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PREFACE

The subject matter of this book has been selected and arranged to meet the needs of junior students. While it is thus particularly suited to the requirements of central and preparatory schools, it will also be found to meet the needs of lower forms in secondary schools and of students in various types of technical schools.

It is divided into three parts, each of which forms approximately a year's course:

PART I: CH. I-VI. GEOMETRICAL IDEAS

Use of instruments: length, angles at a point, perpendiculars and parallels, scale-diagrams.

PART II: CH. VII-XII. IMPORTANT GEOMETRICAL PROPERTIES

Congruence and similarity; angle properties of triangles and circles, areas of parallelograms and similar figures, etc.; Pythagoras theorem.

PART III: CH. XIII-XVIII. FURTHER DEVELOPMENTS

Applications of congruence; intercepts, ratios and similar figures; tangent properties; numerical trigonometry.

By this method of arrangement, those who are limited to a 2-year course will acquire an extensive practical working knowledge of geometry. The 3-year course covers informally the range of plane geometry up to matriculation; it will naturally be followed by a short consolidating review —organised formal geometry—before the matriculation or other first school examination is taken.

The 2-year course corresponds to the standard of the "common entrance" examination, and the 3-year course to the syllabus for entrance scholarships at public schools.

The teaching of geometry in its earliest stages should centre round the use of instruments which give concrete

1 E.g. Durell's Shorter Geometry.
expression to the new and abstract ideas encountered. Instruments should be handed out one by one, not all at once. For example, the beginner should learn the proper way of using a ruler to join two points or to measure the length of a given line before he begins to handle a compass. The object of Part I is to secure a clear idea of the uses that can be made of each instrument and a practical knowledge of the correct manner of handling it. A high degree of technical facility is not of course sought for at this stage, but the foundations of good habits can and should be laid.

An appreciation of the object of the various instruments carries with it a grasp of the fundamental geometrical ideas about length, angles at a point and parallels. Part I is thus a complete commentary on the contents of a box of instruments, and its scope is defined precisely by that theme.

The value of the work done in the class-room is greatly increased if accompanied by some out-door observations and measurements; suggestions for such work are given in a short appendix to Part I.

Part II contains a systematic development of geometrical properties on informal lines. The key-theorems connected with angles of a triangle, congruence and similarity tests, area of parallelogram, the right-angled triangle (Pythagoras' theorem), etc., are introduced by practical experiments, after which methods of "proof" are considered in oral examples arranged in a form suitable for class discussion. It is not desirable that formal proofs should be committed to memory at this stage, but results will become familiar from constant use in practical and numerical work.

In Part III, rather more emphasis is laid on the presentation of formal proofs and standard constructions. The properties of similar figures naturally lead up to the sections on numerical trigonometry which complete the course.

Most of the exercises, and especially those labelled "General Statements," end with a few examples intended only for pupils of special ability.

The authors acknowledge gratefully the help they have received from Miss F. Pycock and from Mr. G. T. Clark, who have read the proofs and have made many valuable suggestions.

C. V. D.

C. O. T.

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PART I

GEOMETRICAL IDEAS

CHAPTER I

LINES, POINTS AND SURFACES

Lines
You use paper on which lines have been ruled in order that you may write straight; the ruled lines are straight lines. The letters which make up a word are represented by lines of different kinds: L is formed by two straight lines, S is a curved line, B is partly straight and partly curved, W is a zig-zag line.

EXERCISE I.a (Oral)
Describe the following lines as (a) straight or (b) curved or (c) zig-zag or (d) partly straight and partly curved:
1. A white thread held loosely against the blackboard.
2. The same thread stretched taut. (Stretched = straight.)
3. The top edge of this page.
4. A plumb line, e.g. the flex supporting an electric lamp.
5. The upper rim of a penny.
6. The edge of your desk.
7. The line in which two walls of the room meet each other.
8. The line in which the ceiling meets a wall.
9. The crack in a pane of glass.
10. The crack between two planks in the floor.
11. The lines on the palm of your hand.
GEOMETRICAL IDEAS

12. The path of the centre of a ball thrown through the air.
13. The crease when a sheet of paper is folded.
14. The letters, E, M, U, D, Z.
16. The edge of a stair carpet laid down a staircase.
17. The outline of a gabled roof.
18. The Equator.

Points
Two lines meet or cross one another at a point.
A point has no size; it marks a position.

Examples should be taken to show that the position of a point is described by naming two or more lines which pass through it or meet at it.

EXERCISE I. b (Oral)
1. Point out two lines which meet at the top outer corner of this page.
2. Point out two or more lines which meet at a corner of the desk.
3. Repeat No. 2 for a corner of the ceiling.
4. Make creases in a sheet of paper to show the position of a point on it.
5. How do the rulings on a sheet of squared paper (or on a squared blackboard) help in describing the position of a point?
6. A policeman is on point-duty where Station Road crosses the High Street. What does this mean?
7. What do you mean by the "points" on a railway line?
8. How can you mark the position of a point on the upper rim of a penny?

LINES, POINTS AND SURFACES

Representation of a Point

The right way
\[ \begin{array}{cccc}
A & B & C & D \\
\end{array} \]

The wrong way
\[ \begin{array}{cccc}
A & B & C & D \\
\end{array} \]

Fig. 1.

The position of the point A is shown by two short cross-lines. If a line BCD is drawn, its ends are points; a short cross-line, as at B, marks the end distinctly; the end labelled D is less clear. To mark a point C between B and D, use a short cross-line.

If a point is represented by a dot, the dot must be very small, in fact only just visible, smaller than this full stop.

Never represent a point by a blob.

The practical work in Ex. I. c should be performed by all, and the results discussed with the whole class.
In many of the exercises labelled "oral", it is desirable that all should write down their answers to a question and the correct answer be given out as soon as this has been done.
This method of teaching will in future be referred to as "class-discussion".

EXERCISE I. c (Oral)
1. Take a piece of paper and fold it so as to show creases like those in Fig. 2.

2. Obtain a square by folding as in Fig. 3 (i) and tearing off the top; use a square sheet obtained in this way for Fig. 3 (ii), (iii).
3. How many points are there in Fig. 3 (iii), whose positions are marked by two creases meeting one another?

4. What is the greatest number of extra points, whose positions will be marked by two creases, if you make one more crease in Fig. 3 (iii)?

5. Use a straight edge to find which sets of three points lie on a straight line in Fig. 4.

6. Mark on your paper points situated roughly as in Fig. 4. Use a straight edge to mark in your own figure a point which is in line both with (i) A, G and B, H; (ii) D, G and B, E; (iii) A, H, and E, F; (iv) B, C and D, E.

The point which is in line both with A, G and with B, H is called the point of intersection of the lines AG and BH.

7. A and B are houses on opposite sides of a river; C is a bridge. Copy Fig. 5 free-hand, and mark neatly on it a footpath for use in walking from A to B.

Mark also a more convenient position for the bridge.

8. There are 4 houses scattered about on a common. Represent them by points on your paper. How many paths may be needed so as to be able to go direct from any one house to any other? Show them in a figure.

How many paths in all if there are 5 houses?

Solids and Surfaces

In geometry, the word solid means anything which takes up room, not only a chair or a brick or a stone, etc., but also a sheet of paper, a coat of paint, a sponge, the water in a glass, the air in a room.

A surface is the boundary between two solids; it has no thickness, just as your shadow has none.

Thus, the surface of a table is not made of wood or of air, it is the boundary between the wood and the air; it marks where the table ends and the air begins. In the same way, if a glass is full of water, the inner surface of the glass is the same as the outer surface of the water where the water touches the glass; the surface is neither glass nor water, but marks where the glass ends and the water begins.

Models should be shown of the following solids:

- Cuboid
- Prism
- Pyramid
- Sphere
- Cylinder
- Cone

Some of these should be constructed, see Appendix, pp. 97-99.
EXERCISE I.d (Oral)

What are the names of the solids which the following objects resemble?

13. How many faces has a brick?

A line in which two faces of a solid meet is called an edge of the solid.

14. How many edges has a brick?

A point at which two or more edges of a solid meet is called a corner or a vertex (plural, vertices) of the solid.

15. How many corners (or vertices) has a brick?

16. Fig. 8 shows the top of a new pencil. Write down the number of the pencil’s (i) faces, (ii) edges, (iii) corners.

17. How many faces of a cube is it possible to see at the same time?

18. Make a table showing the number of faces, corners, and edges of various solids, such as a cuboid, triangular prism, triangular pyramid, 5-sided prism, 4-sided pyramid, a solid L, 4-sided pyramid beheaded, etc. This is easy if you look at a model.

<table>
<thead>
<tr>
<th>Name of Solid</th>
<th>Number of Faces</th>
<th>Number of Corners</th>
<th>Number of Edges</th>
<th>Value of</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>C</td>
<td>E</td>
<td>F + C - E</td>
<td></td>
</tr>
</tbody>
</table>

Can you draw any conclusion from this table?

LINES, POINTS AND SURFACES

Plane Surfaces

A surface is either plane or curved; thus, the surface of a ball is curved, and the surface of a new table is plane or as nearly plane as the carpenter can make it.

The planeness of a surface is tested by using a straight-edge. Apply the edge to the surface; if it fits it for the whole of its length, and in all positions, the surface is plane. This means that, if any two points are taken on the surface, the straight line which passes through the two points lies wholly in the surface; there must not be any light showing between the carpenter’s straight-edge and the surface.

A plane surface is called, for short, a Plane.

Just as we regard a straight line as extending indefinitely each way (to right and left, or up and down, etc.), so we regard a plane as extending indefinitely in the direction of every straight line in the plane. Thus, the top of a table is only a part of a plane surface; the plane itself contains all straight lines drawn on the top of the table and produced indefinitely each way.

Plane Geometry

The greater part of this book deals with lines and points in a plane; this is called “plane geometry”.

Unless the contrary is stated, it will always be understood that the lines and points mentioned lie in one plane.

EXERCISE I.e (Oral)

1. Can a straight-edge be placed so that it fits the curved surface of a cylinder?

2. Two points are taken on the curved surface of a cylinder, and a thread is stretched tight between them. Is the thread straight, (i) always, (ii) ever?

3. Repeat No. 2 for two points on a sphere.

4. Repeat No. 2, for a cone, taking the vertex as one point.

5. Repeat No. 2 for a cone, neither point being the vertex.
6. Can a ship travel in a straight line from Liverpool to New York?

7. A cylinder is placed with its curved surface in contact with a plane. What part of the surface touches the plane? How will the cylinder roll?

8. Repeat No. 7 for a cone.

9. Repeat No. 7 for a sphere.

10. By using a piece of string and looking along it, keeping it straight, find several points on a wall of the room, which are in the same plane as points on the top of your desk.

    In this way, construct the line in which the plane of the top of your desk meets the wall. Is this line straight?

11. Place your book flat on the desk and open the outside cover. Can you open it so that the plane of the cover passes through (i) any one point in the room? (ii) any line drawn on the blackboard?

12. Set up pins of various heights on the desk. Use the cover of your book to see whether a plane can be found passing through (i) any three points, (ii) any four points.

13. Can several planes, one plane, or no plane be found passing through
    (i) 1 straight line; (ii) 2 intersecting straight lines;
    (iii) any 2 straight lines; (iv) 1 straight line and 1 point;
    (v) any 3 straight lines cutting each other at 3 points;
    (vi) 5 points on a straight line; (vii) any 3 points?

14. Can you see in the room 3 lines which meet in a point, but do not lie in one plane?

    How many planes can be found passing through any two of the lines?

15. Can you see in the room 3 planes which meet each other in the same line?

    How do 3 planes generally meet each other?

16. Why is a carpenter’s "plane" so called?
GEOMETRICAL IDEAS

direction from A to B and beyond; similarly "Produce BA" means make the line BA longer beyond A.

Your pencil must have a sharp point.

USE OF RULER

EXERCISE II. a

1. Draw 3 straight lines so that they cut one another in 3 separate points A, B, C. Then draw through A, B, C, 3 lines which cut each other at only 1 point.

2. Draw 4 straight lines so that each line cuts all the others, and so that all the points of intersection are different. How many points of intersection do you get?

3. Mark 3 points A, B, C on a straight line and a point O not on it. Join A, B, and C to O.

4. (i) Draw a figure like Fig. 10, taking any 3 points on each line. Join AQ, BP cutting at S; AR, CP cutting at T; BR, CQ cutting at V.

(ii) Draw two lines ABC, PQR as in Fig. 10 and produce them to meet at O. Take R so that CR passes through the point of intersection of AP, BQ produced. Join as in (i), and test whether O, S, T, V are in a straight line, by joining OV.

5. Draw a diagram with lines arranged as in Fig. 11. First take any 4 points A, B, C, D; then mark any 2 points P, Q on CD. Find whether in your drawing Z is in line with A and B. [The figure is easier to draw if you take B much closer to CD than A is.]

The Circle

Take your compass and draw a circle.

The point O where the steel point of one leg rests on the paper is the centre of the circle. The curved line traced out by the point of the other leg is sometimes called the circumference; but more often the curved line is called a circle, and the word, circumference, is used for its length. The line joining O to any point P on the circle is called a radius (plural, radii). The length of OP is the distance between the compass points, and so all points on the circle are at the same distance from O, that is, all radii of the circle are equal.

How to use a Compass

It needs practice to draw a good circle; the proper way of setting about it is as follows:

(i) Hold the compass at the top, with your thumb and forefinger. Do not hold it by its legs.

(ii) Do not press too hard on the paper. The steel point must not be allowed to make a big hole in the paper.

(iii) Describe your circle clockwise. Do not turn it first one way and then the other.

(iv) Hold the compass nearly upright, but make it lean a little in the direction in which it is moving.

(v) To obtain a given radius, start by opening the compass, rather beyond what is required and then press one leg, gently and steadily, inwards, till the required distance between the points is obtained.

For accurate drawing, it is essential that the pivot of the compass should be fairly stiff, and that the lead (or pencil) should have a sharp point. A lead is better than a pencil; but, if a pencil is used, it should be short; otherwise it will get in the way of your hand; also it must be so arranged that, when the compass is shut, the point of the pencil is very close to the steel point, but projecting just a little beyond it.

Take care of your compass; it will soon be spoilt if you use it for ordinary writing, or for ruling straight lines.

Diameter and Radius

Draw a line AB and mark a point O on it. Use your
compass to draw, with $O$ as centre, and through $A$, the part of the circle $APQ$, which lies above $AB$. This is called a semicircle, and the line $AOQ$ is called a diameter. A diameter consists of two radii in line, and its length is therefore twice that of the radius.

In practice, the sizes of circular objects, such as pipes, gun-barrels, wheels, etc., are usually expressed by stating their diameters; in geometrical drawing it is more convenient to state the radius, because this is the distance between the compass points.

**USE OF COMPASS**

**EXERCISE II. b**

1. (i) The diameter of a wheel is 30 inches; what is its radius?
   (ii) 1 draw a circle of radius 4 cm.; what is its diameter?

2. (i) Mark 3 points $A$, $B$, $C$, not in a straight line. Draw the 2 circles which pass through $C$ and have $A$ and $B$ as their centres.
   (ii) Mark 3 points $D$, $E$, $F$, in this order, on a straight line. Draw the 2 circles which pass through $E$, and have $D$ and $F$ as their centres.
   (iii) Mark 3 points $P$, $Q$, $R$, in this order, on a straight line. Draw the 2 circles which pass through $R$, and have $P$ and $Q$ as their centres.

What are the chief differences between these 3 figures?

3. Mark 3 points $A$, $B$, $C$, in this order, on a straight line. Draw the circle, centre $A$, which passes through $B$, and the circle, centre $A$, which passes through $C$. Draw one diameter of each circle.

   If two circles have the same centre, they are called concentric.

4. Mark 2 points $A$, $B$. Draw, with $A$ and $B$ as centres, 2 circles passing through $B$ and $A$, as in Fig. 14. Why is $CA$ equal to $CB$?

---

5. Draw a circle, centre $O$, and a diameter $AOB$. Draw with $A$ and $B$ as centres, parts of 2 circles, each equal to the first circle, and so copy Fig. 15. If your drawing is good, you will find that $PR$ and $QS$ are each equal to the common radius; use your compass to test this.

6. Draw a circle and mark any point $A$ on it (see Fig. 16). With centre $A$, and the same radius, draw part of a circle cutting the first at $B$, $F$. With centre $B$, and the same radius, mark off the point $C$; with centre $C$, and the same radius, mark off $D$, and similarly $E$. [This is called "stepping" the radius round the circle.] Join alternate points, $AC$, $CE$, $EA$, $BD$, $DF$, $FB$. This figure is Solomon's Seal.

7. Draw a straight line $AB$ and mark a point $X$, not on $AB$. Draw 6 circles each passing through $X$, and such that the centre of each lies on $AB$. What do you notice about them?

**Plane Figures**

If part of a plane is marked off by straight or curved lines drawn in the plane, this part is called a **plane figure**. If it is bounded by straight lines it is called a plane **rectilinear** figure. The word plane, is usually omitted, it being understood that a figure is a plane figure, unless the contrary is stated.

Closed rectilinear figures are named according to the number of their sides. The chief figures are as follows:

- **Triangle** (3 sides)
- **Quadrilateral** (4 sides)
- **Pentagon** (5 sides)
- **Hexagon** (6 sides)
Also, an 8-sided figure is called an **octagon**, and a 10-sided figure is called a **decagon**.

Any rectilineal figure with more than 3 sides may be called a **Polygon**, a many-sided figure; thus a pentagon is a 5-sided polygon.

Any line joining 2 corners of a polygon, which are not consecutive, is called a **diagonal** of the polygon.

The word, **circle**, is most commonly used (see p. 11) to mean a curved line, but it is sometimes used to denote the portion of the plane enclosed by that curved line; in that case, it represents a plane figure. Similarly, the word, semi-circle, is sometimes used to denote the portion of the plane enclosed by the curve and a diameter.

A sector of a circle is the portion of the plane enclosed by 2 radii, OA, OB, and the part of the curve between them (see Fig. 18).

**Rough Figures**

A "rough" figure does not mean an untidy figure, but one not drawn accurately. It may be drawn entirely free-hand, if this can be done neatly; but otherwise straight lines should be ruled. The figures should be large, much larger than those printed in the book.

**POLYGONS**

**EXERCISE II. c**

1. How many corners has (i) a quadrilateral; (ii) a pentagon; (iii) a hexagon; (iv) a decagon?

![Fig. 10.

2. How many diagonals has (i) a quadrilateral; (ii) a pentagon?

3. Draw a pentagon; then draw one diagonal. What two figures are obtained?

4. Into what two figures can a hexagon be divided by a diagonal?

5. Make a table showing the number of compartments F, the number of corners C, the number of sides S for the Figures 19, (i)-(iv), and for one other rectilineal figure you invent for yourself. Is there any simple relation between C + F and S? Compare Ex. I. d, No. 18, p. 6.

6. (i) Draw a 7-sided polygon; then draw all the diagonals through one corner. How many triangles are formed?
   (ii) If all the diagonals through one corner of a 10-sided polygon are drawn, how many triangles are formed? Do not draw the figure.

7. How many diagonals can be drawn through one corner of a hexagon? How many diagonals altogether?

8. Repeat No. 7 for (i) an octagon; (ii) a 10-sided polygon.

9. Draw a triangle ABC; mark a point P on BC and make three circular sectors as in Fig. 20, the centres being the corners of the triangle.

10. Draw a semicircle and its diameter. What two figures are obtained by drawing another radius?

**Measurement of Length**

Look at your ruler. One of its edges is marked off in inches and tenths of an inch, the other edge in centimetres (cm.) and tenths of a centimetre, called millimetres (mm.). Some rulers also show eighths of an inch and twelfths of an inch. In measuring lengths it is best to use the tenths.

To measure the line AB.

Accuracy in drawing and measurement depends on taking trouble and using proper methods.
How to measure accurately

(i) Arrange the paper so that the line to be measured runs across the desk, not up and down it.

(ii) Place the ruler along the line AB and place it so that one of the chief graduations is opposite A, e.g. the 1 inch mark or the 2 inch mark, if the ruler is long enough. Avoid using the zero graduation, if it is worn.

(iii) Place the ruler as close as you can to the line.

(iv) When reading off the length, your eyes should be directly above the point B at which you are looking; do not look at it from one side.

Using your ruler, you will find that the length of AB in Fig. 21 lies between 2·3 in. and 2·4 in.; and with a little practice you will be able to estimate fairly accurately the length of AB to a hundredth of an inch. Try to do so here by imagining that the part of your ruler between 2·3 and 2·4 is divided into 10 equal parts.

Note.—2 in. is often written 2"; similarly 2' denotes 2 ft., and 2" denotes 2 yd.

To draw a Line of given Length

To draw accurately a line, say, 5 cm. long is not as easy as it may sound. The best way of setting about it is as follows:

(i) See that your pencil has a sharp point.

(ii) Using your ruler, draw a line rather longer than 5 cm. (say about 6 cm.) across the paper.

(iii) Mark a point close to the left-hand end by drawing a short cross-line, as at A in Fig. 21.

(iv) Put your ruler as close as you can to the line, with one of the chief graduations, say the 2 cm. mark, opposite A.

To make sure that it is opposite A, look at it from above, not from one side. Also if the ruler has a thick edge, you must stand it on its edge in order to get the graduation-marks close to the line.

(v) Now move your head so that your eye is directly

over the 7-cm. graduation (2 + 5 = 7), and make a very small dot on the line opposite this graduation.

(vi) Turn the paper round and draw a short cross-line through the dot, as at B in Fig. 21.

The dot should be so small that it becomes invisible as soon as the cross-line is drawn.

Fig. 22 shows the correct way of representing a finite straight line, that is a line of definite length. Its extremities must be shown, and this is best done by short cross-lines; it must not be done by blobs.

Right          Wrong

Dividers.—Some boxes of instruments contain a pair of dividers. The use of dividers makes it easier to draw and measure accurately.

To measure a given line AB, open the dividers so that the distance between their points is a little greater than the length of AB. Place one point on A, and press gently and steadily on the other leg, till its point rests on B. Then place the points of the dividers on a graduated scale and read off the required length.

When using dividers to mark off a line of given length, care must be taken that any holes made in the paper are very tiny.

MEASUREMENT OF LENGTH

EXERCISE II. a

1. Measure, as accurately as you can, in inches, and also in centimetres, the lengths of the following lines in Fig. 23, and arrange your answers in a table.

<table>
<thead>
<tr>
<th>PQ</th>
<th>SR</th>
<th>PS</th>
<th>QR</th>
<th>AB</th>
<th>BC</th>
<th>CA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inches .</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Centimetres .</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Copy this table into the end of your notebook, as it will be useful for reference later.
2. Measure in inches, and also in centimetres, the following

lines in Fig. 23: (i) DE; (ii) EF; (iii) FD; (iv) MP; (v) KN.

Write the lengths (i), (ii), (iii), at the end of your notebook, as
they will be useful for reference later.

3. Measure in inches, and also in centimetres, the distances

between the following pairs of points in Fig. 23. Do not join

these points.

(i) K, C; (ii) A, K; (iii) M, Q; (iv) P, C; (v) A, E.

4. Find in cm. the perimeter of the triangle ABC in Fig. 23.

Use the results obtained in No. 1.

The perimeter of a figure is the total length of the boundary

of the figure. To obtain the perimeter of a polygon, measure

the length of each side, and then find the sum of these lengths.

5. Find in inches the perimeter of the quadrilateral PQRS

in Fig. 23. Use the results obtained in No. 1.

6. Which points in Fig. 23 are more than 5 cm. from K ?

Use your compass to find out.

7. Which points in Fig. 23 are less than 1.7 in. from M ?

8. On a piece of squared paper, showing inches and tenths

of an inch, rule lines of the following lengths:

(i) 2 in.; (ii) 2.5 in.; (iii) 1.7 in.; (iv) 2.25 in.

9. Draw (on plain or squared paper) a line of length 5 inches.

Measure it in cm.; then calculate the number of cm. in 1 inch.

10. Draw (on plain or squared paper) a line of length 10 cm.

Measure it in inches; then express 1 cm. in inches.

11. Which of the lines HK, YZ in Fig. 24 do you think is

the longer? Write down your answer, then check by measure-

ment.

12. Which of the lines SM, PQ in Fig. 23 do you think is

the longer? Write down your answer, then check by measure-

ment.

13. Mark a point O on your paper. Then mark 7 other

points each of which is 1.5 inches from O.

Can you save time by using your compass?
14. Guess the length of this page in inches. Write down your answer, then check by measurement.

Can you find a finger joint which nearly measures one inch? If so, this will help you to estimate small lengths, when you have not got a ruler with you.

Find out what distance you can span with your hand; this will help you to estimate larger lengths.

Find out the length of your stride; this will help you to estimate distances by pacing them.

15. Guess, and then measure, the lengths (and breadths, etc.) of various objects, both in inches and also in centimetres; e.g. length, breadth, etc., of box of instruments, of this book, of the top of your desk.

Show the result in a table:

<table>
<thead>
<tr>
<th>Name of Object</th>
<th>Guess (in.)</th>
<th>Guess (cm.)</th>
<th>Measure (in.)</th>
<th>Measure (cm.)</th>
</tr>
</thead>
</table>

Further Practice in using a Compass

For many purposes, it is unnecessary to draw the whole of a circle; and in fact it often makes the figure needlessly confusing to do so.

Any portion of the curve, such as \( \overline{AP} \) in Fig. 25, is called an arc. Also any straight line which joins 2 points on the curve, such as the straight line \( \overline{AB} \) in Fig. 25, is called a chord.

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The plane figure bounded by a chord and the arc of the curve it cuts off is called a segment.
6. Draw the pattern marked out by the thick line in Fig. 28. The dotted parts of the figure are added to show the arrangement and are not to be reproduced. The radii of the three smaller arcs are equal.

7. (i) Draw two lines \(AB, AC\). Construct a figure like Fig. 29, where \(A, P, Q\) are the centres of three arcs, whose radii are equal.

(ii) Repeat (i), taking \(AP = 3\) cm. and the radii of the arcs centres \(P, Q\) each 4 cm.

8. Draw a line \(CD\). Construct a figure like Fig. 30, where \(C, D\) are the centres of two arcs whose radii are equal. Then add to the figure the two circles whose centres are \(C, D\) and whose radii each equal \(CD\).

9. Draw a line \(AB\) and mark a point \(C\) on it. Construct a figure like Fig. 31, where \(C, P, Q\) are the centres of the three arcs. Construct a point \(L\) such that \(LP\) and \(LQ\) are equal and each is less than \(PQ\). Test whether \(L\) is in line with \(KC\).

EXERCISE II. f

1. Draw a line \(AB, 5\) cm. long. Draw two circles, centres \(A, B\), radii 3 cm., 4 cm., and call the points where the circles cut, \(P\) and \(Q\). Write down the lengths of the sides of the triangle \(APB\) and of the triangle \(AQB\).

By drawing one more circle, make a triangle \(ARB\), such that \(AR = 3.5\) cm., \(RB = 4\) cm., \(AB = 5\) cm.

2. Draw enough of two circular arcs, centres \(A, B, 1\) in. apart and with radii 2 in., 2.5 in., to show their points of intersection \(X, Y\).

What are the lengths of the sides of \(\Delta AXY\) and \(\Delta AVB\)?

The symbol, \(\Delta AXY\), means "the triangle" \(AXY\).

3. Draw a triangle \(ABC\), such that the lengths of its sides are 7 cm., 6 cm., 8 cm. Start by drawing a line 7 cm. long, and call it \(BC\). Use your ruler to find the middle point of \(BC\) and call it \(X\). Measure the length of \(AX\).

4. Draw a line \(AB, 4\) cm. long. How can you obtain two triangles, \(\Delta APB\) and \(\Delta AQB\), one on each side of \(AB\), so that \(AP = AQ = 3.5\) cm., and \(BP = BQ = 4.5\) cm. Draw the figure and measure \(PQ\).

5. Construct Figs. 32-34, the units being cm. In each case, measure \(XY\). [Your best plan is to draw \(AB\) first in each case.]
GEOMETRICAL IDEAS

8. Measure the sides of \( \triangle ABC \) on p. 18 in cm. (or use the measurements obtained in Ex. II. d, No. 1).
Construct a triangle with sides of the same length.

9. Repeat No. 8 for \( \triangle PSR \) on p. 18.

A triangle, whose three sides are of equal length, is called equilateral.

10. In Fig. 35, A and B are the centres of the two arcs.

Fig. 35. Join AC, BC. Then \( AC = AB \), why? \( BC = BA \), why?
What follows from these two statements? Draw the figure making \( AB = 1.5 \) inches.

11. Draw a line \( AB \), 5 cm. long. Construct two equilateral triangles \( \triangle ABP, ABQ \).

12. Draw a line \( OA \). On \( OA \) draw an equilateral triangle \( \triangle OAB \); on \( OB \) draw an equilateral triangle \( \triangle OBC \); on \( OC \) draw an equilateral triangle \( \triangle OCD \); and continue this process as often as you can. How many different triangles can be drawn?

PATTERN-DRAWING WITH RULER AND COMPASS

EXERCISE II. g

1. In Fig. 36, \( AB = BC = CD = DE = 2.5 \) cm.; and the centre of each arc lies on \( AE \). Draw the figure.

Fig. 36.

2. In Fig. 37, \( ABC \) is an equilateral triangle, side 2 inches; and \( A, B, C \) are the centres of 3 arcs, whose radii are equal.

Draw the figure.

RULER AND COMPASS

3. Fig. 38 is formed of 3 semicircles; the diameters of the two smaller semicircles are 2.4 cm., 3.8 cm.

Draw the figure.

Fig. 38.

4. Fig. 39 is formed of 5 semicircles; \( AB = CD = DE = 2 \) cm. and \( BC = 1 \) cm.

Draw the figure.

Fig. 39.

5. Draw a line \( AB \), 3 cm. long (see Fig. 40). Construct the positions of the centres \( C \) and \( D \) of the two circular arcs, if their radii are each 2.3 cm. Then draw the arcs.

Fig. 40.

6. Draw the crescent in Fig. 41, given that \( AB = 6 \) cm., and that the radii of the arcs are 4 cm. and 5 cm.

Fig. 41.
CHAPTER III

RIGHT ANGLES

Right-Angled Corners

There are objects all round you with right-angled corners: the outside corner of this page, the corners of the door, the corners of each face of your box of instruments, etc.

All right-angled corners fit one another; also they can be placed side by side in contact with a straight edge; thus two paving stones with right-angled corners can be placed side by side against a straight curb. This illustrates one way of testing whether a corner is right-angled: take (or make) another corner that fits it and see whether the two can be placed in contact with each other and, at the same time, with a straight-edge, as in Fig. 42.

A carpenter tests whether the corners of a block are right-angled by using a try-square (see Fig. 43), which contains a corner ACB, guaranteed to be right-angled by the makers. We say that the angle AGB (angle at corner) is a right angle, and that CA is at right angles to CB or is perpendicular to CB. Equally, CB is at right angles to, or perpendicular to, CA.

For geometrical drawing, the set-square in your box of instruments is more convenient than a try-square. Test whether the corner is right-angled by taking another set-square (it need not be the same shape) and seeing (i) whether the two corners fit and (ii) whether the two can be placed in contact with each other and, at the same time, with a straight-edge.

You can make a right-angled corner without a set-square as follows: take a sheet of paper with a straight edge AB; fold part of this edge CB back along the rest CA, and crease the paper; call the crease CD. When unfolded, AC is again in line with CB; also we know that the two corners ADC, BCD fit each other; therefore each is a right-angled corner, and CD is perpendicular to AB.

EXERCISE III. a (Oral)

1. Test whether the right-angled corner of your set-square fits (i) a corner of this page; (ii) a corner of your desk.
2. Place your set-square ABC, right-angled at C, so that CB lies along an edge XY of your paper. Is ACX also a right-angled corner? Test with another set-square.
3. Hold your set-square ABC upright, with CB resting on the desk, C being the right-angled corner. Now turn it round CA, like a door turning on its hinges. Can you arrange it so that CA remains in the same position, and CB always lies along the desk? If so, CA must be perpendicular to every line through C drawn on the desk, and we say that the line CA is perpendicular to the plane of the desk.
4. Shut up your box of instruments and arrange it so that the upper face is on a slope. Hold 2 set-squares so that one pair of the shorter edges coincide and the other pair rest on the face of the box. Hence obtain a line perpendicular to this face. Why do you need two set-squares?
5. Hold your set-square ABC, right-angled at C, so that the edge CA is perpendicular to the plane of the desk. Then the plane of the set-square is said to be perpendicular to the plane of the desk. By turning the set-square round CA, obtain other planes perpendicular to the plane of the desk.

If a line CA is perpendicular to a plane, any plane through CA is called perpendicular to that plane.

6. Arrange your box of instruments so that the upper face is on a slope. Stand your book, partly open, so that the lower edges of the covers rest on this face and form a V on it. What can you say (i) about the line in which a cover meets the binding of the back; (ii) about the plane of each cover?

7. Point out in the room
   (i) 3 lines perpendicular to the floor;
   (ii) 3 planes perpendicular to the floor.

8. Point out in the room
   (i) 3 lines perpendicular to the wall in front of you;
   (ii) 3 planes perpendicular to this wall.

9. Use your set-square to open the cover of this book so that it is at right angles to the rest of the book.

10. Hold your set-square so that one edge CD is on the desk and another edge CA is perpendicular to the desk. Imagine another line CD drawn across the surface of the set-square. Can you find a line on the desk perpendicular to CD? Try with another set-square. Is there more than one line?

PERPENDICULAR LINES

EXERCISE III. b

1. Make short sentences, using the words "at right angles", saying something about (i) a flagstaff; (ii) the walls of a room;
(iii) cross-roads; (iv) East and West.

2. Make short sentences, using the word "perpendicular", about
   (i) a picture frame; (ii) the letter T;
   (iii) the minute-hand and hour-hand at 3 p.m.
   (iv) the shortest path from a house to a straight road.

3. Take a piece of paper of any shape and fold it; call the crease AB. Fold it again so that A falls on B; call the point, where the creases meet, O. Now open the paper out flat. What is the result?

How many right-angled corners can be fitted together at one point in a plane?

4. What capital letters of the alphabet have all their corners right-angled?

5. Two perpendicular lines OA, OB are drawn on the face of a clock, O being the centre of the face. To what does OB point, if OA points to VI (two answers) ?

   Repeat, if OA points to (i) IV; (ii) XI.

6. In Fig. 46, A, B, C, D represents part of a wall and ABED part of the floor of a room. PQ is a line on the wall perpendicular to AB, and PR is a line on the floor perpendicular to AB.

   (i) Is PQ at right angles to PR?
   (ii) Is PQ at right angles to PF?
   (iii) Is PQ perpendicular to PE?
   (iv) What lines in the figure, through P, are perpendicular to PR?
   (v) Is PF perpendicular to PD?
   (vi) What can you say about a plane through P perpendicular to PF?

Right Angles made with Ruler and Compass

In drawing figures with ruler and compass, you will often get lines at right angles, especially when one-half of the figure can be made to fit the other half, by folding it over, because this is one of the ways in which right-angled corners can be constructed (see p. 27). Look back at Ex. II. e, p. 22. In Fig. 30, the left half of the figure fits the right half, if folded with XY as crease; therefore CD is perpendicular to XY.

EXERCISE III. c

1. Point out lines at right angles, in Fig. 31, p. 22 (see Ex. II. e, No. 9), and give the reason.
2. How can you obtain two perpendicular lines in the following figures? Give the reason in each case.
(i) Fig. 32, p. 23; (ii) Fig. 33, p. 23; (iii) Fig. 40, p. 25.

3. Draw a straight line $AB$ and mark any point $C$ on it. Proceed as in Fig. 47: $P, Q$ lie on a circle, centre $C$; $R$ is a point of intersection of two equal circles, centres $P, Q$. Explain why $CR$ is perpendicular to $AB$. Also test with your set-square.

4. Figs. 48, 49 show two ways of drawing, from a given point $K$, a line $KN$ perpendicular to a given line $AB$.
This is called dropping a perpendicular from $K$ to $AB$.
In Fig. 48, $P, Q$ lie on a circle centre $K$, and $R$ is a point of intersection of two equal circles, centres $P, Q$.
In Fig. 49, $P$ and $Q$ are any points on $AB$, and the circles which meet at $R$ have $P, Q$ as centres and both go through $K$.
Carry out each construction. Explain why $KNR$ is perpendicular to $AB$.
Also test with your set-square.

5. Draw a circle of radius 5 cm. and draw any diameter $PQ$. Use the construction in No. 3 to draw a diameter $RS$ at right angles to $PQ$.
$PQ$ and $RS$ are called perpendicular diameters of the circle.

6. The Clock Face. Draw a circle of radius 3 inches, and two perpendicular diameters; call them XII $C$ VI and IX $C$ III (see Fig. 50).
Step the radius round the circle, starting from XII, and mark the points so obtained, II, IV, VI, VIII, X. Repeat, starting from III, and so obtain the other hour marks.

7. Draw a line $CD$, and construct a line through $C$ perpendicular to $CD$ (see Fig. 51), as follows: Draw any circle with $C$ as centre, cutting $CD$ at $P$. Obtain $H, K$ by stepping the radius round the circle, starting from $P$. The two circles, with centres $H, K$, and the same radius as before, cut at $R$. Join $CR$.

[Do not draw more of the arcs than you need to obtain $R$.]
(i) Where is the other point at which the circles, centres $H, K$, cut?
(ii) What happens to $H$ if the figure is folded about $CR$?
(iii) If this is a clock-face with III at $P$, what other hour marks can you now put in? Does this show why $CR$ is perpendicular to $CP$?

8. Draw a line $AB$ of length 8 cm, and draw a semicircle on $AB$ as diameter. Take any 5 points $P, Q, R, S, T$ on the semicircle and join each to $A$ and to $B$. Use your set-square to discover 5 right angles in the figure. Name them.

9. Draw a line $AB$ and mark any point $C$ on it. Construct a line through $C$ perpendicular to $AB$ (see Fig. 52), as follows: Take any point $O$ outside $AB$, and draw the circle whose centre is $O$ and which passes through $C$. Let $Q$ be the other point at which the circle cuts $AB$, and draw the diameter $QOR$. Join $CR$.
GEOMETRICAL IDEAS

Test with your set-square whether \( GR \) is perpendicular to \( AB \). How is this construction connected with No. 8?

Practice in use of Set-square

Rule a straight line \( XY \) on your paper and place your set-square \( ABC \) in the position as shown in Fig. 53.

If you rule a line along \( AC \), this line is perpendicular to \( XY \). But if you try to rule it right up to the corner \( C \), you will get a "tail" to the line. This must be avoided, so proceed as follows:

To draw a perpendicular to a given line \( XY \) from a given point \( P \) on the line:

(i) Place the set-square with its shortest edge \( CB \) along \( XY \), and so that \( P \) lies between \( B \) and \( C \), but near the right-angled corner \( C \) (see Fig. 54).

(ii) Press down firmly on the set-square, and put a straight-edge, not bevelled (or another set-square), along \( AB \).

(iii) Now press down firmly on the straight-edge and stop pressing on the set-square. Slide the set-square along the straight-edge until the point \( P \) appears just outside the edge \( CA \).

(iv) Now press down firmly on the set-square and leave off the straight-edge; and then rule a straight line through \( P \) along the edge \( CA \), but stop before you reach a corner of the set-square.

To draw a perpendicular to a given line \( XY \) from a given point \( Q \) outside the line:

This is called, dropping a perpendicular from \( Q \) to \( XY \). Place the set-square with its shortest edge \( CB \) along \( XY \), and so that \( Q \) lies under the set-square, but close to the edge \( AC \) (see Fig. 55). Then proceed as before.

N.B.—Never press down firmly on both set-square and straight-edge, at the same time. Press very lightly on the one you are sliding; press heavily on the other.

Alternative Method

The following method gives greater accuracy in drawing, but it is not so easy to explain or to learn.

First place the longest edge \( AB \) of the set-square along \( XY \) and put a straight-edge along \( BC \) (see Fig. 56). Press firmly on the straight-edge and turn the set-square round so that \( AC \) now rests along the straight-edge (see Fig. 57). This makes \( AB \) perpendicular to \( XY \). Slide the set-square along the straight-edge until a line ruled along \( AB \) passes through \( Q \). This is the perpendicular from \( Q \) to \( XY \).

CONSTRUCTION OF FIGURES CONTAINING RIGHT ANGLES

EXERCISE III.d

[Throughout this exercise, right angles should be drawn by using a set-square.]

1. Draw a line \( AB \) 5 cm. long (see Fig. 58). Through \( A \) draw a line perpendicular to \( AB \) and cut off from it a length \( AD \), 3 cm. Through \( B \) and \( D \) draw perpendiculars to \( BA \) and \( DA \) and let them meet at \( G \).

Is the angle \( BCD \) also a right angle? Test with your set-square. Measure \( BC \) and \( CD \).
GEOMETRICAL IDEAS

This figure you have constructed is called a rectangle, sometimes an oblong; its opposite sides are of equal length and all its angles are right angles.

If two consecutive sides of a rectangle are equal (and in this case all its sides are equal), the figure is called a square.

2. Draw a rectangle \( PQRS \) so that \( PQ = 1.5 \) in. and \( QR = 2 \) in. Measure \( PS \) and \( RS \). Measure also the diagonals \( PR \) and \( QS \).

3. Draw a rectangle \( HKMN \) so that \( HK = 3 \) cm. and \( KM = 7 \) cm. Measure the other sides and the two diagonals.

4. Draw a square \( ABCD \) so that \( AB = 1.5 \) in. Draw the two diagonals and measure them. Use your set-square to test whether the diagonals cut at right angles.

5. Draw a circle, centre \( O \), radius \( 5 \) cm., and draw two perpendicular diameters \( AOB, XOY \). What measurements must you make to test whether \( AXY \) is a square? Make them.

6. By using compasses, draw a triangle \( ABC \) so that \( BC = 7 \) cm., \( CA = 6 \) cm., \( AB = 5 \) cm. By using a set-square, draw the line from \( A \) perpendicular to \( BC \). In the same way, drop a perpendicular from \( B \) to \( CA \) and a perpendicular from \( C \) to \( AB \). What do you notice about the three perpendiculars?

The perpendicular from any corner of a triangle to the opposite side is called an altitude of the triangle.

A triangle has three altitudes.

7. Draw a large triangle shaped as in Fig. 59 and draw its three altitudes.

[Notice that the perpendicular from \( A \) to \( BC \) meets \( CB \) produced.]

If you produce all three altitudes, what do you notice about them?

8. Draw a rectangle 4 cm. long, 3 cm. high. Divide it up into squares, such that the sides of each square are 1 cm. long. How many squares are there?

9. Fig. 60 represents a rectangle \( ABCD \), 2\( \frac{1}{2} \) inches long, 1 inch high, divided up into equal squares. What is the side of each square? How many squares are there?

AREA

Measurement of Area

The size of a face of an object is described by comparing it with the size of face of some well-known object. Suppose I take my handkerchief as the standard, I can then find the areas of the faces of various objects in the room by actual measurement with my handkerchief. Thus, for example, I find

my blackboard-area = 12 handkerchief-areas
the door-area = 13 handkerchief-areas

and so on.

The areas of various surfaces in the room should be measured in terms of a unit of this kind.

Choice of Standard

Any shaped figure can be taken as the standard, but the most convenient shape to choose is a square. If each side of a square is 1 inch, we call its area 1 sq. inch; and the area of any figure in sq. inches is the number of such squares it contains. If each side of the square is 1 cm., we call its area 1 sq. cm.; and the area of any figure in sq. cm. is the number of such squares it contains. If a rectangle is divided into squares as in Fig. 62, there is a quick way of counting them which you must find out for yourself.

It is inconvenient to use the same standard for large as for small quantities. Thus large weights are measured in tons, smaller weights in lb. or oz.; large distances are measured in miles, smaller distances in yards, feet, or inches. It is the same for areas. If the area is large, we may take a square whose side is 1 mile as the standard square, and find the area in square miles. For smaller areas, the standard square may be a square foot, a square inch, a square centimetre, etc.

If one of the small squares of a squared blackboard is taken as the standard square (its side need not be an exact number of inches or of cm.), and if various figures are drawn on this
squared blackboard, their areas may be found by counting up the number of squares they contain. This should be done. Also, figures should be drawn on the squared blackboard to illustrate that

1 sq. foot = 144 sq. inches and 1 sq. dm. = 100 sq. cm.

Also chalk lines should be drawn on the floor (or blackboard) to show a square yard divided into square feet.

MEASUREMENT OF AREA

EXERCISE III.

1. In Fig. 61, the shaded square is the standard unit-square. What is the area of the figure? Answer in unit-squares.

2. Repeat No. 1 for Fig. 62. Can you see a method by which the counting of the squares can be done quickly?

3. Draw free-hand a small square, take this as your unit-square. Sketch a rectangle of area (i) 6 unit-squares, (ii) 15 unit-squares.

4. A sheet of postage stamps has 7 rows, and there are 8 stamps in each row. How many stamps are there in the sheet?

5. In Fig. 63, A represents 1 sq. ft.; what does B represent? Answer in two ways.

6. In Fig. 63, A represents a square with side ½ inch long.

(i) What does C represent? What fraction of C is A? What is the area of A in sq. in.?

(ii) What does B represent? How many times is B as big as C? What is the area of B in sq. in.?

7. In Fig. 63, D represents a square with side 1 ft. long.

(i) What does A represent? What is the area of A in sq. ft.? Also in sq. in.?

(ii) Repeat (i) for C.

(iii) Repeat (i) for B.

8. (i) Look at a sheet of squared (inch) paper, and say how many of the small squares there are in 1 sq. inch. What is the area of each small square?

(ii) Draw on the squared paper a rectangle 1·8 in. long, 1·2 in. wide. How many small squares does it contain? What is the area of the rectangle?

9. (i) A room is 18 ft. long and 12 ft. wide; what is the area of its floor?

(ii) A window contains 20 sq. ft. of glass; it is 4 ft. wide; what is its height?

10. (i) Draw a rectangle of area 6 sq. in. and 2 in. high.

(ii) Draw a rectangle of area 5 sq. in. and 2 in. high.

11. A rectangle is 9 in. long and equals in area a square of side 6 in.; what is its breadth?

12. Draw (i) a figure 2 inches square, (ii) a figure of area 2 sq. inches.

13. The scale of a map is 2 inches to the mile. What area is represented by

(i) a square of side 1 inch on the map;

(ii) a square of side ½ inch on the map;

(iii) a rectangle whose area is 1½ sq. inches?

14. Repeat No. 13, if the scale is 4 inches to the mile.

Practical Work out of Doors

Some work of this type is valuable, and may well be introduced here. The kind of work which is suitable at this stage, before the general discussion of angles, is indicated in the Appendix, where it is grouped with other practical work for convenience.
GEOMETRICAL IDEAS

Plans and Scale Drawings

Fig. 64 represents a schoolroom ABCD, with a yard on two sides of it. The diagram is called a plan or a scale-

\[ \begin{array}{c}
\text{A} \\
\text{D} \\
\text{C} \\
\text{B} \\
\text{E} \\
\text{F} \\
\text{G} \\
\text{R} \\
\text{Q} \\
\text{N} \\
\text{S} \\
\text{T} \\
\end{array} \]

drawing, because the actual lengths are all reduced in the same proportion in the drawing, so that the shape remains unaltered. Here each cm. length in the plan represents 10 feet. We therefore say that the scale of the drawing is 1 cm. to 10 ft., written 1 cm. : 10 ft. This means that if we measure any line in the plan in cm., the actual length it represents is the same number of 10-foot lengths.

