 12. 3-83 cm.

REVISION PAPERS 16-20. (pp. 143, 144.)
16. 1. $\frac{1}{2}$. 2. 2 in. 3. $6\frac{1}{2}$, 9 in. 17. 1. $9\frac{1}{4}$, $\frac{1}{4}$ in.
18. 1. 6, 10, 14 in. 19. 2. $6\frac{1}{2}$ in.