Scales are often given as fractions. Thus a scale of \( \frac{1}{10} \) means that 1 inch on the drawing represents 10 inches, or 1 cm. represents 50 cm., etc.

Always write the scale below, or at the side, of the plan.

Two methods of indicating the scale are shown in Fig. 64; use whichever you like.

**EXERCISE III.6 (Oral)**

In Fig. 64, P, Q, R show the positions of doors into the yard, S, T are doors into the schoolroom, and the dotted lines PN, QNCS, SR represent paved paths.

RIGHT ANGLES

Use your ruler to answer the following questions:

1. How long and how wide is the building ABCD?
2. How long are the walls EF and FG of the yard?
3. How far is it from the gate R to the door S?
4. How far is it from the gate P to the door S, going the shortest way you can?
5. What are the distances of C from G and from F?
6. What is the total length of the paved paths?
7. There is a fireplace K in the wall AB, 25 feet from A, not shown in Fig. 64. What will be the length in cm. of AK on the plan? How many feet is the fireplace from G?
8. There is a pole L fixed to the wall EF, 15 feet from F, not shown in Fig. 64. What will be the length in cm. of FL on the plan? How many feet is the pole from B?
9. How could you fix the position of a drain in the yard, which is 40 ft. from E, and 45 ft. from G?

Pattern Drawing

In the following exercise, several of the diagrams contained dotted lines. These are added merely to help you to reproduce the figure. You will obtain the best effect by drawing these parts of the figure as faintly as possible.

**EXERCISE III.7**

Draw the following figures:

1. In Fig. 65, starting from A, the first two lines are 2 cm. long, the next two are 4 cm. long, the next two are 6 cm. long, and so on. All the corners are right-angled. Continue till the lines run off your paper.
GEOMETRICAL IDEAS

2. In Fig. 66, **ABCD** is a square of side, 3 inches. The radius of each arc is 3 inches, and their centres are the corners of the square.

3. In Fig. 67, **AOB, COD** are two perpendicular diameters of a circle of radius 3 cm. Also the centres of the circular arcs lie on these diameters.

4. The capital letters in Figs. 68, 69 have right-angled corners; the measurements are in cm.; draw them and find their areas.

6. Fig. 70 shows the space between two concentric circles, radii 2 cm., 3 cm., divided into four equal portions by lines which meet, if produced, at the centre.

7. In Fig. 71, **ABCD** is a rectangle with sides 6 cm. and 3 cm. long; **C, D** are the centres of two of the arcs and **AB** is a diameter of the third arc.

8. In Fig. 72, the centres of the four equal circular arcs are the corners of the square **ABCD**, side 6 cm.

RIGHT ANGLES

SCALE DIAGRAMS

EXERCISE III. h

[Do not forget to state the scale of your diagram.]

1. Fig. 73 shows the gable-end of a house. Draw it to scale, 1 in. : 10 ft. Find the height of B above AC, by dropping the perpendicular from B to AC.

2. A pole **AB** (see Fig. 74) is kept perpendicular to the ground **CD** by 2 wire ties **AP, AQ**. Find the distances of **P and Q from B**. [Draw, first, the line representing **AB**; scale, 1 cm. : 2 ft.]

3. (i) Fig. 75 shows the distances of two trees **A, B** from a straight road **PQ**. [This means the lengths of the perpendiculars from **A, B** to **PQ**, because the shortest distances are along the perpendiculars.] How far apart are the trees? [Draw first **PMQ**; scale 1 in. : 20 ft.]

(ii) A third tree **C** is 40 ft. from **A and 60 ft. from B**; show on your diagram the two positions it could occupy, and find the distance between them.

4. Fig. 76 shows a load **W** supported by shear-legs, **AC, BC**. The ends **A and B** are on level ground, so that **CW** is perpendicular to **EA** when produced. Find the height of **W** above the level of **AB**.

5. In Fig. 77, **A, B, C, D** represent 4 towns, and the given distances between them are in miles. How far is **B** from **D**?
GEOMETRICAL IDEAS

6. Four trees, A, B, C, D, stand at the corners of a square ABCD of side 50 yards. E is a fifth tree inside the square, 35 yd. from A and 30 yd. from B.

Find how far E is (i) from C, (ii) from D, (iii) from the side CD of the square, (iv) from the side BC of the square.

Another tree F is to be planted 30 yards from A; what is its least possible distance from C? Show it in your plan.

7. Fig. 78 shows a load W being hoisted by a crane. AC is perpendicular to the ground-line AE, and W is half-way between B and AE. Find (i) how high W is above the ground; (ii) how far B is from the line AC, produced.

The crane swings round on the axis AC, just as a door swings on the line of its hinges; what is the path of B? Instead of the word "path," the word "locus" is often used.

8. Fig. 79 shows a mechanism of 6 rods smoothly jointed together; the units are inches. Find how far A is from C, (i) when B is 8 in. from D, (ii) when B is 10 in. from D.

Find also the least possible distance of A from C.

If the ends A and B of the rod AB are both fixed, what is the locus of C? of D? of E?

9. A and B are two trees 100 yards apart; P and Q are two other trees, each of which is 140 yards from A and 140 yards from B. Show the positions of A, B, P, Q on a scale-diagram and find the distance of P from Q in yards.

10. A ladder AB, 20 feet long, leans against the wall of a house, and the end A is 8 feet from the wall. Find how high the other end B is above the ground.

If the end A is moved 2 feet farther from the wall, how much does the other end B descend?

CHAPTER IV

ANGLES

Vertical and Horizontal

Fig. 80 shows a plumb-line. It consists of a cord with a small heavy body attached to one end B. If it is held by the other end A, the line AB of the cord is called a vertical line; when produced, it passes through the centre of the Earth. A builder uses a plumb-line to test whether a wall is upright, i.e. vertical; when properly built, the plane surface of the wall is a vertical plane.

Any plane which contains a vertical line is called a vertical plane.

A plane surface which is perpendicular to a vertical line is called a horizontal plane. The floor of this room ought to be a horizontal plane; you can find out whether it really is so, by using a spirit-level, so called because the liquid in it is spirit. The best natural horizontal plane is the surface of the water in a basin, or in a pond when there are no ripples.

Every straight line drawn on a horizontal plane is called a horizontal line. Thus, if the floor of this room is really horizontal, any line drawn on the floor is a horizontal line. Two points are "at the same level" if the straight line joining them is horizontal. If the top of your desk is a horizontal plane, any two points marked on it are at the same level.
Hold the plumb-line so that B just touches the floor, then the vertical line AB is at right angles to every line through B, drawn on the floor. This fact is expressed by saying, *A vertical line is perpendicular to every horizontal line which meets it.*

But the word "perpendicular" does not mean "vertical".

Fig. 81 shows the telegraph poles at the side of a road running up a steep hill. These poles are *vertical* (or should be), that is, their lines, if produced, pass through the centre of the Earth. But they are *not perpendicular* to the plane surface of the road.

Fig. 82 shows a coach running down a steep scenic railway. The back of the coach is *perpendicular* to the plane surface of the railway track, but it is *not vertical*; that is, it will not, when produced, pass through the centre of the Earth.

**EXERCISE IV. a (Oral)**

[A answers to questions in this Exercise should, where possible, be discovered by experiment, and illustrated practically.]

1. Point (i) vertically downwards, (ii) vertically upwards.
2. Name 3 vertical lines in the room.
3. Name 3 vertical planes in the room.
4. Name 3 horizontal planes in the room.
5. Name 3 horizontal lines in the room.
6. Point horizontally towards (i) the door handle; (ii) a corner of the room.

7. Take a cuboid (e.g., your box of instruments) and hold it so that one of its edges is vertical.
   (i) How many edges altogether are vertical?
   (ii) How many edges are horizontal?
   (iii) How many of its faces are vertical planes?
   (iv) How many of its faces are horizontal planes?

8. (i) Can you hold a cuboid so that one of its edges is horizontal, but none of its edges vertical?
   (ii) If one edge is horizontal, how many edges at least must be horizontal? How many at most can be horizontal?
   (iii) If one edge is horizontal, how many faces at least must be vertical planes? How many faces at most can be vertical planes?
   (iv) Can you hold a cuboid so that one of its edges is a horizontal line, but none of its faces are horizontal planes?

9. Mark a point A on your paper. Then hold the paper so that you could draw on it,
   (i) a vertical line through A;
   (ii) any number of horizontal lines through A;
   (iii) one, and only one, horizontal line through A.

10. (i) The line of the hinges of the door of this room is vertical. What can you say about the plane (front) surface of the door in any position?
    (ii) Imagine there is a trap-door in the floor; the line of its hinges is horizontal. Can you hold the trap-door so that its upper plane surface is (a) vertical, (b) horizontal, (c) neither vertical nor horizontal?

If a line or plane is *neither horizontal, nor vertical, it is called oblique.*

11. The plane of the wall (or blackboard) is vertical; can you draw on it (i) one or many or no horizontal lines; (ii) an oblique line?

12. The surface of the desk is a horizontal plane; can you draw on it (i) a vertical line; (ii) an oblique line?

13. Hold this book so that the surface of the page is an oblique plane. Can you draw on the page (i) a vertical line; (ii) a horizontal line?
GEOMETRICAL IDEAS

14. Open your compasses so that the legs are at right angles.
   (i) If one leg is vertical, what can you say about the other?
   (ii) If one leg is horizontal, must the other leg be vertical?
   (iii) If one leg is oblique, can the other leg be (a) vertical, (b) horizontal?

15. Name two vertical planes in the room, cutting one another. What can you say about their line of intersection?

16. Name in the room a vertical plane cutting a horizontal plane. What can you say about their line of intersection?

17. Hold your book so that the cover and the first page form two different oblique planes. Can two oblique planes intersect in (i) a horizontal line; (ii) a vertical line?

18. Hold your pencil in an oblique line. Can you find (i) a vertical plane, (ii) a horizontal plane, which passes through it?

19. Hold your book so that the plane of the cover is perpendicular to the plane of the first page. Need either plane be vertical?

20. An edge of a chimney of a house should be vertical. Is it usually perpendicular to the roof?

21. The mast of a ship is perpendicular to the deck. Is it always vertical?

Compass Directions and Angles

If you look at a weathercock, you will see two fixed horizontal rods, at right angles to each other, which show the directions North, East, South, West (see Fig. 83).

There is also a movable horizontal rod, which turns round so that it always points in the direction from which the wind is blowing.

ANGLES

The directions North, East, South, West, are the directions of horizontal straight lines.

You can mark out for yourself the direction, North, as follows: On a sunny day, at 12 o'clock, true noon (this is 1 P.M. in "Summer-Time"), hold a straight pole $AB$ so that one end $A$ rests on level ground, and $AB$ is vertical (see Fig. 84). Mark the line $AC$ of the shadow of the pole on the ground. Then $AC$, $A\rightarrow C$, points due North.

If you see the pointer of a weathercock swing round, you know that the direction of the wind has changed. But to describe the change, you must have some way of stating the amount of turning of the pointer, and the direction in which it has turned. The amount of turning is called the size of the angle through which the pointer swings; and the direction of turning is described as clockwise or counterclockwise according as it is the same as or opposite to the way in which the hands of a clock move (see Fig. 85).

If the horizontal lines $ON$, $OE$, $OS$, $OW$ point North, East, South, West, there are 4 right-angled corners at $O$; so, when the pointer $OP$ swings from the position $ON$ to the position $OE$, clockwise, the amount of turning is called one right angle; and if the pointer makes one complete revolution, the amount of turning is four times as much, that is, 4 right angles.

Any amount of turning can therefore be expressed in right angles or fractions of a right angle, where

one right angle = one-quarter of a complete turn.
COMPASS DIRECTIONS

EXERCISE IV. b

[Some of the following examples should be carried out practically.]

What are the angles turned through in Nos. 1-12? Express them in right angles and fractions of a right angle. Use Fig. 86.

1. Face and point E., turn clockwise and point S.
2. Face and point W., turn clockwise and point N.
3. Face and point N., turn clockwise and point W.
4. Face and point E., turn clockwise and point W.
5. Point S., turn counterclockwise and point W.
6. Point N., turn counterclockwise and point E.
7. Point N., turn clockwise and point N.E.
8. Point N.E., turn clockwise and point S.E.
9. Point S., turn counterclockwise and point N.W.
10. Point S.W., turn and point N.E.
11. Point W., turn clockwise and point S.E.
12. Point N.W., turn counterclockwise and point N.E.

What is the smallest angle between the following directions?

What is the final direction after the following turns?
19. Point E., turn clockwise through 1 right angle.
20. Point S., turn clockwise through \( \frac{3}{2} \) right angles.

21. Point W., turn clockwise through 3 right angles.
22. Point N., turn clockwise through 2 right angles.
23. Point S., turn counterclockwise through \( \frac{1}{2} \) right angles.
24. Point N.E., turn counterclockwise through \( \frac{3}{2} \) right angles.
25. Point S., turn clockwise through 1 right angle, and then through \( \frac{1}{2} \) right angles.
26. Point W., turn clockwise through \( \frac{3}{2} \) right angles, and then through \( \frac{1}{2} \) right angles.
27. Point N.W., turn clockwise through \( \frac{3}{2} \) right angles, and then counterclockwise through \( \frac{1}{2} \) right angles.
28. Through what angle do you turn at the command,
   (i) right turn; (ii) about turn;
   (iii) left incline; (iv) form fours?
29. Through what angle does the minute-hand of a clock turn in
   (i) 30 min.; (ii) 15 min.; (iii) 45 min.; (iv) 1 hr.;
   (v) 5 min.; (vi) 25 min.; (vii) \( \frac{1}{2} \) hr.; (viii) 2 \( \frac{1}{4} \) hr.?
30. Through what angle does the hour-hand of a clock turn in
   (i) 12 hr.; (ii) 3 hr.; (iii) 1 hr.; (iv) 2 hr.;
   (v) 5 hr.; (vi) 8 hr.; (vii) 24 hr.; (viii) 1 week?

Naming Angles

If \( \overrightarrow{OA} \), \( \overrightarrow{OB} \) are drawn to show two positions of the pointer of a weathercock (see Fig. 87), the amount the pointer has turned is called the angle \( \angle AOB \) or the angle \( \angle BOA \), written \( \angle AOB \) or \( \angle BOA \).

\( \overrightarrow{OA} \), \( \overrightarrow{OB} \) are called the arms of the angle and \( O \) is called the vertex of the angle.

When an angle is named, as here, by 3 letters, the middle letter is the corner or vertex of the angle, and the outside letters are any points on the two arms. Thus the angle in Fig. 88, which represents the difference of direction of the lines \( \overrightarrow{CN} \), \( \overrightarrow{CQ} \), or the amount of turning, when a pointer swings from the direc-
GEOMETRICAL IDEAS

2. Using only a ruler, draw as well as you can angles of the following sizes; use small arcs to indicate the angle.
   (i) $\frac{1}{2}$ complete turn; (ii) $\frac{1}{4}$ complete turn; (iii) $\frac{1}{2}$ right angle;
   (iv) 2 right angles; (v) $1\frac{1}{2}$ right angles; (vi) $3\frac{1}{4}$ right angles.

ANGLES

Name the arms of each of the following angles, and then name each angle by 3 letters.

3. In Fig. 92, (i) $a$; (ii) $b$; (iii) $c$.
4. In Fig. 93, (i) $a$; (ii) $\beta$; (iii) $\gamma$.

5. Draw any triangle $ABC$. Then name, in two ways, each of the angles of the triangle, (i) angle at corner $A$;
   (ii) angle at corner $B$; (iii) angle at corner $C$.

6. Draw a large quadrilateral $ABCD$; let the diagonals
   $AC$, $BD$ cut at $E$. Put marks on your figure to show that
   (i) $\angle BDA = a$; (ii) $\angle BCD = \beta$; (iii) $\angle CBD = \gamma$;
   (iv) $\angle EAB = \delta$; (v) $\angle BEC = p$; (vi) $\angle AED = q$.

7. Guess the size of the angles in Fig. 94 and describe them.

8. Draw a triangle $ABC$ and produce $AB$ to any point $R$.
   Take any point $P$ on $BC$; let $RP$ produced cut $AC$ at $Q$.
   Mark the figure to show that $\angle PRB = b$ and $\angle PCR = m$.
   Represent by 3 letters in as many ways as you can (i) the angle $m$; (ii) the angle $b$.

9. To change from the direction $OA$ to the direction $OB$, it is necessary to turn clockwise through 1 right angle. What
   angle is described, if you turn counterclockwise? Repeat, if the change of direction from $OA$ to $OB$ is
   (i) $3\frac{1}{2}$ right angles clockwise; (ii) $\frac{1}{2}$ right angle clockwise.
   Draw rough figures.
GEOMETRICAL IDEAS

Addition and Subtraction of Angles

In 3 hours, the hour-hand of a clock moves from the position OA to the position OB (see Fig. 95 (i)). In 3 hours, it turns through \( \angle AOB \), 1 right angle. In 1 hour more, it turns through \( \angle BOC \), 1 right angle (see Fig. 95 (ii)). Altogether, in the 4 hours, it turns through \( \angle AOC \), 1 1/2 right angles (see Fig. 95 (iii)). We say then that the result of adding \( \angle BOC \) to \( \angle AOB \) is \( \angle AOC \), and we write (see Fig. 95 (iii)), \( \angle AOB + \angle BOC = \angle AOC \).

In Fig. 95 (iii), what angle must be added to \( \angle AOB \) to make \( \angle AOC \)? Clearly the answer is \( \angle BOC \). We say therefore that the result of subtracting \( \angle AOB \) from \( \angle AOC \) is \( \angle BOC \), and we write \( \angle AOC - \angle AOB = \angle BOC \).

A Straight Angle

Suppose the pointer of a weathercock is pointing due East along OE (see Fig. 96), and that, owing to the wind changing, it swings round and points along OP; then it has turned through \( \angle EOP \). If it continues to swing round until it points due West along OW, it turns through an extra angle, \( \angle POW \); and the total angle turned through is \( \angle EOP + \angle POW \), which may be written \( \angle EOW \). But OW (due West) is in line with OE (due East), so \( \angle EOW \) is called a straight angle; it is half a complete turn or 2 right angles.

\[ \therefore \angle EOP + \angle POW = 2 \text{ right angles, if } EOW \text{ is a straight line.} \]

ANGLES AT A POINT

EXERCISE IV. d

1. Draw and letter a rough figure showing the minute-hand of a clock at 8 P.M., 8.15 P.M., 8.25 P.M.; name these positions OA, OB, OC. State, both using letters and in right angles, the angle turned through (i) from 8 P.M. to 8.15 P.M., (ii) from 8.15 P.M. to 8.25 P.M., (iii) from 8 P.M. to 8.25 P.M.

2. Draw and letter a rough figure showing the hour-hand of a clock at 3 P.M., 5 P.M., 6 P.M., 7 P.M.; name these positions CP, CQ, CR, CS. State, both using letters and in right angles, the angle turned through from (i) 3 P.M. to 6 P.M., (ii) 6 P.M. to 7 P.M., (iii) 3 P.M. to 7 P.M., (iv) 5 P.M. to 6 P.M., (v) 5 P.M. to 7 P.M.

3. Using only a ruler, draw, as well as you can, angles of the following sizes (see Fig. 97):
   (i) \( a + b \); (ii) \( c - a \); (iii) \( b + d \); (iv) twice \( c \);
   (v) \( d - c \); (vi) \( a + d \); (vii) \( d - b \); (viii) \( b + c - d \).

4. Draw a rough plan of the walk of a man who goes 1 mile N., then 1 mile N.W., then 1 mile W.
   Through what angles does he turn at the various corners?
   Mark them on your plan. What is the total angle turned through?

5. Fig. 98 shows the directions of a man’s walk. Through what angles does he turn at the various corners, clockwise or counterclockwise? Mark them on your own rough figure. What is the total angle turned through?

6. In Fig. 99 (not drawn accurately),
   (i) If \( a = \frac{1}{2} \text{ right angle and } b = \frac{1}{2} \text{ right angle, what is } \angle POR \)?
   (ii) If \( b - c = \frac{1}{2} \text{ right angle, what is } \angle QOS \)?
   (iii) If \( a = b = c = \frac{1}{2} \text{ right angle, what is } \angle POS \)?
   (iv) If \( \angle POR = \frac{1}{2} \text{ right angle and } \angle POQ = \frac{1}{2} \text{ right angle, what is } \angle QOR \)?
   (v) If \( \angle POQ = \frac{1}{2} \text{ right angle and } \angle POR = \frac{1}{2} \text{ right angle and } \angle POS = \frac{1}{2} \text{ right angles, what are } a, b, c \)?

7. In Fig. 99, express simply, both in large letters and in small letters, (i) \( \angle QOR + \angle ROS \); (ii) \( \angle POS - \angle ROS \);
   (iii) \( \angle QOS - \angle ROQ \); (iv) \( \angle POQ + \angle QOR + \angle ROS \).
8. In Fig. 100, \( \text{ACB} \) is a straight line.
   (i) If \( p = \frac{1}{2} \) right angle, what is \( q \) ?
   (ii) If \( q = 1 \frac{1}{2} \) right angles, what is \( p \) ?
   (iii) If \( \angle \text{ACD} = \frac{1}{2} \) right angle, what is \( \angle \text{BDC} \) ?
   (iv) What is \( p + q \) ?
   (v) What is \( p \), if \( q \) is twice \( p \) ?

9. In Fig. 101, (i) If \( a = 1 \frac{1}{2} \) right angles, what is \( b \) ?
   (ii) If \( b = 2 \frac{1}{2} \) right angles, what is \( a \) ?
   (iii) What is \( a + b \) ?

10. In Fig. 102, \( \text{AOB} \) and \( \text{COD} \) are straight lines.
    (i) If \( r = \frac{1}{2} \) right angle, find \( \angle \text{AOC} \), \( \angle \text{AOD} \), \( \angle \text{BOD} \).
    (ii) If \( a = 1 \frac{1}{2} \) right angles, find \( \angle \text{BOC} \), \( \angle \text{AOD} \), \( \angle \text{BOD} \).
    (iii) If \( \angle \text{AOD} \) is \( \frac{1}{4} \) right angle, what is \( \angle \text{BOC} \) ?
    (iv) If \( \angle \text{BOD} \) is \( 1 \frac{1}{2} \) right angles, what is \( \angle \text{AOC} \) ?

CONSTRUCTIONS BY PAPER-FOLDING

EXERCISE IV.e

1. Take any sheet of paper and make a crease slanting across it. Fold again so that, when unfolded, you have two creases at right angles.

2. Take an oblong sheet of paper \( \text{ABCD} \) (see Fig. 103) and fold it so that \( \text{AB} \) falls along \( \text{AD} \). Call the crease \( \text{AP} \).
   (i) What can you say about \( \angle \text{BAP} \) ?
   (ii) Fold again so that the new crease \( \text{AQ} \) *bisects* \( \angle \text{PAB} \), that is, divides \( \angle \text{PAB} \) into two equal angles, so that \( \angle \text{QAB} = \angle \text{QAP} \).
CHAPTER V
THE PROTRACTOR

General Names of Angles
Take your compass (or dividers) and gradually open it out. Keep one arm still and rotate the other. Fig. 106 shows successive positions.

Angle $a$ is less than 1 right angle; it is called an acute angle.
Angle $b$ is 1 right angle.
Angle $c$ is between 1 and 2 right angles; it is called an obtuse angle.
Angle $d$ is 2 right angles; it is sometimes called a straight angle.
Angles $e, f$ are between 2 and 4 right angles; they are called reflex angles.

Right Angles and Degrees
To avoid continually working in fractions of a right angle, we use a smaller unit for measuring angles of ordinary size.
If 1 right angle is divided into 90 equal angles, each is called a degree, written $^\circ$.

1 right angle = 90 degrees = 90°.
And so, 2 right angles = 180°; 4 right angles = 360°; etc.
Also, $\frac{1}{2}$ right angle = 45°; $\frac{3}{4}$ right angle = 2212°;
$\frac{1}{3}$ right angle = 30°; $\frac{3}{8}$ right angle = 60°; etc.

DEGREES AND RIGHT ANGLES

EXERCISE V, a.

1. Express in degrees:
   (i) 3 right angles; (ii) 1½ right angles; (iii) 2½ right angles;
   (iv) ¼ right angle; (v) ½ right angle; (vi) ¾ right angle;
   (vii) ¼ right angle; (viii) 1½ right angles; (ix) 8 right angles.

2. Express in right angles and fractions of a right angle:
   (i) 45°; (ii) 30°; (iii) 75°; (iv) 54°; (v) 135°;
   (vi) 120°; (vii) 100°; (viii) 210°; (ix) 300°; (x) 450°.

3. State whether the various marked angles in Fig. 107 are acute, obtuse or reflex.

4. Using only a ruler, draw, as accurately as you can, the following angles, marking and describing them as in No. 3:
   (i) $a = 80°$; (ii) $b = 100°$; (iii) $c = 240°$;
   (iv) $d = 350°$; (v) $e = 100°$; (vi) $f = 270°$;
   (vii) $g = 200°$; (viii) $h = 60°$; (ix) $k = 540°$.

5. Open your compass to an angle of 70°. What is the reflex angle between the arms?

6. Open your compass to an angle of 80°. Through what further angle must it be opened to bring the arms into line?

7. How many degrees does the hour-hand of a clock turn through in
   (i) 12 hours; (ii) 1 hour; (iii) 4 hours; (iv) 1½ hours?
8. How many degrees does the minute-hand of a clock turn through in
   (i) 30 min.; (ii) 5 min.; (iii) 25 min.; (iv) 40 min.?
   9. Since 2 o'clock, the minute-hand has turned through 130°. Through how much more will it have turned (i) by half-past two; (ii) by 3 o'clock; (iii) by a quarter to three?
   10. Since 3 A.M., the hour-hand has turned through 50°. Through how much more will it have turned (i) by 9 A.M.; (ii) by 3 P.M.; (iii) by 6 A.M.?
   11. What can you say about the two angles through which the hour-hand turns (without calculating them) (i) in $2\frac{1}{2}$ hours and in $3\frac{1}{2}$ hours; (ii) in $4\frac{1}{2}$ hours and $7\frac{1}{2}$ hours?
   12. What can you say about the two angles through which the minute-hand turns (without calculating them) (i) in 12 min. and in 18 min.; (ii) in 4 min. and in 11 min.; (iii) in 22 min. and in 38 min.; (iv) in 17 min. and in 47 min.?
   13. Fig. 108 is formed by two intersecting lines; find the other angles, (i) if $a = 40°$; (ii) if $b = 130°$; (iii) if $c = 45°$.

![Fig. 108.](image)

14. Fig. 109 is formed by two lines meeting a third straight line; find the other angles, (i) if $f = 40°$ and $g = 60°$; (ii) if $f = 50°$ and $g = 60°$; (iii) if $f = g$; (iv) if $g = 80°$ and $f = h$.

**Making a Semicircular Protractor**

The instrument for measuring angles is called a protractor; the best way of learning how to use it is to make one for yourself.

Take a piece of stiff paper, or thin cardboard, e.g. the cover of an old exercise book.

Draw a semicircle, centre $O$, radius 5 cm., and its diameter $AOB$ (see Fig. 110), and construct the positions of the hour-marks of the upper half of a clock-face, as on p. 30.

If two consecutive hour-marks are joined to the centre $O$, the angle between these radii is 30°. So, start by putting 0° at $A$, then put at the successive hour-marks 30°, 60°, 90°, 120°, 150°, and lastly 180° at $B$. Write these graduations along the inside of your semicircle.

An angle is measured by comparing it with an angle of known size; the following small figures show the angles, less than two right angles, at present "in stock".

![Fig. 110.](image)

![Fig. 111.](image)

The difference between these angles is too big; some intermediate graduations, that is more "stock sizes", are needed to make the instrument really useful.

Set your compass so that 3 steps with it along the circle just make up each of the equal arcs; you must do this by trial. You can now divide each of the original arcs into three equal portions.

When you have done this, the angles "in stock" are 10°, 20°, 30°, 40°, 50°, etc. Write these graduations, also, along the inside of the semicircle.

You now have a protractor with which you can measure angles, described counterclockwise from $OA$. It is useful to be able also to read off the sizes of angles described clockwise from $OB$.

To do this, start with 0° at $B$, and graduate the semicircle in the reverse direction, finishing with 180° at $A$. Write these graduations along the outside of the circle.

Each mark has now two graduations opposite it, one inside, and the other outside, the circle. What do you notice about them? What angles do they measure?
Use of Semicircular Protractor

(i) To measure a given angle \( \angle XYZ \).

First arrange your paper so that one arm of the angle (preferably the shorter), say \( YX \), runs across the desk; then look at the angle and see if it is more or less than 90°.

Now place the protractor so that the centre \( O \) coincides with the vertex \( Y \) of the angle, and so that part of the diameter \( BOA \) falls along the arm \( YX \) (see Fig. 112).

There are two graduation marks where \( YZ \) crosses the rim of the protractor; use your common sense to decide which to take.

It may be necessary to produce the arm \( YZ \) to make it cut the rim; if so, do this first of all.

(ii) To draw through a given point \( P \) a line making a given angle with a given line \( PQ \).

First arrange your paper so that \( PQ \) runs across the desk. Place the protractor so that the centre \( O \) coincides with the vertex of the angle you have to draw, in this case \( P \), and so that part of the diameter \( BOA \) falls along \( PQ \) (see Fig. 113).

There are two places on the rim of the protractor where the graduation, corresponding to the given angle, is marked; use your common sense to decide which one you want. (Is the angle you are going to draw acute or obtuse?) Mark a very small dot on your paper opposite the proper graduation; then remove the protractor and rule a line from \( P \) passing through this dot.

Fig. 113 (i) and (ii) show the position of the graduation you ought to select, in order to draw an angle \( \angle QPR \) equal to 70°. After you have drawn the angle, look at it and see whether it is acute or obtuse, so as to check your work.

Making a Rectangular Protractor

You can convert a semicircular protractor (centre \( O \)), into a rectangular protractor as follows:

Take a rectangular sheet of stiff paper or thin cardboard. Mark a point \( O \) at the middle of one of the longer edges, and place the semicircular protractor so that the points labelled \( O \) coincide and \( AB \) lies along the edge of the rectangle, as in Fig. 114.

With your ruler, draw short lines to show where the radii through the graduations on the semicircle meet the edges of the rectangle. Do not draw the whole radius, but only enough of it to show clearly where it meets the edge. Then graduate as before; if you have a rectangular protractor, look and see how the graduations are arranged, and do yours in the same way.

Use of Rectangular Protractor

It is easier to work with a semicircular protractor than a rectangular protractor; but the latter can of course be used in exactly the same way as the former.

There is, however, one advantage in using a rectangular protractor : you can draw with it an angle of given size without having to " make a dot " first.

Given a line \( PQ \), to draw \( PR \) so that \( \angle RPQ \) is 70°.

Arrange the protractor with the centre \( O \) at \( P \), and turned round so that the line \( PQ \) crosses the rim at the 70° graduation (see Fig. 115). Then rule the line \( PR \) along the edge \( OB \) of the protractor.

This method is quite easy when you have got used to it.
MEASURING AND DRAWING ANGLES

EXERCISE V.b

1. Use a protractor (if you have made one for yourself, use that) to measure the acute angles of your set-square.

2. Using only a ruler, draw, as well as you can, angles of $90^\circ$, $30^\circ$, $150^\circ$; label them $a$, $b$, $c$ in this order. Then measure them with a protractor, and write down the actual sizes.

Numbers 3-12 refer to Fig. 23 p. 18, reproduced at end of book; use the protractor in your box of instruments to measure them. [Write the answers to Nos. 3-4 at the end of your notebook, as they will be useful for reference later.]

3. (i) $\angle ABC$; (ii) $\angle ACB$; (iii) $\angle BAC$. What is their sum?

4. (i) $\angle DEF$; (ii) $\angle DFE$; (iii) $\angle EDF$. What is their sum?

5. (i) $\angle PSR$; (ii) $\angle SRQ$; (iii) $\angle RQP$; (iv) $\angle QPS$. What is their sum?

6. (i) $\angle SPR$; (ii) $\angle SPR$; (iii) $\angle PRQ$; (iv) $\angle RPQ$.

7. (i) $\angle PKS$; (ii) $\angle PKQ$: (iii) $\angle QKR$; (iv) $\angle RKS$.

8. (i) $\angle AMS$; (ii) $\angle ANQ$; (iii) $\angle EMS$; (iv) $\angle CNQ$.

9. The acute angle at which SQ cuts DF.

10. The obtuse angle at which AB cuts PR.

11. The obtuse angle at which CA cuts QR.

12. The acute angle at which PS cuts BC.

Use a protractor to draw the following angles. Make them point different ways.

13. (i) $90^\circ$; (ii) $35^\circ$; (iii) $145^\circ$; (iv) $72^\circ$.

14. (i) $160^\circ$; (ii) $20^\circ$; (iii) $63^\circ$; (iv) $126^\circ$.

15. (i) $220^\circ$; (ii) $290^\circ$; (iii) $340^\circ$; (iv) $265^\circ$.

16. Without looking at your protractor, say what other graduation is marked on it opposite (i) $50^\circ$; (ii) $110^\circ$.

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DRAWING TRIANGLES WITH GIVEN ANGLES

EXERCISE V.c

1. Draw the triangle ABC in Fig. 116; and measure BC.

First make an angle of $39^\circ$; then mark off the given distances along its arms. Fig. 116.

Draw the triangles in Nos. 2-4, and measure the third side, the units being cm.

2. 3. 4.

5. Draw the triangle XYZ, given $\angle X = 70^\circ$, $XY = XZ = 2.8$ inches. Measure YZ.

6. Draw the triangle ABC in Fig. 120; and measure AC.

First make AB 7.5 cm., then draw angles at A and B of the given sizes.

7. Draw $\triangle PQR$, given $PQ = 3.5$ inches, $\angle P = 50^\circ$, $\angle Q = 35^\circ$. Measure QR.

8. 9. 10.

11. Draw the triangles in Nos. 8-10, and measure the longest side.
11. Draw a circle, centre O, radius 5 cm., and mark any point A on it. Use your protractor to find points B, C, D, E on the circle so that the angles AOB, BOC, etc., are each \( \frac{1}{3} \) of 360° = 72° (see Fig. 124).

Draw the figure ABCDE. It is called a regular pentagon. Measure AB.

Points of the Compass

Just as the direction midway between North and East is called North-East (NE.), so the direction midway between North and North-East is called North North-East (N.N.E.), and the direction midway between East and North-East is called East North-East (E.N.E.), and similarly for the other midway directions.

The corresponding marks on the circle which show these directions are called the “points of the compass”. [The remaining points of the compass are described in Ex. V. d, Nos. 9-12, but these questions may be omitted at a first reading.]

**EXERCISE V. d**

1. Draw a circle of radius 5 cm. Then mark on it the sixteen “points of the compass” without looking at Fig. 125; so shut up your book. Keep your figure and use it for Nos. 2-8. State in degrees the angles between the following directions:

2. (i) W. and NW.; (ii) W. and N.NW.; (iii) W. and S.SW.
3. (i) SE. and E.N.E.; (ii) SW. and N.N.E.; (iii) W.NW. and NE.
4. (i) S.SE. and E.SE.; (ii) N.NW. and S.SW.; (iii) E.N.E. and W.NW.

What is the final direction in the following cases?

5. Point E.N.E., then turn through 90° clockwise.

---

**THE PROTRACTOR**

6. Point W.N.W., then turn through 90° counterclockwise.
7. Point W.S.W., then turn through 90° counterclockwise.
8. Point N.N.E., then turn through 45° clockwise.

Look at the figure while reading the following remarks:

The direction midway between N. and N.N.E. is called N. by E.

The direction midway between NE. and N.N.E. is called NE. by N.

The direction midway between NE. and E.N.E. is called NE. by E.

The direction midway between E. and E.N.E. is called E. by N.

If you can see what plan has been followed in settling these names, answer the following questions; keep looking at the figure.

9. What would you call the following directions?
   (i) midway between N. and N.N.W.;
   (ii) midway between NW. and N.N.W.;
   (iii) midway between NW. and W.NW.;
   (iv) midway between W. and W.NW.

10. What do the following directions mean?
   (i) E. by S.; (ii) SE. by S.; (iii) SE. by E.;
   (iv) S. by W.; (v) SW. by S.; (vi) W. by S.

11. What is the angle between W. and W. by N.?
12. What is the angle between S. by E. and S. by W.?

Compass Bearings

Any direction may be described conveniently in the following way:

To obtain the direction OP in Fig. 126, start by facing North, then turn through 50° towards the East. The direction OP is therefore called N. 50° E. or 50° E. of N.

Similarly, to obtain the direction OQ, start facing South, then turn through 32° towards the East. The direction OQ is therefore called S. 32° E. or 32° E. of S.
GEOMETRICAL IDEAS

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Directions should always be measured either from the North or from the South, not from the East or West. Thus in Fig. 126, where OR makes 20° with OW, we must say, OR makes 90° + 20° = 70° with OS, and so the direction of OR is S. 70° W. or 70° W. of S. Do not call it W. 20° S.

True Bearings

The Army method of describing directions is to state the angle, described clockwise, which the given direction makes with the north-line ON. This is called the true bearing. Thus, in Fig. 126, the true bearings of P, Q, R from O are respectively 50°, 180° − 32° = 148°, 270° − 20° = 250°. True bearings therefore run from 0° to 360°.

COMPASS AND TRUE BEARINGS

EXERCISE V.e

1. Write shortly the following final directions: draw free-hand figures.
   (i) Point N., turn 30° towards East;
   (ii) Point N., turn 80° towards West;
   (iii) Point S., turn 70° towards West;
   (iv) Point S., turn 100° towards East;
   (v) Point E., turn 10° towards South;
   (vi) Point W., turn 25° towards North;
   (vii) Point S.E., turn 10° towards South;
   (viii) Point S.W., turn 60° towards North;
   (ix) A true bearing of (a) 35°; (b) 165°;
   (c) 240°; (d) 240°.

2. What are the true bearings of the directions in No. 1 (i)-(viii)?

3. By drawing free-hand figures, find the angle AOB, if the directions from O of A and B are
   (i) N. 15° E., N. 60° E.; (ii) N. 40° E., S. 30° E.;
   (iii) N. 25° E., N. 30° W.; (iv) S. 40° W., S. 60° E.;
   (v) S.W., S. 10° E.; (vi) S. 70° W., N. 60° E.

4. Repeat No. 3, if the true bearings of A and B from O are
   (i) 55°, 100°; (ii) 60°, 185°; (iii) 20°, 220°;
   (iv) 40°, 350°; (v) 100°, 300°; (vi) 80°, 260°.

5. In Fig. 127, state the directions of
   (i) P from A; (ii) P from B.
   What is the true bearing of P from B?

                 Fig. 127.

6. In Fig. 128, the direction of Q from C is 40° E. of N. What is ∠QCD?

7. In Fig. 128, the direction of Q from D is 60° W. of N. What is ∠QDC?

8. In Fig. 128, the true bearing of Q from C is 33°. What is ∠QDC?

9. In Fig. 128, the true bearing of Q from D is 305°. What is ∠QDC?

10. Express more simply (i) 160° E. of N.; (ii) 110° W. of S.; (iii) 140° W. of N.

Scale-diagrams

Measurements taken out of doors for finding heights and distances are often recorded on sketch plans, and the required results are then obtained from accurate scale-diagrams made indoors. The next exercise contains drawing problems corresponding to the outdoor work described in the Appendix.

Before starting on the accurate scale-diagram, make a free-hand sketch showing the data, unless one is already provided.

EXERCISE V.f

1. Fig. 129 represents a tower AB, and a point of observation C on the ground. Draw the figure to scale, 1 cm. to 10 ft., and find the height of the tower.
2. Fig. 130 represents the data obtained by observations of a house (H is its front door) from two points P, Q on a straight road. Find from a scale-diagram (i) how far H is from P and Q, (ii) the distance of H from the road.

3. From O (see Fig. 131), C bears due North and D bears N. 28° E. Find from a scale-diagram (i) the distance of D from C, (ii) the bearing of D from C.

4. The angle of elevation of the top of a tree from a point, at the same level as its foot and 25 yards from the foot, is 27°. Find the height of the tree in feet.

5. A kite is flown at the end of a string 300 feet long. What is the height of the kite, when the string makes an angle of 65° with the ground?

6. Guildford is 4 miles north of Godalming, and Compton is 3 miles south-west of Guildford.
   (i) How far is Compton from Godalming?
   (ii) What is the bearing of Compton from Godalming?

7. A is half a mile south of B; a church tower T bears N.W. from A and S. 80° W. from B. Make a plan of A, B, T on a scale of 1 cm. to 100 yards.
   (i) How far is T from the straight road AB?
   (ii) What length of the road between A and B is more than 800 yards from T?
   (iii) There is a windmill K, half-way between A and B. What is the bearing of T from K?

8. In Fig. 132, C and D are points on the ground, 60 feet apart. State what observations have been made and find the height of the tower PQ.

9. A canal has straight banks. A, B are two points 50 yards apart on one bank, and C is a point on the opposite bank; $\angle BAC = 48°$ and $\angle ABC = 32°$. Find the distance of C from A and the breadth of the canal.

10. Draw a triangle OXY so that $OX = OY = 2$ inches and $\angle XOY = 72°$. This represents 3 towns O, X, Y on a scale of 1 inch to 20 miles; also X is due North of O, and Y lies to the eastward.
   (i) How many miles is X from O and from Y?
   (ii) What is the bearing of Y from X?
   Add to the map a town Z which bears N. 35° E. from O and N. 70° E. from X. How many miles is Z from X?

11. Three trees A, B, C stand on the edge of a circular pond; A and B are at opposite ends of a diameter and are 60 yards apart; $\angle CAB = 40°$. Find how far C is from A and from B.

12. Two points O, P on the ground are 50 yards apart and are respectively 70 yards and 40 yards from the foot A of a tower AB. The angle of elevation of the top B of the tower from O is 25° (see Fig. 197, p. 95). Make separate scale-drawings of the triangles OAP, OAB, PAB, and find $\angle OAP$ and the angle of elevation of B from P.

**ADJACENT AND VERTICALLY OPPOSITE ANGLES**

**EXERCISE V. g.**

1. Point North along ON (Fig. 133); turn and point along OR; continue turning and point South along OS.
   (i) What is the total angle, in degrees, turned through?
   (ii) If $x° = 60°$, what is $y°$?
   (iii) If $y° = 130°$, what is $x°$?
   (iv) What equation connects $x$ and $y$?

2. Point West; then turn through $110°$. How much more must you turn to point East? Draw a rough figure.

3. In Fig. 133, $\angle NOR$ is $\frac{1}{4}$ of an "about turn", what is this in degrees? What is $\angle ROS$?
4. In Fig. 134, ON, OS point North and South.

   (i) If OP points N. 55° W., what is ∠POS?
   (ii) If OQ points S. 40° W., what is ∠QON?
   (iii) If ∠POS = 115°, how does OP point?
   (iv) If ∠NOP = 72° and ∠POQ = 64°, what are
       ∠NOQ, ∠QOS?
   (v) If ∠SOQ = 34° and ∠SOP = 118°, what are
       ∠POQ, ∠PON?

Fig. 134.

5. If, in Fig. 134, ∠NOP = ∠POQ = ∠QOS, find the size of each.

6. If, in Fig. 134, ∠NOP = 4x°, ∠POQ = 3x°, ∠QOS = 2x°, find x.

7. Fig. 135 represents two sticks crossing one another at O; 
   ∠DOB shows how much CD must be turned so that OD may
   lie along OB. What other angle also shows this?
   What can you say about ∠AOD?
   The pair of angles AOC, BOD are called vertically opposite
   angles, as also the pair of angles AOD, BOC. The word
   "vertically" means merely that the angles have the same
   vertex; this use of the word has no connection at all with its
   use in the phrase "vertically downwards".

8. I open a pair of scissors so that the angle between
   the blades is 40°, what is the angle between the handles?

9. Draw a rough figure like Fig. 136.
   (i) Mark in it another angle of 30° and another angle
       of 130°.
   (ii) Mark in it the sizes of the remaining angles.
   If two angles have the same vertex and one arm in common and
   lie on opposite sides of that arm, they are called adjacent angles.
   Thus in Fig. 137, ∠p and ∠q are adjacent angles.

Fig. 135. Fig. 136. Fig. 137.

10. Using a ruler, but not a protractor, draw, as well as you
    can, figures showing two adjacent angles, p and q, if
    (i) ∠p = 119°, ∠q = 52°; (ii) ∠p = 81°, ∠q = 105°;
    (iii) ∠p = 133°, ∠q = 47°.

Facts about Angles at a Point

The examples of Ex. V.g illustrate the facts stated
below. Whenever it is necessary to refer to any of these
facts, it saves time to use the abbreviations suggested.

Property of Adjacent Angles

(i) If ∠a and ∠b are adjacent angles formed by one
straight line meeting a second straight line,
∠a + ∠b = 180°.

For reference: adj. ∠s on st. line.

(ii) If ∠POQ and ∠QOR are adjacent angles and if
∠POQ + ∠QOR = 180°, then POR is a straight line.

For reference: adj. ∠s 180°.

Property of Vertically Opposite Angles

If two straight lines intersect, then the vertically oppo-
site angles are equal.
In Fig. 140, a = c and b = d.

For reference: vert. opp. ∠s.

Two Important Phrases

Two angles which together make up 180° are said to
be supplementary or to be supplements of each other.
E.g. 70° and 110° are supplementary angles.
Two angles which together make up 90° are said to be
complementary or to be complements of each other.
E.g. 70° and 20° are complementary angles.
CALCULATION OF ANGLES AT A POINT

EXERCISE V. h

Calculate the unknown angles in the following figures:

1. 

\[ \begin{align*}
&\text{Fig. 141.} \\
&\text{(i)} \\
&\text{(ii)} \\
&\text{(iii)} \\
&42^\circ, 100^\circ, a \\
&60^\circ, 80^\circ, b \\
&140^\circ, 160^\circ, c
\end{align*} \]

2. 

\[ \begin{align*}
&\text{Fig. 142.} \\
&\text{(i)} \\
&\text{(ii)} \\
&\text{(iii)} \\
&210^\circ, d \\
&x^\circ, x^\circ, e \\
&140^\circ, 22^\circ
\end{align*} \]

3. 

\[ \begin{align*}
&\text{Fig. 143.} \\
&\text{(i)} \\
&\text{(ii)} \\
&\text{(iii)} \\
&90^\circ, 140^\circ, 40^\circ, f \\
&140^\circ, 90^\circ, 140^\circ
\end{align*} \]

4. 

\[ \begin{align*}
&\text{Fig. 144.} \\
&\text{(i)} \\
&\text{(ii)} \\
&\text{(iii)} \\
&2^\circ, 3^\circ, 2^\circ \\
&10^\circ, 35^\circ, m \\
&52^\circ, 52^\circ, 52^\circ
\end{align*} \]

5. What are the supplements of (i) 35°, (ii) 155°?

6. What are the complements of (i) 35°, (ii) 75°?

7. State the property of adjacent angles, using the word "supplementary".

\[ \begin{align*}
&\text{Fig. 145.} \\
&\text{Fig. 146.}
\end{align*} \]

8. A wheel has 6 spokes; what is the angle between two adjacent spokes?

9. Using your protractor and compass, draw a figure to represent a wheel with 5 spokes.

10. What is \( x \) if \( \overrightarrow{N} \) E is the same direction as S. 50° E.? 

11. What angle equals 4 times its supplement?

12. What angle equals 5 times its complement?

**General Statements**

All the examples in the last exercise dealt with angles whose sizes were given or could be calculated. We can often obtain general results by using letters instead of special numbers in the data, just as is done in Algebra.

**EXERCISE V. j**

1. In Fig. 145, ON bisects \( \angle POQ \). Calculate \( \angle RON \), if 
   (i) \( \angle ROP = 94^\circ \) and \( \angle ROQ = 62^\circ \); 
   (ii) \( \angle ROP = 2x^\circ \) and \( \angle ROQ = 2y^\circ \).

Now prove that \( \angle RON \) is always half the sum of \( \angle s ROP, ROQ \).

2. In Fig. 146, \( \overrightarrow{AOB} \) is a straight line; also \( \overrightarrow{OH}, \overrightarrow{OK} \) bisect the angles \( \angle AOC, \angle BOC \).

Calculate \( \angle MOD \), if 
   (i) \( \angle BOC = 70^\circ \); (ii) \( \angle AOC = 130^\circ \); (iii) \( \angle BOC = 2x^\circ \).

Now make and prove a general statement about this figure.
3. In Fig. 147, $\angle BAC$ and $\angle PAQ$ are right angles. Calculate $\angle QAC$, if
   (i) $\angle BAP = 25^\circ$; (ii) $\angle BAP = y^\circ$.
   Make and prove a general statement about this figure.

4. In Fig. 147, $\angle BAC$ and $\angle PAQ$ are right angles. Calculate $\angle BAQ$, if
   (i) $\angle CAP = 68^\circ$; (ii) $\angle CAP = z^\circ$.
   Prove that the angles $\angle CAP, \angle BAQ$ are supplementary.

5. In Fig. 148, OP bisects $\angle AOB$ and OQ bisects $\angle BOC$.
   (i) Prove that $\angle POQ = \frac{1}{2} \angle AOC$.
   (ii) Prove that $\angle AOQ + \angle POQ = 3 \angle POQ$.

6. In Fig. 149, one straight line is cut by two other straight lines. Give reasons for the following:
   (i) If $a = p$, then $a = r$.
   (ii) If $b = s$, then $d = q$.
   (iii) If $b = q$, then $b + p = 180^\circ$.
   (iv) If $c + s = 180^\circ$, then $c = p$.

---

**CHAPTER VI**

**PARALLELS**

Parallel Lines

On an ordinary sheet of ruled paper, the printed lines are parallel. **Two lines are called parallel, if a plane can be found which contains them and if also the lines never meet however far they are produced.**

The symbol for "parallel" is $\parallel$.

Fig. 150 represents a box. The edges $AB, DC$ are parallel lines; they lie in a plane, the top of the box, and never meet however far they are produced. Hold the box so that $AB, HG$ lie in a horizontal plane, say, the plane (produced) of the top of your desk, or in the surface of the water in a bowl. This shows how a plane can be found containing $AB$ and $HG$; also they never meet, however far they are produced; they are therefore parallel.

$AB$ and $DH$ also never meet however far they are produced, but they are not parallel because no plane can be found which contains both of them. Hold the box so that $AB$ and the corner $D$ lie on the top of the desk, then $DH$ lies outside this plane.

**Facts about Parallel Lines**

*Lines at right angles to the same line are parallel, if they lie in a plane.*

The lines on a sheet of ruled paper are at right angles to one pair of edges.
Use your ruler to draw the straight line \( PQ \). Place your set-square \( ABC \) against the ruler, as in Fig. 151, and rule along \( AC \). Then slide the set-square into the position \( A'B'C' \), keeping the ruler firm, and rule along \( A'C' \).

\( AC, A'C' \) are both at right angles to \( PQ \) and are parallel lines.

*Lines at a given angle to the same line are parallel, if they lie in a plane.*

In Fig. 151, rule along \( AB \) and along \( A'B' \).

\( BA, B'A' \) make equal angles with \( PQ \) and are parallel lines.

Make any crease in a sheet of ruled paper; the ruled lines all make equal angles with the line of the crease.

*Lines in the same direction are parallel, if they lie in a plane.*

In Fig. 152, the line \( ON \) points due north, and the parallel lines \( AB, CD, HK \) each make \( 30^\circ \) with \( ON \) as shown. Then the direction of each of these lines is \( N. 30^\circ \) E.

*Parallel lines are everywhere the same distance apart.*

Two of the lines on a sheet of ruled paper are the same distance apart on the left of the paper as on the right, and as in the middle.

If the edges of a path are parallel lines, the width of the path is the same all the way along it.

If a number of pins are put in the blackboard at a given distance from the top edge, they trace out on the board a straight line parallel to the top edge.

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**PARALLEL LINES AND PLANES**

**EXERCISE VI. a (Oral)**

1. Point out pairs of parallel lines in the room, (i) on your desk, (ii) on this book, (iii) on the door, (iv) on the floor.

2. Explain why the following may or may not be parallel lines, (i) the two edges of a path, (ii) the lines of a railway track.

3. Draw two circles with the same centre. (i) Are the curves everywhere the same distance apart? (ii) Are they parallel lines?

4. Can you draw parallel lines, one on the top and (a) the other on the bottom, (b) the other on the side of the box? Can you draw parallel lines on the curved surface (a) of a cylinder, (b) of a cone?

5. If a sheet of ruled paper is wrapped round a cylinder, so that the ruled lines remain straight, will they remain (i) the same distance apart, (ii) parallel?

6. If a sheet of ruled paper is wrapped round a cylinder so that the ruled lines are no longer straight, do they remain the same distance apart?

7. Are any two slats of the cover of a roll-top desk parallel? Do they remain parallel when the desk is being opened?

*Lines directed towards the same point are not parallel; but if the point is sufficiently far away, they are very nearly parallel and may be regarded as parallel.*

8. [i] All the boys in a class point at one corner of the blackboard; are their arms parallel?

(ii) Two telescopes are pointed at the same star; are they parallel?

9. [i] Two roads in a town run due north; are they parallel? (ii) Two roads, one in England, the other in America, run due north; are they parallel?

10. [i] In Fig. 150, p. 75, state which edges of the box are parallel to \( EA \).

(ii) Name 4 edges which cannot meet \( GH \), however far produced, but which are not parallel to it.
Planes like the upper and lower surface of a plank, which will never meet however far they are produced, are called parallel planes.

11. Point out 3 planes in the room which are parallel to the floor. Are all horizontal planes parallel?

12. Point out a wall whose plane is parallel to that of the door, when shut. Are these planes parallel when the door is open? Are all vertical planes parallel?

13. Hold a book obliquely. Are the front and back covers parallel planes when the book is (i) shut, (ii) open?

14. (i) If two planes are parallel, must they have equal slopes?
   (ii) If two planes have equal slopes, must they be parallel?

15. (i) Point out 3 pairs of vertical lines in the room. Are the lines in each pair parallel?
   (ii) Point out 3 pairs of horizontal lines in the room. Must two horizontal lines be parallel?

16. Can you draw on your desk lines parallel to every line marked on the floor? Can you do so if your desk is tilted?

Use of Set-square for drawing Parallels

By sliding a set-square along a ruler, as indicated on p. 76, a line can be drawn parallel to any given line and passing through any given point. The process must be practised; the essential thing is to hold the ruler firmly, while sliding the set-square along it.

Parallelogram

If a 4-sided figure ABCD is formed by drawing two pairs of parallel lines (see Fig. 153), it is called a parallelogram. This may be written for short, par. The figure is usually drawn with the sides AB, BC, CD, DA not produced beyond the corners.

CONSTRUCTION OF FIGURES CONTAINING PARALLEL LINES

EXERCISE VI. b

[Use a set-square whenever you draw parallel lines in this exercise.]

1. Rule a line and mark 3 points at different distances from it. Through each point draw a line parallel to the first line.

2. Rule a line; use your set-square to draw 5 lines perpendicular to it. Are these 5 lines parallel?

3. Draw a figure showing a road running south and two roads crossing it at right angles. What are their directions?

4. In Fig. 154, \( \angle ABC \) and \( \angle DCB \) are right angles. (i) Is AB parallel to BC? (ii) Is AD parallel to BC?
   (iii) Is the perpendicular from D to AB parallel to BC?

5. Draw a figure showing two roads running north and a road crossing them running southeast. Show the sizes of the angles in your figure.

6. Make a sheet of plain paper into a sheet of ruled paper. [Start by marking off equal distances along a line drawn up the paper; then draw perpendiculars to this line.]

7. Draw two parallelograms of different shapes. Measure their sides in cm.

8. Draw any \( \triangle ABC \); through A, B, C draw lines \( ZAY, XAZ, YCZ \parallel BC, CA, AB \), to form \( \triangle XYZ \). Measure \( YA, AZ, ZB, BX \).

9. Draw a par. \( \triangle ABC \) with \( \angle ABC = 90^\circ \). What are the other angles? Measure \( AC \) and \( BD \). What is the name of this special kind of par.?

10. Draw \( \triangle ABC \) such that \( AB = 4 \text{ cm}, BC = 3 \text{ cm}, \angle ABC = 90^\circ \). Complete the par. \( \triangle ABC \). [This means, draw lines through A, C parallel to BC, BA to meet at D.] Measure \( AC, BD \).
PARALLEL LINES AND A LINE CUTTING THEM

Corresponding Angles

In Fig. 155, $AB$ runs north; $CD$, $EF$ are parallel lines in the direction $N$, $62^\circ$ E. The angles $\angle BPD$, $\angle BQF$ are called corresponding angles; each is $62^\circ$. The angles $\angle AQF$, $\angle APD$ are also called corresponding angles; each is $(180^\circ - 62^\circ)$.

Name the angle which corresponds (i) to $\angle BQE$, (ii) to $\angle AQE$. What are the sizes of these angles? In Fig. 151, p. 76, the angles $\angle PBA$, $\angle PA'B'$ correspond because their arms $BA$, $B'A'$ are ruled along the same side of the set-square. These examples illustrate the following statement:

If a line cuts two parallel lines, the corresponding angles are equal.

For reference: corresp. $\angle s$, $CD$, $EF$ ||.

CORRESPONDING ANGLES

EXERCISE VI.c

[Use a set-square, whenever you draw parallel lines in this exercise.]

1. A man walks due north along $AB$ (see Fig. 156), then turns and walks in the direction $N$, $53^\circ$ E, along $DD$; at $D$ he turns and walks due north along $DQ$.

(i) Find all the angles in Fig. 156.

(ii) Through what angle, in degrees, right-handed or left-handed, does he turn at $B$? at $D$?

(iii) Using the letters in Fig. 156, name pairs of corresponding angles.

PARALLELS

2. A man walks along the zig-zag path $APQD$ in Fig. 157.

(i) What is the size of the angle, right-handed or left-handed, through which he turns at $P$? at $Q$?

(ii) What follows if $APB$, $CQD$ are lines in the same direction?

3. Repeat No. 2 for the zig-zag path $CQPB$.

4. Draw two parallel lines and three lines crossing them (see Fig. 158). The letters $a$, $b$ show two corresponding angles, so do $b$, $b'$; mark in the same way other pairs of corresponding angles $c$, $c'$; $d$, $d'$; etc. You should get as far as $l$, $l$.

5. If in Fig. 158, $a=105^\circ$, $b=70^\circ$, $c=125^\circ$, find all the other angles; mark them on your own figure.

6. Draw the parallel $ABCD$ in Fig. 159. Find the other angles in the figure. Use the letters in Fig. 159 to name all the pairs of corresponding angles.

Alternate Angles

In a capital $Z$, there are two parallel lines and a line meeting them; the two angles on opposite sides and at opposite ends of the meeting line are called alternate.

In Fig. 160, $AB$ represents a pencil lying on a table, with its point at $A$. Turn it about $B$ through an angle $x^\circ$ clockwise, its new position is $A'B'$; next turn it about $A'$ through an angle $y^\circ$ counterclockwise, its final position is $A''B'$. If the final direction $A''B'$ in which the pencil points is the same as the first direction $AB$, the two rotations, $x^\circ$ clockwise and $y^\circ$ counterclockwise, balance each other, and so $x^\circ = y^\circ$. 
Take a sheet of paper with parallel edges CD, EF and make a crease PQ (Fig. 161). Divide the paper into two pieces by tearing along the crease and show that the two pieces fit each other. Mark the equal angles α, α; β, β; then replace the pieces in their original position. These examples illustrate the following statement.

If a line meets two parallel lines, the alternate angles are equal.

*For reference:* alt. ∠s, CD, EF ⊥.

Thus in Fig. 161, if CD is parallel to EF, ∠CPQ = ∠PQF, alternate angles.

In this figure, ∠DPQ and ∠PQE are also on opposite sides and at opposite ends of PQ, and are therefore called alternate angles. If ∠CPQ = 130° = ∠PQF, what is ∠DPQ? What is ∠PQE?

**ALTERNATE AND INTERIOR ANGLES**

**EXERCISE VI. d**

*Use a set-square, whenever you draw parallel lines in this exercise.*

1. In Fig. 162, HK is drawn parallel to YZ, and α = 65°. (i) What is β? (ii) What is γ? (iii) What is δ? Give reasons.

2. In Fig. 162, HK is parallel to YZ and β = 70°. (i) What is δ? (ii) What is α? (iii) What is γ? Give reasons.

**PARALLELS**

3. Fig. 163 contains two parallel lines. (i) Find e, f, g if d = 120°. (ii) Find d, f, g if e = 115°. Give reasons.

4. Draw a large capital N and mark the alternate angles.

5. What are the names of the following pairs of angles in Fig. 163? (i) c, f; (ii) f, g; (iii) d, g; (iv) d, f; (v) d, e.

6. Draw a figure like Fig. 164, formed by two pairs of parallel lines. Mark on your own figure,
   (i) an angle alternate to p, call it q1;
   (ii) two angles alternate to q, call them q1 and q2;
   (iii) two angles alternate to r, call them r1 and r2.
Now name the angles which correspond to q.

7. Draw a figure like Fig. 164, formed by two pairs of parallel lines. Mark on your own figure,
   (i) two angles corresponding to p, call them p1 and p2;
   (ii) two angles corresponding to r, call them r1 and r2.
Now name the angles which are alternate to q.

8. Draw a figure like Fig. 164, formed by two pairs of parallel lines. Mark in your figure the sizes of all the angles,
   (i) if p = 72°, (ii) if r = 110°.

9. In Fig. 165, the arrows denote || lines.
   (i) If a = 70°, find q, p, b; find also a + b and p + q.
   (ii) If b = 115°, find p, a, q; find also a + b and p + q.

In Fig. 165, p and q are called interior angles on the same side of the cutting line, sometimes, simply, interior angles; a and b are also interior angles.
GEOMETRICAL IDEAS

10. In Fig. 166, the arrows denote \( \parallel \) lines.
(i) If \( a = 65^\circ \), find \( b, c \) and \( b + c \).
(ii) If \( f = 80^\circ \), find \( d, e \) and \( e + f \).
(iii) If \( b = 130^\circ \), find \( a, c \) and \( b + c \).

11. In Fig. 166, the arrows denote \( \parallel \) lines.
If \( a = x^\circ \), find in terms of \( x \), giving reasons, (i) \( c \), (ii) \( b \), (iii) \( b + c \).

What can you say about the interior angles on the same side of a line cutting two \( \parallel \) lines?

12. In Fig. 166, the arrows denote \( \parallel \) lines.
(i) If \( c = 35^\circ \), what is \( b \)?
(ii) If \( e = 95^\circ \), what is \( f \)?

Corresponding, Alternate, and Interior Angles

The facts which have been illustrated in Ex. VI. \( c, d \) are as follows:

In Fig. 167, if \( PQ \) is parallel to \( RS \),
(i) \( a = d \), corresponding angles;
(ii) \( b = e \), alternate angles;
(iii) \( c + d = 180^\circ \), interior angles on same side of cutting line.

For reference: corresp. \( \angle s, PQ, RS \parallel \); alt. \( \angle s, PQ, RS \parallel \); sum of int. \( \angle s 180^\circ, PQ, RS \parallel \).

Transversal

A line which cuts two or more other lines is called a transversal (trans = across). Thus in Fig. 167, \( LM \) is a transversal of the \( \parallel \) lines \( PQ, RS \).

ANGLES IN FIGURES CONTAINING PARALLEL LINES

EXERCISE VI. e

In this exercise, arrows in the diagrams indicate that lines are given parallel. Dotted lines are suggestions for a construction.
17. In Fig. 178, name (i) two equal alternate angles, (ii) two equal corresponding angles.

If $\angle ABC = 50^\circ$ and $\angle BAC = 60^\circ$, find $\angle ACD$ and $\angle ACB$.

What is $\angle A + \angle B + \angle ACB$?

18. In Fig. 178, $\angle ACK = 36^\circ$ and $\angle KCD = 70^\circ$, find each angle of $\triangle ABC$, and then find their sum.

Parallel Lines contrasted with Lines not parallel

Fig. 179 shows the angles which a transversal makes with any two lines. The names, corresponding, alternate, interior, may still be used to describe the angles in this figure.

Thus $b$ and $f$ are corresponding angles, but they are not equal, because $PQ$ is not parallel to $RS$.

For the same reason, the alternate angles $d$ and $f$ are not equal; and the sum of the interior angles $d$ and $e$ is not $180^\circ$. If we then know what angles a transversal makes with a pair of lines, we can say whether the lines are parallel or not, by using the following tests:

Angle Tests for Parallel Lines

In Fig. 179,

(i) If $b = f$, corresponding angles, then $PQ$ is parallel to $RS$.

(ii) If $d = f$, alternate angles, then $PQ$ is parallel to $RS$.

(iii) If $e + f = 180^\circ$, interior angles, then $PQ$ is parallel to $RS$.

EXERCISE VI.1

1. In Fig. 180, $ABN$ points north, $AP$ points N. $63^\circ$ E., $BQ$ points N. $65^\circ$ E. Draw a rough figure and show on it the sizes of all the angles. Show also a line $AQ$ parallel to $BQ$. On which side of $AB$ do the lines $AP, BQ$ meet when produced? Give reasons.

2. Repeat No. 1, supposing that $AP$ points N. $75^\circ$ E. and that $BQ$ points N. $70^\circ$ E.

3. In Fig. 181, state whether $EP$ is $\parallel$ to $FQ$, with the following data. If $EP$ is not parallel to $FQ$, on which side of $EF$ will they meet when produced?

   (i) $b = 80^\circ$, $c = 95^\circ$; (ii) $b = 80^\circ$, $d = 78^\circ$;

   (iii) $b = 75^\circ$, $c = 105^\circ$; (iv) $b = 75^\circ$, $d = 78^\circ$.

4. What can you say about the angles $b, c$ in Fig. 181 if $PE$ and $QF$ meet below $EF$, when produced?

5. Using a protractor, find the angle at which the line $PQ$ would cut the line $RS$ in Fig. 179, if produced, without producing them.
6. Fig. 182 shows two straight roads AB, CD crossed by 3 other straight roads. What is the difference of direction of AB and CD? Hence find (i) $x$, (ii) $y$.

Fig. 182.

Find out whether Figs. 183-186 contain parallel lines; give reasons.

7.  

8.  

9.  

10.  

Fig. 183.  
Fig. 184.  
Fig. 185.  
Fig. 186.

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**GENERAL STATEMENTS**

**EXERCISE VI. G**

In this Exercise, arrows indicate that lines are given parallel. Dotted lines are suggestions for a construction.

_Give reasons, using abbreviated references._

1. In Fig. 187, prove that $a = b$.
2. In Fig. 187, prove that $a + c = 180^\circ$.

Fig. 187.

3. In Fig. 188, prove that $d = e$.
4. In Fig. 189, PQRS is a parallelogram, prove that $a = b$.

Fig. 188.

5. In Fig. 190, PQ is $\parallel$ to RS and QR is $\parallel$ to ST, prove that $\angle PQR = \angle RST$.
6. In Fig. 191, prove that $b + c = d$.

Fig. 189.

Fig. 190.

7. In Fig. 192, prove that $\angle ACD = \angle A + \angle B$.
What result is obtained by adding $\angle ACB$ to each?
8. In Fig. 192, if CK bisects $\angle ACD$, prove that $\angle A = \angle B$.  

Fig. 191.

Fig. 192.
APPENDIX (Part I)

PRACTICAL WORK

If time is available, some outdoor work should be done concurrently with the indoor experimental work of Part I. The following sections are labelled by chapters to show the standard of knowledge involved.

I. SURVEYING WITHOUT ANGLE-MEASUREMENT
   (Ch. II)

Apparatus: (i) A measuring tape or chain; 
(ii) A plane-table and spirit-level.

A plane-table can be made by placing a drawing-board on a tripod; a spirit-level is needed for setting it up horizontally.

Example.—Make a plan or map, showing the relative positions of various objects, trees, posts, corners of a field, etc.

First make a rough free-hand sketch of the objects to be mapped, using letters P, Q, R, etc., to name them. Keep this sketch with you while you are making your observations.

Measure out a base-line AB of any convenient length. Set up the plan-table at A and level it; fix a pole in the ground at B. Draw a line ab on a sheet of paper to represent AB on some convenient scale and pin the paper to the drawing-board by a pin at a. Turn the paper round till ab is in line with B; then pin it also at b.

Look at P, and put a pin in the paper, near its far edge, on the line AP, and write → P near the pin. Repeat this for the lines AQ, AR, etc., and then rule ap, aq, ar, etc.

Next set up a pole at A and move the plane-table to B; repeat the process and rule the lines bp, bq, etc., giving the points p, q, etc., which represent the objects.

Measure the lines ap, bp, etc., in your plan and then calculate the actual distances AP, BP, etc. Check by measuring some of these distances or by moving the plane-table to P and repeating the process with PA as base-line. This is called checking by cross-bearing.

Other Typical Problems

1. Find the distance of an object which is visible but not easily accessible, e.g. a tree T on the other side of a river. [Take a base-line AB and map A, B, T.]

2. Find the distance between two objects, P and Q, out of sight of one another.

In Fig. 194, P and Q are both visible from O, and the lengths OP, OQ are measured.

(i) Set up the plane-table at O, draw the lines OP, OQ in the directions OP, OQ; then draw ΔOPQ to scale.

(ii) Draw ΔOPQ to scale on the ground itself; e.g. put pegs in the ground at p, q half-way from O to P, Q, then ΔOPQ is a scale drawing of ΔOPQ, half-scale, so pq = 1/2PQ. Or take OP = 1/2OP and OQ = 1/2OQ, etc.

II. PRACTICAL LOCI (Ch. III)

Meaning of a Locus

The search for "buried treasure" is the best practical way of explaining what a "locus" is.

To discover the position on the ground of an object O, it is necessary to have two clues to its position. If only one clue is given, the object can lie anywhere on some line or curve, which is called its locus (plural, loci). The word, locus, means place; the line or curve contains all the possible places where the object can be.
An Important Locus

A number of chairs are placed so that each is 5 feet from the side wall of a room, to leave a passage. Then all the chairs are in a straight line, and we say that the locus of all points which are at a given distance from a given straight line, and on the same side of it, is a straight line.

Oral Work.—What can you say about the position of an object O,

(i) if O is in line with a post P and a tree T ?
(ii) if O is 20 yd. from a flagstaff F ?
(iii) if O is 10 yd. from the wall of a house ?
(iv) if O is 10 yd. from the side line of a football ground ?

Intersecting Loci

If a second clue is given to the position of O, there is a second line or curve on which O must lie, that is, a second locus of O. The position of O is therefore found by looking to see where the two loci cross one another. They may cross at more than one point ; if so, there is more than one possible position for O, but usually not many. We call these positions points of intersection of the loci, and say that O is found by means of intersecting loci.

Oral Work.—Name intersecting loci for finding O, and state the number of possible positions,

(i) if O is in line with two posts P, Q and also in line with two trees S, T ;
(ii) if O is in line with two posts P, Q and 20 yd. from a tree T ;
(iii) if O is 20 yd. from a tree T and 30 yd. from a tree S ;
(iv) if O is 30 yd. from a tree T and 20 yd. from a straight wall AB.

Typical Problems

[When there is more than one possible position of the buried object, search should be made in each place simultaneously.]

1. Treasure is buried 10 yd. from a straight fence AB and 8 yd. from a post C ; C is 15 yd. from AB.

2. Treasure is buried 20 yd. from a tree T and in line with two posts A, B.

3. A, B, C, D are posts at the corners of a quadrilateral with sides of various lengths up to 30 yd. An object is buried in line with A, C and also in line with B, D.

4. A, B, C are 3 posts. An object is buried the same distance from A as B is, and in line with B, C.

Examples on Loci for class discussion (Ch. V)

1. Name the intersecting loci for finding the vertex A of a triangle ABC if BC = 7 cm. and if (i) ∠B = 73°, ∠C = 40°, (ii) AB = AC = 7 cm., (iii) AB = 6 cm., AC = 9 cm.

2. Take two points A, B, 6 cm. apart. Draw and label the locus of O if (i) O is 4 cm. from A ; (ii) O is 5 cm. from B ; (iii) AO makes 40° with AB. What can you say about the points (a) where locus (i) crosses locus (ii) ; (b) where locus (ii) crosses locus (iii) ?

3. Take two points C, D, 2 in. apart. Draw the locus of Q if CQ makes 130° with CD ; draw the locus of a point 2.5 in. from D. What can you say about the points of intersection of the loci ?

4. Draw a rectangle ABCD such that AB = 4 cm., BC = 3 cm. Draw and label the locus of O if (i) O is in line with A, C, (ii) O is 4.5 cm. from B, (iii) DO makes 10° with DC. What can you say about the points (a) where locus (i) crosses locus (ii) ; (b) where locus (ii) crosses locus (iii) ; (c) where locus (i) crosses locus (iii) ?

5. Take two points A, B 5 cm. apart. Draw the locus of a point (i) 3 cm. from the line AB, (ii) 4 cm. from A. What can you say about the points of intersection of the loci ?
III. CONSTRUCTION OF RIGHT ANGLES OUT OF DOORS (Ch. III-IV)

String Set-square

Take a piece of string about 13 yards long. Attach one end of it to a thin stick or a long knitting needle, A; attach another stick B to the string, 3 yards from A, by a couple of half-hitches, to avoid slipping; attach in the same way sticks C and D, so that BC is 4 yards and CD is 5 yards. When D is brought close to A, and the string is stretched taut (see Fig. 195), the corner ABC is right-angled. Test this on the corner of a table.

The ancient Egyptians (1300 B.C.) used this method of constructing right angles, in marking out boundaries which had been swept away by the overflowing of the Nile.

Typical Problems

1. Mark the outside corners of a tennis court, that is, the corners of a rectangle 26 yd. long, 12 yd. wide. Check by finding whether the diagonals are equal.

2. P is a post some distance from the straight edge XY of a path. Construct the perpendicular from P to XY as follows: use a long string, with one end fastened to P, to find two points Q, R on XY such that PQ = PR. Find the mid-point of QR, call it N. Use the string set-square to test whether L PNX is a right angle.

   Use the long string to trace out a circle, centre P, radius PN; what do you notice about it? What is the shortest way from P to the path XY?

3. Set up a vertical pole AB and at true noon put a peg in the ground at C so that AC marks the shadow of AB. Then AC points due north. Use the string set-square to mark on the ground a line running east and west. Check this if you can; is there a weathercock or a church in sight?

IV. HEIGHTS AND DISTANCES BY ANGLE MEASUREMENT (Ch. V)

Apparatus: (i) a plane-table with fitted protractor; (ii) a clinometer.

The Plane-table

Make a circular protractor on a large sheet of stiff paper, graduate it from 0° to 360° (see Fig. 196), and mount it on a drawing-board. Make a pointer out of thin wood or cardboard and pivot it on an axis through the centre. Place the drawing-board on a tripod and level it. It is then possible to measure angles in a horizontal plane. Greater accuracy is secured by attaching "sights" to the pointer.

Angles of Elevation and Depression

Stand at O and point horizontally along ON; then move your arm in a vertical plane until you are pointing upwards at B along OB [see Fig. 197 (i)]; the angle NOB through which your arm has turned is called the angle of elevation of B from O.

Fig. 196.

If, when you are standing at O, an object C is below you [see Fig. 197 (ii)], first point horizontally along ON, then move your arm in a vertical plane till you are pointing downwards at C along OC; the angle NOC, through which your arm has turned, is called the angle of depression of C from O.
GEOMETRICAL IDEAS

The Clinometer

The object of a clinometer is to measure angles of elevation or depression.

On thin wood or thick cardboard, make a large semi-circular protractor, centre O, diameter AB (see Fig. 198). Mark the mid-point E of the arc AB, 0°, and graduate from 0° to 90°, both along EA and along EB.

It is a convenience to shape the instrument as in Fig. 198, where CD is parallel to AB.

Make a small hole at O, and suspend a plumb-line from it. With this apparatus, we can measure angles in a vertical plane.

Hold the instrument in a vertical plane with one eye at O, looking along CD; if the plumb-line cuts the rim at E, CD must be horizontal. If CD is now tilted up through any angle, OE moves away from the plumb-line through the same angle; an observer, standing at the side, can then read off this angle by noting the graduation where the plumb-line cuts the rim. Care is needed to keep the instrument in a vertical plane throughout the process.

The graduations on the arc OB are used when CD is pointed downwards at some object below the observer, that is for angles of depression.

Greater accuracy can be obtained by fixing "sights" to CD.

Typical Problems

1. The Problems in Section I. (p. 91) may now be repeated with the use of a protractor.

2. Find the height of a building PQ (see Fig. 199).

(i) Observe the angle of elevation of Q from A, and measure AP. Then make a scale-diagram.

(ii) It may be impracticable to measure from A right up to P. Take a point B on the line AP; observe the angles of elevation of Q from A and from B; measure AB. Then make a scale-diagram.

(iii) Find the length of the shadow thrown by PQ, and at the same time the length of the shadow of an upright stick, say 6 ft. high. Then work by proportion.

3. Standing at an upstairs window Q, find its height PQ above the ground (Fig. 199) when it is inconvenient, or not practicable, merely to drop a plumb-line. (i) Measure the angle of depression of A from Q and measure AP. (ii) Measure the angles of depression of A and B from Q and measure AB.

4. C and D are two posts about 20 yd. apart; an object Q is buried so that CO makes 45° with CD, and DO makes 60° with DC.

5. C and D are two posts about 20 yd. apart; an object Q is buried so that it is 15 yd. from D and so that CO makes 25° with CD.

V. MODELS OF SOLIDS (Ch. III-VI)

Any well-made model should be preserved, as duplicates are useful for demonstration.

Example.—Take a box made of thin cardboard, and open it out flat, cutting along the edges where necessary.

The result (Fig. 200) is called the net of the box. Treat in the same way a triangular prism and a pyramid on a square base.

Construction of Models

Start by drawing the net of the solid. If cardboard is used, the edges may be fastened with thin strapping or even stamp paper. If, however, stiff paper is used, better results are obtained by leaving flaps along suitable edges of the net. These can be gummed when the paper is folded to form the surface of the solid.
GEOMETRICAL IDEAS

Make models of the following solids:

The dotted lines in the figures indicate the creases. Edges, which may conveniently have flaps (if required), are marked F.

1. (i) A cube, each side 2½ inches.
   (ii) A cuboid, 3 inches by 2 inches by 1½ inches (Fig. 201).

![Fig. 201](image)

2. A triangular prism, height 5 inches, each side of base 2 inches (Fig. 202).

3. A pyramid on a triangular base, each side of the base and each edge to be 2½ inches (Fig. 203).
   This solid is called a regular tetrahedron (4 faces).

![Fig. 203](image)

4. A pyramid on a square base; side of base, 2 inches, length of each slant edge 3 inches (Fig. 204).

5. An open box, 4 in. long, 3 in. wide, 1½ in. high.
   On a rectangular sheet of paper, 7 in. long, 6 in. wide, mark at each corner squares of side 1½ in. Then cut away these squares, except what you need for flaps (see Fig. 205). What sized sheet would you need to make an open box, 5 in. long, 3 in. wide, 2 in. high?

![Fig. 204](image)

![Fig. 205](image)

PRACTICAL WORK

6. A solid bounded by 8 faces, each of which is an equilateral triangle, side 2 in. long. This solid is called a regular octahedron (8 faces); its net is shown in Fig. 206.

![Fig. 206](image)

7. Make a hexagonal pyramid (a pyramid on a 6-sided base); each side of base to be 3 cm. and each slant edge to be 5 cm.

8. Make a circular cylinder.
   Take a rectangular sheet of paper, 6½ in. by 5 in.; put some gum along one of the 6½ in. edges and bend it to form a circular cylinder, 5 in. high, with open ends.
   Lastly, cut out 2 circular discs, with flaps, and use them to close the ends of the cylinder.

9. Make a circular cone.
   Cut out a circle of radius 3 in. and draw on it two radii at an angle of 120°. Cut away the part of the circle between these radii (i.e. a sector). Put some gum along one of the bounding radii of the remainder, the larger portion of the circle, and bend it to form a circular cone, slant edge 3 in., with open base.
   Lastly, cut out a circular disc, with flaps, and use it to close the base of the cone.

10. A solid bounded by 20 faces each of which is an equilateral triangle, side 3 cm. long.
    This solid is called a regular icosahedron (20 faces); its net is shown in Fig. 207.

![Fig. 207](image)
PART II

IMPORTANT GEOMETRICAL PROPERTIES

CHAPTER VII

ANGLES OF PLANE FIGURES

Exterior Angle of a Triangle

If any side BC of a triangle ABC is produced to D, \( \angle ACD \) is called an exterior angle of \( \triangle ABC \).

![Figure 208](image)

In Fig. 208, where \( \angle ACD \) is an exterior angle of \( \triangle ABC \), we call the angles at A and B interior opposite angles.

Examples for Class Discussion

1. (i) In Fig. 209 (i), CX is drawn \( \parallel \) to BA. Copy and complete the following, with reasons:

\[ \angle ACX = \ldots ; \angle XCD = \ldots \]

\[ \therefore \angle ACD = \ldots \]

(ii) In Fig. 209 (ii), AK is drawn \( \parallel \) to BC. Copy and complete, with reasons:

\[ \angle BAK = \ldots ; \angle KAE = \ldots ; \angle BAE = \ldots \]

(iii) In Fig. 209 (ii), \( \angle EAB \) is an exterior angle of \( \triangle ABC \), name the interior opposite angles.

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2. Draw $\triangle PQR$; produce $PQ$ to $K$, $PR$ to $L$, $RP$ to $M$.
If $\angle KQR$ is taken as ext. $\angle$, name the int. opp. $\angle$s.
If $\angle PQR$ and $\angle QPR$ are taken as a pair of int. $\angle$s, name the opposite ext. $\angle$.

What are the int. opp. $\angle$s for ext. $\angle QPM$?

3. How many "exterior angles" of different sizes can be formed for a triangle? How many altogether? Show on a figure which exterior angles must be equal, marking them $x^\circ$, $y^\circ$, $z^\circ$.

**EXERCISE VII. a (Oral)**

Find the unknown marked angles in Figs. 210-212; give reasons. Arrows indicate that lines are given parallel.

1. [Fig. 210]

2. [Fig. 211]

3. [Fig. 212]

4. In Fig. 213, $CK$ is drawn $\parallel$ to $BA$. Copy and complete, with reasons:

   $\angle ACK = \ldots \ldots ; \angle KCD = \ldots \ldots ; \angle ACD = \ldots \ldots$

The fact illustrated by these examples is stated as follows:
The exterior angle of any triangle is equal to the sum of the two interior opposite angles.

For reference:

ext. $\angle$ of $\Delta = \text{sum int. opp. } \angle$s.

Thus, in Fig. 213, $\angle ACD = \angle ABC + \angle BAC$.

**EXERCISE VII. b**

Find the unknown marked angles in Figs. 214-216; give reasons. Arrows indicate that lines are given parallel.

1. [Fig. 214]

2. [Fig. 215]

3. [Fig. 216]
SUM OF THE ANGLES OF A TRIANGLE

Examples for Class Discussion

1. (i) Find the remaining angle of \( \triangle PQR \) in Fig. 219 (i). Draw \( RX \parallel PQ \), then copy and complete, with reasons:
   \[ \angle PRX = \ldots \quad \angle XRS = \ldots \quad \therefore \angle PRS = \ldots \]
   But \( \angle PRS + \angle PRQ = \ldots \quad \therefore \angle PRQ = \ldots \]

(ii) In Fig. 219 (ii), \( CK \) is drawn \( \parallel BA \). Answer, with reasons:
   What other angle equals \( x^\circ \)? What equals \( y^\circ \)?
   What do you know about \( x^\circ, y^\circ, z^\circ \)?
   What is \( \angle A + \angle B + \angle ACB \)?

2. Cut out a large triangle \( ABC \) in stiff paper or thin cardboard (e.g., the cover of an old exercise book). Cut off the corners at \( A \) and \( B \), fit them together at \( C \) in the positions of \( \angle ACK \) and \( \angle KCD \) in Fig. 219 (ii).

3. Draw any large triangle \( ABC \). Place your pencil along \( BC \) and note which way it points (\( B \rightarrow C \) or \( C \rightarrow B \)).
   Hold it at \( B \) and turn it through \( \angle ABC \) so that it lies along \( BA \) (is this clockwise or counterclockwise ?); then hold it at \( A \) and turn it through \( \angle BAC \) so that it lies along \( AC \) (is this \( c \) or \( c.c. \) ?); then hold it at \( C \) and turn it through \( \angle ACB \) so that it lies along \( CB \) (is this \( c \) or \( c.c. \) ?).
   Which way is it now pointing (\( B \rightarrow C \) or \( C \rightarrow B \) ? What is the total angle turned through? What is the sum of the separate angles turned through?
   The fact illustrated by these examples is stated as follows:
   The sum of the three angles of any triangle is two right angles.
   For reference: \( \angle \text{sum of } \Delta = 180^\circ \).

EXERCISE VII. c

1. Find the remaining angle of a triangle if two of its angles are (i) \( 50^\circ, 55^\circ \); (ii) \( 117^\circ, 49^\circ \); (iii) \( 12^\circ, 14^\circ \).

2. In \( \triangle ABC \), (i) if \( \angle A = \angle B = 70^\circ \), find \( \angle C \);
   (ii) if \( \angle B = \angle C \) and \( \angle A = 80^\circ \), find \( \angle B \).

3. If the angles of a triangle are equal, find the size of each.

4. In a right-angled triangle, one angle is \( 53^\circ \); find the other acute angle.

5. The angles of a triangle are \( 2x^\circ, 3x^\circ, 4x^\circ \). Find \( x \).

6. Find the angles of a triangle if each base angle is double the vertical angle.

7. What do you know about the two acute angles in any right-angled triangle?

8. Can you draw a triangle
   (i) so that its angles are (a) \( 48^\circ, 59^\circ, 73^\circ \); (b) \( 45^\circ, 70^\circ, 75^\circ \);
   (ii) so that two of its angles are \( 65^\circ \) and \( 105^\circ \)?
9. Use your protractor to measure the 3 angles of \( \triangle ABC \) in Fig. 23 (see end of book). What is the error (if any) in the sum of your measurements?

10. Repeat No. 9 for \( \triangle DEF \) in Fig. 23 (see end of book). Calculate the unknown marked angles in Figs. 220-223; arrows denote that lines are given parallel. Give short reasons.

11. Fig. 220.

(i) \( 72\, ^\circ \)

(ii) \( 32\, ^\circ \)

12. Fig. 221.

(i) \( 144\, ^\circ \)

(ii) \( 32\, ^\circ \)

13. Fig. 222.

(i) \( 100\, ^\circ \)

(ii) \( 40\, ^\circ \)

14. Fig. 223.

(i) \( 20\, ^\circ \)

(ii) \( 30\, ^\circ \)

15. A ladder rests against a wall. It makes an angle of 65° with the ground. What angle does it make with the wall?

16. In Fig. 224, \( PB, PC \) bisect \( \angle ABC, \angle ACB \). Find \( \angle BPC \)

(i) if \( \angle ABC = 30^\circ, \angle ACB = 70^\circ \);

(ii) if \( \angle ABC = 35^\circ, \angle BAC = 84^\circ \).

17. In Fig. 224, \( BN \) bisects \( \angle ABC \). Find \( \angle BNC \), if \( \angle BCA = 70^\circ \) and \( \angle BAC = 64^\circ \).

18. In \( \triangle ABC \), \( \angle B = 50^\circ \) and \( \angle C = 70^\circ \); also \( BE \) is the perpendicular from \( B \) to \( AC \), and the line which bisects \( \angle BAC \) cuts \( BE \) at \( K \). Find \( \angle ABE \) and \( \angle AKE \).

THE ANGLES OF A POLYGON

Examples for Class Discussion

The questions in Ex. VII.d are intended for class discussion, the answer to each question being given out before the next question is asked.

EXERCISE VII.d

Interior Angles of a Polygon

1. Find the sum of the interior angles of the pentagon \( ABCDE \) in Fig. 225. Take any point \( O \) inside the pentagon and join it to each corner. Copy and complete the following: The pentagon contains \( \ldots \) triangles, and the sum of all the angles in these \( \ldots \) triangles is \( \ldots \) right angles. Subtract all the angles at \( O \), namely \( \ldots \) right angles. \( \therefore \) the sum of the interior \( \angle s \) of \( ABCDE \) is \( \ldots \) right angles.

2. Draw any hexagon (6 sides) and repeat the argument of No. 1. Work in right angles, not in degrees.

3. Draw any 7-sided polygon and repeat the argument of No. 1.
4. Without drawing a figure, work through the argument of No. 1, p. 107, for (i) a 20-sided polygon, (ii) a 100-sided polygon.

Now make a general statement, the sum of the interior angles of an \( n \)-sided polygon is . . . . . .

5. What is the quickest way of finding the sum of the interior angles of a quadrilateral?

**Exterior Angles of a Polygon**

6. Draw on the floor a large triangle ABC, and produce the sides "in order" (see Fig. 226) to form the exterior angles, \( p^\circ, q^\circ, r^\circ \).

![Fig. 226](image)

![Fig. 227](image)

Start from any point \( \mathbf{K} \) on \( \mathbf{BC} \) and walk round the figure till you get back to \( \mathbf{K} \); your path is \( \mathbf{KCABK} \). What are the sizes, in degrees, of the angles through which you turn at \( \mathbf{C} \) at \( \mathbf{A} \) at \( \mathbf{B} \)?

What follows from the fact that you have made one complete turn?

7. Draw a large quadrilateral \( \mathbf{ABCD} \) on the floor and produce the sides in order (see Fig. 227). Start at any point \( \mathbf{K} \) on \( \mathbf{AB} \) and walk all round the figure, \( \mathbf{KBDAEK} \).

What can you say about \( q^\circ + r^\circ + s^\circ + p^\circ \)?

8. Repeat No. 7 for a polygon with 7 sides. Check your answer as follows:

The interior angle + the exterior angle at each corner = 2 right angles; · · · the sum of all the interior angles and all the exterior angles is . . . . . ; but the sum of the interior angles is . . . . . , and the sum of the exterior angles is . . . . .

9. Draw on the floor a 4-sided figure \( \mathbf{ABCD} \) as in Fig. 228. This is not a convex polygon because one of its interior angles is reflex. We say it has a re-entrant corner (at \( \mathbf{D} \)). Walk all round this figure, starting from \( \mathbf{K} \). The angles through which you turn are \( g^\circ, r^\circ, s^\circ, p^\circ \). Why is it no longer possible to find the value of \( q^\circ + r^\circ + s^\circ + p^\circ \)?

![Fig. 228](image)

10. Draw on the floor a 5-sided polygon \( \mathbf{ABCDE} \) which crosses itself as in Fig. 229. Start at some point \( \mathbf{K} \) on \( \mathbf{AB} \) and walk all round the figure, \( \mathbf{KBCDEAK} \). Mark on your figure the exterior angles through which you turn at \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E} \). What is the sum of all these angles? [Walk round the figure before you answer.] Use this result to find the sum of the marked interior angles of the pentagon. Try to find another way of working out this result to check it.

If a polygon has no re-entrant corners and does not cross itself, it is called convex.

The facts illustrated above are stated as follows:

(i) The sum of the interior angles of a convex polygon with \( n \) sides is \( (2n - 4) \) right angles.

(ii) If the sides of a convex polygon are produced in order, the sum of the exterior angles is \( 4 \) right angles.

**EXERCISE VII.e**

1. Find in right angles the angle sum of a convex polygon (i) with 8 sides, (ii) with 9 sides, (iii) with 12 sides.

2. Find the unknown marked angles in Fig. 230.
3. Find the unknown marked angles in Fig. 231 (i), (ii).

4. All the exterior angles of a hexagon are equal. Express in degrees (i) each exterior angle, (ii) each interior angle.

5. All the angles of a 10-sided convex polygon are equal. Express each angle in degrees.

6. One angle of a quadrilateral is 120° and the other three are equal; how large are the equal angles?

7. (i) Each exterior angle of a polygon is 40°. How many sides has it?
   (ii) Each interior angle of a polygon is 160°. How many sides has it?

8. Prove that the angle sum for an 8-sided polygon is double that for a pentagon.

9. How many sides has a polygon if its angle sum is double that for a 7-sided polygon?

10. One angle of a pentagon is half each of the other four angles. Find the smallest angle.

GENERAL STATEMENTS

It is suggested that Nos. 1-5 of Ex. VII. f should be taken orally. Any or all of the remainder may be omitted at a first reading.

Many examples will of course have been omitted previously for lack of time; suggestions for omission, made in this and other similar notes, are on the ground of difficulty, an oral treatment being desirable for the more important questions.

EXERCISE VII. f

[Remember to give a reason for each step in your argument.]

1. If, in Fig. 232, \( \angle BAD = \angle BCD \), prove that \( \angle ABC = \angle ADC \).

2. If, in Fig. 233, \( \angle ATN = \angle ABC \), find another pair of equal angles.

3. In Fig. 234, \( \angle BAC \) is a right angle, and \( \overline{AN} \) is perpendicular to \( \overline{BC} \). Prove that \( \angle NAC = \angle ABC \). Also find another pair of equal angles in the figure.

4. In Fig. 235, \( \overline{AX} \) is parallel to \( \overline{BY} \); also \( \overline{PA}, \overline{PB} \) bisect \( \angle XAB, \angle YBA \). Prove that \( \angle APB = 90° \).

5. Express \( \angle t \) in Fig. 236 (i), (ii) in two different ways in terms of the other marked angles, and so find a relation between these other angles.

6. If, in Fig. 232, \( \angle AQC = 2 \angle BAQ \), find an angle equal to \( \angle BAQ \).

7. In Fig. 237, prove that \( \angle ABC = \angle ADC = \angle DAB + \angle DCB \). [Join \( \overline{DB} \) and produce it to \( \overline{K} \).]

8. In Fig. 238, \( \overline{AP} \) is parallel to \( \overline{CQ} \), prove that

\[ \angle ABC = \angle PAB - \angle QCB. \]
CHAPTER VIII

SYMmetry, CONGRUENCE AND SIMILARITY

Symmetry about a Line

The best way of understanding precisely what is meant by symmetry about an axis and of seeing what tests are required to establish symmetry is by practical illustrations from paper-folding.

Examples for Class Discussion

1. Take a piece of paper, fold it, and call the crease EF. Mark any three points A, B, C on one side of the crease (see Fig. 230). With the paper folded, prick through the points A, B, C and call the new points on the other side of the crease, P, Q, R. Then spread the paper out flat and draw \( \triangle ABC, \triangle PQR \).

The method of construction shows that the crease EF bisects and is at right angles to the lines joining each pair of corresponding points A, P; B, Q; C, R; that is, EF is the perpendicular bisector of AP and of BQ and of CR.

We therefore say that the figure is symmetrical about EF, and the line EF is called the axis of symmetry. If a looking-glass is held along EF and perpendicular to the paper, PQR is the reflection of ABC; the point P is called the image of A in EF and QR is the image of BC.

(i) With what does AC coincide, when the paper is folded?

(ii) What does BC equal?

(iii) Name a line equal to AQ, to BR, to FP.

(iv) Name an angle equal to \( \angle ACB \), and one equal to \( \angle ABQ \).

(v) Why must the lines AQ and PB cut one another on EF?

(vi) What angle equals \( \angle CQR \)?

Since the triangles ABC, PQR fit one another exactly, when the paper is folded over, every side and angle of \( \triangle ABC \) equals the corresponding side and angle of \( \triangle PQR \). We therefore say that \( \triangle ABC \) is congruent to \( \triangle PQR \), and we write \( \triangle ABC = \triangle PQR \).

2. Fold a piece of paper and call the crease EF. On one side of EF, draw any quadrilateral ABCD. Construct its image in EF by pricking through, call it PQRS. Then fold the paper out flat.

(i) Measure AC and PH.

(ii) Let AC cut BD at H; mark in your figure the image K of H.

(iii) Name lines which have EF as perpendicular bisector.

(iv) Name lines equal to BD, EC, HA, FR, CP, BK.

(v) Name angles equal to \( \angle ADC \), \( \angle DBC \), \( \angle EKF \), \( \angle PHB \), \( \angle ARQ \).

(vi) Name triangles congruent to \( \triangle ABC \), \( \triangle AHM \), \( \triangle PQS \), \( \triangle AHE \).

EXERCISE VIII. a

1. Draw any line ENF. Then draw a line ANB perpendicular to EF and make AN = NB. Check the accuracy of your drawing by folding the paper, with EF as crease, and pricking through. Mark any point P on the paper, not on EF; then construct the image of P in EF. Check by pricking through.
2. Fold a piece of paper and cut a figure out of the doubled sheet, starting from a point on the crease and ending at another point on the crease. Make it a complicated shape. Open it out and test the symmetry of the figure in 3 places.

Draw figures like Figs. 240-241, and mark in any axis of symmetry.

3. 

\[
\begin{array}{c}
\text{(i)} \\
\text{(ii)} \\
\text{(iii)} \\
\end{array}
\]

Fig. 240.

4. 

\[
\begin{array}{c}
\text{(i)} \\
\text{(ii)} \\
\text{(iii)} \\
\end{array}
\]

Fig. 241.

5. Draw axes of symmetry for the capital letters H, K.

6. Fold a piece of paper, pressing the crease firmly. Then open the paper and place several drops of ink on one side of the crease and not very near it. Fold again, and press firmly to spread out the ink. Then open and blot carefully. Test the symmetry of the figure in 3 places.

7. Repeat No. 6, putting the ink in the crease.

8. Repeat Nos. 6, 7, using both red and black ink in the same figure.

9. Draw a triangle ABC, such that the angles at B and C are unequal and acute. Without folding the paper, construct the image K of A in BC; join KB, KC. The figure ABKC is called a kite.

(i) What axis of symmetry has it got?
(ii) Name pairs of equal lines and pairs of equal angles.
(iii) What can you say about the diagonals of a kite?
(iv) Check your drawing by folding and pricking through.

Symmetry, Congruence and Similarity

The Principle of Symmetry

If a figure is symmetrical about a straight line, one half of the figure can be made to coincide with the other half, by folding with this line as crease.

Therefore any line, angle, or triangle in the one half of the figure is equal to the corresponding line, angle, or triangle in the other half of the figure.

This property is very useful, but, before it can be employed, it is of course necessary to be sure that the figure really is symmetrical. The most important cases of symmetry about a line in geometrical figures will now be discussed. When you recognise any one of them, you may safely use the principle of symmetry.

Note.—Symmetry about an axis plays a far more important part in elementary geometry than symmetry about a centre; and a clearer grasp of the principle of symmetry is obtained if, at this stage, attention is confined to the former. Arguments from symmetry must not be vague. This danger is avoided if statements that two things are equal by symmetry are associated consciously with the existence of an axis of symmetry, the leading ideas being “folding and pricking through” or “folding and fitting”.

Symmetrical Figures

Examples for Class Discussion

The paper-folding described in Nos. 1-3 should be done by everyone.

1. Any line AB is symmetrical about its perpendicular bisector, ENF.

In Fig. 242, N is the mid-point of AB and ENF is perpendicular to AB; P is any point on EF. (i) The paper is folded with EF as crease. Why does NA fall along NB? Why does A coincide with B?

(ii) What can you say about PA and PB?

(iii) Copy and complete the sentence: If Q is any point on the perpendicular bisector of AB, then QA . . . . . .
2. Any angle $\angle BAC$ is symmetrical about its bisector, $AN$.
In Fig. 243, $AN$ bisects $\angle BAC$ and $AQ = AR$.
(i) The paper is folded with $AN$ as crease. Why does $AB$ fall along $AC$? Why does $Q$ coincide with $R$?
(ii) What can you say about the diagonals of the quad. $AQPR$? What kind of figure is $AQPR$?
(iii) What angle is equal to $\angle APQ$? to $\angle PQR$?
If two sides of a triangle are equal, the triangle is called isosceles, and the third side is called its base.
Thus in Fig. 243, where $AQ = AR$, $\triangle AQR$ is isosceles and its base is $QR$.
(iv) Name another isosceles triangle in Fig. 243.

![Fig. 243](image1)

3. An isosceles triangle $ABC$ is symmetrical about the line $AN$, which bisects the angle between the equal sides $AB$, $AC$.
In Fig. 244, $AB = AC$, and $AN$ bisects $\angle BAC$.
(i) The paper is folded with $AN$ as crease. Why does $AB$ fall along $AC$? Why does $B$ coincide with $C$?
(ii) Use (i) to show that $\angle ANB$ is a right angle. What can you say about $BN$ and $\angle ABN$?
(iii) Copy and complete the sentence: If two sides of a triangle are equal, then the angles opposite those sides . . . . , and the line, which bisects the angle between the equal sides, . . . .

![Fig. 244](image2)

4. A circle is symmetrical about any one of its diameters.
In Fig. 245, $HK$ is a diameter of a circle, centre $O$.
The distance from $O$ of every point on the semicircle $HAK$ is equal to the radius; therefore if the paper is folded with $HK$ as crease, all points on $HAK$ in their new positions are still at a distance from $O$ equal to the radius and therefore coincide with points on $HBK$.
: $HAK$ coincides with $HBK$, and so the circle is symmetrical about $HK$.
(i) If $AB$ is a chord perpendicular to $HK$ and cutting it at $N$, what can you say about $AN$?
(ii) Copy and complete the sentence: If a chord of a circle is drawn perpendicular to a diameter, then . . . .

![Fig. 245](image3)

5. A figure consisting of two circles is symmetrical about the line joining their centres.
In Fig. 246, $A$ and $B$ are the centres of the circles.
By No. 4, each circle is symmetrical about the line $EABF$.
(i) What can you say about the common chord $PQ$?
(ii) By joining points named in Fig. 246, obtain 4 isosceles triangles; in each case name the equal sides.
(iii) Name angles equal to $\angle APB$, $\angle AEP$, $\angle BQF$.
(iv) Is there symmetry if the circles do not intersect?
USE OF SYMMETRY

EXERCISE VIII. b

1. Use a penny, or something larger, to draw a circle. Cut the circle out, and find its centre by folding.

2. In Fig. 247, the lines \( HK \) and \( LM \) bisect each other at right angles.

(i) What symmetry is there in this figure?

(ii) Name all equal lines in the figure.

(iii) Name all equal angles in the figure.

(iv) Why do these results show that the figure is a parallelogram with all its sides equal? What is the name for such a parallem.

3. Draw any line and mark two points \( A \), \( B \) on it; take any point \( C \) outside the line. Draw the circles, centres \( A \) and \( B \), which pass through \( C \). Use this figure to construct the line from \( C \) perpendicular to \( AB \); give reasons.

Draw two figures, making \( \angle ABC \) acute in one figure and obtuse in the other.

Give reasons why the triangles in Figs. 248-251 are congruent.

4. 

5. 

6. 

7. 

Fig. 248. 

Fig. 249. 

Fig. 250. 

Fig. 251.

Construction of Triangles

In Exercises VIII. c, e, g, individual drawing should be accompanied by class discussion to emphasise the facts which are illustrated. Numerical results obtained from individual measurements should be compared.

EXERCISE VIII. c

Dimensions stated numerically

1. Draw each of the triangles shown in Fig. 253, and measure the indicated sides and angles; the unit is 1 cm.

Fig. 253.

Are you sure that the data are sufficient to fix the size and shape of these triangles? Look at No. 2 and Fig. 254 on next page before you answer.

Mark the data on your own drawings and use them for Nos. 3, 4, 5.
2. Fig. 254 shows 4 triangles drawn in different positions from the data of Fig. 253 (i).
   Point out two axes of symmetry for this figure. Are all four triangles congruent?
3. Draw 4 triangles from the data of Fig. 253 (ii), arranged like Fig. 254. Point out the axes of symmetry. What follows?
4. Repeat No. 3 for Fig. 253 (iii).

   ![Diagram](image)

5. Is the triangle in Fig. 255 congruent to that in Fig. 253 (ii), the unit being 1 cm. for each? In what order should you draw the various lines to construct Fig. 255?
6. Make a rough figure showing \( \triangle ABC \) with \( AB = 6 \text{ cm.} \), \( \angle B = 50^\circ \), \( \angle C = 73^\circ \). What calculation ought you to make before starting to draw an accurate figure?
7. Can you draw a triangle with angles \( 47^\circ, 68^\circ, 65^\circ \)? Draw two such triangles of different sizes, if you can.

**Dimensions obtained by Measurement**

Nos. S-10 refer to Fig. 23 (see end of book).

8. Copy \( \triangle ABC \) in Fig. 23, by measuring two sides in cm., and the included angle. Compare, with compass or dividers, the length of the third side with that of the printed figure.
9. Copy \( \triangle PQR \) in Fig. 23, by measuring two angles and (in cm.) the side between them. Compare, with compass or dividers, the lengths of the other two sides with those of the printed figure.
10. Copy \( \triangle DEF \) in Fig. 23, by measuring its three sides in cm. Measure the largest angle of your figure and the corresponding angle of the printed figure.

**General Tests for Congruent Triangles**

1. Two triangles are congruent if any two sides of the first are equal to any two sides of the second, and if the included angles are also equal.
   *For reference*: 2 sides, inc. \( \angle \); or S.A.S.

2. Two triangles are congruent if any two angles of the first are equal to any two angles of the second, and if any side of the first is equal to the corresponding side of the second.
   *For reference*: 2 \( \angle s \), corresp. side; or A.S.A.

3. Two triangles are congruent if the three sides of the first are equal to the three sides of the second.
   *For reference*: 3 sides; or S.S.S.

**Order of Letters**

The statement that \( \triangle ABC \) is congruent to \( \triangle PQR \), written \( \triangle ABC = \triangle PQR \), should mean that \( \angle A = \angle P, \angle B = \angle Q, \angle C = \angle R \), etc., the order of the letters showing which sides and which angles are equal. The order need not be alphabetical: thus we might have \( \triangle AZT = \triangle NCB \); this would mean \( \angle A = \angle N, AZ = NC, \angle T = \angle B \), etc.
Necessary Data for a Triangle

The previous examples have illustrated the fact that you cannot draw a triangle of definite size and shape unless you are told three independent facts about it. Thus the statement that the 3 angles are $47^\circ$, $72^\circ$, $61^\circ$ is merely equivalent to the statement that two of the angles are $47^\circ$, $72^\circ$, because the angle sum for every triangle is $180^\circ$; it therefore contains only two independent facts.

In Ex. VIII. c, triangles have been constructed which fit uniquely three independent measurements; the examples in Ex. VIII. e show other possibilities, of which account must be taken.

CONGRUENT TRIANGLES

EXERCISE VIII. d

[Draw rough figures for yourself and mark the data on them.]

The two triangles in Fig. 256 are given congruent. With the data of Nos. 1-3, state this in the form $\triangle ABC = \ldots$; and show on your own figures the sizes of the remaining sides and angles of $\triangle KQN$.

1. (i) $\angle K = 60^\circ$, $\angle N = 56^\circ$; (ii) $\angle Q = 60^\circ$, $\angle N = 56^\circ$; (iii) $QK = 2.6''$, $KN = 2.5''$.

Fig. 256.

2. (i) $\angle Q = 56^\circ$, $\angle K = 64^\circ$; (ii) $\angle Q = 64^\circ$, $\angle K = 56^\circ$; (iii) $QN = 2.7''$, $KQ = 2.5''$.

3. (i) $QK = 2.5''$, $\angle Q = 60^\circ$; (ii) $QN = 2.5''$, $\angle N = 60^\circ$; (iii) $KN > QN > KQ$.

4. If the triangles in Fig. 256 are congruent,
   (i) can $KN$ equal 2.7'' when $\angle Q = 60^\circ$ ?
   (ii) can $QN$ equal 2.5'' when $\angle N = 56^\circ$ ?
   (iii) can $\angle K$ be the greatest angle if $KN$ is the greatest side ?

9. If $\triangle FGH$ is congruent to either triangle in Fig. 257, what can you say (i) about $\angle G$ if $FH = 7.1$, (ii) about $FG$ if $\angle F = 62^\circ$ ?

10. If $\triangle XYZ$ is congruent to either triangle in Fig. 258, what can you say (i) about $\angle Y$ if $XZ = 6$, (ii) about $XY$ if $\angle Y = 79^\circ$ ?

Find out whether the triangles in Figs. 257-260 are congruent, and, if so, name them with letters in the proper order, and show on your own figures the sizes of the remaining sides and angles. Give reasons.

The unit for length is 1 cm.

5.

Fig. 257.

6.

Fig. 258.

7.

Fig. 259.

8.

Fig. 260.
CONSTRUCTION OF TRIANGLES WITH THREE INDEPENDENT DATA

EXERCISE VIII. c

The Data may be inconsistent
1. Can you draw a triangle ABC so that \( \angle A = 30^\circ, \angle B = 60^\circ, \angle C = 90^\circ \) ? If not, why not?

2. A boy measures the sides of \( \triangle DEF \) and says that \( DE = 7 \text{ cm}, \ EF = 2.5 \text{ cm}, \ FD = 3.5 \text{ cm} \). Try to draw it from these measurements. What is wrong?

3. Draw, if you can, \( \triangle XYZ \) and measure \( \angle XYZ \), given that
   (i) \( XY = 7 \text{ cm}, \ YZ = 2.5 \text{ cm}, \ ZX = 10 \text{ cm} \);
   (ii) \( XY = 7 \text{ cm}, \ YZ = 2.5 \text{ cm}, \ ZX = 8 \text{ cm} \);
   (iii) \( XY = 7 \text{ cm}, \ YZ = 2.5 \text{ cm}, \ ZX = 9.5 \text{ cm} \).
   If you cannot draw it, explain why not.

4. A boy, measuring \( \triangle PQR \), says that \( PQ = 5 \text{ cm}, \ \angle PQR = 70^\circ, \ \angle QPR = 120^\circ \). Try to draw it. What is wrong?
   What is a likely mistake for the boy to have made?

5. Draw, if you can, \( \triangle ABC \) and measure \( BC \), given that
   (i) \( AB = 4 \text{ cm}, \ AC = 5 \text{ cm}, \ \angle ABC = 90^\circ \);
   (ii) \( AB = 4 \text{ cm}, \ AC = 3 \text{ cm}, \ \angle ABC = 90^\circ \).
   If you cannot draw it, explain why not.

The Data may be insufficient
6. Draw a triangle \( \triangle ABC \), given that \( AB = 3 \text{ in}, \ AC = 1.8 \text{ in}, \ \angle ABC = 32^\circ \).
   Measure \( \angle ACB \).
   [First draw \( \angle AXY = 32^\circ \), then mark off \( BA \) along \( BX = 3 \text{ in} \); use your compass to find the position of \( C \) on \( BY \), since \( AC = 1.8 \text{ in} \), see Fig. 266, not drawn accurately.]
   The data in No. 6 consist of the lengths of two sides and the size of a not-included angle. Fig. 266 shows why two triangles of different sizes can be drawn to fit this set of data, although there are three independent facts.

7. In your figure for No. 6, mark another point \( K \) on \( BY \) so that \( AK = 3 \text{ in} \), and join \( AK \).
   (i) What kind of triangle is \( \triangle ABK \) and how much is \( \angle AKB \)?
   (ii) What agreement as regards sides and angles is there between \( \triangle ABK \) and \( \triangle AKC \)?
   (iii) Draw for yourself two triangles which are obviously not congruent but agree as regards the measurements of two sides and a not-included angle.
8. Draw a triangle $ABC$, given that
\[ AB = 4.2 \text{ cm, } AC = 5 \text{ cm, } \angle ABC = 90^\circ. \]
Measure $BC$.

[First draw $\angle XBY = 90^\circ$ and proceed as in No. 6.]

The data in this question consist of the lengths of two sides and the statement that a not-included angle is a right angle. Fig. 207 shows two triangles $ABC_1$, $ABC_2$ which fit this set of data.

(i) What kind of triangle is $\triangle AC_1C_2$? What can you say about $\triangle AC_2B$? What then about $\triangle BAC_2$?

(ii) What axis of symmetry is there in the figure?

(iii) Is the size and shape of a triangle fixed if the lengths of two sides are given and if a not-included angle is a right angle?

9. (i) Draw a triangle $DEF$, given that $DE = 7 \text{ cm, } DF = 5.5 \text{ cm, } \angle DEF = 35^\circ$. If you can draw triangles of different sizes to fit the data, do so. Measure $EF$.

(ii) Repeat (i) for $\triangle PQR$ where $PQ = 7 \text{ cm, } PR = 8 \text{ cm, } \angle PQR = 35^\circ$. Measure $QR$.

10. Try to draw a triangle $ABC$ so that $AB = 5 \text{ cm, } AC = 3.5 \text{ cm, } \angle ABC = 55^\circ$. What is wrong?

**Two sides and a not-included Angle**

Some of the examples in Ex. VIII.6 show that, when the measurements of two sides and a not-included angle are given, it may be possible to draw two triangles of different sizes to fit the data. This is known as the "ambiguous case".

If, however, the given not-included angle is a right angle, the size and shape of the triangle is fixed completely, cf. Ex. VIII.6, No. 8.

**Symmetry, Congruence and Similarity**

If a triangle is right-angled, the side opposite the right angle is called the hypotenuse. A special test for congruence is therefore as follows:

**Two right-angled triangles are congruent, if their hypotenuses are equal, and if one other side of the first triangle is equal to one other side of the second triangle.**

*For reference*: Rt. $\angle$, hyp., 1 side; or $90^\circ$, H.S.

**Use of the Congruence Tests**

It is often possible to show that triangles are congruent although no actual lengths in inches or cm., or sizes of angles in degrees, are given.

Thus in Ex. VIII.d, No. 11, we can prove $\triangle CAB = \triangle KAP$ without knowing that $\triangle BAC = 48^\circ$, because $\angle BAC = \angle PAK$, vertically opposite, in any case. Again, in Ex. VIII.d, No. 12, the side $BD$ is "common" to the two triangles and so is the same for both, whatever its length may happen to be.

**Exercise VIII.f**

The data in Nos. 1-6 refer to two triangles $ABC, XYZ$. In each case, draw a rough figure and show the data on it in some distinctive way.

Do the data make the triangles congruent? If so, state briefly the test used, and name the triangles properly.

1. (i) $AB = YZ, AC = XZ, \angle A = \angle Z$;
   (ii) $AB = XY, \angle A = \angle X, \angle B = \angle Z$.

2. (i) $AB = XY, \angle A = \angle X, \angle C = \angle Z$;
   (ii) $AC = XZ, AB = XY, \angle C = \angle Z$.

3. (i) $AB = YZ, \angle A = \angle Z, \angle B = \angle Y$;
   (ii) $\angle A = \angle Z, \angle C = \angle X, AC = XY$.

4. (i) $BC = YZ, CA = XY, \angle C = \angle X$;
   (ii) $BC = XY, CA = YZ, AB = ZX$.

5. (i) $\angle B = 90^\circ = \angle Y, AB = YZ, BC = ZX$;
   (ii) $\angle A = 90^\circ = \angle X, AB = XZ, BC = YZ$.

6. (i) $BC = ZX, \angle C = \angle X, \angle A = \angle Y$;
   (ii) $AB = AC, XY = XZ, \angle A = \angle X$. 
7. If in $\triangle ABC$, $DEF$, it is given that $\angle B = \angle D$ and $\angle C = \angle E$, what additional fact will secure that the triangles are congruent? Give three answers.

8. Repeat No. 7, if $AB = DF$ and $AC = EF$. [2 answers.]

Look for congruent triangles in the figures of Nos. 9-14; give reasons. Name them properly, and write down what other lines or angles are equal.

9. In Fig. 268, BD bisects $\angle ABC$ and $\angle ADC$.

![Fig. 268]

10. In Fig. 269, PQ bisects $AB$ at right angles.

11. In Fig. 270, $AB$ is parallel to $DC$, and $AD$ to $BC$.

12. In Fig. 271, $EF$ is equal and parallel to $GH$.

13. In Fig. 272, $AB = AC$ and $AN$ bisects $\angle BAC$.

14. In Fig. 273, $O$ is the centre of the circle and $ON$ is perpendicular to the chord $AB$.

15. Two unequal circles, centres $A$ and $B$, intersect at $P, Q$. Prove that $\triangle APB = \triangle AQB$. What follows?

**Congruence and Similarity**

If a photograph is printed by putting the paper in contact with the film, the original and the print agree both in shape and size; that is, they are congruent. But if a photograph is projected on a screen, as at a cinema, the original and the projected picture agree only in shape, not in size; the two figures are then called similar; one

**Exercise VIII. g**

(i) The shape of a triangle is fixed by the ratio of two sides and the size of the included angle.

1. A, B, C are 3 villages; B is 5 miles due north of A, and C is 4 miles N. 40° E. from A. Make a map of $A, B, C$ on the scale

(i) $\frac{1}{2}$ inch to the mile; (ii) 2 cm. to the mile.

On map (i), what are the lengths of $AB, AC$? What is $\frac{AB}{AC}$?

On map (ii), what are the lengths of $AB, AC$? What is $\frac{AB}{AC}$?

By measurement, find for each map $BC$, $\frac{BC}{AB} \angle ABC$.

2. If $DE = \frac{1}{2} DF$, what is $DE$ if (i) $DF = 12\text{ cm.}$, (ii) $DF = 10\text{ cm.}$, (iii) $DF = 8\text{ cm.}$, (iv) $DF = 6\text{ cm.}$? What is $\frac{DE}{DF}$?

If $\frac{AB}{AC} = \frac{1}{3}$, what is $AB$ if $AC = 4\text{ inches}$? What is $AC$ if $AB = 7.5\text{ inches}$?

K
3. **PQR** is a triangle such that $\angle QPR = 105^\circ$ and $\frac{PQ}{PR} = \frac{3}{2}$. Draw two triangles of different sizes to fit the data. In each case find $\angle QPR$ and $\frac{QR}{PR}$.

(II) The shape of a triangle is fixed by the sizes of two angles.

4. A, B, C are 3 houses. A is due south of B, and the bearings of C from A and B are N. 47$^\circ$ E. and S. 58$^\circ$ E. Draw two plans of different sizes to fit the data. On each plan measure $CA$, $CB$, and work out the ratio $\frac{CA}{CB}$.

5. Two of the angles of a triangle are 97$^\circ$ and 40$^\circ$. From a scale-drawing find the ratio of the longest to the shortest side. [The answers obtained by different members of the class should be compared.]

(III) The shape of a triangle is fixed if the ratios of two of the sides to the third side are known.

6. Draw a triangle $ABC$ so that $\frac{AB}{BC} = \frac{3}{4}$ and $\frac{AC}{BC} = \frac{1}{4}$. Measure the largest angle. [Compare answers as in No. 5.]

7. Measure in cm. the sides of $\triangle ABC$ in Fig. 23 (see end of book); draw a triangle $XYZ$ so that the lengths of its sides are three-fifths of those of $\triangle ABC$. Measure the largest angle of $\triangle ABC$ and of $\triangle XYZ$.

(IV) The shape of a right-angled triangle is fixed by the ratio of one side to the hypotenuse.

8. Draw a right-angled triangle so that one side is half the hypotenuse. Measure the largest acute angle. [Compare answers as in No. 5.]

9. Fig. 274 shows a ladder $AB$ resting against a wall. If $\angle BAN = 5\degree$, find from a scale-diagram the ratio $\frac{AN}{AB}$.

10. Draw two differently shaped triangles $ABC$ to fit the data, $\angle ABC = 32^\circ$, $\frac{AC}{AB} = \frac{3}{2}$. Measure $\angle ACB$ in each. [If you find this difficult, look at Fig. 266, p. 125.]

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**USE OF SIMILARITY TESTS**

**EXERCISE VIII. h**

Are the triangles in Fig. 275 similar? If so, state the test used, name the triangles properly, and write down the magnitudes of the unmarked sides and angles; the unit of length is 1 cm.

1. ~

2. **XYZ** is a triangle similar to the smaller triangle in Fig. 275.

   (i) If $\angle Y = 61^\circ$ and $XZ = 4\cdot4$ cm., what are $XY$ and $YZ$?

   (ii) If $XZ = 2\cdot2$ in. and $YZ = 1\cdot8$ in., what are $\angle X$ and $\angle Y$?
3. Repeat No. 1 for Figs. 276, 277.

4. 

5. PQR is a triangle similar to the larger triangle in Fig. 277.
   (i) If its smallest side is 1 ft., find in inches the lengths of its other sides.
   (ii) If\( \frac{PQ}{PR} = \frac{3}{5} \), find \( \angle P \) and \( \angle R \).

Find pairs of similar triangles in Figs. 278-281; state the test used, name them properly, and write down the lengths of the unmarked lines. [The result of No. 7 can be used for No. 9.]

Arrows denote that lines are given parallel; the unit of length is 1 cm.

6. 

7. 

8. 

9. 

The Parallelogram

The definition of a parallelogram is a quadrilateral whose opposite sides are parallel. Starting from this statement, various properties can be proved.

(i) The opposite sides of a parallelogram are equal.
   Given \( ABCD \) is a parallelogram.
   To prove \( AB = DC \) and \( AD = BC \).
   Construction. Join \( DB \).
   Proof. In \( \triangle ABD \), \( CDB \)
   \( \angle ABD = \angle CDB \), alt. \( \angle s, AB, DC \parallel \).
   \( \angle ADB = \angle CBD \), alt. \( \angle s, AD, BC \parallel \).
   \( BD \) is common.
   \( \therefore \ Triangle ABD \equiv Triangle CDB \), 2 \( \angle s \) and corresp. side.
   \therefore \( AB = CD \) and \( AD = BC \), corresp. sides.

For reference: opp. sides par\( \parallel \).

The proper way of setting out the proof of a general statement or theorem must be noted carefully.

(i) First state what is given, using letters.
(ii) Next state what is to be proved.
(iii) Next state the construction, if any.
(iv) Lastly, give the proof, putting in all the reasons.

Reasons may be stated briefly; the kind of abbreviation to be used has been indicated under the heading "For Reference"; but any form of abbreviation which is clear is permissible. From the proof just given, we obtain two other important properties of the parallelogram:

(ii) Each diagonal bisects the parallelogram.
(iii) The opposite angles of a parallelogram are equal.

For we have proved, in Fig. 282, that \( \angle ABD = \angle CDB \).
\( \therefore \angle BAD = \angle DCB \).
Similarly \( \triangle ABC \equiv \triangle CDA \), and \( \angle ABC = \angle CDA \).
There is another property of the parallelogram which should be known.

(iii) The diagonals of a parallelogram bisect one another.

Given $ABCD$ is a parallelogram, and its diagonals cut at $K$.

To prove $AK = CK$ and $BK = DK$.

Proof. In $\triangle ABK$, $CDK$

\[ \angle KAB = \angle KCD, \text{alt. } \angle s, AB, DC \parallel \]
\[ \angle KBA = \angle KDC, \text{alt. } \angle s, AB, DC \parallel \]
\[ AB = CD \text{ opp. sides par}^m. \]

\[ \therefore \triangle ABK = \triangle CDK, 2 \angle s \text{ and corresp. side.} \]
\[ \therefore AK = CK \text{ and } BK = DK, \text{ corresp. sides.} \]

The Isosceles Triangle

The chief property of the isosceles triangle was deduced from principles of symmetry on p. 116. The proof may also be set out as follows:

The angles at the base of an isosceles triangle are equal.

Given $ABC$ is a triangle having $AB = AC$.

To prove $\angle ABC = \angle ACB$.

Construction. Draw $AE$ to bisect $\angle BAC$ and to cut $BC$ at $E$.

Proof. In $\triangle ABE$, $ACE$

$AB = AC$, given.

$AE$ is common.

\[ \angle BAE = \angle CAE, \text{ constr.} \]

\[ \therefore \triangle ABE = \triangle ACE, 2 \text{ sides and inc. } \angle. \]
\[ \therefore \angle ABE = \angle ACE. \]
7. In Fig. 288, \( AB \) and \( YZ \) are the diameters of two concentric circles. Find an angle equal to \( \angle OAY \); give reasons. What follows?

8. \( ABCD \) is a parallelogram. If \( \angle ABC \) is a right angle, prove that \( AC = DB \).

9. \( ABCD \) is a parallelogram. If \( BD \) bisects \( \angle ABC \), prove that all the sides of \( ABCD \) are equal.

10. In Fig. 290, \( ABCD \) is a square and \( BCK \) is an equilateral triangle. Find in degrees (i) \( \angle BDC \), (ii) \( \angle BCK \), (iii) \( \angle DCK \), (iv) \( \angle CDK \).
Prove that \( \angle BDK = \frac{1}{2} \angle BCK \).

11. \( AB \) is any chord of a circle, centre \( O \) (see Fig. 273, p. 128). If \( N \) is the midpoint of \( AB \), prove that \( \angle ANO = 90^\circ \).

12. \( AB \) is any chord of a circle, centre \( O \) (see Fig. 273, p. 128). If \( ON \) is the perpendicular from \( O \) to \( AB \), prove that \( AN = NB \).

13. If, in Fig. 285, \( PYQZ \) is a parallelogram, and if any line \( MON \) through \( O \) cuts \( PQ \) at \( M, N \), prove that \( OM = ON \).

14. In Fig. 290, \( ABCD \) is a parallelogram, and \( QA, QB \) bisect \( \angle DAB, \angle CBA \); prove that \( AB = 2AD \).

15. \( ABCD \) is a parallelogram. If \( AB = BC \), prove that \( AC \) is perpendicular to \( BD \).

CHAPTER IX

AREA

Area of a Rectangle

The meaning of Area has been explained in Chapter III. Examples were taken to show that a rectangle \( x \) units long, \( y \) units wide, where \( x \) and \( y \) are any whole numbers, can be divided into \( xy \) unit squares, that is, squares whose sides are of unit length. The area of each of these squares is called a unit of area, and therefore the area of the whole rectangle is \( xy \) units of area.

Examples for Class Discussion

What is the area of a rectangle 2\( \frac{1}{2} \) inches long, 1\( \frac{1}{2} \) inches wide? This rectangle cannot be divided into a whole number of 1 inch squares.

Take the unit of length to be 3\( \frac{1}{2} \) inch and copy and complete the following sentences:

The length of the rect. = \( 2 \frac{1}{2} \) in. = \( \frac{5}{2} \) in. = \( \frac{3}{2} \) in. = \( \ldots \) units.

The breadth of the rect. = \( 1 \frac{1}{2} \) in. = \( \ldots \) = \( \ldots \) = \( \ldots \) units.

\( \therefore \) area of rect. = \( \ldots \) = \( \ldots \) = \( \ldots \) units.

But 1 sq. in. is \( \ldots \) units long and \( \ldots \) units wide.

\( \therefore \) 1 sq. in. = \( \ldots \) units of area.

\( \therefore \) area of rect. = \( \frac{15 \times 8}{6 \times 6} \) sq. in. = \( \frac{15}{6} \) sq. in. = \( \frac{5}{2} \times \frac{1}{2} \) sq. in. = \( 2 \frac{1}{4} \times 1 \frac{1}{2} \) sq. in.
This example shows why the area of a rectangle is
found by multiplying the number of inches (or cm., etc.)
in its length by the number of inches (or cm., etc.) in its
breadth, whether these numbers are whole numbers or
fractions. This process is called, for short, "multiplying
the length by the breadth".

Instead of the length and breadth, we often speak of
the base and height; hence the following rule:

Area of Rectangle = length \times breadth
= base \times height.

Hence we have the following formula:

If the base of a rectangle is \( b \) in., and if its height is
\( h \) in., its area is \( A \) sq. in., where

\[
A = bh.
\]

Figures composed of Rectangles

The area of a figure may often be found by regarding
it either as the sum or as the difference of two rectangles.

**EXERCISE IX. a**

[In Figs. 291-297, all corners are right-angled and the unit
of length is 1 foot.]

Find the areas of Figs. 291, 292.

1. \( \text{Fig. 291 (i)} \)

2. \( \text{Fig. 291 (ii)} \)

3. \( \text{Fig. 291 (iii)} \)

4. \( \text{Fig. 292 (i)} \)

5. \( \text{Fig. 292 (ii)} \)

6. \( \text{Fig. 292 (iii)} \)

7. Find the area of Fig. 291 (ii) by producing CD, AF to
meet at K, and using the subtraction method; area equals
rect. ABCK - rect. FEDK.

8. Find the area of Fig. 291 (iii) by the subtraction method.

9. Enclose Fig. 292 (i) in a square of side 12 units; then
find the area of the cross by the subtraction method.

10. Find the unshaded area in

\( \text{Fig. 293} \)

11. A room is 20 ft. long, 14 ft. wide, 11 ft. high. Draw
a rough figure showing the 4 walls of the room folded out flat,
and find the total area of the walls.

12. Fig. 294 represents a rectangle \((a + b)\) in. high and \(x\) in.
wide, divided into two compartments. The area of this rectangle
is \((a + b)x\) sq. in., and the areas of the compartments are
\(ax\) sq. in. and \(bx\) sq. in.; therefore \((a + b)x = ax + bx\). Draw
a figure to illustrate in the same way that \((3 + 2)9 = 3 \times 9 + 2 \times 9\).

13. If in Fig. 294 we replace \(x\) by \(c + d\), we obtain Fig. 295.
What does this tell you about \((a + b)(c + d)\) ?

14. Copy Figs. 296 (i), (ii), and insert
in your copies in each compartment its area,

\( a \)

\( b \)

\( ab \)

\( a^2 \)

\( b^2 \)

as in Fig. 295. State in each case the area of the whole figure.
Then state the algebraic relation which the figure illustrates.

15. Illustrate by a figure

(i) \((2a)^2 = 4a^2\);
(ii) \((3b)^2 = 9b^2\).

16. What relation is illustrated by Fig. 297?
17. A stamp measures 2.3 cm. by 1.9 cm. If the unit of length is 1 mm., what are the dimensions of the stamp, and what is its area? Then find the area in sq. cm.

18. A rectangle is 2 1/2 in. long, 1 3/4 in. wide; choose a unit such that the length and breadth are represented by whole numbers and find the number of units of area in the rectangle. Then find the area of the rectangle in sq. inches.

19. Find the area of a figure whose corners (on squared paper) are the points, (3, 1); (3, 7); (8, 1); (8, 7).

20. The curved line in Fig. 298 is the map of the coast line of an island.

(i) What area is represented by 1 sq. in. on the map?
(ii) What is the area represented by the shaded figure?
(iii) What is approximately the area of the island?

[In counting small squares, omit any portions less than half a square, and count any portion more than half a square as a whole.]

21. Repeat No. 15, if the scale is 10 miles to the inch, instead of that shown on the figure.

**Area of Right-angled Triangle**

To find the area of one face of your set-square, take another equal set-square and fit them together (see Fig. 299), so that they form a rectangle.

Measure $AB$ and $BC$ in cm. What is the area of $ABCD$? What is therefore the area of $\triangle ABC$?

---

**EXERCISE IX. b**

1. In Fig. 299, $AB = 8$ in., $BC = 6$ in. What is the area of (i) rect. $ABCD$, (ii) $\triangle ABC$?

2. Find the area of $\triangle ABC$ in Fig. 299, if
   (i) $AB = 6$ cm., $BC = 4$ cm.; (ii) $AB = 2\frac{1}{2}$ ft., $BC = 1\frac{1}{2}$ ft.;
   (iii) $AB = 1\frac{1}{2}$ yd., $BC = 2$ ft.; (iv) $AB = h$ in., $BC = b$ in.

3. (i) Use the Construction indicated in Fig. 300 to find the area of $\triangle EFGH$; the unit is 1 inch.
   (ii) Sketch a figure which shows two quadrilaterals congruent to $\triangle EFGH$ fitted together to form a rectangle. What is the length and the height of this rectangle? Hence find the area of $\triangle EFGH$ in a second way.

4. In Fig. 291 (ii), p. 138, if $FD$ is joined, find the area of the whole figure thus formed.

5. In Fig. 291 (ii), p. 138, if $BE$ is joined, find the area of $\triangle BDC$.

6. In Fig. 301, the units are cm.; find the area of $\triangle ABC$.

7. In Fig. 302, the units are inches; find the area of $\triangle PQR$ if $PK = QR$.

8. In Fig. 303, the units are feet; find the area of $\triangle XYZ$.

9. In Fig. 299, $BC$ is 5 in. and the area of $\triangle ABC$ is 20 sq. in. find the length of $AB$.

10. $ABC$ is a triangle and $AN$ is the perpendicular from $A$ to $BC$, as in Fig. 301.
   (i) If $BN = 5$ in., $NC = 3$ in., and area of $\triangle ABC = 20$ sq. in., find $AN$.
   (ii) If $BN = 7$ cm., $AN = 6$ cm., and area of $\triangle ABC = 36$ sq. cm., find $NC$. 

---

**Fig. 298.**

**Scale of Miles**

**Fig. 299.**

**Fig. 300.**

**Fig. 301.**

**Fig. 302.**

**Fig. 303.**
11. In Fig. 304, ABCD is a rectangle. Find the area of ΔPBC.

12. In Fig. 304, mark a point Q on AB so that AQ = 2"; find the area of ΔPQB.

Area of Parallelogram

Draw any parallelogram ABCD on stiff paper, and draw the perpendicular AH from A to CD produced (see Fig. 305 (i)).

Cut out the figure ABCH (see Fig. 305 (ii)), and also a triangle congruent to ΔADH, and shade it (see Fig. 305 (iii)).

First place (iii) on (ii) as shown in (i); this leaves uncovered the parΔm. ABCD.

Then slide the triangle into the position shown in (iv); this leaves uncovered the rect. ABNH.

\[ \text{area of parΔm. } ABCD = \text{area of rect. ABNH.} \]

Base and Height

If we call AB the base of the parΔm. ABCD, then AH, the perpendicular distance between the parallel sides, AB and DC, is called the height or altitude. It is also the height of rect. ABNH, corresponding to AB as base. We therefore say that the parΔm. ABCD and the rect. ABNH are "on the same base AB and have the same height AH".

EXERCISE IX. c

1. In Fig. 305, if AB = 4 in., AH = 7 in., CH = 10 in., find the area of (i) ΔAHD, ΔBNC, (ii) rect. ABNH, (iii) quad. ABCH, (iv) parΔm. ABCD.

2. (i) Measure, in cm., AB and AH in Fig. 306; then find the area of parΔm. ABCD in sq. cm.

(ii) Fig. 307 is a copy of parΔm. ABCD in Fig. 306, with BC drawn across the page. Measure, in cm., BC and the distance between the parallels BC, AD; then find the area of parΔm. ABCD in a second way. Compare with (i).

(iii) Sketch the parΔm. ABCD and show on your figure two rectangles equal in area to it, one with AB as base and the other with BC as base.

3. Draw a parΔm. ABCD such that AB = 3 in., AD = 2 in., ∠BAD = 65°. Construct two rectangles equal in area to it, one with AB as base, and the other with AD as base. Measure the heights of the rectangles and then find the area of ABCD in two ways.

4. In Fig. 308, ABCD is a parΔm.

(i) Find the area of ABCD.

(ii) Find the distance between the parallels BC, AD.
5. Draw a par. $ABCD$ and draw $BP, BQ$ perpendicular to $DA, DC$. If $AD = 9$ cm., $BP = 5$ cm., $BQ = 7.5$ cm., calculate (i) area of $ABCD$, (ii) length of $AB$.

6. Use instruments to draw a rectangle $ABCD$ of area 30 sq. cm., with $AB = 6$ cm. Then draw a par. $ABPQ$ of equal area, with $AQ = 6.5$ cm., and measure $\angle QAB$. Then draw a rect. $BPXY$ of equal area and measure $PX$. How can you check the result?

7. (i) The base of a par. is 8 cm., and its height is 9 cm.; find its area.
   (ii) The base of a par. is 7 cm., and its area is 42 sq. cm.; find its height.
   (iii) A par. is $6\frac{1}{2}$ in. high; its area is 10 sq. in.; find its base.

8. Draw a par. $ABCD$, so that $AB = 6$ cm., $AD = 5$ cm., and of area 24 sq. cm.; measure $\angle DAB$.
   Construct a rhombus $ABHK$ of the same area; measure $\angle KAB$. [A rhombus is a par. with two adjacent sides equal.]

9. Fig. 309 shows two par.s $ABCD$, $ABXY$ on the same base $AB$ and such that the sides $DC, YX$, parallel to $AB$, form a straight line $DCYX$.

   Such par.s are said to be between the same parallels.

   (i) How can you draw a rectangle in the figure so that it is equal to each par.?
   (ii) Name two congruent triangles in the figure; then use the method of p. 142, Fig. 305, to prove that the par.s are equal in area.

10. The two par.s $ABCD$ in Fig. 310 (i), (ii) are given congruent, and the dotted lines represent their heights.
    (i) What can you say about the rectangles $CDLM$, $BCHK$? (ii) Can you check the data?

11. In Fig. 309, if $AY$ is longer than $AD$, which of the pairs of parallel lines is the farther apart, and why?

Area of Triangle

Draw any triangle $ABC$. Through $A$ draw a line parallel to $BC$, through $C$ draw a line parallel to $BA$, let these lines cut at $X$ (see Fig. 311).

Then $ABCX$ is a parallelogram. This process is called, completing the par. $ABCX$.

Since a parallelogram is bisected by a diagonal, the area of $\triangle ABC$ is half the area of par. $ABCX$.

Examples for Class Discussion

1. Fig. 312 shows a triangle $ABC$, a par. $ABCX$, a rect. $BCYZ$, all on the same base and between the same parallels.

   Answer, with reasons, the following:
   (i) What is the area of par. $ABCX$? (ii) What is the area of $\triangle ABC$? (iii) What can you say about the areas of $\triangle ABC$ and rect. $BCYZ$? (iv) What is the area of $\triangle ABC$?
2. Draw accurately $\triangle ABC$, making $BC = 7$ cm., $CA = 5$ cm., $AB = 6$ cm.; and draw rect. $BCYZ$ as in Fig. 313.

Now use your own figure for (i)-(iv).

(i) Complete the par. $ABCD$ and then explain the relation between the areas of $\triangle ABC$ and rect. $ZBCY$.

(ii) Find in sq. cm., making any necessary measurements, the areas of rect. $ZBCY$ and $\triangle ABC$.

(iii) Make an accurate copy of $\triangle ABC$ by pricking through on to another sheet. Take $AC$ as base of $\triangle ABC$ (turn the paper round) and draw a rect. $CAPQ$ so that $PQ$ passes through $B$. Find in sq. cm., making any necessary measurements, the area of rect. $CAPQ$ and $\triangle CAB$. Compare with (ii).

(iv) If you have time, do this again making a rect. $ABMN$ on the same base $AB$ and of the same height as $\triangle ABC$. Take the average of your three results for the area of $\triangle ABC$.

These examples illustrate the following fact:

Area of Triangle = Half of area of Rectangle on same base and with same height.

This is often quoted in the form:

$\text{Area of Triangle} = \frac{1}{2} \text{ base} \times \text{height}$

Hence we have the following formula:

If the base of a triangle is $b$ in., and if its height is $h$ in., its area is $A$ sq. in., where

$A = \frac{1}{2}bh$.

**EXERCISE IX. d**

1. Show, on a rough sketch, data such that area of $\triangle ABC$ is 12 sq. cm., (i) if $BC = 6$ cm., (ii) if $AC = 8$ cm., (iii) if one altitude is 2 cm.

2. In Fig. 313, where $ZBCY$ is a rectangle, if $ZA = 5$ in., $AY = 2$ in., $BZ = 4$ in., find the area of (i) rect. $ZBCY$, (ii) $\triangle ZAB$, (iii) $\triangle YAC$. Find, by subtraction, the area of $\triangle ABC$. How can you check the answer?

3. In Fig. 314, $ABMN$ is a rectangle. If $AB = 8$ cm., and if area of $\triangle ABC = 24$ sq. cm., find (i) area of rect. $ABMN$, (ii) length of $AN$, (iii) length of perpendicular from $C$ to $AB$.

Draw a rough figure and sketch in it a rectangle with $BC$ as base and equal in area to rect. $ABMN$. If $BC = 7.5$ cm., find the length of the perpendicular from $A$ to $BC$.

4. In Fig. 315, $LP = 4$ cm. and the area of $\triangle TLP$ is 18 sq. cm. Find (i) the area of rect. $LMNP$, (ii) the length of $LM$, (iii) the length of the perpendicular from $T$ to $LP$ produced.

If $LT = 12$ cm., find the length of the perpendicular from $P$ to $LT$.

5. In Fig. 314, the area of rect. $ABMN$ is 30 sq. cm., and the length of the perpendicular from $B$ to $AC$ is 12 cm. Find the length of $AC$.

6. Construct a triangle so that two of its sides are 6 cm., 6.5 cm., and so that its area is 15 sq. cm. Measure the third side. Is there more than one answer?

7. Fig. 316 shows two triangles $PBC$, $QBC$ on the same base $BC$ and such that the line joining their vertices $P$, $Q$ is parallel to $BC$.

Such triangles are said to be between the same parallels.

(i) How can you draw a rectangle in the figure so that its area is double that of $\triangle PBC$ or $\triangle QBC$.

(ii) Complete the par., $BXP$, $BQY$, and use them to prove that $\triangle BPC$ and $\triangle BQO$ are equal in area.

This fact is stated as follows:

Triangles on the same base and between the same parallels (or with equal heights) are equal in area.
8. Draw figures showing
   (i) 3 parallelograms on the same base and between the
       same parallels;
   (ii) 3 triangles on the same base and between the same
       parallels;
   (iii) a triangle and a parallelogram on the same base and
       between the same parallels.

What can you say about the areas?

The relation which exists in No. 8 (iii) is important. It is stated as follows:
If a triangle and a parallelogram are on the same base and
between the same parallels, the area of the triangle is half the
area of the parallelogram.

**Base and Height of a Triangle**

Any side of a triangle may be taken as base; the perpendicular to this side from the opposite vertex is the corresponding height.

The examples in Ex. IX.c illustrate the fact that a parallelogram can be regarded as having two distinct bases and two corresponding heights; its area can therefore be found by two different calculations.

Similarly Example 2 on p. 146 shows that a triangle can be regarded as having three distinct bases and three corresponding heights; its area can therefore be found by three different calculations.

Fig. 317 shows the three heights, or altitudes, AD, BE, CF of \( \triangle ABC \) corresponding to the bases BC, CA, AB.

![Fig. 317](image)

If the triangle is obtuse-angled, as in Fig. 318, two of the altitudes fall outside the triangle.

![Fig. 318](image)

**AREA**

The rule or formula for the area of \( \triangle ABC \) is often written, as follows:

\[
\triangle ABC = \frac{1}{2} BC \cdot AD = \frac{1}{2} CA \cdot BE = \frac{1}{2} AB \cdot CF.
\]

**Area of any Quadrilateral**

Fig. 319 shows any quadrilateral \( ABCD \). By joining \( AC \), we obtain two triangles, whose areas can be calculated separately.

**Oral Example.**—In Fig. 319, BP and DQ are the perpendiculars from B, D to AC. Find the area of \( ABCD \) if \( AC = 6 \text{ in.}, BP = 3 \text{ in.}, DQ = 5 \text{ in.} \)

![Fig. 319](image)

![Fig. 320](image)

[The perpendiculars from the corners of a polygon to a diagonal are called offsets.]

**Reducing a Quadrilateral to a Triangle of Equal Area**

Another method for finding the area of a quadrilateral is to start by converting it into a triangle of equal area. Let \( ABCD \) be the given quadrilateral (see Fig. 320). Draw the diagonal \( AC \). *Leave \( \triangle ABC \) alone*, but replace \( \triangle ADC \) by another triangle of the same area.

Draw \( DX \) parallel to \( AC \) to cut \( BC \) produced at \( X \); join \( AX \). Complete the sentence:

\( \triangle ADC \) and \( \triangle AXC \) are on the same base .... and between the same parallels .... What follows?

Now simplify, \( \triangle ABC + \triangle ADC = \triangle ABC + \triangle AXC \).

This argument shows that the area of the quad. \( ABCD \) is equal to the area of the triangle \( ABX \).
Equivalent Figures

The word "equivalent" really means "equal in weight"; it is used in geometry to mean equal in area.

Equivalent triangles need not be congruent, although, of course, congruent triangles, being equal in all respects, are equal in area, that is, equivalent. The words "congruent" and "equivalent" are therefore represented by different symbols, = for congruent, = for equivalent.

If in a problem you are given that $\triangle PQR = \triangle XYZ$, you must not assume that these triangles are congruent, but only that they are equal in area.

**EXERCISE IX.e**

1. Find in sq. cm. the area of $\triangle ABC$, Fig. 23 (see end of book).

2. Find in sq. cm. the area of quad. PQRS, Fig. 23 (see end of book).

3. Draw Fig. 321 accurately, the units being cm. Then replace $\triangle ADC$ (i) by an equivalent $\triangle AXC$ with $X$ on $BC$ produced, (ii) by an equivalent $\triangle AYC$ with $Y$ on $BA$ produced. Make any necessary measurements and work out the area of quad. $ABCD$ in two ways.

**Definition.**—A quadrilateral, with one pair of opposite sides parallel, is called a trapezium.

4. The parallel sides $BC, AD$ of a trapezium $ABCD$, Fig. 322, are of lengths 9 cm., 5 cm., and are 6 cm. apart. Find the area of $ABCD$, without drawing an accurate figure,

(a) by calculating the areas of $\triangle ABC$, $\triangle CDA$, separately (Fig. 322 (i));

(b) by reducing $ABCD$ to the equivalent triangle $ABX$ (see Fig. 322 (ii)). [What do you know about $CX$?]

5. Take two trapezia congruent to the trapezium $ABCD$ in No. 4, and fit them together as in Fig. 323, not drawn to scale. Can you see why they form a parallelogram? Use this fact to find the area of $ABCD$.

6. With the data of Fig. 324, find the area of (i) $\triangle ABC$, (ii) $\triangle ADC$, (iii) quad. $ABCD$.

7. With the data of Fig. 324, find the area of $\triangle ADB$.

It can be proved that $BD = 5''$; use this fact to find the length of the perpendicular from $A$ to $BD$.

8. Draw a circle of radius 4 cm. By "stepping the radius" round the circle, find the vertices of a regular hexagon inscribed in the circle. Find by any method the area, in sq. cm., of the hexagon.

**GENERAL STATEMENTS**

It is suggested that Nos. 1, 3, 4, 5, 6 of Ex. IX.f should be taken orally. Any or all of the remainder may be omitted at a first reading.

**EXERCISE IX.f**

[Remember to give a reason for each step in your argument.]

1. In Fig. 325, $ABC$ is any triangle and $D$ is the mid-point of $BC$; also $BDAP$ and $CDAP$ are parallelograms; prove that

(a) $\triangle ADB = \triangle APD$;

(b) $\triangle ADB = \triangle ADC$. 

Fig. 322. Fig. 323.
2. PBC and QBC are two triangles of equal area on the same side of BC; PH and QK are altitudes. Prove that (i) PH - QK, (ii) PQ is parallel to BC.

Fig. 326.  

Fig. 327.  

3. In Fig. 326, AB is parallel to DC. Name triangles which have the same area as (i) \( \triangle DAB \), (ii) \( \triangle SDC \), (iii) \( \triangle DKA \). Give reasons.

4. In Fig. 326, what construction should you make in order to obtain
   (i) \( \triangle AXB \) equal in area to \( \triangle AKB \) with X on BC?
   (ii) \( \triangle DYA \) equal in area to \( \triangle DKA \) with Y on AB?
   Give reasons.

5. If in Fig. 327, HK is parallel to BC, prove that
   \( \triangle AHC = \triangle AKB \).

6. If in Fig. 327, H and K are the mid-points of AB and AC, find two triangles equal in area to \( \triangle AHK \).
   Give reasons. What fact about HK can now be proved?

7. In Fig. 328, F is the mid-point of AB and Z is the mid-point of CF. Prove that
   (i) \( \triangle AZC = \triangle BZF \), (ii) \( \triangle BAZ = \frac{1}{2} \triangle BAC \).

8. Draw any parallelogram ABCD and take any points P, Q on CD, DA respectively. Prove that
   \( \triangle APQ = \triangle BQP \).

9. In Fig. 329, if the triangles \( \triangle ADK \), \( \triangle BCK \) are equal in area, prove that the triangles \( \triangle ABK \), \( \triangle CDK \) are equiangular.

CHAPTER X

ANGLE PROPERTIES OF A CIRCLE

Isosceles Triangles

The examples in Ex. X. 7 illustrate the relations between the interior and exterior angles of an isosceles triangle which are due to the fact (see p. 134), that the angles at the base of an isosceles triangle are equal.

Try to make frequent use of the fact,
   exterior angle of \( \triangle = \) sum of interior opposite angles.

You will be doing much unnecessary work if you always start from the fact that the angle sum of a triangle is two right angles.

Converse Properties

If you are given that £1 is worth 120 francs, you can prove that 1 franc is worth 2 pence.

Conversely, if you are given that 1 franc is worth 2 pence, you can prove that £1 is worth 120 francs.

If in any statement you interchange what is given and what follows from the data, the new statement is called the converse of the old one. The fact that a statement is true does not imply that the converse is true, although it often is so.

It is true to say,

If a man has only one leg, he cannot run fast.

It is not true to say,

Conversely, if a man cannot run fast, he has only one leg.

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It has been proved on p. 134 that
If, in \( \triangle ABC \), \( AB = AC \), then \( \angle ABC = \angle ACB \).

The converse property is
If, in \( \triangle ABC \), \( \angle ABC = \angle ACB \), then \( AB = AC \).

You must not assume that this converse is true, unless you can prove it. But if you did Ex. VIII. 9, No. 2, p. 135, you have already proved it; and in future you may make use of it, e.g. in Ex. X. a, Nos. 3, 4, 12, etc.

EXERCISE X. a (Oral)

All questions in this Exercise refer to Fig. 329.

1. What angles are equal if \( CA = CB \)? Name two pairs.
2. What angle equals \( \angle h \) if \( AB = BC \)?
3. What sides are equal if \( \angle r = \angle v \)?
4. What sides are equal if \( \angle f = \angle g \)?
5. If \( AB = AC \) and \( \angle t = 57^\circ \), find \( \angle f \).
6. If \( AB = AC \) and \( \angle f = 108^\circ \), find \( \angle v \).
7. If \( AB = AC \) and \( \angle r = 64^\circ \), find \( \angle t \).
8. If \( AB = AC \) and \( \angle t = 52^\circ \), find \( \angle r \).
9. If \( AB = BC \) and \( \angle g = 134^\circ \), find \( \angle v \).
10. If \( AC = BC \) and \( \angle g = 124^\circ \), find \( \angle v \).
11. If \( AC = BC \) and \( \angle r = 66^\circ \), find \( \angle h \).
12. If \( \angle f = 100^\circ \) and \( \angle t = 50^\circ \), find \( \angle r \). What follows?
13. If \( \angle t = 76^\circ \) and \( \angle v = 28^\circ \), find \( \angle r \). What follows?
14. If \( \angle g = 128^\circ \) and \( \angle r = 64^\circ \), what follows?
15. If \( AB = BC \), connect \( \angle h \) and \( \angle r \).
16. What follows if (i) \( \angle f = 2\angle t \), (ii) \( \angle g = 2\angle v \)?
17. What follows if (i) \( \angle h = \angle r + \angle v \), (ii) \( \angle r = \angle v + \angle t \)?
18. If \( AB = AC \) and \( \angle t = 60^\circ \), what follows?
19. If \( \angle g = 120^\circ \) and \( \angle r = 60^\circ \), what follows?
20. If \( \angle f - \angle t = \angle h - \angle r \), prove \( \triangle ABC \) is isosceles.

Numerical Work

It is often a good plan to take a letter to represent the number of degrees in an angle whose size is unknown, and then form an equation. In such cases, distinguish carefully between the two kinds of notation: the angle \( \theta \), written \( \theta \), and the angle whose size is \( x \) degrees, written \( x^\circ \); \( \theta \) is simply the name of an angle, \( x^\circ \) represents the size of the angle in degrees.

The caution that time is often saved by using the property of the exterior angle of a triangle instead of relying upon the property of the angle sum should be repeated.

Examples for Class Discussion

Draw a rough figure and show the data on it. Copy and complete the following; give reasons.

1. In Fig. 330, \( AB = AC \); if \( \angle KAC = 148^\circ \), find \( \angle ABC \).
   Let \( \angle ABC = x^\circ \), then \( \angle ACB = \ldots \).
   But \( \angle ABC + \angle ACB = \text{ext.} \angle ABC \).
   \( \therefore x^\circ + 5x^\circ = \ldots \);
   \( \therefore 2x^\circ = \ldots \);
   \( \therefore \angle ABC = \ldots \).

2. In Fig. 330, \( AB = AC \); if \( \angle ACE = 5\angle BAC \), find \( \angle BAC \).
   Let \( \angle BAC = x^\circ \), then \( \angle ACE = \ldots \).
   But \( \angle ABC + \angle BAC = \text{ext.} \angle ABC \).
   \( \therefore \angle ABC + x^\circ = 5x^\circ \);
   \( \therefore \angle ABC = \ldots \);
   \( \angle ACB = \ldots \);
   \( \therefore x^\circ = 180^\circ \);
   \( \therefore x = \ldots \);
   \( \therefore \angle BAC = \ldots \).

EXERCISE X. b

1. One base angle of an isosceles triangle is \( 38^\circ \); what is the vertical angle?
2. The vertical angle of an isosceles triangle is \( 44^\circ \); what is each base angle?
3. Find the remaining angles of an isosceles triangle if one angle is \(80^\circ\). There are two sets of answers.

4. Find the remaining angles of an isosceles triangle if one angle is (i) \(130^\circ\), (ii) \(70^\circ\), (iii) \(60^\circ\). Give all possible answers.

5. Find the angles of an isosceles triangle if one exterior angle is \(140^\circ\). There are two sets of answers.

6. Find the angles of an isosceles triangle if one exterior angle is (i) \(50^\circ\), (ii) \(108^\circ\), (iii) \(120^\circ\). Give all possible answers.

7. In Fig. 331, \(AB = AC\).
   (i) If \(y = 40\), find \(x\).
   (ii) If \(x = 70\), find \(y\).
   (iii) If \(\angle ACB = 25^\circ\), find \(x\).
   (iv) Find an equation between \(x\) and \(y\).

8. Find the angles of an isosceles triangle if the vertical angle is (i) double, (ii) treble each base angle.

9. Find the angles of an isosceles triangle if each base angle is (i) double, (ii) four times the vertical angle.

10. In Fig. 332, \(AB = AC\).
    (i) If \(x = 100\), find \(y\).
    (ii) If \(y = 140\), find \(x\).
    (iii) If \(\angle ABC = 72^\circ\), find \(y\).
    (iv) Find an equation between \(x\) and \(y\).
    (v) Find \(\angle BAC\) if \(\angle EAC\) exceeds \(\angle DCA\) by half a right angle.

11. In Fig. 333, \(OA = OP = OB\). Find \(\angle OPA\), \(\angle OPB\), and compare \(\angle AOB\) with \(\angle APB\), if
    (i) \(x = 50\), \(y = 36\); (ii) \(x = 68\), \(y = 52\).
    What can you say about the circle, centre \(O\), radius \(OA\)?

12. In Fig. 333, where \(OA = OP = OB\), find the equation which connects \(x\) and \(y\). What angles in the figure are \(45^\circ\) ? What can you say about \(\angle AOB\)?

13. In Fig. 334, \(OC = OD = OR\). Find \(\angle COS\), \(\angle DOS\), and compare \(\angle COD\) with \(\angle CRD\), if \(\angle ORC = 34^\circ\) and \(\angle ORD = 56^\circ\).
    What can you say about the circle, centre \(O\), radius \(OC\)?

14. In Fig. 334, where \(OC = OD = OR\), if \(\angle SOC = 30^\circ\) and \(\angle COD = 80^\circ\), find \(\angle ORC\) and \(\angle CRD\).

15. In Fig. 334, where \(OC = OD = OR\), find, with reasons, an angle equal to (i) \(2\angle ORC\), (ii) \(2\angle ORD\), (iii) \(2\angle CRD\).

### ANGLES AT THE CENTRE OF A CIRCLE

The fundamental property has been indicated in Ex. X. 6. It should be illustrated by further oral work (Ex. X. c) before it is stated explicitly.

### EXERCISE X. c (Oral)

#### Examples for Class Discussion

1. In Fig. 335, the vertex \(O\) of \(\angle POQ\) is the centre of the circle; \(QO\) is produced to cut the circle again at \(R\).
   (i) What angles in the figure are equal?
   (ii) If \(\angle ORA = 25^\circ\), find \(\angle POQ\).
   (iii) If \(\angle AOB = 40^\circ\), find \(\angle ARR\).
   (iv) If \(\angle ARB = 5^\circ\), what is \(\angle AOB\)? Give reasons.

2. Draw a figure like Fig. 335, but make \(\angle POQ\) obtuse.
   (i) Write more simply, in 3 ways, \(\angle OAR + \angle ORA\).
   (ii) If \(\angle ORA = 70^\circ\), find \(\angle POQ\).
   (iii) If \(\angle AOB = 125^\circ\), find \(\angle ARB\).
   (iv) If \(\angle ARB = 9^\circ\), what is \(\angle AOB\)? Give reasons.
3. In Fig. 336, O is the centre of the circle.
   (i) Name two isosceles triangles in the figure.
   (ii) If $\angle QRA = 31^\circ$ and $\angle QRC = 23^\circ$, find $\angle QOA$ and $\angle QOC$. Then find $\angle AOC$ and $\angle ARC$.
   (iii) If $\angle QRA = x^\circ$ and $\angle QRC = y^\circ$, find $\angle QOA$ and $\angle QOC$. What can you say about $\angle AOC$ and $\angle ARC$?

![Fig. 336.](image1)

![Fig. 337.](image2)

![Fig. 338.](image3)

4. In Fig. 337, O is the centre of the circle.
   (i) Name two isosceles triangles in the figure.
   (ii) If $\angle QRA = 21^\circ$ and $\angle QRC = 54^\circ$, find $\angle QOA$ and $\angle QOC$. Then find $\angle AOC$ and $\angle ARC$.
   (iii) If $\angle QRA = x^\circ$ and $\angle QRC = y^\circ$, find $\angle QOA$ and $\angle QOC$. What can you say about $\angle AOC$ and $\angle ARC$?

What is the difference between Fig. 336 and Fig. 337?

5. Draw a figure like Fig. 336, but make each of the angles $\angle AOQ$, $\angle COQ$ obtuse.
   (i) Give the reasons why $\angle AOQ = 2\angle ARQ$.
   (ii) What can you say about $\angle COQ$?
   (iii) How do you describe $\angle AOQ + \angle COQ$, and what can you say about it?

6. Fig. 338 shows the same construction as Fig. 336, but $\angle AOC$ is now a diameter.
   (i) What angles in the figure are equal?
   (ii) If $\angle QOA = 50^\circ$, what is $\angle QOC$? Then find $\angle QRA$ and $\angle QRC$. What do you notice about them?
   (iii) If $\angle QRA = x^\circ$ and $\angle QRC = y^\circ$, what are $\angle QOA$, $\angle QOC$? Hence obtain an equation between $x$ and $y$ and simplify it. What follows?

ANGLERS AT THE CENTRE AND CIRCUMFERENCE

In Fig. 339, $\angle POQ$ is an angle whose vertex is at the centre $O$ of the circle; it is called an angle at the centre.

The part of the circle cut off between the arms of the angle is the arc $AEB$, so $\angle POQ$ is said to stand on the arc $AEB$. Three letters are used to name the arc $AEB$, because the arc $AB$ might mean either arc $AEB$ or arc $AHB$, although it is often taken to mean the shorter of the two arcs. Thus $\angle POQ$ is the angle at the centre standing on the arc $AEB$.

$\angle XKY$ is an angle whose vertex $K$ is on the circle; it is called an angle at the circumference. Thus the arc $AEB$ is the part of the circle cut off between the arms of $\angle XKY$, so $\angle XKY$ is said to stand on the arc $AEB$. Thus $\angle XKY$ is an angle at the circumference standing on the arc $AEB$.

In Fig. 339, $\angle AOB$ and $\angle AKB$ are angles standing on the same arc, one at the centre and the other at the circumference. There is, of course, only one angle at the centre standing on a given arc, but there are any number of angles at the circumference standing on it. Thus $\angle AKB$ also stands on arc $AEB$, so does $\angle AHB$, if $HA$, $HB$ are joined. Notice, however, that, if $EA$, $EB$ are joined, $\angle AEB$ stands on the arc $AHB$, not on arc $AEB$, and the angle at the centre standing on the arc $AHB$ is the reflex angle $AOB$.

The fact which is illustrated in Ex. X. c may now be stated as follows:

The angle at the centre of a circle is double any angle at the circumference which stands on the same arc.

For reference: $\angle$ at centre $= 2\angle$ at $O$. 

\[ \text{For reference: } \angle \text{ at centre } = 2\angle \text{ at } O. \]
Angles and the Arcs on which they stand

It is important to learn how to see at a glance the arc on which an angle stands and to be able to recognize other angles standing on the same arc.

To find the arc, look at the arms of the angle, follow them from the angle till they cut the circle again in two points, and see what arc lies between these points.

The following example gives the kind of practice that is necessary.

Example for Class Discussion

Draw on the blackboard a large circle and mark 8 points, A, B, C, D, E, F, G, H, at irregular intervals on it. Join every pair of points.

(i) Name (with three letters) the arcs on which the following angles stand: \( \angle BCD \), \( \angle BEH \), \( \angle ACE \), \( \angle HFE \), \( \angle CBA \), etc.

(ii) Name all the angles at the circumference which stand on the following arcs:
- arc \( ACD \), arc \( HFC \), arc \( BAF \), arc \( BDF \), arc \( ABC \), arc \( GED \), etc.

(iii) Name all the angles at the circumference which stand on the same arc as \( \angle ACG \), \( \angle EHB \), \( \angle CDF \), \( \angle CAF \), \( \angle HFE \), etc.

(iv) Mark the centre \( O \) and insert in a different colour any necessary radii.

Name the angles which stand on the same arc as \( \angle AOC \). What do you know about them? Repeat for \( \angle DOF \) and for the reflex angle \( COE \).

EXERCISE X. d (Oral)

All questions in this exercise refer to Fig. 340, where \( O \) is the centre of the circle. Give short reasons for your answers.

1. If \( \angle p = 140^\circ \), find \( \angle a \), \( \angle b \) and \( \angle c \).

2. If \( \angle b = 60^\circ \), find \( \angle p \) and \( \angle c \).

3. If \( \angle a = 55^\circ \), find \( \angle b \).

4. If \( \angle f = 110^\circ \), find \( \angle q \) and \( \angle g \).

5. If \( \angle q = 240^\circ \), find \( \angle f \) and \( \angle g \).

6. If \( \angle b = 80^\circ \), find \( \angle p \); then find \( \angle q \) and \( \angle f \).

7. If \( \angle p = 150^\circ \), find \( \angle c \) and \( \angle g \).

8. If \( \angle c = 65^\circ \), find \( \angle f \).

9. If \( \angle q = 105^\circ \), find \( \angle b \).

10. If \( \angle c \) were \( 90^\circ \), what would \( \angle p \) be? What would follow?

11. If \( \angle f \) were \( 90^\circ \), what would \( \angle q \) be? What would follow?

12. Copy and complete the following: the \( \angle s a, b, c \) all stand on the arc \( \ldots \ldots \) and are equal because each is half \( \ldots \ldots \).

13. Copy and complete the following: the two \( \angle s f, g \) each stand on the arc \( \ldots \ldots \) and are equal because each is half \( \ldots \ldots \).

14. How can you make an angle double \( \angle MHN \)? Name an angle equal to \( \angle MHN \).

15. Name two angles each equal to \( \frac{1}{2} \angle TOS \).

Angles standing on the same Arc

In Fig. 340, each of the angles \( \angle HLK, \angle HMN, \angle HNK \) is half \( \angle HOE \), the angle at the centre standing on arc \( HTK \); \( \therefore \angle HLK = \angle HMN = \angle HNK \).

This fact may be stated as follows:

Angles at the circumference, standing on the same arc, are equal.
Angles in a Segment

A circle is divided by a chord into two parts, called segments; the larger part is called the major segment, and the smaller the minor segment. Thus, in Fig. 341, $\triangle ABH$ is a major segment and $\triangle AEB$ is a minor segment.

Draw any number of angles standing on the arc $\triangle AEB$ (see Fig. 342). These angles $\triangle APB$, $\triangle AQB$, $\triangle ARB$, etc., are said to be in the segment $\triangle AHB$.

In particular, if $\triangle AB$ is a diameter, the segment is a semicircle, and the angles are called angles in a semicircle.

If the whole circle is drawn, it is usually best to look at the arc on which an angle at the circumference stands, but otherwise the phrase, "in the segment," is convenient.

Fig. 343 shows three segments, a major segment, a semicircle, and a minor segment. In each case, angles in the segment are drawn, also the angle at the circumference which is double each of them. The property, stated on p. 161, may therefore be expressed as follows:

**Angles in the same segment of a circle are equal.**

For reference: $\angle$s on same arc or $\angle$s in same seg.

Angle in a Semicircle

Look at Fig. 343, and then copy and complete the following sentences:

(i) In a major segment, the angle at the centre, $\angle AOB$, is less than 2 right angles, therefore any angle at the circumference, $\angle APB$, is ...... [Fig. 343 (i)].

(ii) In a minor segment, the angle at the centre, $\angle AOB$, is more than 2 right angles, therefore any angle at the circumference, $\angle AXB$, is ...... [Fig. 343 (iii)].

(iii) In a semicircle, the angle at the centre, $\angle AOB$, is equal to ......, therefore any angle at the circumference, $\angle AB$, is ...... [Fig. 343 (ii)].

The property, just proved, is stated as follows:

The angle in a semicircle is a right angle.

*For reference:* $\angle$ in semicircle, 90°.

This fact was discovered by Thales (600 B.C.), one of the "Seven Wise Men"; he is called the "Father of Geometry".

Angles of a Cyclic Quadrilateral

A quadrilateral, whose four vertices lie on a circle, is called cyclic.

Examples for Class Discussion

1. In Fig. 344, $\triangle$ is the centre of the circle.

   (i) Which angle equals $\angle ADC$ ? Which equals $\angle ABC$ ?

   (ii) If $\angle C = 140^\circ$, what is $\angle ABC$ ? What are $\angle ABC$, $\angle ADC$ ? What is $\angle ABC + \angle ADC$ ?

   (iii) If $\angle ADC = 100^\circ$, find in succession $\angle p$, $\angle q$, $\angle ABC$. What do you notice about $\angle ADC$ and $\angle ABC$ ?

   (iv) Copy and complete the following argument:

   $\triangle\angle ADC = \frac{1}{2}\angle ABC$ and $\angle ABC = \frac{1}{2}\angle ABC$;

   $\therefore \angle ADC + \angle ABC = \ldots$; but $\angle p + \angle q = \ldots$;

   $\therefore \angle ADC + \angle ABC = \ldots$
2. In Fig. 344, CB is produced to N to form the exterior angle ABN.

(i) If \( \angle ADC = 130^\circ \), what is \( \angle ABC + \angle ABN \)?
(ii) If \( \angle ADC = 110^\circ \), what is \( \angle ABC + \angle ABN \)?
(iii) Copy and complete the following:
\[ \angle ABN + \angle ABC = \ldots \ldots \text{ because } \ldots \ldots \]
\[ \angle ADC + \angle ABC = \ldots \ldots \text{ because } \ldots \ldots \]
\[ \ldots \ldots \ldots \]
\[ \angle ABN = \ldots \ldots \ldots \]

The facts proved in these examples are stated as follows:

The sum of the opposite angles of a cyclic quadrilateral is two right angles.

For reference: opp. \( \angle s \) cyclic quad., \( 180^\circ \).

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

For reference: ext. \( \angle \) cyclic quad. = int. opp. \( \angle \).

**EXERCISE X.e (Oral)**

State a suitable reference to any angle property you use.

Numbers 1-10 refer to Fig. 345. Draw rough figures for yourself.

1. Which angle equals (i) \( \angle ADB \), (ii) \( \angle ABD \)?
2. Which angle equals (i) \( \angle ADG \), (ii) \( \angle BAD \)?
3. What follows if BD is a diameter?
4. Which angle equals (i) \( \angle ADC \), (ii) \( \angle BDC \)?
5. Find all the angles you can, if \( \angle ACD = 30^\circ \) and \( \angle BCF = 80^\circ \).
6. Find all the angles you can, if \( \angle ADB = 50^\circ \) and \( \angle CAB = 55^\circ \).
7. Find all the angles you can, if \( \angle KAD = 100^\circ \) and \( \angle BDC = 50^\circ \).

8. Find the remaining angles, if \( \angle ADG = 80^\circ \), \( \angle ACB = 60^\circ \), \( \angle ANB = 95^\circ \).
9. Find the remaining angles, if \( \angle KAD = 95^\circ \), \( \angle EBC = 110^\circ \), \( \angle ACD = 35^\circ \).
10. If \( \angle ADB = 55^\circ \) and \( \angle ACD = 35^\circ \), find \( \angle BAD \). What follows?

**Fig. 344.**

**Draw a large circle and mark any four points, A, B, C, D, in order on it; use it for Nos. 11-16.**

11. Mark a point L on the circle so that \( \angle BLD = \angle BCD \).
12. Mark a point M on the circle so that \( \angle BMD \), \( \angle BCD \) are supplementary.
13. Mark a point N on the circle so that \( \angle DNG = \angle DNG \).
14. Make two exterior angles equal to \( \angle BCD \).
15. Make an angle in the minor segment cut off by AB, label it \( r \).
16. Make an angle in the major segment cut off by CD, label it \( t \).

**ANGLES IN A CIRCLE**

**EXERCISE X.f (Numerical)**

Give short reasons for each step.

1. In Fig. 346, O is the centre of the circle. Find (i) \( \angle AOC \), (ii) \( \angle ABC \), (iii) \( \angle APC \).

If also BOP is a straight line, find \( \angle OCP \).

**Fig. 346.**

2. In Fig. 347, AB and CD are perpendicular chords; find \( \angle ABD \).

If O is the centre, prove that \( \angle AOD + \angle BOC = 180^\circ \).
3. In Fig. 348, find (i) \( \angle CBE \), (ii) the angle at the centre standing on the arc CPA, (iii) the acute angle at which BC cuts AD.

![Fig. 348](image)

4. In Fig. 349, AB is a diameter;
   (i) if \( \angle PAB = 43^\circ \), find \( \angle AQP \);
   (ii) if \( \angle BPQ = 28^\circ \), find \( \angle ABQ \).

5. In Fig. 349, O is the centre of the circle. If \( \angle POB = 75^\circ \) and \( \angle AOQ = 125^\circ \), find \( \angle PAQ \).

6. In Fig. 350, O is the centre of the circle.
   (i) If \( x = 100 \), find \( y \).
   (ii) If \( y = 55 \), find \( x \).
   (iii) Find \( x \) in terms of \( y \).

7. In Fig. 351, \( \angle BAC = 42^\circ \), \( \angle ADB = 33^\circ \), \( \angle BKC = 114^\circ \), find \( \angle BCD \).

![Fig. 350](image)

8. In Fig. 351, \( \angle ADC = 95^\circ \), \( \angle ACB = 45^\circ \), \( \angle DBF = 130^\circ \); find (i) \( \angle BAC \), (ii) \( \angle CAD \), (iii) \( \angle AKB \).

9. ABCDE is a regular pentagon inscribed in a circle, centre O. Find, in degrees, (i) \( \angle DOC \), (ii) \( \angle DAC \), (iii) \( \angle BCD \).

10. With the data of No. 9, find in degrees
   (i) the angle in the major segment cut off by AC,
   (ii) the angle in the minor segment cut off by CD.

![Fig. 351](image)
8. In Fig. 357, O is the centre of the circle; prove that
\[ \angle AOB = 2 \angle BPQ. \]

9. In Fig. 349, p. 166, where AB is a diameter, prove that
\[ \angle PAB + \angle AQP = 90^\circ. \]

10. In Fig. 358, PAQR and HBK are straight lines; prove that
\[ \angle KQR = \angle HPA. \]

What follows?

11. In Fig. 357, where O is the centre, prove that
\[ \angle AOB + 2 \angle APB = 360^\circ. \]

12. ABCD is a cyclic quadrilateral in which AC bisects \( \angle BAD \) and also bisects \( \angle BCD \). Prove that \( \angle ABC = 90^\circ \).

13. Draw any two circles cutting each other at A, B, and draw the diameters AL, AM of the two circles. Prove that
\[ \angle LBM \text{ is a straight line.} \]

14. Carry out and prove correct the following construction for drawing a perpendicular to a given line XY at a given point A on it.

Take any point P; draw the circle centre P, radius PA, cutting XY again at Q. Draw the diameter QPR of the circle. Join AR. Then AR is the required perpendicular at A to XY.

15. Draw a circle and mark any 6 points A, B, C, D, E, F, in order on it. Prove that
\[ \angle ABC + \angle CDE + \angle EFA = 360^\circ. \]

16. In Fig. 358, where PAQ and HBK are straight lines, prove that
\[ \angle HAK = \angle PBQ. \]

CHAPTER XI

PYTHAGORAS’ THEOREM

What Pythagoras discovered

If you look at a floor, paved with equal tiles, each of which is half a square, you will see squares made up of two tiles and squares made up of four tiles, so that one of the larger squares is equal to the sum of two of the smaller squares.

Fig. 360 shows a right-angled triangle such that the square on its hypotenuse is one of the larger squares, and the squares on the other two sides are the smaller squares.

Therefore, in this special case, the area of the square on the hypotenuse is equal to the sum of the areas on the other two sides.

Pythagoras (or one of his pupils, who formed a secret society) in the sixth century B.C., discovered that this is true for every right-angled triangle. You can test it by the following experiment, which was probably the kind of method Pythagoras used. The proof, outlined on pp. 170, 171, was invented by Euclid, 300 years later.

Draw any right-angled triangle, and call the lengths of the two shorter sides \( a \) inches, \( b \) inches (see Fig. 361).
Then you want
(i) eight duplicates of this triangle, cut out in stiff white paper,
(ii) two squares, each side \(a + b\) inches, either shaded or coloured.

Arrange four of the triangles on each square in the way shown in Figs. 362, 363.

In Fig. 362, the uncovered shaded area is the square on the hypotenuse of the triangle, labelled 1; in Fig. 363, the uncovered shaded area is made up of the squares on the other two sides of the triangle, labelled 1.

What does this experiment show? We call it an experiment, not a proof. But you can make it into a proof, if you show that the shaded areas are really squares. It is not difficult to do this.

**Class Discussion of Euclid’s Proof**

In Fig. 364, \(\triangle ABC\) is a triangle, right-angled at \(A\), with squares drawn on its three sides; \(AX\) is drawn at right angles to \(BC\) and is produced so as to divide the square on \(BC\) into two parts. We shall show that one of these parts equals the square on \(AB\), and that the other equals the square on \(AC\).

(i) Look at the shaded triangle \(\triangle HBC\), Fig. 364 (i); imagine it is on a pivot at \(B\); spin it through a right angle clockwise.

What is the new position of \(BH\)? of \(BC\)? of \(\triangle HBC\)?

(ii) What can you say about the two shaded triangles in Fig. 364 (i), (ii)? Obtain this result in another way.

(iii) Prove that \(CA\) is in line with \(AK\). Then copy and complete the following:

\(\triangle HBC\) and sq. \(HBAK\) are on the same base \(\ldots \ldots \ldots \ldots\), and between the same parallels \(\ldots \ldots \ldots \ldots\), and so the area of \(\triangle HBC\) equals \(\ldots \ldots \ldots \ldots\).

Also \(\triangle ABQ\) and rect. \(BQXY\) are \(\ldots \ldots \ldots \ldots\), and so the area of \(\triangle ABQ\) equals \(\ldots \ldots \ldots \ldots\); but \(\triangle HBC\) = \(\ldots \ldots \ldots \ldots\); \(\therefore\) square \(HBAK\) = \(\ldots \ldots \ldots \ldots\).

(iv) Now draw your own figure of the right-angled triangle, the three squares, and the altitude \(AXY\). Draw and shade \(\triangle MCB\).

If \(\triangle MCB\) turns round \(C\) as a pivot through \(90^\circ\) counterclockwise, what is its new position? Draw it in and shade it.

Copy and complete the following:

\(\triangle MCB\) and sq. \(MCAN\) are \(\ldots \ldots \ldots \ldots\), and so \(\triangle MCB\) = \(\ldots \ldots \ldots \ldots\); \(\triangle CPA\) and rect. \(CPYX\) are \(\ldots \ldots \ldots \ldots\), and so \(\triangle CPA\) = \(\ldots \ldots \ldots \ldots\); but \(\triangle MCB\) = \(\ldots \ldots \ldots \ldots\); \(\therefore\) sq. \(MCAN\) = \(\ldots \ldots \ldots \ldots\).

\(\therefore\) the square on \(AB\) + the square on \(AC\) = \(\ldots \ldots \ldots \ldots\).

(v) For what part of this proof is it necessary to know that \(\angle BAC\) is a right angle?
Proofs are necessary. If you merely judge by appearances, you may be misled; look at Fig. 365.

A Jig-saw Puzzle

Draw for yourself on squared paper, with ½ inch as unit, (i) a square, side 8 units long, (ii) a rectangle 13 units long, 5 units high. Cut out the square and cut it up into the four pieces marked out in Fig. 365. Place these pieces on the rectangle in the positions indicated. Do they fit? If so, this shows that the area of the square equals the area of the rectangle. Is this true?

Euclid’s Proof of Pythagoras’ Theorem

This proof should not be learnt at a first reading, but the method should be discussed owing to the importance of the results obtained incidentally in the proof.

Pythagoras’ theorem is stated as follows:

In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

For reference: Pythagoras.

Notation for Areas of Squares and Rectangles

The area of the square BCPQ in Fig. 364, whose side is BC, is usually written BC².

Thus Pythagoras’ theorem is often stated as follows:

If, in ΔABC, ∠BAC = 90°, then BC² = BA² + AC².

The area of the rectangle BQYX in Fig. 364 may be written BQ . BX; but BQ = BC, it can therefore also be written BC . BX. Similarly BC . CX means the area of a rectangle, whose length and breadth equal BC and CX, and we speak of this rectangle as contained by BC and CX. Thus in Fig. 364, BC . CX represents an area equal to that of rect. PCXY.

Again BX . XC means the area of a rectangle whose two adjacent sides equal BX and XC; no such rectangle is drawn in Fig. 364, but it could easily be drawn, if required.

Euclid’s proof of Pythagoras’ theorem shows that the area of the square BAKH equals the area of the rectangle BQYX.

This is written more shortly as follows:

BA² = BQ . BX or BA² = BC . BX.

Similarly, CA² = CP . CX or CA² = CB . CX.

The real meaning of Pythagoras’ theorem will be overlooked, unless at first special emphasis is laid on the idea of area. Direct applications of the theorem, where it is unnecessary to draw the actual squares, should be postponed until the relations between the areas of the squares and rectangles in Fig. 364 are fully realised, that is, the reasoning should at first be made to depend on the geometrical idea of area, instead of being merely arithmetical.

The following oral example and the questions in Ex. XI. a should therefore be done by all.

Example for Class Discussion

Draw a rough figure like Fig. 364 (i) without the shaded triangle. Let AB = 4 in., AC = 3 in., ∠BAC = 90°.

(ii) Show on your figure the areas of sq. BK, sq. CN, rect. BY, rect. CY.

(iii) What is the area of sq. BP? What is the length of BC?

(iv) Copy and complete the following:

area of rect. BQYX = ..., but BQ = ...

⇒ BX = ...

area of rect. CPYX = ..., but CP = ...

⇒ CX = ...

How can you check your answers?
EXERCISE XI. a (Oral)

[All the questions in this exercise refer to Fig. 364, where \( \angle BAC = 90^\circ \).]

1. If \( AB = 8 \text{ cm}, \ AC = 6 \text{ cm}, \) mark on your own (rough) figure the areas of sq. BK, sq. CN, rect. BY, rect. CY, sq. BP. Then calculate the lengths of BC, BX, XC.
2. If \( AB = 12 \text{ cm}, \ BC = 13 \text{ cm}, \) mark on your own (rough) figure the areas of sq. BK, sq. BP, rect. BY, rect. CY, sq. CN. Then calculate the lengths of AC, BX, XC.
3. If \( AC = 7 \text{ in}, \ BC = 25 \text{ in}, \) mark on your own (rough) figure the areas of sq. CN, sq. CQ, rect. CY, rect. BY, sq. BK. Then calculate the lengths of AB, CX, BX.
4. If \( AC = 2 \text{ in}, \ AB = 3 \text{ in}, \) find the areas of sq. CN, sq. BK, sq. BP. Then find the length of BC to the nearest \( \frac{1}{2} \text{ in}. \)
5. If \( AB = 15 \text{ cm}, \ BC = 17 \text{ cm}, \) find the areas of sq. BK, sq. BP, sq. CN. Then find the length of AC.
6. If area of rect. BY = 144 sq. cm., and area of rect. CY = 51 sq. cm., find area of sq. BP, sq. BK, sq. CN.

What are the lengths of BC, AB, CA?

The result of Pythagoras’ theorem can often be employed numerically, without the squares on the sides being drawn.

Examples for Class Discussion

1. With the data of Fig. 366 (i), find the length of AB.
   Let \( AB = x \text{ in}. \)
   Write down in sq. inches the areas of the squares on AB, BC, AC.

2. Find the area of the isosceles triangle ABC in Fig. 366 (ii).

PYTHAGORAS’ THEOREM

Draw AN perpendicular to BC. Why does \( \triangle ANB = \triangle ANC \)? What is the length of BN?

Let \( AN = h \text{ in}. \) From the right-angled triangle ANB, obtain an equation for \( h \) and solve it. Now use the formula, area of \( \triangle = \frac{1}{2} \text{ height} \times \text{base}. \)

3. In Fig. 367, O is the centre of a circle of radius 10 cm. and PQ is a chord of length 12 cm. What is the distance of O from PQ?
   Draw ON perpendicular to PQ.
   Let ON = \( h \text{ cm}. \); then obtain an equation for \( h \) and solve it.

SIMPLE NUMERICAL APPLICATIONS

EXERCISE XI. b

1. In Fig. 368, PQ = 5 in., QR = 12 in., find PR.
2. In Fig. 368, PR = 10 cm., QR = 8 cm., find PQ.

3. In Fig. 368, PQ = 4 cm., QR = 5 cm., find PR to the nearest mm.

4. Find the hypotenuse of a right-angled triangle if the other two sides are 2 in., 2.1 in.

5. Find the third side of a right-angled triangle if the hypotenuse is 6.8 cm., and one side is 6 cm.

6. In Fig. 369, if \( AB = AC = 2.5 \text{ in}, \ BC = 3 \text{ in}, \) find the altitude AN and the area of \( \triangle ABC \).

7. In Fig. 367, where O is the centre of the circle, if PQ = 8 in. and ON = 3 in., find the radius.
8. Find the length of the chord of a circle of radius 3·9 cm., if its distance from the centre of the circle is 1·5 cm.

9. A ladder 21·4 feet long rests against the wall of a house, with one end 10 feet from the wall. How high up the wall does the other end reach?

10. In Fig. 368, \(PQ = 4\) in., \(PR = 7\) in., find \(QR\) to the nearest \(\frac{1}{2}\) in.

11. Find, to the nearest \(\frac{1}{6}\) in., the length of a diagonal of a square of side 2 inches.

12. In Fig. 369, \(AB = BC = CA = 6\) cm., find the altitude \(AN\), correct to the nearest mm.

FURTHER NUMERICAL APPLICATIONS

**EXERCISE XI. c**

1. Draw, with instruments, a triangle \(PQR\) such that \(PQ = 2\) in., \(QR = 3\) in., \(\angle PQR = 90^\circ\). Use this figure to construct a square of area 13 sq. in. Measure a side of the square; what does this tell you about \(\sqrt{13}\)?

2. Construct a square of area (i) 10 sq. inches, (ii) 14 sq. inches. Measure the side of each square. What does this tell you about \(\sqrt{10}\) and \(\sqrt{14}\)?

3. Use the relations, \(1^2 + 2^2 = 5\) and \(3^2 + 2^2 = 5\), to find geometrically, in two ways, an approximate value of \(\sqrt{5}\).

4. Draw a parallelogram \(ABCD\) such that \(AB = BC\) (i.e. a rhombus). It can be proved that all its sides are equal and that its diagonals cut at right angles; draw them.
   (i) If \(AB = 5\) in. and \(AC = 8\) in., find \(BD\).
   (ii) If \(AC = 6\) cm. and \(BD = 4\) cm., find \(AB\) to nearest mm.

5. Draw a semicircle on \(AB\) as diameter and draw a chord \(AP\). If \(AB = 2·5\) in. and \(AP = 2\) in., find the distance of \(P\) from \(B\).

6. In Fig. 370, the units are inches; find the length of \(BC\).

Find the distance of \(D\) from \(A\) in Figs. 371-374; the units are inches. The dotted lines are inserted merely to help you.

7. 

8. 

9. 

10. 

11. Fig. 375 represents a box with rectangular faces; the units are inches. Find the lengths of (i) \(PD\), (ii) \(PR\), (iii) \(PC\), (iv) \(SQ\), (v) \(SB\).

12. A hall is 48 ft. long, 36 ft. wide, 25 ft. high. Find the length of (i) a diagonal of the floor, (ii) a diagonal of the hall.
Examples for Class Discussion

1. \( AD \) is an altitude of the equilateral triangle \( ABC \). Prove that \( AD^2 = \frac{1}{2} BC^2 \).

Conclude your own figure.
Let \( BC = x \) inches, then \( BD = \ldots \); also \( AB = \ldots \).
\[ \therefore \text{by Pythagoras, } AD^2 + \ldots = \ldots; \]
\[ \therefore AD = \ldots = \ldots \]

2. \( AB \) is a diameter of the circle \( APBQ \). Prove that \( AP^2 - AQ^2 = BQ^2 - BP^2 \).

Draw your own figure.
Then \( AP^2 + \ldots = \ldots \) because \ldots
and \( AQ^2 + \ldots = \ldots \) because \ldots
\[ \therefore AP^2 - AQ^2 = BQ^2 - BP^2; \] why does this give what is required?

GENERAL STATEMENTS

**EXERCISE XI. d**

Give short reasons for each step.

1. In Fig. 376, prove that \( AB^2 - AC^2 = BD^2 - DC^2 \).

![Fig. 376](image)

2. \( EFGH \) is a quadrilateral such that \( \angle EFG \) and \( \angle EHG \) are right angles; prove that \( EF^2 + FG^2 = GH^2 + HE^2 \).

3. In Fig. 377, prove that \( PQ^2 + RS^2 = QR^2 + PS^2 \).

![Fig. 377](image)

4. In Fig. 378, \( AB \) is parallel to \( DC \); prove that \( AB^2 + BC^2 = AD^2 + DC^2 \).

5. Draw a triangle \( ABC \), right-angled at \( A \), and mark any two points \( P, Q \) on \( AB, AC \). Prove that \( PQ^2 + BC^2 = PC^2 + QB^2 \).

Any or all of the rest of this exercise may be omitted at a first reading.

6. In Fig. 376, take any point \( R \) on \( AD \); then prove that \( AB^2 + CR^2 = AC^2 + BR^2 \).

![Fig. 376](image)

7. In Fig. 379, \( AB = BC = 2CD \); prove that \( AD = 3CD \). [Let \( CD = x \) in.]

8. In Fig. 380, \( P, Q \) are the mid-points of \( AB, AC \). Prove that \( BP^2 + PQ^2 + QC^2 = \frac{1}{2} BC^2 \). [Let \( AB = 2y \) in., \( AC = 2z \) in.]

![Fig. 380](image)

9. In Fig. 380, if \( Q \) is the mid-point of \( AC \), prove that \( BC^2 = BQ^2 + 3QC^2 \). [Let \( AC = 2x \) in.]

10. In Fig. 380, where \( P, Q \) are the mid-points of \( AB, AC \), prove that \( BQ^2 + CP^2 = 5PQ^2 \).

11. In Fig. 380, if \( AQ = 2QC \), prove that \( BC^2 = BQ^2 + 5QC^2 \).

12. With the data of Fig. 364, p. 171, prove that \( AB^2 + AM^2 = BP^2 \).
CHAPTER XII

SIMILAR FIGURES AND SOLIDS

The tests for recognising similar triangles were discussed in Chapter VIII. The exercises in this chapter revise and extend the previous work.

Similarity

If we call two figures similar, we mean that their shapes are the same; for example, a photograph is similar to an enlargement of itself, an estate is similar to the reduced plan, or map, which represents it. The usefulness of enlargements and reductions depends on two facts:

(i) All lengths are altered in the same ratio.

If the length of a man’s arm in an enlargement is 3 times its length in the photograph, then the length of his leg is also 3 times as much.

(ii) The sizes of all angles are unaltered.

If the lines, which represent on a map two fences of a field, meet at 60°, then the fences themselves meet at 60°.

These facts are assumed in all solutions of problems by scale-diagrams.

EXERCISE XII. a (Oral)

1. A page of this book is a rectangle 18 cm. high, 12 cm. wide. What are the dimensions of figures the same shape as this page,
   (i) half as high; (ii) longest side 6 cm.;
   (iii) shortest side 8 in.; (iv) perimeter 6 cm.;
   (v) if each side is reduced in the ratio \( \frac{1}{3} \);
   (vi) if each side is increased in the ratio \( \frac{1}{3} \).

2. A page of this book is a rectangle 18 cm. high, 12 cm. wide. Which of the following rectangles are similar to it?
   (i) the floor of a room, 24 ft. long, 16 ft. wide;
   (ii) a lawn, 90 yd. broad, 120 yd. long;
   (iii) the top of a table, 10 ft. by 15 ft.;
   (iv) a stamp, 1-6 cm. wide, 2-4 cm. high.

3. A concert-hall is \( x \) yards long, \( y \) yards wide. What can you say about \( x \) and \( y \), if the hall is the same shape as a sheet of paper 3 in. by 1-2 in.?

4. Fig. 381 represents three rectangles, not drawn to scale.

![Fig. 381](image)

Are any two of them, or all of them, the same shape? Is it necessary to know the units?

5. Repeat No. 4 for the rectangles in Fig. 382.

![Fig. 382](image)

6. What are the dimensions of a rug, the same shape as a rectangular mat, 15 ft. by 6 ft., if one of its sides is 10 ft.? Illustrate your answers by rough figures.

7. Are any two or all of the triangles in Fig. 383 the same shape? Is it necessary to know the units?

![Fig. 383](image)

8. What are the sides of a triangle the same shape as \( \triangle DEF \) in Fig. 383, (i) if its hypotenuse is 15 in.; (ii) if its shortest side is 2 miles?

9. Which of the following pairs of figures must be similar?
   (i) two squares; (ii) two semicircles; (iii) two rectangles;
   (iv) two equilateral triangles; (v) two rhombuses; (vi) two isosceles right-angled triangles.
10. In Fig. 384, PQ is parallel to CD. Give the reasons why the quadrilaterals $ABPQ, ABCD$ (i) must be equiangular, (ii) cannot be similar.

11. (i) Sketch two parallelograms which are equiangular but obviously not the same shape.

(ii) Sketch two parallelograms such that the sides of one are proportional to the sides of the other but yet are obviously not the same shape.

**EXERCISE XII. b (Written)**

1. Fig. 385 shows the dimensions of a rectangular hall with a semicircular end. What is $z$?

Draw rough figures showing the dimensions for a scale-diagram, (i) scale, 1 cm. to 12 ft.; (ii) scale, 1 in. to 20 ft.

2. A, B, C are 3 towns; C is 24 miles north of A; B is 30 miles north-east of A. Use instruments to draw 3 plans, scales (i) 1 cm. to 6 miles; (ii) 1 cm. to 10 miles; (iii) 1 in. to 12 miles.

Are the 3 plans the same shape? Use each plan separately to find the distance and bearing of B from C.

3. What can you say about the three triangles in Fig. 386?

(i) What follows if $QR = 4$ inches?

(ii) What follows if $TX = 4$ inches?

(iii) Use a ruler and protractor to draw a triangle similar to $\triangle ABC$, with its longest side 5 cm.; measure and calculate the other sides.

4. Fig. 387 represents three parallelograms.

(i) Are any two or all of them the same shape?

(ii) Show on the scale, 1 cm. to 1 inch, all three paral- lelograms, in one figure, so that each has one corner at $A$ and two of its sides along $AB, AD$, the longer along $AB$.

5. A rectangular photograph $PQRS$ (see Fig. 388) is mounted centrally on a card $ABCD$ of the same shape; $PQ = 4$ in., $QR = 9$ in., and the distance between the parallels $QR, BC$ is $\frac{3}{4}$ in. Find the length and breadth of the card.

Draw the figure on the scale, 1 cm. to 1 inch.

6. Fig. 389 shows a rectangle divided into two smaller rectangles. Is either portion the same shape as the original?

7. Draw a rough figure showing a rectangular lawn, 20 yd. long, 16 yd. wide, with a path 2 yd. wide all round it. Is the rectangle formed by the outer edge of the path the same shape as the lawn? Give reasons.

8. A rectangular sheet of paper is 5 in. long, 2 in. wide. Show how to cut it up into two sheets similar to each other (but not congruent).

9. In Fig. 390, $PQ$ and $XY$ are each parallel to $BC$. Name three similar triangles.

If $AP = 3PX = XB = 1\frac{1}{2}PQ$, what can you say about other lengths in the figure?

10. With the data of No. 9, name, with reasons, three quadrilaterals in Fig. 390 which are equiangular to one another. Are all or any of them similar?
Similar Solids

For these, as for plane figures,
(i) any angle in the first is equal to the corresponding angle in the second;
(ii) the ratio of the lengths of any two lines in the first is equal to the ratio of the lengths of the corresponding lines in the second.

For example, if a statue is the same shape as the man it represents, angles are unchanged, but lengths are all increased, or decreased, in the same ratio.

EXERCISE XII. C

1. A box measures 30 in. by 20 in. by 12 in. What are the dimensions of a box of the same shape, (i) if its longest edge is 3 ft.; (ii) if its shortest edge is 8 cm.?

2. The measurements of two cuboids are 2 in. by 4 in. by 8 in. and 2 in. by 4 in. by 1 in. Are they the same shape?

3. The measurements of a cuboid are 12 cm. by 8 cm. by 6 cm.; another cuboid is the same shape and has one edge of length 4 cm. What can you say about the lengths of the two other adjacent edges?

4. An oil drum is 18 in. in diameter and 30 in. high. What are the dimensions of a drum the same shape, (i) if its diameter is 2 ft.; (ii) if its height is 2 ft.?

Any or all of the rest of this exercise may be omitted at a first reading.

5. A brick is 9 in. long, 4½ in. wide, and 2½ in. high. Another brick is 15 cm. long, 7½ cm. wide, 4 cm. high. Are these bricks the same shape? Is any face of the first brick similar to a face of the second brick?

6. How can you find out whether a penny and a halfpenny are similar solids, if large numbers of each kind of coin are available?
5. Use your instruments to draw a triangle with sides 6 cm., 9 cm., 12 cm., and divide it into triangles with sides 2 cm., 3 cm., 4 cm. How many of these triangles are there? Are they similar to the original triangle? What is the ratio of the areas of one of the small triangles and the original triangle?

6. Give examples of two similar figures such that the area of the larger is 16 times the area of the smaller, if each figure is (i) a square; (ii) a rectangle; (iii) a triangle; (iv) a circle.

7. Give possible dimensions of two triangles such that the ratio of their bases is \( \frac{1}{2} \), and the ratio of their heights is also \( \frac{1}{2} \). What is the ratio of their areas?

8. In Fig. 392, \( \triangle ABC \) is similar to \( \triangle XYZ \), and \( AH \), \( XK \) are corresponding altitudes.

Fig. 392. Fig. 393.

(i) Name two other pairs of similar triangles in this figure.
(ii) If \( BC = 5 \text{ cm.} \), \( AH = 4 \text{ cm.} \), \( YZ = 10 \text{ cm.} \), find \( XK \). Also find the ratio of the areas \( \triangle ABC : \triangle XYZ \).
(iii) Repeat (ii) if \( BC = 5 \text{ cm.} \), \( AH = 4 \text{ cm.} \), \( YZ = 7 \text{ cm.} \).
(iv) If \( YZ = \frac{1}{2} \), what is \( AH \)? What is \( \triangle ABC : \triangle XYZ \)?

9. Draw on squared paper two figures similar to Fig. 393, so that \( \text{ABCD} \) is a square of side (i) 1 inch, (ii) 3 inches. What is the ratio of the areas of the two polygons, \( \text{MNPQRS} \)?

10. The sides of three equilateral triangles are 3 cm., 4 cm., 5 cm.; what connection is there between their areas?

The examples in Ex. XII. d illustrate the following statement:

The ratio of the areas of two similar figures equals the square of the ratio of the lengths of any two corresponding lines, that is, equals the square of the ratio of their linear dimensions.

VOLUMES OF SIMILAR SOLIDS

EXERCISE XII. e (Oral)

1. What is the ratio of the volumes of two cubes if the lengths of their edges are (i) 3 cm., 6 cm.; (ii) 2 cm., 6 cm.; (iii) 3 in., 9 in.?

What is the ratio of the areas of their total surfaces?

2. A cuboid is 5 in. by 4 in. by 3 in. How is its volume changed if each edge is (i) doubled, (ii) trebled, (iii) divided by 10, (iv) multiplied by \( x \)?

How is the area of the total surface changed?

3. Give possible dimensions for two cuboids such that each edge of the first is three-fifths of the corresponding edge of the second. What is the ratio of their volumes?

4. How many cubes of edge 1 inch can be cut out of a cube of edge 2 inches? Make a sketch.

5. Give possible dimensions of two similar solids such that the volume of the larger is 8 times that of the smaller, if each is (i) a cube, (ii) a cuboid, (iii) a sphere, (iv) a cylinder.

6. A brick is 9 in. long, 4 1/2 in. wide, 3 in. high. The largest edge of a similar brick is 6 in.; what is its width and height? What is the ratio of (i) the volumes, (ii) the areas of the total surfaces?

7. A model of a statue is made on a (linear) scale of \( \frac{1}{4} \). What is the ratio of the volume of the model to that of the statue?

8. The ratio of the volumes of two similar kettles is \( 1:8 \); what is the ratio of the areas of their total surfaces?

The examples in Ex. XII. e illustrate the following statement:

The ratio of the volumes of two similar solids equals the cube of the ratio of the lengths of any two corresponding lines, that is, equals the cube of the ratio of their linear dimensions.
SIMILAR OBJECTS: AREAS AND VOLUMES

EXERCISE XII.

1. A ground plan of a house is drawn on a scale of \( \frac{1}{10} \). What is the ratio of areas on the plan to corresponding actual areas? How many sq. inches of area on the plan correspond to the floor of a room of actual area 240 sq. ft. ?

2. A screen 6 ft. high (not necessarily rectangular) requires 54 sq. ft. of material for covering; how much is needed for a screen of the same shape 4 ft. high ?

3. How many times can a cylindrical tumbler, 4 in. high and of diameter 3 in., be filled from a cylindrical cask, 40 in. high and of diameter 30 in. ?

4. A solid metal sphere, radius 3 in., weighs 8 lb.; find the weight of a solid sphere of the same material, if its radius is (i) 6 in., (ii) 9 in., (iii) 1 foot.

Any or all of the rest of this exercise may be omitted at first reading.

5. If it costs £4 to gild a sphere of radius 4 ft., what will it cost to gild a sphere of radius 6 ft. ?

6. Two hot-water cans are the same shape; the smaller is 10 in. high and holds a quart; the larger is 15 in. high; how much will it hold ? They are made from the same tin-sheeting; the larger weighs 12 oz.; what does the smaller weigh, both being empty ?

7. A sphere of lead of radius 4 in. is melted down and cast into spherical bullets each of radius 0.1 in. How many bullets will there be ?

8. A lodger pays 8 pence for a scuttle of coal, the scuttle being 20 in. deep. What should he pay if the scuttle was the same shape and 2 \( \frac{1}{2} \) ft. deep ?

9. Two models of the same statue are made of the same substance; one is 3 in. high and weighs 8 oz.; the other weighs 4 lb.; what is its height ?

10. Two tin kettles are the same shape, the smaller being 6 in. high and having 108 sq. in. of surface. The larger is 8 in. high; what is its surface area ?

Water boils in the first kettle at a temperature of 212° F.; at what temperature does it boil in the second ?

PART III

FURTHER DEVELOPMENTS

CHAPTER XIII

APPLICATIONS OF CONGRUENCE

PARALLELOGRAM, RECTANGLE, SQUARE, RHOMBUS

It is important to distinguish carefully between the definition of a figure and the facts which can then be proved about it. In Chapter VIII, a parallelogram was defined as a 4-sided figure with its opposite sides parallel, and it was then proved (see pp. 133, 134) that

(i) the opposite sides are equal;
(ii) the opposite angles are equal;
(iii) each diagonal bisects the parallelogram;
(iv) the diagonals bisect each other.

If you are given that a quad. is a par\( ^{\text{a}} \), you can assume these facts about it. If, however, you are asked to prove that a quad. is a par\( ^{\text{a}} \), you should prove that the conditions in the definition are satisfied, that is, you should prove that its opposite sides are parallel.
FURTHER DEVELOPMENTS

Examples for Class Discussion

Facts about a Rectangle

Definition.—If one angle of a parallelogram is a right angle, the figure is called a rectangle.

1. In Fig. 394, ABCD is a \( \text{par}^m \), such that \( \angle \text{DAB} = 90^\circ \).

   (i) Why does \( \angle \text{ABC} \) equal \( 90^\circ \)? What can you say about \( \angle \text{ADC} \) and \( \angle \text{BCD} \)?
   Give reasons.

   (ii) What triangles can you prove congruent, which will show that \( \text{AC} = \text{BD} \)?
   Give the reasons why they are congruent.

   (iii) Copy and complete the sentences:
   (a) If one \( \angle \) of a \( \text{par}^m \), is a right angle, then all the \( \angle \)s . . . .
   (b) By definition, a rect. is a \( \text{par}^m \), in which . . . . . . . . . . . . ; we have now proved that all the angles of a rect. . . . . and that its diagonals . . . .

Facts about a Square

Definition.—If two adjacent sides of a rectangle are equal, the figure is called a square.

2. In Fig. 395, EFGH is a rectangle, such that \( EF = EH \).

   (i) Why does \( EF = FG \)? What can you say about \( GH \)?
   (ii) Give reasons why \( \triangle \text{ENF} = \triangle \text{ENH} \)
   What follows about \( \angle \text{ENF} \)?
   (iii) Copy and complete the sentences:
   (a) Since EFGH is a rectangle, all its angles . . . . . . . . and its diagonals . . . .
   (b) By definition, a square is a rect. in which . . . . . . . . ; we have now proved that all the sides of a square . . . . . . . . . . . . . . and that the diagonals of a square . . . .

Facts about a Rhombus

Definition.—If two adjacent sides of a parallelogram are equal, the figure is called a rhombus.

[It is understood that none of its angles are right angles, except in the special case when it would be called a square.]

3. In Fig. 396, PQR is a \( \text{par}^m \), such that \( \text{PQ} = \text{PS} \).

   (i) Why does \( \text{PQ} = \text{QR} \)? What can you say about \( \text{RS} \)?
   (ii) Give reasons why \( \triangle \text{PKQ} = \triangle \text{PKS} \).
   What follows about \( \angle \text{PKQ} \)?
   (iii) What angles in the figure equal \( \angle \text{PSQ} \)?
   (iv) Copy and complete the sentence:
   By definition, a rhombus is a \( \text{par}^m \), in which . . . . . . . . . ; we have now proved that all its sides . . . . . . . . . . . . . . , and that its diagonals . . . .

Facts about Diagonals

The different kinds of parallelograms can be distinguished by the properties of their diagonals as follows:

In any parallelogram, the diagonals bisect each other.
In a rectangle, the diagonals bisect each other and are equal.
In a rhombus, the diagonals bisect each other at right angles.
In a square, the diagonals bisect each other at right angles, and are equal.

EXERCISE XIII. a

Special Quadrilaterals

1. ABCD is a quad. such that \( AB = DC \) and \( AD = BC \); prove that ABCD is a \( \text{par}^m \). This means, prove that \( AB \parallel DC \), and \( AD \parallel BC \).

2. ABCD is a quad. such that \( \angle A = \angle C \) and \( \angle B = \angle D \); prove that (i) \( \angle A + \angle B = 180^\circ \), (ii) ABCD is a \( \text{par}^m \).

3. ABCD is a quad. such that \( AB \) is equal and parallel to \( DC \); prove that \( AD \) is equal and parallel to \( BC \).
   What kind of quad. is ABCD?
4. In Fig. 397, if \( HT = TK \) and \( LT = TM \), prove that \( HMKL \) is a parallelogram. This means, prove that \( HM \parallel KL \) and \( HL \parallel MK \).

5. In Fig. 397, if \( TH = TK - TL = TM \), prove that \( HMKL \) is a rectangle. This means, prove \( HMKL \) is a parallelogram, and prove \( H \angle LHM = 90^\circ \).

6. In Fig. 397, if \( HT = TK \) and \( LT = TM \) and \( \angle HTL = 90^\circ \), prove that \( HMKL \) is a rhombus. This means, prove \( HMKL \) is a parallelogram, and prove \( HM = HL \).

**Numerical Examples**

7. In Fig. 394, where \( ABCD \) is a rectangle, (i) if \( \angle BKC = 40^\circ \), find \( \angle DAC \); (ii) if \( \angle ADB = 75^\circ \), find \( \angle CKD \).

8. In Fig. 395, where \( EFGH \) is a square, draw \( HQ \) to bisect \( \angle EHF \). (i) Find \( \angle EqF \); (ii) prove that \( \angle GHQ = 3 \angle QHE \).

9. In Fig. 396, where \( PQRS \) is a rhombus, (i) if \( \angle SPQ = 80^\circ \), find \( \angle SPR \); (ii) if \( \angle PRQ = 35^\circ \), find \( \angle PSR \).

**Constructions**

10. Construct a parallelogram \( ABCD \), such that \( AC = 8 \) cm., \( BD = 6 \) cm., \( AD = 3 \) cm. Measure \( AB \).

11. Construct a rectangle \( ABCD \), such that \( AB = 4.5 \) cm., \( AC = 5 \) cm. Measure \( BD \) and \( BC \).

12. Construct a rhombus \( PQRS \), such that \( PR = 3 \) in., \( QS = 2 \) in. Measure \( PQ \).

13. The diagonals of a parallelogram are 10 cm. and 7 cm. Can it have a side 9 cm. long? a side 1 cm. long?

   What can you say about the possible lengths of its sides?

   Find by measurement and calculation the lengths of the sides if the parallelogram is a rhombus.

14. Construct a rhombus \( ABCD \), such that \( AB = 5 \) cm., \( AC = 6 \) cm. Measure \( \angle DAB \). Measure and calculate the length of \( BD \).

15. Construct a rhombus \( PQRS \), such that \( PR = 2 \) in., \( \angle QPS = 75^\circ \). Measure \( QS \). Find the area of the rhombus.

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**Tests for a Parallelogram**

By definition, a quadrilateral is a parallelogram if its opposite sides are equal. Ex. XIII. a, Nos. 1-4, supply four other tests, as follows:

- A quadrilateral is a parallelogram,
  (i) if both pairs of opposite sides are equal;
  (ii) if both pairs of opposite angles are equal;
  (iii) if one pair of opposite sides is equal and parallel;
  (iv) if the diagonals bisect each other.

**Ruler and Compass Constructions**

Practice has been given in various previous exercises in performing certain fundamental constructions. These are grouped together in Ex. XIII. b, Nos. 1-5, and should be worked through with instruments by all. The remaining examples in the exercise illustrate the use of these constructions.

**EXERCISE XIII. b**

Carry out the constructions in Nos. 1-5. State shortly what you do and show that the required result is obtained.

1. Bisect a given angle \( \angle BAC \) (see Fig. 398). [\( P, Q \) lie on a circle, centre \( A \); two equal circles, centres \( P, Q \), cut at \( R \).]

2. Bisect a given straight line \( AB \) (see Fig. 399). [Two equal circles, centres \( A, B \), cut at \( P, Q \).] This construction gives the perpendicular bisector of \( AB \).

3. \( C \) is a given point on \( AB \). Construct at \( C \) the line perpendicular to \( AB \) (see Fig. 400). [\( P, Q \) lie on a circle, centre \( C \); two equal circles, centres \( P, Q \), cut at \( R \). For the proof, join \( RP, RQ \).]
4. K is a given point not on AB. Construct the perpendicular from K to AB (see Fig. 401).

(4, Q lie on a circle, centre K; two equal circles, centres P, Q cut at R. For the proof, join KP, KQ, PR, QR.)

5. Given an angle BAC and a line DE, construct at D an angle EDF equal to angle BAC (see Fig. 402).

(3, Q lie on any circle, centre A. Now copy the triangle APQ so that the side equal to AP is along DE.)

6. Draw any line AB; then construct two equilateral triangles, ABP and ABQ. What do you know about AB and PQ? Find in degrees the sizes of ∠APQ and ∠PAQ.

7. Draw a large triangle ABC, not isosceles. Construct the perpendicular bisectors of AB and BC and CA.

8. Draw a large triangle ABC and construct its three medians. [The line joining any vertex of a triangle to the middle point of the opposite side is called a median.]

9. Draw a large acute-angled triangle, not isosceles. Construct each altitude of the triangle.

10. Repeat No. 9 for an obtuse-angled triangle.

11. Draw a line and mark 2 points A and B on it. Take any point C outside the line. Construct a line CP so that ∠ABC and ∠BCP are alternate and equal. What do you know about CP?

Chords of a Circle

The following properties have occurred in previous examples; their proofs should be known as they are required so often.

Examples for Class Discussion

1. In Fig. 403, AB is a chord of a circle, centre O; ON is the mid-point of AB. Prove that ON is perpendicular to AB.

(i) Give in full the reasons why ∆ONA = ∆ONB.

(ii) Explain why it follows that ∼ONA = 90°.

2. In Fig. 403, AB is a chord of a circle, centre O; ON is the perpendicular from O to AB. Prove that AN = NB.

Give in full the reasons why ∆ONA = ∆ONB.

3. (i) If AB is a chord of a circle, what can you say about the perpendicular bisector of AB?

(ii) If you are given an arc of a circle, how can you construct the centre of the circle?

The facts established in these examples are required for proving the constructions in Ex. XIII.c. They may be stated as follows:

(i) The line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

(ii) The line drawn from the centre of a circle perpendicular to a chord bisects the chord.

(iii) The perpendicular bisector of a chord passes through the centre of the circle.
CONSTRUCTION OF CIRCLES

EXERCISE XIII. c

1. Use a coin to draw a circle. Find its centre by drawing the perpendicular bisectors of two chords.

2. (i) Draw a large triangle ABC, and construct the perpendicular bisectors of AB and AC, and let them cut at O. Now draw the circle which passes through A and B and C.
   This circle is called the circumscribed circle of the triangle ABC or, for short, the circumscribed circle of \( \triangle ABC \), and its centre O is called the circumcentre of \( \triangle ABC \).
   (ii) Give reasons why the perpendicular bisectors of AB and BC and CA all pass through one point.
   (iii) Can you draw through any 3 points (a) one circle, (b) more than one circle? Can you draw a circle through any 4 points?

3. (i) Draw the circumcircle of a triangle whose sides are 3 cm., 4 cm., 5 cm. Measure its radius.
   This is called the circumradius of the triangle.
   (ii) Repeat for a triangle whose sides are 3 cm., 4 cm., 6 cm.
   (iii) Repeat for a triangle whose sides are 3 cm., 4 cm., 6 cm.
   What do you notice about the positions of the circumcentres in the three cases? Can you explain it by using the property about the angle at the centre of a circle?

4. Draw a rectangle ABCD so that \( AB = 4 \cdot 5 \text{ cm.}, BC = 6 \text{ cm.} \) Draw the circumcircle of \( \triangle ABC \). What is its radius? Why must the circle also pass through D?

5. Draw any straight line AB and mark any two points P, Q, not on AB. How can you construct a circle to pass through P and Q and have its centre somewhere on AB? What happens if the line joining P and Q is perpendicular to AB?

6. (i) Fig. 404 shows two circles cutting at A and B. What can you say about the perpendicular bisector of AB?
   AB is called the common chord of the two circles.
   (ii) Construct two circles of radii 4 cm. and 5 cm., and so that their common chord is of length 6 cm. Measure the distance between their centres. [There are two answers; find both.]

APPLICATIONS OF PYTHAGORAS' THEOREM

EXERCISE XIII. d

[Note.—In this exercise, Nos. 1-6 should be done by all; any or all of the remaining examples may be omitted at a first reading.]

1. A chord of a circle of radius 6 cm. is 8 cm. long. Calculate its distance from the centre.
2. A chord of a circle of radius 7 cm. is at a distance of 5 cm. from the centre. Calculate its length to the nearest mm.
3. A chord of length 10 cm. is at a distance of 12 cm. from the centre of the circle. Calculate the radius.

LENGTHS OF CHORDS AND RADI

Pythagoras' theorem can often be used to find the lengths of lines connected with a circle.

Examples for Class Discussion

1. In Fig. 405, NO bisects AB at right angles; also AB = 12 cm., CN = 4 cm. Find the radius of the circle.

Since ON is the perp. bisector of AB, it passes, when produced, through the centre O.

Let the radius = r cm.

What is ON in terms of r?

By Pythagoras, from \( \triangle ONA \),

\[ r^2 = (r - 4)^2 + 6^2. \]

Solve this equation.

2. In Fig. 405, ND bisects AB at right angles; also AB = 8 cm., ND = 12 cm. Find the radius of the circle.

(i) Why does the centre O lie on ND?
(ii) Let the radius = r cm. What is ON in terms of r?
Use Pythagoras to complete the equation, \( r^2 = \ldots \ldots \); then solve it.
FURTHER DEVELOPMENTS

4. What is the radius of a circle if a 4 inch chord is at a distance of 1½ inches from the centre?

5. In Fig. 406, CD is a chord perpendicular to the diameter AB; AP = 12 cm., PB = 3 cm.; calculate the length of CD.

6. In a circle of radius 4 cm., AB is a diameter and AC is a chord of length 6 cm.; find the length of BC to the nearest mm.

7. In a circle of radius 5 in., two parallel chords are of lengths 8 in., 6 in. What is the distance between them? [Two answers.]

8. In Fig. 405, CN bisects AB at right angles. If AB = 6 in. and CN = 1 in., calculate the radius of the circle.

9. In Fig. 405, ND is the perpendicular bisector of AB. If AB = 10 cm. and ND = 12 ½ cm., find the radius of the circle.

10. In Fig. 407, O is the centre of each circle; their radii are 5 cm. and 7 cm.; also FG = 8 cm. Find the length of EF to the nearest mm.

11. Fig. 408 represents a hemispherical basin of radius 9 inches, containing water. If the greatest depth of the water is 5 inches, find, to the nearest ½ inch, the diameter PQ of the surface of the water. [Draw a figure showing a vertical section through the centre of the hemisphere.]

12. A cylindrical jug of diameter 8 cm. stands on a horizontal shelf, and a ball of diameter 10 cm. rests on the rim of its open end. What is the height of the top of the ball above the level of the rim of the jug? [Draw a figure showing a vertical section through the centre of the ball.]

APPLICATIONS OF CONGRUENCE

GENERAL STATEMENTS

It is suggested that Nos. 1-4 of Ex. XIII.e should be taken orally; any or all of the remainder may be omitted at first reading.

EXERCISE XIII.e

Give short reasons for each step.

1. In Fig. 409, ABCD and ABPQ are parallelograms. Prove that CP is parallel to DQ.

2. With the data of No. 1, prove that the sum of the areas of ABPQ and QPDC equals the area of ABCD.

3. In Fig. 410, O is the centre of the circle and OH, OK are perpendicular to AB, CD. If AB = CD, prove that OH = OK.

4. In Fig. 407, O is the centre of each circle. Prove that EF = GH. [From O, draw ON perpendicular to EH; make no more construction.]

5. In Fig. 411, ABCD is a parallelogram and CP = PD. Prove that BQ = 2AD.

6. In Fig. 412, A, B are the centres of the circles and PCQ is parallel to AB. Prove that PQ = 2AB.
7. In Fig. 413, $\triangle ABC$ is any triangle; lines are drawn through each corner parallel to the opposite side and form the triangle $XYZ$. The altitudes $BE, CF$ of $\triangle ABC$ meet at $H$.

![Fig. 413.](image)

Prove that (i) $CX = CY$; (ii) $CF$ is the perpendicular bisector of $XY$.

What can you now say about $BE$ and the point $H$ where $CF$ cuts $BE$? What can you say about the third altitude $AD$ of $\triangle ABC$?

8. In Fig. 414, $AB = BC = CD$ and $BCQP$ is a rhombus. Prove that $\angle AND = 90^\circ$.

![Fig. 414.](image)

9. In Fig. 415, $ABCD$ is a square and $AP = AK$. Prove that $\angle AKP = 3\angle BKP$.

![Fig. 415.](image)

10. In Fig. 416, $PCQ$ is parallel to $RDS$; prove that $PQ = RS$. [Draw the perpendiculars from the centres to $PQ$ and produce them backwards to meet $RS$.]

![Fig. 416.](image)

CHAPTER XIV

INTERCEPTS AND RATIOS

Intercepts by Equidistant Parallels

Take an ordinary sheet of ruled paper. The ruled lines are parallel and at equal distances apart. Now draw any line obliquely across them (see Fig. 417). The parts into which this line is cut by the ruled lines are called the intercepts made by the parallel lines.

![Fig. 417.](image)

It is easy to prove that the intercepts are equal if the parallel lines are equidistant.

Examples for Class Discussion

1. In Fig. 418, $AP, BQ, CR$ are parallel, also $AB = BC$; prove that $PQ = QR$.

Draw $PX$ and $QY$ parallel to $ABC$.

(i) What can you say about the quad. $APXB$? and about $BCYQ$?

(ii) What follows about $PX$ and $QY$? Give reasons.

(iii) Give the reasons why $\triangle APX = \triangle QYR$. What follows?

![Fig. 418.](image)
(iv) In Fig. 419 (i), \(AP, BQ, CR, DS, ET\) are parallel and \(AB = BC = CD = DE\). What follows?

If we draw Fig. 419 (i) so that \(P\) is the same point as \(A\), we obtain Fig. 419 (ii). What follows? Make a proper sentence.

Fig. 419

2. Use a sheet of ruled paper to divide a given line \(AB\) into 5 equal parts.

If you need paper with the rulings close together, use squared paper.

(i) Suppose that the given line \(AB\) is equal to (say) 7 times the distance between two adjacent rulings. Mark any point \(O\) on a ruled line and draw a circle, centre \(O\), radius equal to \(AB\). In Fig. 420 the rulings divide \(OP\) into 5 equal parts. Why is this? But \(OP = AB\), so mark off these equal lengths along \(AB\).

(ii) In Fig. 420, what fraction is \(OL\) of \(OP\)? Mark points \(E, F\) on \(AB\) so that \(AE = \frac{1}{5} AB, AF = \frac{2}{5} AB\).

(iii) In Fig. 420, what fraction is \(OM\) of \(OQ\)? Mark points \(H, K\) on \(AB\) so that \(AH = \frac{3}{5} AB, AK = \frac{4}{5} AB\).

What is \(\frac{3}{5} - \frac{4}{5}\)? What line represents \(\frac{1}{2} AB\)?

3. By using set-square and compass, divide a given line \(AB\) into 7 equal parts.

(i) Draw the line \(AB\). Draw any line \(AK\); start from \(A\) and with your compass cut off along it 7 equal parts of any convenient length (see Fig. 421). Join \(PB\) and draw parallels to it through the other points marked on \(AP\). Why does this do what is required?

(ii) Repeat (i), but do not draw in the parallel lines; mark only where they cross \(AB\). Why is this better?

(iii) Mark on \(AB\) points \(R, S\) so that \(AR = \frac{1}{5} AB, AS = \frac{2}{5} AB\).

EXERCISE XIV. a

Accurate figures are required in this Exercise.

1. Using ruled paper, draw \(AB\) equal to 4 times the distance between two rulings and then divide it into three equal parts.

2. Mark off a length of 10 cm. along the edge of a sheet of paper; use a sheet of ruled paper or squared (inch) paper to divide it into 7 equal parts.

3. Show, as in Fig. 420, using squared (inch) paper, lines of length 8.5 cm., divided into 3, 4, 5, 6, 7, 8, 9, 10 equal parts. Mark the points of division clearly.

4. Draw a large triangle \(ABC\); use your set-squares and compass to divide each side into 5 equal parts.

(i) Mark \(P, Q\) on \(AB\) so that \(AP = \frac{1}{5} AB\) and \(BQ = \frac{2}{5} BA\).

(ii) Mark \(R, S\) on \(BC\) so that \(BR = \frac{1}{5} BC\) and \(CS = \frac{2}{5} CE\).

(iii) Mark \(T\) on \(AC\) so that \(AT = \frac{3}{5} AC\); what fraction is \(TC\) of \(AC\)?

5. Fig. 422 shows a method of dividing \(AB\) into 3 equal parts. \(AX, BY\) are any two parallel lines; \(AF, FG, BK, KL\) are all equal (any convenient length).

(i) Carry out this construction.

(ii) State two facts about the lines \(FQ, LK\); why does this prove that \(GK\) and \(FL\) are parallel?

(iii) Why does \(AP\) equal \(QB\)?
FURTHER DEVELOPMENTS

6. Draw a line 5 cm. long. Use the method of No. 5 to divide it into 4 equal parts.

7. In Fig. 423, AX and BY are perpendicular to AB; also AH = 3 cm. and BK = 4 cm. What fraction is AP of AB? What fraction is PB of AB? [You need not draw an accurate figure.]

8. Draw a figure like Fig. 423 without any of the dotted lines. Make AH = 3 cm. and BK = 4 cm. What fraction is AP of AB? What fraction is PB of AB?

Fig. 423.

INTERCEPTS AND RATIOS

Examples for Class Discussion

EXERCISE XIV. b (Oral)

All questions in this exercise refer to Fig. 424, where the line AB is divided into 12 equal parts.

1. Find the values of the following ratios:
   (i) \( \frac{AG}{GB} \) ; (ii) \( \frac{AS}{SB} \) ; (iii) \( \frac{AM}{MB} \) ; (iv) \( \frac{AP}{PB} \);
   (v) \( \frac{AH}{HS} \) ; (vi) \( \frac{AQ}{QT} \) ; (vii) \( \frac{AG}{GM} \) ; (viii) \( \frac{AR}{RS} \).

2. Find the ratio in which (i) Q divides AR; (ii) G divides FR; (iii) H divides FN; (iv) T divides SB.

3. Name the point which divides (i) AH in ratio 2 : 3; (ii) AT in ratio 2 : 3; (iii) AM in ratio 2 : 1; (iv) AM in ratio 1 : 2.

[Note.—The rest of this exercise may be omitted at first reading.]

If M is any point on PR between P and R, we can say that M divides PR internally in the ratio PM : MR.

If, however, M is any point on PR produced either way we say that M divides PR externally in the ratio PM : MR.

4. Find the ratio in which
   (i) R divides AP externally; (ii) N divides PS externally;
   (iii) Q divides SN externally; (iv) P divides SB externally.

5. Name the point which divides
   (i) AQ in ratio 2 : 1, internally, externally;
   (ii) QR in ratio 3 : 2, externally;
   (iii) QR in ratio 2 : 3, externally.

EXERCISE XIV. c

1. Draw a line AB and use instruments to divide it at P in the ratio 2 : 3 and at Q in the ratio 3 : 2.

What are the values (i) of \( \frac{AP}{AB} \)? (ii) of \( \frac{AQ}{AB} \)?
2. Draw a line CD and construct points H, K on it so that $CH : HD = 4 : 3$ and $CK = \frac{1}{2}KD$. What are the values of (i) $\frac{CH}{CD}$, (ii) $\frac{HD}{CD}$, (iii) $\frac{KD}{CD}$, (iv) $\frac{KH}{CD}$?

3. In Fig. 425,

(i) if $\frac{AQ}{QB} = \frac{2}{3}$, find $\frac{AQ}{AB}$;
(ii) if $\frac{AR}{RB} = \frac{2}{3}$, find $\frac{AR}{AB}$;
(iii) if $\frac{AP}{PB} = \frac{3}{4}$, find $\frac{AP}{AB}$.

4. In Fig. 425, if AB is 8 inches long
(i) find AQ if $\frac{AQ}{QB} = 3 : 2$;
(ii) find AR if $\frac{AR}{RB} = 7 : 3$;
(iii) find AP if $\frac{AP}{PB} = 3 : 8$.

Equal Intercepts along the Sides of a Triangle

Examples for Class Discussion

1. In Fig. 426, $AE = EF = FB$; also EP, FQ are parallel to BC, and PX, QY are parallel to AB. [Do not letter any other points in the figure.]

(i) Name two parms. in the figure.

INTERCEPTS AND RATIOS

(iii) What lines in the figure equal $\frac{AP}{AB}$? What are the values of $\frac{AQ}{AC}$, $\frac{AF}{AB}$? What follows?

(iii) Name 3 lines in the figure equal to EP. Give reasons. What is the value of $\frac{EP}{BC}$?

Copy and complete, $\frac{AE}{AB} = \frac{AQ}{AC} = \frac{AF}{AB}$.

(iv) What line in the figure equals FQ? Give a reason.
What is the value of $\frac{FQ}{BC}$? Give a reason.
Copy and complete, $\frac{AE}{AB} = \frac{AQ}{AC} = \frac{AF}{AB}$.

2. In Fig. 427, $ABC$ is any triangle and $AB$ is divided into 5 equal parts at E, F, G, H. Parallels are then drawn to BC and to AB as was done in Fig. 426. [Do not letter any other points in the figure.]

(i) What lines in the figure equal $\frac{AP}{AB}$? Give reasons.

(ii) What lines in the figure equal $\frac{EP}{BC}$? Give reasons.

What is the value of $\frac{EP}{BC}$? Give a reason.
Copy and complete, $\frac{AE}{AB} = \frac{AQ}{AC} = \frac{AF}{AB}$.

(iii) What are the values of $\frac{FQ}{BC}$, $\frac{GR}{BC}$, $\frac{HS}{BC}$? Give reasons.

(iv) Copy and complete, $\frac{AE}{AB} = \frac{AR}{AB}$.

(v) What are the values of $\frac{GR}{CA}$ and $\frac{RZ}{AB}$? Give reasons.
**Mid-point Theorem**

3. In Fig. 428, \(ABC\) is any triangle and \(E\) is the mid-point of \(AB\); \(EP\) and \(PX\) are drawn parallel to \(BC\) and \(AB\).

   (i) What do you know about the point \(P\)? What about the point \(X\)?

   (ii) Name 2 lines in the figure equal to \(EP\). Give reasons. What is the value of \(\frac{EP}{BC}\)?

   (iii) What is the value of \(\frac{PX}{AB}\)? Give reasons.

This discussion shows that if a line is drawn through the mid-point \(E\) of \(AB\), parallel to \(BC\), then it bisects \(AC\); it follows that the line joining the mid-points \(E, P\) of \(AB\), \(AC\) must be parallel to \(BC\). Also we have seen that \(EP = \frac{1}{2}BC\). These results are stated as follows:

The line joining the mid-points of the sides of a triangle is parallel to the base and equal to half the base.

4. In Fig. 429, \(AB\) is divided into 5 equal parts and parallels are drawn to \(BC\), as shown.

   (i) If \(BC = 1\) in., what are the lengths of \(EP, FQ, GR, HS\)?

   (ii) If \(BC = \frac{1}{5}\) in., what are the lengths of \(EP, FQ, GR, HS\)?

   (iii) Repeat (ii) if \(BC = \frac{1}{3}\) in., if \(BC = 1\) cm.

The Diagonal Scale

With a ruler graduated in inches and tenths of an inch, you can measure lengths to the nearest tenth of an inch, and you can make a fair guess of the number of hundredths of an inch. It is possible to measure more accurately by means of a diagonal scale, if one is printed on your ruler, but dividers should be used; compasses are a poor substitute.

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**Examples for Class Discussion**

Draw accurately Fig. 430, which shows a diagonal scale for measuring to \(\frac{1}{8}\) inch.

---

The upper line is graduated in inches, the zero being put one inch from the left. The first inch is then divided into 5 equal parts as shown. Draw a perpendicular line down the page and mark off 5 equal parts along it; then draw parallels and divide the first inch of the lower line also into 5 equal parts as shown; the 'diagonal scale' is completed by drawing the oblique lines.

(i) What is the length of \(TR, P1, R3, S4\)?

(ii) What is the length of \(X1, Y2, Z3\)?

(iii) What is the length of \(AX, BY, CZ\)?

(iv) On your copy of the figure mark two points \(E, F\) \(\frac{1}{8}\) in. apart.

(v) \(\frac{3}{8} + \frac{1}{8} + \frac{2}{8} = \frac{5}{8}; \) express in the same way \(\frac{1}{8}, \frac{2}{8}, \frac{3}{8}\). Mark two points \(GH\) on your figure \(\frac{1}{8}\) in. apart.

(vi) Draw on squared paper a square of side 2 in.; use your diagonal scale to measure its diagonal to the nearest \(\frac{1}{8}\) in.

(vii) Repeat (vi) for a square of side 2.5 inches.
EXERCISE XIV. d

1. Make on squared paper a diagonal scale for reading distances to \( \frac{1}{10} \) inch. Make it in the form of an oblong 2 inches high.

Draw, on squared paper, squares of sides 3 inches, 2-4 inches, 1-3 inches and measure their diagonals correct to \( \frac{1}{10} \) inch.

2. Draw as accurately as possible lines of length 4 cm. and 7 cm. Measure each in inches to \( \frac{1}{10} \) inch, and, from each result, express 1 cm. in inches.

3. Draw a triangle \( \triangle ABC \), making \( AB \) 3 inches. Construct points \( P \) and \( Q \) on \( AB \) and \( AC \) so that \( PQ \) is parallel to \( BC \) and \( PQ \) equals \( \frac{1}{2}BC \). What is the length of \( PB \)?

Nos. 4-8 refer to Fig. 431, where \( AP \), \( BQ \), \( CR \) are perpendicular to \( XY \). The line \( AKE \) is drawn parallel to \( XY \) to help you.

4. If \( AC = CB \), what do you know about \( CK \)? Hence find \( CR \) if (i) \( AP = 5 \) cm., \( BQ = 9 \) cm.; (ii) \( AP = 1-6 \) in., \( BQ = 2-2 \) in.

5. If \( AC = CB \), find \( BQ \) if (i) \( AP = 7 \) cm., \( CR = 10 \) cm.; (ii) \( AP = 3 \) in., \( CR = 3-6 \) in.

6. If \( AC = AB \), what do you know about \( CK \)?

If \( AP = 5 \) cm. and \( BQ = 7 \) cm., find \( CR \).

7. If \( AC : CB = 2 : 3 \), what is the ratio of \( CK \) to \( BE \)?

If \( AP = 7 \) cm. and \( BQ = 10 \) cm., find \( CR \).

8. If \( AP = 10 \) cm., \( CR = 12 \) cm., \( BQ = 16 \) cm., find the ratio \( AC : CB \).

[Note.—The rest of this exercise may be omitted at a first reading.]

9. (i) If \( AB \) is divided internally at \( Q \) in the ratio 5 : 6, is \( Q \) nearer to \( A \) or to \( B \)? Draw a rough figure.

(ii) If \( AB \) is divided externally at \( S \) in the ratio 2 : 3 is \( S \) in \( AB \) produced or in \( BA \) produced? Draw a rough figure.

10. Draw a line \( AB \) and construct points \( P \), \( Q \), \( R \) on it, so that \( AP : PQ : QR : RB = 2 : 1 : 3 : 4 \).

INTERCEPTS AND RATIOS

What are the values of (i) \( AP : PR \); (ii) \( QR : PB \); (iii) \( PR : AB \)?

If \( QR = 4 \) in., what are the lengths of \( RB \) and \( AB \)?

11. Draw a line \( CD \) and divide it externally at \( K \) in the ratio 7 : 4. What is the value of \( CK : CD \) ?

Equal Ratios in Similar Figures

Suppose, in Fig. 432, that \( \frac{AH}{AB} = \frac{7}{11} \). This means that if \( AB \) is divided into 11 equal parts, then \( AH \) contains exactly 7 of them.

Through each point of division, imagine a line drawn parallel to \( BC \), just as is done in Fig. 427, and let \( HK \) be the parallel through \( K \). Then these lines make 11 equal intercepts on \( AC \), of which \( AK \) contains exactly 7:

\[
\frac{AK}{AC} = \frac{7}{11} = \frac{AH}{AB}
\]

Again, through each point of division of \( AC \), imagine a line drawn parallel to \( AB \), and let \( KL \) be the parallel through \( K \). Then these lines make 11 equal intercepts on \( BC \), of which \( BL \) contains exactly 7:

\[
\frac{BL}{BC} = \frac{7}{11} = \frac{HK}{BC}
\]

But \( BLKH \) is a paral.

\[
\frac{AH}{AB} = \frac{7}{11} = \frac{HK}{BC}
\]

\[
\frac{AH}{AB} = \frac{7}{11} = \frac{HK}{BC}
\]

We can repeat this argument, whatever the value of the ratio \( \frac{AH}{AB} \). But, since \( HK \) is parallel to \( BC \), \( \triangle AHK \) is equiangular to \( \triangle ABC \).

:: for the equiangular triangles \( \triangle AHK \), \( \triangle ABC \) the ratio of any side of \( \triangle AHK \) to the corresponding side of \( \triangle ABC \).
is the same, whichever pair of corresponding sides is chosen; and this can easily be shown to be true for any two equiangular triangles.

This is the way to prove the statement in Chapter VIII that triangles which are equiangular are the same shape, and to show how it is that problems can be solved by scale-drawings.

Examples for Class Discussion

1. In Fig. 433, \( \angle B = \angle Q \), \( \angle C = \angle R \), and the units are cm.
   
   (i) What are the lengths of \( PQ \), \( PR \) ?
   
   (ii) What are the values of the ratios \( \frac{PQ}{PR} \) ?
   
   (iii) Name a ratio equal to \( \frac{AB}{BC} \); name one equal to \( \frac{PR}{PQ'} \).
   
   (iv) If \( AD \) and \( PS \) are altitudes, copy and complete

   \[
   \frac{AD}{BC} = \quad \frac{RS}{PQ} = \quad \frac{PR}{PQ'} = \quad \frac{AB}{PQ} = \quad \frac{AD}{DC} = \quad \frac{PR}{PQ}.
   \]

   (v) Copy and complete

   \[
   \frac{PR}{PQ} = \quad \frac{QR}{PR} = \quad \frac{QR}{PQ} = \quad \frac{AC}{AB}.
   \]

   This example shows the different ways equal ratios are obtained by comparing similar figures:

   (a) The ratio of any two lines in the first is equal to the ratio of the corresponding lines in the second.

   (b) The ratio of any line in the first to the corresponding line in the second is the same, whatever pair of corresponding lines is chosen.

2. Enlarge the quadrilateral \( ABCD \) in the ratio 5 : 2. Draw any quad. \( ABCD \) and perform the construction shown in Fig. 434. Take any point \( O \). Join \( OA \) and produce it to \( A' \) so that \( OA' : OA = 5 : 2 \). Join \( OB, OC, OD \)

   and produce them. \( A'B', B'C', C'D' \) are drawn parallel to \( AB, BC, CD \).

   (i) Why is \( \frac{A'B'}{AB} = \frac{5}{2} \)?

   (ii) Why is \( \frac{B'C'}{BC} = \frac{5}{2} \)? How much is \( \frac{C'D'}{CD} \)?

   (iii) Why is \( \frac{OD'}{OA} = \frac{OA'}{OD} \)? What follows?

   (iv) Why is \( A'B'C'D' \) equiangular to \( ABCD \).

   *This construction can be used to enlarge or reduce any figure in any given ratio.*

EXERCISE XIV e

1. In Fig. 433, \( \angle B = \angle Q \), \( \angle C - \angle R \). If \( QR \) equals 12 cm. (instead of 15 cm. as printed), find the lengths of \( PQ \) and \( PR \).

   What are the values of \( \frac{PR}{PQ} \) and \( \frac{AC}{AB} \)?
2. In Fig. 435, PQ is parallel to AB.
   (i) If $PQ = 2^\circ$, find CP and CQ. What are the values of $\frac{CP}{CA}$ and $\frac{CP}{PA}$?
   (ii) If $CP = 3.2^\circ$, find PQ and CQ. What are the values of $\frac{PQ}{AB}$ and $\frac{CP}{PA}$?

3. In Fig. 435, $\angle Y = \angle B$ and $\angle Z = \angle C$.
   (i) If $XY = 1.5^\circ$, find XZ and YZ.
   (ii) If $YZ = 3.6^\circ$, find XY and XZ.
   (iii) Find the values of the ratios $\frac{XY}{XZ}$ and $\frac{XY}{YZ}$.

4. In Fig. 436, BPQR is a parallelogram, and the units are cm.
   (i) What is the value of $\frac{CP}{PQ}$? If $PQ = 6$ cm., find CP; then find PB, QR and AQ.
   (ii) If $QR = 2.4$ cm., find PQ and AQ.
   (iii) If $AQ = 4.8$ cm., find PQ and QR.

[Note.—The rest of this exercise may be omitted at a first reading.]

5. In Fig. 436, BPQR is a paralellogram. If $QR = x$ cm. and $QP = y$ cm., find an equation between $x$ and $y$.

6. In Fig. 437,
   (i) if $AP = 5$ in., $BQ = 3$ in., $AB = 10$ in., find AR;
   (ii) if $AR = 6$ cm., $AB = 10$ cm., find the ratio of $AP : BQ$.

7. Show that the line joining (1, 2) to (4, 3) is parallel to, and half of, the line joining (0, 0) to (6, 2).

8. A light is 9 ft. above the floor; a ruler, 8 in. long, is held horizontally under the light and 4 ft. above the floor. Find the length of the shadow on the floor.

9. Draw a triangle with sides of length 2 in., 3 in., 4 in. Then enlarge this triangle in the ratio 11 : 7.

10. Fig. 438 shows a small portion of a piece of squared paper, with the point A (4, 2) joined to the point B (0, -3) by a line cutting the x-axis OC at P.
   (i) Enlarge the figure by drawing it on squared paper with 1 inch as unit, and read off the length of OP.
   (ii) What can you say about the ratio $\frac{OP}{LA}$? Hence calculate the length of OP.

11. With the data of Fig. 438,
   (i) find the ratio $\frac{AP}{AB}$;
   (ii) find the ratio in which FH divides QD.

12. Fig. 438 shows a graph passing through F, G, H, L. Enlarge the graph by drawing it on squared paper with 1 inch as unit, and read off from your figure as accurately as you can the distance from Q of the point where the graph crosses QD.
GENERAL STATEMENTS

The examples in Exercise XIV.1 illustrate the "mid-point theorem"; it is suggested that Nos. 1-3 should be taken orally; any or all of the remainder may be omitted at a first reading.

EXERCISE XIV.1

Give short reasons for each step.

1. P, Q, R, S are the mid-points of the sides AB, BC, CD, DA of the quadrilateral ABCD. Prove that PQ is equal and parallel to SR. [Hint. Join AC.]

What can you say about PQ?

2. The mid-points of the sides of a triangle are joined. Prove that the figure is divided into four congruent triangles.

3. In Fig. 431, p. 210, where AC = CB and AP, BQ, CR are parallel, prove that \[ CR = \frac{1}{2} (AP + BQ). \]

4. P is any point inside a triangle ABC; H, K, X, Y are the mid-points of AB, AC, PB, PC respectively. What can you say aboutHX and KY?

5. Prove that the mid-points of the sides of a rectangle are the corners of a rhombus.

What can you say about the mid-points of the sides of a rhombus?

6. If, in Fig. 437, p. 215, S is the mid-point of PQ, and if SC is the perpendicular from S to AB, prove that \[ SC = \frac{1}{2} (AP - BQ). \]

7. In Fig. 422, p. 203, AP = PQ = QB and AX is parallel to BY. If G, P, Y are in a straight line, prove that BY = 2AG. [Hint. Draw a parallel to BY through Q.]

8. Y, Z are the mid-points of the sides AB, AC of \( \triangle ABC \); BZ cuts CY at G; AG is joined and produced to H, so that \( AG = GH \). Prove that (i) BH is parallel to CGY, (ii) BHCG is a paralellogram, (iii) AGH bisects BC, (iv) \( GY = \frac{1}{2} CG = \frac{1}{2} CY \).
that $OQ = ON$; join $NQ$. Then $\angle OQN$ is greater than $\angle ONP$; therefore $\angle OQN$ is greater than $90^\circ$. But $OQ = ON$; what therefore can you say about $\angle OQN$, and then about $\angle ON + \angle OQN$? Why is the result false?

Since the conclusion is untrue, it is wrong to suppose that there is a point $P$ on $AB$ such that $OP$ is less than $ON$.

Next, if possible, suppose there is a point $P$ on $AB$ such that $OP$ equals $ON$.

If so, what can you say about $\angle OPN$, and then about $\angle ONP + \angle OPN$? Why is this result false?

Since the conclusion is untrue, it is also wrong to suppose that there is a point $P$ on $AB$ such that $OP$ equals $ON$.

Therefore, because it is wrong to suppose that there is a point $P$ on $AB$ such that either $OP$ is less than $ON$, or $OP$ equals $ON$, it follows that, for every position of $P$ on $AB$ (except $N$), $OP$ is greater than $ON$, and so the original statement is true.

This method of proof is called "Reductio ad absurdum" or reduction to absurdity, because it consists in showing that all other alternatives are wrong (or absurd).

**Tangents**

Draw any circle centre $O$ and mark any point $A$ on it. Through $A$ draw the line $PAQ$ perpendicular to $OA$. Since $OA$ is the perpendicular from $O$ to $PQ$, it is the shortest distance from $O$ to $PQ$; therefore $A$ is the point on $PQ$ which is nearest $O$.

But $A$ is on the circle; therefore every other point on $PAQ$ is outside the circle, because its distance from $O$ is greater than the radius $OA$.

Thus $PAQ$ is a line such that it has one point $A$ on the circle and every other point outside the circle. This is what is meant when $PAQ$ is called a tangent to the circle; and we say that $PAQ$ touches the circle and that $A$ is the point of contact.

---

**Construction of Tangents**

The following constructions should be carried out by everyone, and their proofs discussed.

**Examples for Class Discussion**

1. Draw a circle, centre $O$, and take any point $A$ on it. Construct the tangent at $A$ to the circle.
   - Join $OA$. What must you do next, and why?

2. Draw a circle, centre $O$, and take any point $T$ outside it. Construct the tangents from $T$ to the circle.
   - Draw the circle on $OT$ as diameter, cutting the given circle at $P, Q$. How is this done? What do you know about $\angle TPO$ and why? What follows?
   - Why is $TQ$ a tangent?

3. Draw a straight line $XY$ and take a point $O$ outside it. Construct a circle, with $O$ as centre, to touch $XY$.
   - From $O$, draw the perpendicular $OA$ to $XY$. How is this done?
   - With centre $O$, radius $OA$, draw a circle. Why does this touch $XY$?
4. Draw a straight line $XY$ and take a point $A$ on it. Construct a circle, of radius $3\text{ cm}$, to touch $XY$ at $A$. At $A$, draw $AC$ perpendicular to $XY$. How is this done? From $AC$, cut off $AO$ equal to $3\text{ cm}$. With centre $O$, radius $OA$, draw a circle. Why does this touch $XY$?

Now draw another circle, also of radius $3\text{ cm}$, to touch $XY$ at $A$.

**EXERCISE XV. a**

[Use instruments in the following examples.]

1. Draw a circle of radius $2\text{ cm}$, centre $O$, and take points $A$, $B$, $C$ on the circle so that $\angle AOB = 130^\circ$, $\angle BOC = 15^\circ$ and $\angle COA = 80^\circ$. Draw the tangents at $A$, $B$, $C$ so as to form the triangle $XYZ$. Calculate the angles of $\triangle XYZ$.

2. Draw a circle of radius $4\text{ cm}$, centre $O$, and take points $A$, $B$, $C$ on the circle so that $\angle AOB = 40^\circ$, $\angle BOC = 55^\circ$ and $\angle COA = 95^\circ$. Draw the tangents at $A$, $B$, $C$ so as to form the triangle $XYZ$. Calculate the angles of $\triangle XYZ$.

3. Draw a circle of radius $3.5\text{ cm}$ and draw a diameter of this circle. At each end of the diameter draw a tangent. Why are the tangents parallel?

4. (i) Draw a circle of radius $2\text{ cm}$, centre $O$, and draw two diameters $AOB$, $COD$ so that $\angle AOC = 35^\circ$. Draw the four tangents at $A$, $B$, $C$, $D$. What can you say about the quadrilateral formed by the tangents? What are its angles? (ii) How can you construct a parallelogram of tangents with one angle $60^\circ$?

5. $T$ is a point outside a circle, centre $O$; construct the two tangents from $T$ to the circle, if the radius is $3\text{ cm}$ and $OT = 5\text{ cm}$.

If $P$, $Q$ are the points of contact, find by calculation and measurement the lengths of $TP$, $TQ$.

6. Repeat No. 5 if the radius is $5\text{ cm}$ and $OT = 7\text{ cm}$.

7. Draw a straight line $XY$ and take a point $A$ on it. Draw circles of radii $3.5\text{ cm}$ and $2.5\text{ cm}$ to touch $XY$ at $A$ on opposite sides of $XY$.

What is the distance between their centres?

---

8. Draw a straight line $AB$ and take a point $N$ on it. Draw circles of radii $4\text{ cm}$ and $2.5\text{ cm}$ to touch $AB$ at $N$ on the same side of $AB$.

What is the distance between their centres?

9. Draw $\triangle ABP$ so that $AB = 6.5\text{ cm}$, $AP = 5\text{ cm}$, $BP = 3\text{ cm}$; then construct the circle, centre $P$, which touches $AB$.

From $A$ and $B$, draw the other tangents $AH$, $BK$ to this circle. If $H$, $K$ are the points of contact, measure $AH$ and $BK$; what is their sum?

**Lengths of the Tangents**

Take any point $T$ outside a circle, centre $O$, and draw the tangents $TP$, $TQ$ from $T$, as in Fig. 444.

Copy and complete the following arguments, stating the reasons carefully.

In the triangles $TOP$, $TOQ$,

\[ OP = OQ \]

\[ OT \]

Also $\angle OPT$ and $\angle OQT$ are \[ \therefore \triangle TOP = \triangle TOQ \]

\[ \therefore TP = \]

The distance of $T$ from the point of contact $P$ (or $Q$) of the tangent $TP$ (or $TQ$) is called the **length of the tangent** from $T$ to the circle. It is unnecessary to say which tangent, $TP$ or $TQ$, is to be taken because we have just proved them equal. This result is stated as follows:

**The lengths of the tangents to a circle from any external point are equal.**

From the same congruent triangles, $\angle OTP = \angle OTQ$, and so $OT$ bisects $\angle PTQ$. This result may be stated:

**The line joining any point outside a circle to the centre of the circle bisects the angle between the tangents from the point to the circle.**
Examples for Class Discussion

The following questions refer to Fig. 444, where TP and TQ are the tangents from T to a circle, centre O.

1. If the radius is 4 cm. and if OT = 6 cm., find TP.
   The distance of T from the centre.
   Let TP = z cm. Copy and complete the following:
   By Pythagoras, since ∠OPT = 90°, because z = ...
   √z = ...; ∴ ∠PTQ = ...
   TP = ..., correct to nearest mm.

2. If the radius is 4 cm. and the length of the tangent from T is 5 cm., find the distance of T from the centre.
   Let OT = y cm. Copy and complete the following:
   By Pythagoras, since ∠OPT = 90°,
   y² = ...
   √y = ...; ∴ y = ...
   OT = ..., correct to nearest mm.

3. If ∠POQ = 130°, calculate ∠PTQ and ∠OTQ.
   The sum of the angles of quad. TPOQ is ...
   But ∠OPT = ... and ∠OQT = ...
   ∠PTQ + ∠POQ = ...
   But OT bisects ∠PTQ, ∴ ∠OTQ = ...

EXERCISE XV. b

1. In Fig. 445, SAT is a tangent to the circle, centre O.
   (i) If ∠TAB = 35°, find ∠OAB and ∠AOB.
   (ii) If ∠OAB = 50°, find ∠BAT and ∠AOB.
   (iii) If ∠AOB = 110°, find ∠OAB and ∠BAT.

2. In Fig. 444, TP, TQ are tangents to the circle, centre O.
   (i) If ∠POT = 65°, find ∠OTP.
   (ii) If ∠PTQ = 40°, find ∠POQ.

3. In Fig. 445, SAT is a tangent to the circle, centre O.
   (i) If OA = 5 cm. and OS = 13 cm., find SA.
   (ii) If OA = 8 in. and SA = 15 in., find SO.

4. In Fig. 446, O is the centre of the semicircle APB and PT is a tangent.

5. In Fig. 447, AB is a diameter of a circle and AT is a tangent.
   (i) If TA = 12 in. and TB = 3 in., find TP.
   (ii) If TO = 13 cm. and TP = 12 cm., find TA and TB.

6. Draw Fig. 444, where TP, TQ are tangents to the circle, centre O, and join PQ. If ∠OPQ = 25°, find ∠PTQ.

7. In Fig. 448, O is the centre of each circle, and the radii are 10 cm., 6 cm. Calculate the length of the chord PQ which touches the smaller circle.

8. Fig. 449 shows a triangle ABC with its sides touching a circle at P, Q, R. If AB = 7 cm., BC = 9 cm., CA = 5 cm., find the length of AQ. [Let AQ = x cm., then CP = CQ = (5 - x) cm.; also express AR, BR, BP in terms of x; then form an equation.]

9. In Fig. 449, the sides of ∆ABC touch the circle at P, Q, R. If BC = 2 in., CA = 1-4 in., AB = 2-2 in., find BP and AQ.
10. In Fig. 450, the sides of $\triangle ABC$, produced where necessary, touch the circle at $X, Y, Z$.
   (i) If $BC = 6 \text{ cm}, \ CA = 7 \text{ cm}, \ AB = 9 \text{ cm}$, find $BX$.
   (ii) If $BC = 2 \text{ in}, \ CA = 2.4 \text{ in}, \ AB = 2.6 \text{ in}$, find $AY$.

11. Fig. 451 shows a garden-roller resting against a step $AB$.
    If $AB = 9 \text{ in}, \ AP = 15 \text{ in} \text{ and } \angle BAD = 90^\circ$, find the diameter of the roller.
    [Draw $BN$ perpendicular to the radius $OP$.]

12. In Fig. 452, $PQ$ touches each of the circles, centres $A, B$,
    radii 12 cm, 5 cm; also $AB = 25 \text{ cm}$.
    $BN$ is drawn perpendicular to $AP$.
    (i) What sort of quadrilateral is $BNPQ$? Give reasons.
    (ii) What is the length of $PN$ of $AN$?
    (iii) Calculate the length of $BN$. What is the length of the "common tangent" $PQ$?

Circles touching each other

If two circles touch the same line at the same point, they are said to touch each other, and the line is called a common tangent.

Look at Fig. 453.

In Fig. 453 (i), the two circles touch the common tangent $XY$ at $N$, on opposite sides of $XY$; they are said to touch externally.

In Fig. 453 (ii), the two circles touch $XY$ at $N$, on the same side; they are said to touch internally.

In each figure, $AN$ and $BN$, being radii, are at right angles to the tangent $XNY$, and so are parts of the same straight line. Another way of putting this is to say:

If two circles touch each other, the line joining their centres (produced if necessary) goes through the point of contact.

If the contact is external, the distance between the centres is equal to the sum of the radii.

In Fig. 453 (i), $AB = AN + NB$.

If the contact is internal, the distance between the centres is equal to the difference of the radii.

In Fig. 453 (ii), $AB = AN - BN$.

CONSTRUCTION OF CIRCLES IN CONTACT

Examples for Class Discussion

1. Fig. 454 shows 3 circles, centres $A, B, C$, radii 4 cm, 3 cm, 2 cm, touching each other externally.

2. Fig. 455 shows 3 circles, centres $D, E, F$, radii 5 cm, 3 cm, 1.5 cm, touching each other.

(i) Which contacts are internal?
(ii) What are the lengths of $EF, FD, DE$? Give reasons.

(iii) Draw the figure accurately. First construct $\triangle DEF$ after that is done, draw the circles.
3. In Fig. 454, A, B, C are the centres of three circles touching each other externally. If BC = 6 cm., CA = 8 cm., AB = 9 cm., find the radii of the circles.

Let the radius of the circle, centre A, be \( z \) cm.

Since \( AB = 9 \) cm., radius of circle, centre B, is ....

Since \( AC = 8 \) cm., radius of circle, centre C, is ....

But \( BC = 6 \) cm., so form an equation for \( z \) and solve it.

**EXERCISE XV. C**

*Accurate figures are required only in numbers marked with an asterisk, e.g. 3*.

1. A, B (not marked) are the centres of the two tangent areas in Fig. 456.

   (i) What is \( AB \) if the radii are 3 cm. and 7 cm.?

   (ii) If \( AB = 12 \) cm., find the radii if one radius is twice the other.

![Fig. 456.](image)

2. C, D (not marked) are the centres of the two tangent areas in Fig. 457.

   (i) What is \( CD \) if the radii are 5 cm. and 11 cm.?

   (ii) If \( CD = 2 \) in., and one radius is 0-5 in., find the other.

   (iii) If \( CD = 2 \) in., and if one radius is 3 in., find the other. Is there more than one answer?

3*. Use your instruments to draw two circles,

   (i) radii 4 cm., 3 cm. touching externally;

   (ii) radii 5 cm., 3 cm. touching internally.

4*. Draw a line \( AB \), 6 cm. long, and a circle, centre A, radius 4 cm. Calculate the radii of the two circles which have 8 as centre and touch this circle. Also draw part of each circle.

---

5*. In Fig. 454, A, B, C are the centres of circles of radii 3.5 cm., 2.5 cm., 2 cm., touching one another externally. Find the lengths of the sides of \( \triangle ABC \), and draw an accurate figure.

6. \( AB \) is 4 cm.; two equal circles, with A and B as centres, touch one another. If a circle of radius 5 cm. touches each of these circles (i) externally, (ii) internally, how far is its centre from A?

7. In Fig. 458, \( AB = 2.3 \) in., \( BC = 1.9 \) in., \( AC = 2 \) in.; \( PQR \) is a circle, centre B, radius 1.1 in.

   Show that circles, centres A and C, can be drawn to touch the circle \( PQR \) and also touch each other.

![Fig. 458.](image)

8. In Fig. 458, \( AB = 5.6 \) cm., \( BC = 4.4 \) cm., \( CA = 4.8 \) cm., and B is the centre of circle \( PQR \). If the circle, centre A, radius AP, touches the circle, centre C, radius CQ, calculate the length of \( BP \).

9*. In Fig. 454, where A, B, C are the centres of the circles which touch one another, if \( AB = 4.6 \) cm., \( AC = 4.4 \) cm., \( BC = 3.8 \) cm., calculate the radii of the circles. Then draw the figure.

10*. In Fig. 455, D, E, F are the centres of circles of radii 5 cm., 3.4 cm., 1.2 cm., touching one another as shown. Calculate the sides of \( \triangle DEF \). Then draw the figure.

11*. \( AB \) is 1.6 cm.; and there are two circles, centres A, B, radii 4 cm., 2.4 cm. respectively. Represent the data on a neat, not accurate, figure. (i) Why do these circles touch? (ii) What can you say about the position of the centre C of a circle of radius 1 cm. which touches both circles?

   Finally draw the figure accurately.

12. In Fig. 459, \( AD = 7 \) cm., \( DC = 3 \) cm.; also \( AB \) and \( BC \) are quadrants (quarters) of circles touching at \( B \). Calculate the radius of each circle.
13*. Draw the pattern in Fig. 460 formed by three arcs, each of radius 4 cm., touching one another at A, B, C, and D. [It will help you to draw first a rough figure showing the whole of each circle and its centre.]

14*. Draw two circles of radii 3 cm. and 2 cm. so that their centres are 6 cm. apart. Draw a circle of radius 5 cm. to touch each of these circles. Make a separate figure for each different possible solution; the contacts may be internal or external.

**ANGLES BETWEEN A TANGENT AND A CHORD THROUGH THE POINT OF CONTACT**

**Examples for Class Discussion**

1. In Fig. 461, AP is a chord through the point of contact A of the tangent SAT, and AB is a diameter; also \( \angle PAT = 55^\circ \). Answer the following, with reasons:

   (i) How much is \( \angle PAB \)?
   (ii) How much is \( \angle APB \)?
   (iii) How much is \( \angle ABP \)?
   (iv) Mark any point Q on the (major) arc ABP and join QA, QP (see Fig. 462). How much is \( \angle AQP \)?
   (v) Mark any point R on the minor arc AP, and join RA, RP (see Fig. 462). How much is \( \angle ARP \)?

2. Repeat No. 1, taking \( \angle PAT \) equal to 72°.

3. In Fig. 463, AP is any chord through the point of contact A of the tangent SAT, and AB is a diameter.

Copy and complete, with reasons, the following:

(i) \( x^2 + y^2 = \ldots \); but \( \angle APB = \ldots \);
   \( y^2 + z^2 = \ldots \); \( \angle x^2 + y^2 - y^2 + 2z \);
   \( x^2 = \ldots \); i.e. \( \angle PAT = \angle APB \);
   but \( \angle AQP = \ldots \); \( \angle PAT = \angle AQP \).

(ii) In Fig. 462, \( \angle PAT + \angle PAS = \ldots \)
   and \( \angle AQP + \angle ARP = \ldots \);
   \( \angle PAT + \angle PAS = \angle AQP + \angle ARP \);
   but \( \angle PAT = \ldots \); \( \angle PAS = \ldots \).

You will get a clear idea of the facts established in these examples if you look at Fig. 464.

The chord AP makes with the tangent SAT the acute angle \( \angle PAT \) and the obtuse angle \( \angle PAS \), and we have shown that

(i) \( \angle PAT = \angle AQP \); (ii) \( \angle PAS = \angle ARP \).

If \( \angle PAT \) is on the right of the chord AP, it is equal to the angle in the segment on the left of the chord, \( \angle AQP \).

If \( \angle PAS \) is on the left of the chord AP, it is equal to the angle in the segment on the right of the chord, \( \angle ARP \).

Thus the angle which the tangent makes with any chord, drawn through the point of contact, on either side, is equal to the angle in the segment on the opposite side of the chord; this is called, for short, the alternate segment; and the property is stated as follows (note the plurals, angles, segments):

The angles which a tangent makes with any chord through the point of contact are equal to the angles in the alternate segments.

For reference: \( \angle \) in alt. seg.
EXERCISE XV.d

Give short reasons for each step.

[In Nos. 1-3, AT is a tangent and A is its point of contact.]

1. In Fig. 465, \( \angle BAT = 62^\circ, \angle CAS = 68^\circ \). Find the angles of \( \triangle ABC \).

Fig. 465.

2. In Fig. 465, \( \angle CAT = 110^\circ \). Find \( \angle ABC \).

3. In Fig. 466, find \( \angle APQ \) and \( \angle AQP \).

4. In Fig. 467, the sides of \( \triangle ABC \) touch the circle; if \( \angle XYZ = 55^\circ \) and \( \angle XZY = 48^\circ \), find \( \angle ABC \) and \( \angle BAC \). [First find the angles at X.]

5. In Fig. 467, the sides of \( \triangle ABC \) touch the circle; if \( \angle B = 56^\circ \) and \( \angle C = 72^\circ \), find \( \angle XYZ \) and \( \angle XZY \).

6. In Fig. 468, TP and TQ are tangents.
   (i) If \( \angle PTQ = 54^\circ \), find \( \angle PRQ \).
   (ii) If \( \angle PRQ = 112^\circ \), find \( \angle PTQ \).

7. In Fig. 468, TPX, TQY are tangents.
   If \( \angle XPR = 146^\circ \) and \( \angle YQR = 128^\circ \), find \( \angle PRQ \) and \( \angle PTQ \).

Fig. 468.

8. In Fig. 469, AH, BK, HK are tangents to the circle, centre O.
   If \( \angle OHK = 35^\circ \), find \( \angle OKH \) and \( \angle BOK \).

Fig. 469.

9. In Fig. 450, p. 224, if \( \angle ABC = 55^\circ \) and \( \angle ACB = 75^\circ \), find \( \angle ZXY \).

10. Draw a circle of radius 2 inches and mark any point A on it. Find points B, C on the circle so that the angles of \( \triangle ABC \) are 74°, 38°, 68°. [First draw the tangent TAS at A, compare No. 1.]
    The triangle \( \triangle ABC \) is said to be inscribed in the circle.

11. Draw a line \( AB \) 4 cm. long. Construct a segment of a circle \( APB \) so that \( \angle APB = 55^\circ \) (see Fig. 470).
    First make \( \angle BAT = 55^\circ \) and arrange so that \( AT \) is a tangent and \( AB \) a chord.
    Draw a line at \( A \) perp. to \( AT \); draw the perp. bisector of \( AB \). These cut at the required centre O.
    Give reasons and perform the construction. Measure the radius.

12. Draw a line \( AB \) 5 cm. long. Construct a segment of a circle \( AQB \) so that \( \angle AQB = 121^\circ \). Measure the radius. Use the method of No. 11.

Loci and their Use in Constructions

In a search for “treasure” buried in a field (see Appendix, Part I) two clues are required. In the same way, to fix the position of a point \( P \) in a plane, we must be told two facts about it. Thus, if one fact is that \( P \) is 2 in. from a given point \( A \), then \( P \) lies somewhere on the circle, centre \( A \), radius 2 in., and we call this circle the locus of \( P \) because it contains all the possible positions of \( P \), which fit this particular condition. Suppose we are also told that \( P \) is 1 in. from a given line \( BC \); then \( P \) lies somewhere on one of the 2 lines parallel to \( BC \), on either side of it, and 1 in. from it. This pair of lines is the locus of \( P \) which fits the second condition. To find the position of \( P \) which fits both conditions, we must take the point (or points) where the two loci intersect.
Centres of Circles touching Two Lines

In Fig. 471, the circle, centre A, touches the lines OP, OQ; \( \therefore OA \) bisects \( \angle POQ \) (see p. 221); \( \therefore A \) lies on the line bisecting \( \angle POQ \).

![Diagram showing circles touching two lines]

Fig. 471.

What can you say about the centres B, C, D of the other circles in Fig. 471?

The fact illustrated by Fig. 471 is expressed as follows:

If a circle touches two given (intersecting) straight lines, the locus of its centre is the pair of lines which bisect the angles between the given lines.

LOCI

EXERCISE XV. c (Oral)

1. State the locus of a point P moving in a given plane, subject to the following conditions:
   (i) A is a given point; \( AP = 3 \) cm.
   (ii) AB is a given line; \( P \) is 5 cm. from AB.
   (iii) C, D are given points; \( PC = PD \).
   (iv) OE, OF are given lines; \( P \) is equidistant from OE and OF, produced if necessary.

What are the answers if P moves in space, subject to these conditions?
CONSTRUCTIONS

EXERCISE XV.f

1. Draw a triangle ABC and construct a circle to touch the three sides and lie inside the triangle.

   (i) Since the circle is to touch BA and BC, on what line must its centre lie?

   (ii) But the circle is to touch CA and CB, so what can you say about its centre?

   (iii) If the centre is I, how can you find the radius? Now draw the circle.

   (iv) What do you know about the line IA? Complete the sentence, “the three lines which .... meet in a point”.

   This circle is called the \textit{inscribed circle} of the triangle ABC, or for short the \textit{in-circle}. Its centre I is called the \textit{in-centre} and its radius the \textit{in-radius}.

2. (i) Draw a triangle ABC; produce AB, AC to X, Y; construct a circle to touch BC, BX, CY.

   Use the method of No. 1 (see Fig. 472). This circle is called an \textit{escribed circle} of the triangle ABC, or for short an \textit{ex-circle}, and its centre I, is called an \textit{ex-centre}.

   For every triangle, there are \textit{three} ex-circles and \textit{one} in-circle.

   (ii) Construct the two other ex-circles of the triangle ABC.

3. Draw a triangle ABC, such that AB = 5 cm., BC = 6 cm., CA = 7 cm. \textit{Construct}

   (i) a point P on AC equidistant from AB and BC;

   (ii) a point Q on AC equidistant from B and C;

   (iii) two points R, S, 3 cm. from BC, and equidistant from A and C.

   (iv) two points G, H, 5 cm. from C, and equidistant from A and B.

4. Draw $\angle BAC$ equal to 70°. Construct all possible positions of a point, whose distances from the lines AB, AC, produced if necessary, are 3 cm., 4 cm. respectively.

5. Draw $\angle BAC$ equal to 55°. One end P of a line PQ, 3 cm. long, moves along AB, and $\angle QPA$ is always 60°. What is the locus of Q?

   Construct points H, K on AB, AC such that $\angle AHK = 60°$ and HK = 3 cm.

6. In Fig. 473, the units are cm. The point A and the line BC are given; draw them. After you have done so, construct the positions of P and Q. Measure DQ.

7. Draw an equilateral triangle of side 3 cm. Draw one of its ex-circles and measure its radius.

8. Draw a triangle whose sides are 6 cm., 8 cm., 10 cm.; construct its in-circle and measure its radius.

   Draw the circumcircle of this triangle. Where is its circumcentre? What is its circumradius?

9. Draw a line BC and take a point A, distant 1·3 in. from BC. Construct a circle of radius 1 in. to pass through A and to touch BC.

   (i) Why does the centre lie somewhere on a circle, centre A, radius 1 in.?

   (ii) Why does the centre lie somewhere on a line parallel to BC at a distance 1 in. from it?

   There are two circles satisfying the data; draw both. Measure the distance between their centres; find it also by calculation.

10. Draw a circle, centre O, radius 2 cm. and take any point A on it. Take a point B so that $\angle OAB = 145°$ and AB = 3 cm. Construct a circle to pass through B and to touch the given circle at A, and measure its radius. [Fig. 474 suggests the construction you must make to obtain the required centre P. Give reasons.]
FURTHER DEVELOPMENTS

11. Draw a circle of radius 2.5 in., and two radii OA, OB such that ∠BOA = 50°. Construct a circle to touch the given circle at A and to pass through the mid-point of OB.

12. Draw a line AB, 8 cm. long, and a semicircle having AB as diameter. Construct a circle of radius 1.5 cm. to touch AB and to touch the semicircle internally.

GENERAL STATEMENTS

Most of the examples in Ex. XV, g illustrate the "alternate segment theorem". It is suggested that Nos. 1, 3, 5, 7 should be taken orally. Any or all of the remainder may be omitted at a first reading.

EXERCISE XV, g

Give short reasons for each step.

1. In Fig. 475, where TA is a tangent, prove that ∠TCA = ∠TAB.

2. In Fig. 475, where TA is a tangent, if AC bisects ∠BAT, prove that AC = CB.

3. In Fig. 476, where AT is a tangent and PQ is a diameter, prove that ∠TAP + ∠APQ = 90°.

4. In Fig. 475, draw BK parallel to CA to meet the tangent TA, produced, in K; prove that ∠AKB = ∠ABC.

5. In Fig. 477, where AP, AQ are tangents, prove that ∠ABP = ∠ABQ.

6. In Fig. 477, where AP, AQ are tangents, prove that ∠PBQ = ∠PAQ.

7. In Fig. 478, PQ and TR are common tangents; prove that TP = TQ.

8. If in Fig. 475, AT is a tangent and BC a diameter, prove that ∠CTA + 2∠CAT = 90°.

9. In Fig. 479, the circles touch internally at A. Prove that PX is parallel to QY.

10. In Fig. 478, where PQ and TR are common tangents, prove that ∠PRQ = 90°.

11. In Fig. 480, O is the centre and BT is a tangent; prove that ∠AOP = 2∠ATB.

12. A circle is drawn inside a quadrilateral ABCD so that it touches all four sides, prove that AB + CD = BC + AD.
What follows if ABCD is a parallelogram!
CHAPTER XVI

USE OF SIMILAR TRIANGLES

General Tests

The tests for similar triangles were discussed in Chapter VIII, and some indication was given in Chapter XIV (see p. 211) of the way in which these tests can be "proved". Of the three general tests, the angle-test is much the most useful.

Map-making

When a surveyor makes a map of a country, he starts by measuring very carefully the length of a single selected straight-line, his "base-line". All the other measurements he makes out of doors are measurements of angles. The length of the base-line enables him to state the scale of his map; the angles fix the positions on the map of the landmarks observed.

Suppose the base-line is actually 2 miles long and is represented by the line $AB$, 1 inch long, in Fig. 481, then the scale of the map is "2 miles to the inch". To fix on the map the position of a church-tower $P$, the surveyor measures $\angle PAB$ and $\angle PDA$; he can then map $\triangle PAB$. Other landmarks $Q, R, S$, etc., are mapped in the same way.

In Fig. 482, the same base-line is represented by $A'B'$, $\frac{1}{4}$ inch long, so the scale of this map is "4 miles to the inch"; the same angle-measurements fix $P', Q', R'$, etc., as were used to fix $P, Q, R$, etc., in Fig. 481.

Figs. 481, 482 are therefore two maps (on different scales) of the same piece of country. The angle-test for similar triangles states that the shapes of the two maps are the same as each other and as the country so mapped.

Suppose the church-tower $P$ is 1$\frac{1}{4}$ miles from $A$, then the actual distance of $P$ from $A$ is $\frac{1}{4}$ of the actual distance of $B$ from $A$. Therefore on the map, in Fig. 481, $AP' = \frac{1}{2} AB$ and $AP = \frac{3}{4}$; on the map in Fig. 482, $A'P' = \frac{3}{4} A'B'$ and $A'B' = \frac{3}{4}$.

Thus $\frac{AP}{AB} = \frac{A'P'}{A'B'}$ and, in the same way, $\frac{BP}{B'A'} = \frac{B'P'}{B'A'}$.

There is another equally useful way of expressing these facts. If we compare one map directly with the other, we have, for the equiangular triangles $\triangle ABP, \triangle A'B'P'$,

$\frac{AP}{AB} = \frac{BP}{B'P'} = \frac{A'B'}{A'B'} = \frac{A'B'}{B'P'}$.

The angle-test, on which the map-maker relies, may be stated as follows:

If two triangles are equiangular, then the ratios of their corresponding sides are equal, and the triangles are similar.

It is obvious that two quadrilaterals can be equiangular to one another without being similar, e.g., two rectangles, 2 cm. by 1 cm. and 4 cm. by 1 cm. We can map $\triangle ABC$ if we know two (and $\therefore$ all) its angles; but we cannot map quad. $\triangle ABC$ if we only know three (and $\therefore$ all) its angles. With the given base line $AB$, two angle-measurements are needed to fix $P$, and two more to fix $Q$, that is, four independent measurements in all; three angle-measurements are not enough.

Naming Similar Triangles

The letters used in naming similar triangles should be arranged in corresponding orders. Thus, if in the triangles $\triangle ABC, \triangle PQR$, $\angle A = \angle Q$ and $\angle C = \angle P$, then $\angle B = \angle R$ and we say $\triangle ABC$ is similar to $\triangle QRP$, not to $\triangle QPR$. 
ANGLE TEST FOR SIMILAR TRIANGLES

EXERCISE XVI.\(a\)

1. Find the lengths of the unmarked lines in Fig. 483, not drawn to scale.

2. In Fig. 484, PQ is parallel to BC.
   (i) Why is \(\triangle APQ\) similar to \(\triangle ABC\)?
   (ii) If \(AP = 3\) cm., find \(AQ\) and \(PQ\).
   (iii) If \(AQ = 4\) cm., find \(AP\) and \(PQ\).
   (iv) If \(PQ = 3\) cm., find \(AP\) and \(AQ\).

3. In Fig. 485, KQ is parallel to BC.
   Prove that the two triangles are similar and put marks on your figure to show which pairs of sides correspond.
   (i) If \(AK = 6\)", find \(AQ\) and \(KQ\).
   (ii) If \(KQ = 3\)", find \(AQ\) and \(AK\).
   (iii) If \(AQ = 7-2\)", find \(AK\) and \(KQ\).

4. In Fig. 486, HK is parallel to BC.
   (i) If \(HK = 6\)", find \(KC\) and \(BC\).
   (ii) If \(BC = 10\frac{1}{2}\) in., find \(HK\).

5. The shadow of a tower is 60 feet long, and at the same moment the shadow of a 4-foot pole, held upright, is 5 feet long. How high is the tower?

6. A flagstaff, 20 ft. high, is set up on the top of a tower. The shadow of the flagstaff is 18 ft. long when the shadow of the tower and flagstaff is 72 ft. long. Represent the data on a rough figure and find the height of the tower.

7. Fig. 487 represents a light \(L\), hanging directly above the edge \(AB\) of a table; \(PQ\) is the shadow of \(AB\) on the floor. If \(LX = 5\) ft. and \(XY = 3\) ft., what can you say about the ratios (i) \(LA : AP\); (ii) \(LA : LP\); (iii) \(AB : PQ\)?
   If \(AB = 7\frac{1}{2}\) ft., how long is \(PQ\)?

8. In Fig. 487, \(AB\) is parallel to \(PQ\). What ratios are equal to \(\frac{AB}{PQ}\)?
   If \(AB = 9\) cm., \(PQ = 15\) cm., (i) find \(XY\) if \(LX = 6\) cm.;
   (ii) find \(LX\) if \(XY = 3\) cm.

9. How far in front of a pinhole camera must a man \(AB\), 6 ft. high, stand (see Fig. 488), in order that a full-length photograph may be taken on a film \(PQ\), 2\(\frac{1}{2}\) in. high, if \(PQ\) is 2\(\frac{1}{2}\) in. from the pinhole \(L\)?

10. Houses on opposite sides of a road are 40 ft. apart. How far back from a window 3 ft. wide must you be, so as to see (with one eye shut) a width of 27 ft. of the wall of the opposite house?

11. Fig. 489 shows a cone of height 15 in. and base-diameter 9 in. Find the diameter of a section \(PQ\) parallel to the base and 5 in. from it.

12. The diameter of the base of a cone is 8 in., and the diameter of a parallel section 1\(\frac{1}{2}\) in. from the base is 6 in. Find the height of the cone.
FURTHER DEVELOPMENTS

Ratio Tests

The two other general tests for similar triangles, given in Chapter VIII, are as follows:

If two triangles have an angle of one equal to an angle of the other, and if the ratios of the sides containing the equal angles are the same, then the triangles are similar.

Thus, in Fig. 490,

\[ \frac{\angle A}{\angle P} = \frac{AB}{AP} \text{ and if } \frac{AB}{AP} = \frac{PQ}{PR}, \]

then \( \triangle ABC \) is similar to \( \triangle PQR \).

If the ratios of the corresponding sides of two triangles are equal, then the triangles are similar.

Thus, in Fig. 490,

\[ \frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}, \text{ then } \triangle ABC \text{ is similar to } \triangle PQR. \]

The Fourth Proportional

If, in Fig. 490, \( \triangle ABC \) is similar to \( \triangle PQR \), the angles \( A \), \( B \) being equal to the angles \( P \), \( Q \) respectively, then

\[ \frac{AB}{PQ} = \frac{AC}{PR}. \]

If the lengths of \( AB \), \( AC \), \( PQ \) are given, the length of \( PR \) can be found from this relation; it is called the fourth proportional to the three given lengths \( AB \), \( AC \), \( PQ \).

By calculations, if the length of \( PR \) is \( x \) units, we have

\[ \begin{align*}
1.6 & = \frac{2.6}{x} ; \\
2.4 & = \frac{6}{x} \;
\end{align*} \]

\[ \therefore 1.6x = 2.6 \times 2.4 ; \]

\[ \therefore x = \frac{2.6 \times 2.4}{1.6} = 3.9. \]

EXERCISE XVI. b

1. In Fig. 492, \( AP = \frac{1}{2} AB \) and \( AQ = \frac{1}{2} AC \). What follows? Give reasons.

2. In Fig. 492, \( AP = \frac{1}{2} AB \) and \( AQ = \frac{1}{2} AC \). What follows? Give reasons.

3. In Fig. 492, if \( PQ \) is parallel to \( BC \), complete the equations, \( \frac{AB}{AC} = \frac{PQ}{BC} = \frac{AP}{BC} = \frac{AQ}{BC} = \frac{1}{2} \).

Name three lengths for which \( AB \) is the fourth proportional.

Name three lengths for which \( AC \) is the fourth proportional.

4. By drawing and measurement, find the fourth proportional to \( (i) \) 1-6, 2-4, 3-7; \( (ii) \) 14, 19, 31; \( (iii) \) 7-3, 6-1, 1-7.

Explain the construction for \( (iii) \). [Take as units: for \( (i) \) 1 inch, for \( (ii) \) 0-1 inch, for \( (iii) \) 1 cm.]
5. Find simple whole numbers proportional to 2·1, 2·8, 3·5. What can you say about the triangle whose sides are 2·1 cm., 2·8 cm., 3·5 cm.?  

6. Are two triangles similar if their sides are (i) 5, 6, 7 cm. and 7·5, 9, 10·5 cm.; (ii) 5, 6, 7 cm. and 10, 11, 12 cm.; (iii) 5, 6, 7 cm. and 10, 14, 12 cm.; (iv) 4, 7, 9 cm. and 4, 7, 9 in.?  

7. Fig. 493 represents an instrument for measuring the thickness of a plate. The two arms are equal and are pivoted at A. Also the distances of A from P, Q are one-fifth of the distances of A from B, C. What follows?  

8. By drawing and measurement, with 1 cm. as unit, find the value of x if:  

(i) \( \frac{2.7}{3.6} = \frac{3.5}{x} \); (ii) \( \frac{3}{3.8} = \frac{x}{4.7} \); (iii) \( \frac{x}{1.7} = \frac{4.3}{6} \)  

Check your answers algebraically.  

9. AB and AC are two given lengths. Construct a length PQ such that \( \frac{AB}{AC} = \frac{AP}{PQ} \). Take AB = 3 cm., AC = 4·2 cm. and measure PQ. The length PQ is called the third proportional to AB, AC.  

10. Find by drawing and measurement the third proportional to 45, 24.  

11. In Fig. 494, AB = 9 in., AC = 12 in., AP = 4 in., AR = 3 in. Why are there two similar triangles in the figure? Name them carefully. What is PR if BC = 6 in.?  

12. In Fig. 494, with the data of No. 11, if AK = 6\( \frac{2}{3} \) in., prove that \( \triangle AKB \) is similar to \( \triangle ABC \), and find BK.  

13. In Fig. 495, two lines, AB, CD cut at P. Why are there two similar triangles in the figure? Name them carefully. What angles are equal? What follows as to ACBD?  

14. Name carefully a second pair of similar triangles in Fig. 495, if AD and BC are joined. Give reasons.  

15. In Fig. 496, show that two pairs of similar triangles can be named without using any more letters. What angles are equal? What statement can you make about quad. PRSQ?  

Further Uses of Similar Triangles  
Most of the equiangular triangles in Ex. XVI a were formed by a pair of parallel lines cutting two intersecting lines. But there are many other important geometrical figures containing equiangular triangles. 

EXERCISE XVI. c  
1. Fig. 497 shows two chords AB, CD of a circle intersecting at P.  

(i) Give the reasons why the two triangles in the figure are similar; name them carefully.  

(ii) Copy and complete \( \frac{AP}{PD} = \frac{DA}{DB} \).  

(iii) If AP = 6 cm., DP = 3 cm., PC = 4 cm., find PB.  

(iv) If AP = 3", PD = 1.5", DA = 2", PB = 1.2", find PC, BC.
2. Fig. 498 shows two chords $AB$, $CD$ of a circle, meeting, when produced, at $Q$.
   (i) Give the reasons why the two triangles in the figure are similar; name them carefully.
   \[ \frac{AQ}{QD} = \frac{DA}{QD} \]
   (ii) Copy and complete \[ \ldots \ldots \ldots \]
   (iii) If $AQ = 6 \text{ cm}$, $BQ = 4 \text{ cm}$, $DQ = 3 \text{ cm}$, find $CQ$.
   (iv) If $AQ = 12 \text{ cm}$, $QD = 7.5 \text{ cm}$, $DA = 8 \text{ cm}$, $EQ = 9 \text{ cm}$, find $QC$, $BC$.

3. In Fig. 498, join $AC$ and $BD$; draw your own figure.
   (i) What triangle is similar to $\triangle AQC$? Name it carefully. Give reasons.
   (ii) Name two ratios equal to $AC : BD$.
   (iii) If $AQ = 9 \text{ cm}$, $CQ = 12 \text{ cm}$, $CD = 7.5 \text{ cm}$, $AC = 5 \text{ cm}$, find $AB$, $BD$.
   (iv) What triangle in Fig. 499 is similar to $\triangle PQB$. Give reasons.
   (v) If $PQ = 7 \text{ in}$, $QB = 10 \text{ in}$, $AB = 15 \text{ in}$, find $AC$. By using Pythagoras, find $PB$ to the nearest $\frac{1}{15} \text{ in}$. What does the ratio $BC : BP$ equal?

5. The triangle $ABC$ is right-angled at $A$, and $AN$ is an altitude.
   (i) Give reasons why $\triangle ANB$ is similar to $\triangle ACB$.
   (ii) Name carefully another triangle similar to $\triangle ACB$.
   (iii) Copy and complete, $\triangle ABC$ is similar to $\triangle \ldots \ldots \ldots$ and to $\triangle \ldots \ldots \ldots$.
   (iv) Copy and complete in two different ways, $\frac{BC}{CA} = \frac{AB}{\ldots \ldots \ldots}$
   (v) If $AB = 8 \text{ cm}$, and $AC = 6 \text{ cm}$, prove that $BC = 10 \text{ cm}$ and then find $AN$ and $BN$.
   (vi) If $AN = 3 \text{ in}$, and $BN = 4 \text{ in}$, prove that $AB = 5 \text{ in}$ and then find $AC$ and $CN$.

6. In Fig. 500, $PT$ is a tangent.
   (i) Name carefully a triangle similar to $\triangle PBT$. Give reasons.
   (ii) Name two ratios equal to $ST : TA$.
   (iii) If $PT = 6 \text{ cm}$, $TB = 4.5 \text{ cm}$, $BP = 4 \text{ cm}$, find $TA$ and $AB$.
   (iv) If $PA = l \text{ inches}$ and $PB = m \text{ inches}$, find $PT$ in terms of $l$, $m$.
   (v) If $TA = c \text{ inches}$ and $TB = d \text{ inches}$, find the ratio $\frac{PA}{PB}$ in terms of $c$, $d$.

7. $AB$ is the diameter of a semicircle $APB$, and $PN$ is the perpendicular from $P$ to $AB$.
   (i) Name carefully two triangles similar to $\triangle ANP$. Give reasons.
   (ii) If $AN = 8 \text{ cm}$, and $NB = 4.5 \text{ cm}$, find $PN$, $AP$, $BP$.

8. Fig. 501 represents a sphere of radius $1.5 \text{ in}$, resting inside a conical funnel. The diameter $PQ$ of the open end is $7.2 \text{ in}$; find the distance of the centre $O$ of the sphere from the vertex $V$ of the funnel.

9. A nut, 1 inch in diameter, is placed between the arms $VP$, $VQ$ of a pair of nutcrackers, with its nearer end $K$, 1 inch from the apex $V$ (ep. Fig. 501). Each of the arms $VP$, $VQ$ is 6 inches long. Find $PQ$.

10. $ABC$ is a triangle such that $AB = AC$; and $P$ is a point on $AC$ such that $BP = BC$.
    (i) Name carefully two similar triangles. Give reasons.
    (ii) If $AB = 9 \text{ cm}$ and $BC = 6 \text{ cm}$, find $CP$.
    (iii) If $AB = l \text{ inches}$ and $BC = m \text{ inches}$, find $CP$ in terms of $l$, $m$. 

[Diagram Fig. 501]
RATIOS AND PRODUCTS

In algebra, if \( \frac{a}{b} = \frac{c}{d} \), then \( ad = bc \).

From the two equal ratios, we obtain two equal products. Conversely, if \( ad = bc \), then \( \frac{a}{b} = \frac{c}{d} \) and \( \frac{a}{c} = \frac{b}{d} \). Thus the statements,

\[
9 \times 8 = 12 \times 6, \quad \frac{9}{12} = \frac{6}{8} \quad \frac{9}{12} = \frac{6}{8},
\]

are all equivalent to one another.

In geometry, a ratio is represented most simply by the comparison of the lengths of two lines, and a product most simply by the area of a rectangle.

\[\text{Fig. 502.} \quad \text{Fig. 503.}\]

In Fig. 502, the ratios \( \frac{9}{12} \) and \( \frac{6}{8} \) are represented by \( \frac{AH}{AB} = \frac{AK}{AC} \). If \( \frac{AH}{AB} = \frac{AK}{AC} \), then \( HK \) is parallel to \( BC \), and conversely.

In Fig. 503, the products \( 9 \times 8 \), \( 12 \times 6 \) are represented by the areas of rect. \( ACXY \), rect. \( ABPQ \), where \( AY = AH \) and \( AQ = AK \).

If \( \frac{AH}{AB} = \frac{AK}{AC} \), then rect. \( ACXY \), rect. \( ABPQ \) are equal in area, and conversely.

Thus a statement that two ratios are equal can be transformed into a statement that two rectangles are equal in area, and vice versa.

Notation

Be careful not to confuse the notation for a ratio with that for the area of a rectangle.

\[\text{EF : GH means the ratio of \( \text{EF} \) to \( \text{GH} \); it is measured by the quotient \( \frac{\text{EF}}{\text{GH}} \).}\]

\[\text{KL} : \text{MN} \text{ means the area of the rectangle " contained by } \text{KL} \text{ and } \text{MN} \text{ " (see Chapter XI, p. 172), that is, a rectangle such that two adjacent sides equal } \text{KL} \text{ and } \text{MN}; \text{ it is measured by the product, } \text{KL} \times \text{MN}.\]

EXERCISE XVI. d (Oral)

1. Sketch a figure to illustrate \( \frac{7}{10} = \frac{14}{20} \). What equal products does this give? Illustrate them.

2. Sketch a figure to illustrate \( 6 \times 10 = 15 \times 4 \). What equal ratios does this give (two answers)? Illustrate them.

3. In \( \triangle ABC \), a line parallel to \( AC \) meets \( BA, BC \) at \( X, Y \).
   (i) What ratio equals \( \frac{BY}{AX} \)? What areas are therefore equal?
   (ii) Copy and complete, \( \frac{XY}{AX} = \frac{BY}{AX} \). What areas are therefore equal?

4. Sketch two rectangles \( ABCD, PQRS \) which are equal in area, but not congruent. Name ratios equal to \( \frac{AB}{PQ}; \ \frac{AB}{PS}; \ \frac{PQ}{AD} \).

5. Sketch a rectangle \( ABCD \) equal to a square \( EFGH \). Represent the data (i) in product form, (ii) in ratio form (two ways).

6. What can you say about the rectangles \( ABCD, KLMN \) with the following data:
   (i) \( AB = KN \); (ii) \( DA = AB \);
   (iii) \( AB : BC = KL : LM \); (iv) \( AB \cdot AD = KL \cdot KN \).

7. In Fig. 497, p. 245, name a ratio equal to \( \frac{AP}{PD} \); give reasons. What areas are therefore equal?
8. In Fig. 498, p. 246, name a ratio equal to \( \frac{QA}{QC} \); give reasons. What areas are therefore equal?

9. In Fig. 500, p. 247, where \( PT \) is a tangent, name a ratio equal to \( \frac{PA}{PT} \); give reasons. What areas are therefore equal?

10. \( AN \) is an altitude of \( \triangle ABC \), and \( \angle BAC = 90^\circ \). Explain why \( \frac{BN}{BA} = \frac{BA}{BC} \). What areas are therefore equal? Illustrate them by a rough figure. How does Euclid’s proof of Pythagoras’ theorem show they are equal?

**Rectangle Properties**

Several of the results obtained in the last exercise are useful general properties.

**The Right-angled Triangle**

In Fig. 504, \( AN \) is an altitude of \( \triangle ABC \), right-angled at \( A \).

(i) Why is \( \triangle ABN \) similar to \( \triangle CBA \)?

(ii) Why is \( \angle ANB \) similar to \( \angle ACB \)?

**Mean Proportional**

In Fig. 504, where \( \angle BAC \) and \( \angle ANB \) are right angles, we have seen that

\[
\frac{BN}{BA} = \frac{BA}{BC}
\]

or

\[
BN : BA = BA : BC.
\]

As \( BA \) appears twice in the middle of this proportion, it is called the mean proportional between \( BN \) and \( BC \).

In algebra, if \( a : x = x : b \) or \( \frac{a}{x} = \frac{x}{b} \) or \( ab = x^2 \), then \( x \) is the mean proportional between \( a \) and \( b \).
Chord and Tangent

In Fig. 507, the tangent at any point T and any chord AB meet at P.

Why is ΔAPT similar to ΔTPB? What does \( \frac{PA}{PT} \) equal?

Hence \( PA \cdot PB = PT^2 \).

EXERCISE XVI. e

[In this exercise, assume the facts stated on pp. 250-252.]

1. AN is an altitude of ΔABC and \( \angle BAC = 90^\circ \). If BN = 16 cm. and NC = 9 cm., find, as shortly as possible, the lengths of AN, AB, AC.

2. In Fig. 508, BC is the diameter of the semicircle. If BN = 1.44 in. and CN = 0.25 in., find, as shortly as possible, the lengths of AN, AB.

3. Construct Fig. 508, making BC = 4.1 cm., BN = 2.9 cm. Hence find by measurement the mean proportional between 2.9 and 4.1. Check your answer arithmetically. Measure AN; between what lengths is AN the mean proportional? Check arithmetically.

4. Construct Fig. 508, making BN = 2.8 cm., NC = 1 cm. Hence find by measurement \( \sqrt{2.8} \), and check arithmetically.

Find also, from the same figure, \( \sqrt{3.8} \) and check.

5. Calculate the mean proportional between (i) 9 and 16, (ii) 2 and 3, (iii) 1 and 7, (iv) 3 and 4.

Draw a rectangle 4 cm. long, 3 cm. wide and then construct a square of equal area.

6. Find \( \sqrt{43} \) by drawing and measurement. Do not construct the mean proportional between 1 and 43, as it is difficult
to do this with reasonable accuracy. Take numbers closer together; since \( 43 + 5 = 8 \cdot 6 \), it will do to take 5 and 8.6.

7. Draw a rectangle 7 cm. long, 3 cm. wide. Construct a square of equal area and measure its side.

8. Draw an equilateral triangle of side 5 cm. Construct a rectangle equal in area to the triangle. Then construct a square equal in area to this rectangle, and measure its side.

9. Make a rough copy of Fig. 506 (i), p. 251.

(i) If AP = 6 in., PB = 2 in., PD = 3 in., find PC.
(ii) If AB = 11 cm., PB = 3 cm., PD = 4 cm., find CD.
(iii) If AP = 12 in., PB = 3 in., CP = PD, find CD.

10. Make a rough copy of Fig. 506 (ii), p. 251.

(i) If CD = 8 in., DP = 4 in., BP = 6 in., find AB.
(ii) If AP = \( \frac{1}{2} \) CP and BP = 2 in., find DP.

11. Make a rough copy of Fig. 507, p. 252.

(i) If AB = 5 in., BP = 4 in., find PT.
(ii) If PT = 12 in., PB = 9 in., find AB.
(iii) If PA = 8 cm., PB = 2 cm., AT = 7 cm., find BT.
(iv) If PA = 9 cm., AT = 6 cm., BT = 5 cm., find BP.

12. In Fig. 509, O is the centre of the circle, and PT is a tangent.

(i) If AP = 9 cm., AB = 5 cm., OP = 6.5 cm., find PT; then find the radius OT.
(ii) Why is \( PA \cdot PB \) always equal to \( OP^2 - \text{(radius)}^2 \)?

13. In Fig. 510, O is the centre of the circle, and OPD is a chord drawn perpendicular to OP.

(i) What do you know about CP and PD?
(ii) If AP = 8 cm., PB = 4 cm., OP = 2 cm., find CP and radius OC.
(iii) Why is \( AP \cdot PB \) always equal to \( \text{(radius)}^2 - OP^2 \)?
14. The roadway BAC of a bridge (see Fig. 511), is a circular arc, resting on supports B, C at the same level; the highest point A of the road is 4 feet above BC, and BC is 8 yards. Find the diameter of the arc.

![Figure 511](image)

![Figure 512](image)

15. In Fig. 511, AB = AC = 10 cm., AP = 6 cm. Find the length of the diameter of the circle ABC.

**GENERAL STATEMENTS**

**EXERCISE XVI.**

*Give short reasons for each step.*

1. In Fig. 512, the circles touch at A; prove that $\frac{AP}{AX} = \frac{AQ}{AY}$.

   What rectangle property follows? [Draw the tangent at A.]

2. The diagonals of a cyclic quadrilateral ABCD intersect at P; prove that $AD \cdot PC = BC \cdot PD$.

3. In Fig. 513, PX, PY are tangents; prove that $PX = PY$.

4. If in Fig. 513 where PX is a tangent, $PD = 2PX$, prove that $CD = 3CP$.

![Figure 513](image)

![Figure 514](image)

5. In Fig. 514, PX, PY are parallel to AC, AD; prove that $BX = BY$.

   What follows?

6. In Fig. 515, AQR bisects $\angle BAC$; prove that $\frac{AQ}{AB} = \frac{AC}{AR}$.

   What rectangle property follows?

7. ABC is any triangle inscribed in a circle; AD is an altitude of the triangle; AP is a diameter of the circle. Prove that $\frac{AP \cdot AD}{AB \cdot AC}$.

   *Any or all of the rest of this exercise may be omitted at a first reading.*

8. AN is an altitude of the triangle ABC. If $\angle BAC = 90^\circ$, and if $AB = 2AC$, prove that $BN = 4CN$.

9. In Fig. 516, AD bisects $\angle BAC$; prove that $\frac{BD}{BA} = \frac{DC}{AC}$.

   [Draw CP parallel to DA and first prove $AP = AC$.]

![Figure 516](image)

![Figure 517](image)

10. In Fig. 517, AD bisects $\angle BAC$ externally, that is, AD bisects $\angle EAC$; prove that $\frac{BD}{BA} = \frac{DC}{AC}$.

   [Draw CP parallel to DA.] Can you use exactly the same words for proving No. 9 as for proving No. 10? What is the object of inserting the point E in Fig. 516?

11. In $\triangle ABC$, $AB = 6$ cm., $BC = 5$ cm., $CA = 4$ cm.; the internal and external bisectors of $\angle BAC$ cut $BC$ and $BC$ produced at P and Q respectively. Use the facts in Nos. 9, 10 to find the lengths of BP and BQ.

12. In $\triangle ABC$, $AB = 4$ cm., $BC = 3$ cm., $CA = 5$ cm.; what do you know about $\angle ABC$? (The internal bisector of $\angle ABC$ meets $AB$ at D. Use No. 9 to find the length of BD. Then find CD to the nearest mm.)
CHAPTER XVII

TRIGONOMETRY OF THE RIGHT-ANGLED TRIANGLE

The Tangent of an Angle

Draw two equal angles, A and B, and from points P, Q on one arm of each angle draw perpendiculars PM, QN to the other arm, as in Fig. 518.

Then the triangles AMP, BNQ are equiangular and therefore similar.

\[
\frac{MP}{AM} = \frac{NQ}{BN}
\]

\[
\therefore \text{ the value of each of these ratios depends only on the size of the angle } \angle PAM; \text{ it does not depend on the size of the triangle.}
\]

For example if \( \angle A = 39^\circ \) and \( \angle B = 39^\circ \), we find by measurement that \( \frac{MP}{AM} \) and \( \frac{NQ}{BN} \) each equal 0-81 approximately.

The ratio \( \frac{MP}{AM} \) is called the tangent of the angle \( \angle PAM \) and is written \( \tan \angle PAM \).

In the triangle \( \triangle PAM \), the side \( PM \), opposite \( \angle PAM \), is called the "opposite side". Of the sides \( AP, AM \), which contain \( \angle PAM \), \( AP \) is the "hypotenuse" and \( AM \) is called the "adjacent side, not hypotenuse" or, for short, the "adjacent side".

The general definition is as follows:

If a perpendicular is drawn from any point in either arm of an angle \( \theta \) to the other arm,

\[
\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}
\]

The tangents of acute angles have been calculated and are given in books of tables. Their approximate values can be found by measurement as follows:

- Draw a circle, centre O, of unit radius (see Fig. 519); draw the tangent AT at any point A on the circle. Draw a line OP making any given angle \( \theta \) with OA and cutting AT at P. Then \( \tan \theta = \frac{AP}{OA} \). If \( OA = 1 \) in. and \( AP = t \) in., then \( \tan \theta = \frac{t}{1} = t \). The name "tangent of an angle" is derived from this connection with a tangent to a circle.

Examples for Class Discussion

1. Find, by drawing, the values of \( \tan 25^\circ \), \( \tan 40^\circ \), \( \tan 50^\circ \), \( \tan 65^\circ \). It saves time to use squared paper.

Compare the results with the values given in the tables.

2. Find, by drawing, the angles whose tangents are (i) \( 1 \); (ii) 1-8. Compare the results with the angles found from the tables.

3. In Fig. 518, what is \( \tan \angle PAM \) if \( PM = 4-5 \) cm. and \( AM = 5 \) cm.? Use the tables to find \( \angle PAM \) to the nearest degree.

Repeat this (i) if \( PM = 8 \) cm., \( AM = 5 \) cm.; (ii) if \( PM = 4-1 \) in., \( AM = 2 \) in.

4. In Fig. 518, if \( AM = 10 \) cm. and \( \angle PAM = 35^\circ \), copy and complete the following:

\[
\tan \angle PAM = \frac{PM}{AM} = \tan \theta \] ; \( \therefore PM = \ldots \ldots \ldots \); \( \therefore \angle PAM = \ldots \ldots \ldots \)

Repeat this (i) if \( AM = 3 \) in. and \( \angle PAM = 73^\circ \); (ii) if \( AM = 4 \) in. and \( \angle PAM = 42^\circ \); but \( \tan 35^\circ = 0-7002 \); \( \therefore PM = \ldots \ldots \ldots \)
5. In Fig. 518, if $PM = 10$ cm. and $\angle PAM = 33^\circ$, find $AM$.

Let $AM = x$ cm., then $\tan 33^\circ = \frac{10}{x}$;

$\therefore \ x \times 0.6494 = 10 \therefore x = \frac{10}{0.6494}.$

Now find $x$ by long division.

*But it is easier to say*

$\tan \angle PAM = \tan 33^\circ = 0.574 ; \therefore x = \frac{10}{0.574} = 15.4$ cm.

**Detailed Use of Tables**

The general method of obtaining from the tables (i) the tangent of a given angle, (ii) the angle whose tangent is given, should be explained orally.

**EXERCISE XVII. a (Oral)**

1. Read off from the tables the values of
   (i) $\tan 40^\circ$; (ii) $\tan 50^\circ$; (iii) $\tan 65^\circ$; (iv) $\tan 82^\circ$;
   (v) $\tan 24^\circ 30'$; (vi) $\tan 37^\circ 24'$; (vii) $\tan 49^\circ 54'$;
   (viii) $\tan 63^\circ 12'$; (ix) $\tan 64^\circ 2'$; (x) $\tan 28^\circ 40'$;
   (xi) $\tan 53^\circ 57'$; (xii) $\tan 46^\circ 29'$.

2. Find from the tables, in degrees and minutes, the angles whose tangents are
   (i) 0.3249; (ii) 1.1504; (iii) 2.4751; (iv) 1;
   (v) 0.3172; (vi) 0.804; (vii) 1.2046; (viii) 3.4197;
   (ix) 0.3647; (x) 0.3749; (xi) 0.3732; (xii) 0.3685;
   (xiii) 1.0147; (xiv) 1.4677; (xv) 2.0323; (xvi) 2.5527.

3. Find from the tables, in degrees and minutes, the angles whose tangents are
   (i) 0.5; (ii) 1.5; (iii) 2.75; (iv) 10; (v) $\frac{1}{2}$;
   (vi) $\frac{1}{3}$; (vii) $\frac{3}{2}$; (viii) $\frac{3}{4}$; (ix) $\frac{1}{3}$; (x) $\sqrt{3}$.

**EXERCISE XVII. b**

1. In Fig. 520, $\angle A = 90^\circ$. Write down the value of $\tan B$, and then use the tables to express $\angle B$ in degrees and minutes to the nearest minute.

2. In Fig. 521, $\angle A = 90^\circ$. Find the length of $AB$.

3. In Fig. 522, $\angle D = 90^\circ$. Find $\angle CAB$ in degrees and minutes.

4. In Fig. 523, $\angle D = 90^\circ$. Find the length of $AB$. 

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**THE RIGHT-ANGLED TRIANGLE**

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FURTHER DEVELOPMENTS

5. From a point on the ground 100 yards from a tower the angle of elevation of the top of the tower is $32^\circ$. Find the height of the tower in feet.

6. The shadow of a vertical pole 12 ft. high is 15 ft. long; find the altitude of the sun.

7. A ladder, leaning against a vertical wall, makes an angle of $21^\circ$ with the wall, and the foot of the ladder is 6 ft. from the wall. How high up the wall does the ladder reach?

8. $AD$ is an altitude of $\triangle ABC$, and $AB = AC$.
   (i) If $B = 64^\circ$ and $BC = 8$ cm., find $AD$.
   (ii) If $B = 56^\circ$ and $AD = 10$ cm., find $BC$.

9. What is the angle of elevation of the top of a spire, 270 ft. high, from a point on the ground 150 yd. from the foot of the tower?

10. A chord of a circle is 4 in. long and subtends an angle of $110^\circ$ at the centre. Find the distance of the chord from the centre.

11. TP, TQ are the tangents from T to a circle of radius 4 cm. Find the length of each tangent if $\angle PTQ = 50^\circ$.

12. The pole of a bell tent is 8 ft. high, and the diameter of the base of the tent is 14 ft. What angle does the slant side of the tent make with the ground?

13. $ABC$ is a triangle such that $AB = AC$, $\angle BAC = 40^\circ$ and $BC = 6$ cm. Find the area of $\triangle ABC$.

14. The diagonals of a rhombus are 6 cm. and 8 cm. long. Find the angles of the rhombus.

The Sine and Cosine of an Angle

Draw the same figure as on p. 256, now called Fig. 524.

If $\angle A = \angle B$, the triangles $\triangle AMP$, $\triangle BNP$ are similar.

$\therefore \frac{MP}{AP} = \frac{NQ}{BQ}$

$\therefore$ the value of each of these ratios depends only on the size of the angle $\angle PAM$, not on the size of the triangle.

The ratio $\frac{MP}{AP}$ is called the sine of the angle $\angle PAM$ and is written $\sin \angle PAM$.

We also know that $\frac{AM}{BN}$, $\frac{AP}{BQ}$, so the value of each ratio depends only on the size of the angle $\angle PAM$.

The ratio $\frac{AM}{AP}$ is called the cosine of the angle $\angle PAM$ and is written $\cos \angle PAM$.

The general definitions are as follows:

If a perpendicular is drawn from any point in either arm of an angle $\theta^\circ$ to the other arm,

\[
\sin \theta^\circ = \frac{\text{opposite side}}{\text{hypotenuse}}
\]

\[
\cos \theta^\circ = \frac{\text{adjacent side}}{\text{hypotenuse}}
\]

Since the hypotenuse is greater than either of the other two sides, neither the sine nor the cosine of an angle can exceed the value, one.

Also, if in Fig. 524, $\angle PAM = \theta^\circ$, then $\angle PAM = 90^\circ - \theta^\circ$.

But $\frac{AM}{AP}$ equals $\cos \angle PAM$ and also equals $\sin \angle APM$.

\[\therefore \cos \theta^\circ = \sin (90^\circ - \theta^\circ)\]

In words, the cosine of any angle is equal to the sine of its complement; cosine is short for complement-sine, i.e. sine of complement. Thus $\cos 70^\circ = \sin 20^\circ$, $\cos 40^\circ = \sin 50^\circ$, etc.

Approximate values of sines and cosines of acute angles can be found by measurement as follows:

Draw a circle, centre $O$, of unit radius (see Fig. 525); draw a radius $OA$ and then a line making any given angle $\theta^\circ$ with $OA$ and cutting the circle at $Q$; draw $QN$ perpendicular to $ON$.
If the radius is 1 in., $OQ = 1$ in.; if $ON = x$ in. and $NQ = y$ in.,

then

$$\sin \theta = \frac{NQ}{OQ} = \frac{y}{1} = y;$$

and

$$\cos \theta = \frac{ON}{OQ} = \frac{x}{1} = x.$$

It saves time to use squared paper: take 5 inches as unit, and draw only a quadrant of the circle.

The sine, cosine and tangent are called Trigonometrical Ratios, and in learning how to use them, you are learning "Trigonometry".

Examples for Class Discussion

1. Draw on squared paper a quadrant of a circle of radius 5 inches. Use it to find the values of $\sin 25^\circ$, $\cos 25^\circ$, $\sin 39^\circ$, $\cos 39^\circ$, $\sin 51^\circ$, $\cos 51^\circ$.

What do you think are the values of $\sin 90^\circ$, $\cos 90^\circ$; $\sin 0^\circ$, $\cos 0^\circ$?

Compare all the results with the values given in the tables.

2. Find, by drawing, the angle (i) whose sine equals $\frac{1}{2}$, (ii) whose cosine equals $\frac{1}{2}$. Check by using tables.

3. In Fig. 524, what is $\sin PAM$ if $PM = 4.7$ in. and $AP = 10$ in.? Use the tables to find $\angle PAM$ to the nearest degree.

Repeat this (i) if $PM = 1.7$ in., $AP = 2$ in.;
(ii) if $PM = 2$ cm., $AP = 3$ cm.

4. In Fig. 524, if $AP = 2$ in. and $\angle PAM = 43^\circ$, copy and complete the following:

$$\frac{PM}{\sin \theta} = \frac{AM}{\cos \theta} = 1.36 \text{ in.}$$

Repeat (i) if $AP = 6$ cm. and $\angle PAM = 54^\circ$;
(ii) if $AP = 10$ cm. and $\angle PAM = 72^\circ$.

Detailed Use of Tables

The sine and cosine tables are used in the same way as the tangent tables. But because the cosine of an angle decreases when the angle increases, the numbers in the difference columns of the cosine tables must be subtracted.

EXERCISE XVII. c (Oral)

1. Read off from the tables the values of
   (i) $\sin 22^\circ$; (ii) $\sin 30^\circ$; (iii) $\sin 68^\circ$; (iv) $\sin 88^\circ$;
   (v) $\sin 25^\circ 24'$; (vi) $\sin 45^\circ 42'$; (vii) $\sin 74^\circ 12'$;
   (viii) $\sin 24^\circ 44'$; (ix) $\sin 55^\circ 5'$; (x) $\sin 30^\circ 41'$.

2. Read off from the tables the values of
   (i) $\cos 22^\circ$; (ii) $\cos 30^\circ$; (iii) $\cos 68^\circ$; (iv) $\cos 88^\circ$;
   (v) $\cos 34^\circ 36'$; (vi) $\cos 63^\circ 42'$; (vii) $\cos 63^\circ 44'$;
   (viii) $\cos 28^\circ 15'$; (ix) $\cos 55^\circ 17'$; (x) $\cos 48^\circ 46'$.

3. Find from the tables, in degrees and minutes, the angles whose sines are
   (i) 0.5150; (ii) 0.8290; (iii) 0.6639; (iv) 0.7169;
   (v) 0.3162; (vi) 0.9358; (vii) 0.9; (viii) 0.4504.

4. Find from the tables, in degrees and minutes, the angles whose cosines are
   (i) 0.2419; (ii) 0.5736; (iii) 0.7955; (iv) 0.9354;
   (v) 0.3719; (vi) 0.5573; (vii) 0.8276; (viii) 0.9504;
   (ix) 0.4908; (x) 0.8194; (xi) 0.8237; (xii) 0.4902.

EXERCISE XVII. d

1. In Fig. 526, the triangles are right-angled. Why is this?
Write down as fractions the sine, cosine, tangent of each marked angle.

Fig. 526.
Nos. 2-3 refer to Fig. 327. Write the fractions as trigonometrical ratios, in more than one way when possible:

2. (i) \( \frac{AC}{BC} \), (ii) \( \frac{PQ}{PR} \), (iii) \( \frac{EF}{YX} \), (iv) \( \frac{XY}{YF} \), (v) \( \frac{QR}{PR} \)

3. (i) \( \frac{AC}{AB} \), (ii) \( \frac{YZ}{XZ} \), (iii) \( \frac{EG}{GF} \), (iv) \( \frac{AB}{BC} \), (v) \( \frac{QR}{QP} \)

![Fig. 327.](image)

Nos. 4-6 refer to Fig. 327.

4. (i) \( BC = 10 \), \( \angle C = 25^\circ \), find \( AB, AC \); (ii) \( PR = 2 \), \( \angle R = 21^\circ 18' \), find \( PQ, QR \).

5. \( EF = 5 \), \( \angle E = 73^\circ 20' \), find \( EG, GF \).

6. \( XY = 10 \), \( \angle X = 24^\circ \), find \( YZ, XZ \).

7. \( B \) is 5000 yards N. 28° E. from \( A \). How far is \( B \) (i) East of \( A \), (ii) North of \( A \) ?

8. The string of a kite is 600 ft. long and makes an angle of 37° with the horizontal. Find the height of the kite.

9. In Fig. 327, solve the following: (i) \( BC = 10 \), \( BA = 5 \), find \( \angle C \); (ii) \( PQ = 3 \), \( PR = 5 \), find \( \angle P \); (iii) \( EG = 4 \), \( GF = 8 \), find \( \angle E \); (iv) \( XY = 25 \), \( XZ = 40 \), find \( \angle X \).

10. A ladder, 12 ft. long, leans with one end against the wall and the other end on the ground 3 ft. from the wall. What angle does the ladder make with the wall?

11. A chord of a circle of radius 6 cm. is 5 cm. long. Calculate the angle the chord subtends at the centre.

12. The legs of a pair of dividers are each 12 cm. long and they are opened to an angle of 33°. Find the distance between their points.

---

13. A regular pentagon is inscribed in a circle of radius 5 cm. What is the length of its side?

14. The slant edge of a cone is 4 in. and the diameter of the base is 5 in. Find the vertical angle of the cone.

15. \( ABCD \) is a parallelogram; \( AB = 4^\circ \), \( AD = 5^\circ \), \( \angle BAD = 51^\circ \). Find the area of \( ABCD \).

16. In Fig. 328, \( AB \) is a diameter of the circle \( AQP \). If \( AB = 8 \) cm., \( AP = 7 \) cm., \( BQ = 6 \) cm., find \( \angle ABQ \) and \( \angle PKB \).

### The Cosecant, Secant and Cotangent

Names are also given to the reciprocals of the sine, cosine and tangent of an angle; they are called respectively the cosecant, secant and cotangent of the angle, written for short, cosec, sec and cot.

\[
\text{cosec} \theta = \frac{1}{\sin \theta} \\
\sec \theta = \frac{1}{\cos \theta} \\
\cot \theta = \frac{1}{\tan \theta}
\]

Thus

\[
\frac{z}{x} = \text{cosec} \theta \text{ and } \frac{z}{x} = \sec (90^\circ - \theta)
\]

\( z \) : the cosecant of any angle equals the secant of its complement.

Also

\[
\frac{y}{x} = \cot \theta \text{ and } \frac{y}{x} = \tan (90^\circ - \theta)
\]

\( y \) : the cotangent of any angle equals the tangent of its complement.

Note.—Just as "cosine" is short for "complement-sine", so "cosecant" is short for "complement-secant", and "cotangent" is short for "complement-tangent". And we can say that the "co-ratio" of any angle is equal to the corresponding ratio of its complement.
FURTHER DEVELOPMENTS

Thus, \( \cos 20^\circ = \sin 70^\circ \); cosec \( 20^\circ = \sec 70^\circ \); cot \( 20^\circ = \tan 70^\circ \); etc.

In the tables for each of the co-ratios, the numbers in the difference columns must be *subtracted*, that is, for cosines, cosecants, cotangents.

EXERCISE XVII.

1. Use tables to write down the values of
   (i) cosec \( 37^\circ \); cosec \( 38^\circ \); cosec \( 37^\circ 24' \); cosec \( 37^\circ 26' \);
   (ii) sec \( 46^\circ \); sec \( 47^\circ \); sec \( 46^\circ 18' \); sec \( 46^\circ 22' \);
   (iii) cot \( 51^\circ \); cot \( 52^\circ \); cot \( 51^\circ 36' \); cot \( 51^\circ 39' \).

2. Use tables to find \( \angle A \) if
   (i) cosec \( A = 1.0913 \); cosec \( A = 1.0910 \); cosec \( A = 2.0205 \);
   (ii) sec \( A = 1.8020 \); sec \( A = 1.8055 \); sec \( A = 3.2808 \);
   (iii) cot \( A = 1.6128 \); cot \( A = 1.6149 \); cot \( A = 0.4613 \).

3. In Fig. 530, \( \angle A = 90^\circ \), find \( \angle B \).

![Fig. 530](image)

4. In Fig. 531, \( \angle A = 90^\circ \), find \( CA \) and \( CB \).

![Fig. 531](image)

5. A kite is flying at a height of 400 ft. above the ground, at the end of a string making \( 37^\circ \) with the vertical. Find the length of the string.

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6. A chord of length 6 cm. subtends an angle of \( 100^\circ \) at the centre of a circle. What is the radius?

7. The tangents from a point \( A \) to a circle are 2.5 in. long and contain an angle of \( 103^\circ \). Find the distance of \( A \) from the centre.

8. One angle of a rhombus is \( 42^\circ \), and the shorter diagonal is 8 cm. Find the length of a side.

9. A taut elastic string joins two points \( A, B \), 20" apart and at the same level. When a body is attached to the midpoint \( C \) of the string, \( AC \) and \( BC \) make angles of \( 90^\circ 20' \) with the horizontal. How much has the string stretched?

10. The vertical angle of a cone is \( 66^\circ \) and the diameter of the base is 7 cm.; find the height of the cone and the length of its slant edge.

11. In Fig. 532, the triangle \( ABC \) is inscribed in the rectangle \( APQR \). Calculate the sides and angles of \( \triangle ABC \).

![Fig. 532](image)

12. In Fig. 533, \( AB \) is a diameter of the circle \( APB \) and \( NB = 2 \) in.; find the lengths of \( NP, AP, AB \).

13. In Fig. 533, \( AB \) is a diameter and \( BQ \) is a tangent. If \( AB = 5 \) cm., find \( PQ \).

14. The pilot of an aeroplane, flying horizontally at a height of 3000 feet, sees a church at an angle of depression of \( 62^\circ \). 12 seconds later, the church is vertically below him. Find his speed in feet per second.

15. \( ABC \) is a triangle such that \( \angle B = \angle C = 70^\circ \) and \( BC = 6 \) cm. Find the diameter of the circle which passes through \( A, B, C \).
PROBLEMS IN SOLID GEOMETRY

Intersecting Planes

Open a book at any angle; the two pages, you now see, form two planes intersecting in a straight line, the line of the binding, \( AB \).

Or, take a sheet of paper, and fold it, as in Fig. 534, with \( AB \) as crease; then the two portions of the sheet form two planes intersecting in the line \( AB \).

From any point \( O \) on \( AB \), draw \( OP, OQ \) perpendicular to \( AB \), one line in each plane. The size of \( \angle POQ \) does not depend on the position of \( O \) on \( AB \), and it is called the angle between the two planes.

If the plane \( ABQ \) is horizontal, the line \( OP \) is called a line of greatest slope of the plane \( ABP \).

Intersecting Line and Plane

Let any line \( OP \) cut a plane \( ABCD \) at \( O \). From \( P \) draw \( PN \) perpendicular to the plane \( ABCD \), and join \( ON \). Then \( ON \) is called the projection of \( OP \) on the plane \( ABCD \), and \( \angle PON \) is called the angle between the line \( OP \) and the plane \( ABCD \).

EXERCISE XVII. I

[Numbers 1 and 2 are intended for class discussion.]

1. Fig. 536 shows a pyramid of height 7 cm. on a square base \( ABCD \) of side 12 cm. The perpendicular from the vertex \( O \) to the base meets it at the centre \( N \) of the base.

THE RIGHT-ANGLED TRIANGLE

(i) By using Pythagoras, find the lengths of \( NB \) and \( OB \).
(ii) What angle does \( OB \) make with the base?
(iii) What angle does the plane \( OBC \) make with the base?
(iv) What angle does \( OB \) make with \( OC \)?
(v) What angle does the plane \( OBC \) make with \( OAD \)?

2. Fig. 537 shows a box with rectangular faces; \( AB = 12^\circ \), \( AE = 8^\circ \), \( AD = 9^\circ \).

(i) By using Pythagoras, find the lengths of \( AC, AG \).
(ii) What angle does \( AG \) make with the plane \( ABCD \)?
(iii) What angle does \( AG \) make with the plane \( ADHE \)?
(iv) If \( AQ = 3^\circ \), what angle does \( HQ \) make with the plane \( ABCD \)?
(v) What angle does the plane \( ABHG \) make with the plane \( ABCD \)?

(vi) At what angles do the lines \( AG, BH \) cut one another?

3. \( ABCD \) is a rectangular courtyard, \( AB = 160 \text{ ft.}, BC = 120 \text{ ft.} \); there is a vertical flagstaff \( AP \) at \( A \). The angle of elevation of \( P \) from \( B \) is \( 15^\circ \), find the angles of elevation of \( P \) from \( D \) and from \( C \).

4. The base of a pyramid is a regular hexagon of side 8 cm. and its height is 6 cm, and all its slant edges are equal. Find (i) the angle which each slant edge makes with the base, (ii) the angle which each face makes with the base.

5. Fig. 538 represents a rectangular sheet of paper \( ABCD \) pinned to a drawing-board which makes an angle of \( 40^\circ \) with the horizontal plane \( ABEF \). A line \( AP \) is ruled on the paper so that \( \angle PAB = 65^\circ \).

(i) What is the height of \( C \) above \( ABEF \)?
(ii) What is the angle of slope of \( AC \)?
(iii) What is the length of \( AP \) and the slope of \( AP \)?
(iv) A line \( AQ \) is drawn on the paper so that its slope is \( 35^\circ \). How far from \( D \) does it cut \( DC \)?
6. Fig. 539 represents a door opening through an angle of 52°, AD being the line of the hinges; AB = 4, AD = 8. Find
the angle between AC and AF.

*Any or all of the rest of this exercise may be omitted at a first
reading.*

7. The base BC of the triangle ABC is horizontal; AB = AC
and ∠BAC = 42°. The plane ABC makes 65° with the
horizontal; find the angle of slope of AB.

8. A pyramid stands on a square base of side 6 cm., and
each face is an isosceles triangle with the angle at its vertex
equal to 40°. Find (i) the angle each face makes with the
base, (ii) the length of each slant edge, (iii) the angle each slant
edge makes with the base.

9. (i) Fig. 540 (i) shows an equilateral triangle ABC, side
6 cm., inscribed in a circle, centre O. Calculate the radius of
the circle.

(ii) Fig. 540 (ii) shows a regular tetrahedron, ABCD, each
edge being of length 6 cm. (Each face is an equilateral
triangle.) Find (a) the angle which AD makes with the base
ABC; (b) the height OD; (c) the angle which the face BDC
makes with the base ABC.

10. (i) A regular pentagon ABCDE, side 5 in., is inscribed
in a circle, centre O. Calculate the radius of the circle.

(ii) A pyramid, vertex V, has a regular pentagon ABCDE
as base; also if O is the centre of the base, VO is perpendicular
to the base. If AB = 5 in. and AV = 8 in., find (a) the angle
which VA makes with the base; (b) the height VO; (c) the
angle which the face VAB makes with the base.

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CHAPTER XVIII

TRIGONOMETRY OF THE GENERAL TRIANGLE

Notation

If ABC is any triangle, it is customary to use A, B, C to
denote the sizes of the angles and a, b, c the lengths of the
sides opposite those angles, as shown in Fig. 541.

Area of any Triangle

If AD is the altitude of ∆ABC,
then area of ∆ABC = \frac{1}{2}AB \cdot BC.

In Fig. 542 (i), \( \frac{AD}{AC} = \sin\ ACB, \quad \frac{AD}{b} = \sin\ C; \)

\[ \therefore \quad \frac{AD}{b} = \sin\ C; \]

\[ \therefore \quad \text{area of } \triangle ABC = \frac{1}{2}b \sin\ C. \]

In Fig. 542 (ii), where ∠ACB is obtuse,

\[ \frac{AD}{AC} = \sin\ ACD, \quad \frac{AD}{b} = \sin\ (180° - C); \]

\[ \therefore \quad \frac{AD}{b} = \sin\ (180° - C). \]

\[ \therefore \quad \text{area of } \triangle ABC = \frac{1}{2}b \sin\ (180° - C). \]
So far, nothing has been said about the meaning of the sine of an obtuse angle. If we define the sine of an obtuse angle to be equal to the sine of its supplement, that is, if we take \( \sin C \) equal to \( \sin (180° - C) \), the formula for the area of an obtuse-angled triangle will be the same as the formula for the area of an acute-angled triangle. And so we can say, in all cases,

\[
\text{area of } \triangle ABC = \frac{1}{2}ab \sin C.
\]

Examples for Class Discussion

1. What is the area of \( \triangle ABC \) in Fig. 542, if
   (i) \( a = 4 \text{ in.}, b = 5 \text{ in.}, C = 70° \);
   (ii) \( a = 4 \text{ in.}, b = 5 \text{ in.}, C = 110° \);
   (iii) \( a = 2 \text{ cm.}, b = 3 \text{ cm.}, C = 57° \);
   (iv) \( a = 2 \text{ cm.}, b = 3 \text{ cm.}, C = 123° \)?

2. Use tables to write down the values of
   (i) \( \sin 130° \);
   (ii) \( \sin 155° \);
   (iii) \( \sin 117° 24' \);
   (iv) \( \sin 147° 46' \).

3. Use tables to find pairs of angles whose sines equal
   (i) \( 0.8910 \);
   (ii) \( 0.3057 \);
   (iii) \( 0.6351 \);
   (iv) \( 0.5802 \).

The Sine Formula

In the same way, we can show that the area of \( \triangle ABC \) equals \( \frac{1}{2}bc \sin A \) and \( \frac{1}{4}ab \sin B \).

\[
\therefore \quad \text{bc sin } A = \text{ca sin } B = \text{ab sin } C;
\]

Divide by \( abc \),

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};
\]

\[
\therefore \quad \frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin C}{c};
\]

This is called the **sine formula** for solving a triangle.

We can obtain it direct from Fig. 542 as follows:

As on p. 271, in Fig. 542 (i), \( AD = b \sin C \),

and in Fig. 542 (ii), \( AD = b \sin (180° - C) = b \sin C \).

Also in each figure, \( \frac{AD}{AB} = \sin B \), \( \therefore \quad AD = c \sin B \).

\[
\therefore \quad b \sin C = c \sin B; \quad \therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C};
\]

**Example 1.**—In \( \triangle ABC \),

\[
a = 5, \quad b = 5, \quad C = 40°; \quad \text{find } B \text{ and } c.
\]

From the sine formula,

\[
\frac{6}{\sin 40°} = \frac{5}{\sin B}; \quad \therefore \quad \sin B = \frac{5 \sin 40°}{6} = \frac{3.214}{6} = 0.5357;
\]

From the tables, \( 0.5357 = \sin 32° 24' \), to nearest minute;

\( \therefore \quad \text{also } 0.5357 = \sin 147° 36' \);

\( \therefore \quad B = 32° 24' \) or \( 147° 36' \).

But \( B = 147° 36' \) is impossible, because this value would make \( A + B \) more than \( 180° \).

\( \therefore \quad B = 32° 24' \) is the only solution.

**Note.**—The value of \( C \) is \( 180° - 40° - 32° 24' = 107° 36' \).

We can now find \( c \) from the equation,

\[
\frac{c}{\sin 107° 36'} = \frac{6}{\sin 40°};
\]

**Use of Logarithms.**—Those who know how to use logarithms, should employ them for finding the value of \( c \), as it shortens the arithmetical work.

\[
\text{Sin } 107° 36' = \sin 72° 24'; \quad \log 6 = 0.7782 \]

\[
\text{sin } 40° = 0.7989 \quad \log \text{sin } 72° 24' = 1.8081
\]

\[
\therefore \quad \log \text{numerator} = 0.7574
\]

\( \therefore \quad c = 8.90, \text{ to 3 figures.} \)

\[
\log c = 0.9493
\]
FURTHER DEVELOPMENTS

Example 2.—(The ambiguous case.) In \( \triangle ABC \),
\[ a = 4, \quad b = 5, \quad A = 40^\circ; \] find \( B \).

\[ \frac{4}{\sin 40^\circ} = \frac{5}{\sin B} \]
\[ \therefore \sin B = \frac{5 \sin 40^\circ}{4} = 0.8035. \]

From the tables, \( \sin 53^\circ 28' = 0.8035 \);
\[ \therefore \text{also } \sin 126^\circ 32' = 0.8035 \]
\[ \therefore B = 53^\circ 28' \text{ or } 126^\circ 32'. \]

Here, both solutions are possible (see Fig. 544); two triangles of different sizes can be drawn to fit the data (2 sides and a not-included angle). This is the Ambiguous Case (see p. 126).

EXERCISE XVIII. a

1. Use tables to find the values of the sines of
   (i) \( 160^\circ \); (ii) \( 110^\circ \); (iii) \( 91^\circ \); (iv) \( 137^\circ \); (v) \( 173^\circ \);
   (vi) \( 102^\circ 24' \); (vii) \( 125^\circ 44' \); (viii) \( 114^\circ 32' \).

2. Find pairs of angles whose sines equal
   (i) \( 0.4067 \); (ii) \( 0.7531 \); (iii) \( 0.4797 \); (iv) \( 0.9320 \).

3. If \( B = 47^\circ, \quad C = 53^\circ, \quad b = 4 \), find \( c \).

4. If \( A = 110^\circ, \quad B = 29^\circ, \quad a = 10 \), find \( b \).

5. If \( b = 12, \quad c = 9, \quad B = 55^\circ \), find \( C \).

6. If \( a = 9, \quad b = 7, \quad A = 113^\circ \), find \( b \).

7. If \( B = 33^\circ, \quad C = 95^\circ 36', \quad b = 5.271 \), find \( c \) and \( a \).

8. If \( a = 6, \quad b = 8, \quad A = 41^\circ \), find \( B \) and \( C \). Draw rough figures to illustrate your answers.

9. If \( b = 8, \quad c = 7, \quad B = 39^\circ \), find \( C \) and \( a \).

10. If \( b = 4, \quad c = 3, \quad B = 127^\circ 20' \), find \( C \) and \( a \).

11. If \( a = 9.24, \quad c = 7.48, \quad A = 37^\circ 40' \), find \( C \) and \( b \).

TRIGONOMETRY OF GENERAL TRIANGLE

Example 2.—(The ambiguous case.) In \( \triangle ABC \),
\[ a = 4, \quad b = 5, \quad A = 40^\circ; \] find \( B \).

\[ \frac{4}{\sin 40^\circ} = \frac{5}{\sin B} \]
\[ \therefore \sin B = \frac{5 \sin 40^\circ}{4} = 0.8035. \]

From the tables, \( \sin 53^\circ 28' = 0.8035 \);
\[ \therefore \text{also } \sin 126^\circ 32' = 0.8035 \]
\[ \therefore B = 53^\circ 28' \text{ or } 126^\circ 32'. \]

Here, both solutions are possible (see Fig. 544); two triangles of different sizes can be drawn to fit the data (2 sides and a not-included angle). This is the Ambiguous Case (see p. 126).

EXERCISE XVIII. a

1. Use tables to find the values of the sines of
   (i) \( 160^\circ \); (ii) \( 110^\circ \); (iii) \( 91^\circ \); (iv) \( 137^\circ \); (v) \( 173^\circ \);
   (vi) \( 102^\circ 24' \); (vii) \( 125^\circ 44' \); (viii) \( 114^\circ 32' \).

2. Find pairs of angles whose sines equal
   (i) \( 0.4067 \); (ii) \( 0.7531 \); (iii) \( 0.4797 \); (iv) \( 0.9320 \).

3. If \( B = 47^\circ, \quad C = 53^\circ, \quad b = 4 \), find \( c \).

4. If \( A = 110^\circ, \quad B = 29^\circ, \quad a = 10 \), find \( b \).

5. If \( b = 12, \quad c = 9, \quad B = 55^\circ \), find \( C \).

6. If \( a = 9, \quad b = 7, \quad A = 113^\circ \), find \( b \).

7. If \( B = 33^\circ, \quad C = 95^\circ 36', \quad b = 5.271 \), find \( c \) and \( a \).

8. If \( a = 6, \quad b = 8, \quad A = 41^\circ \), find \( B \) and \( C \). Draw rough figures to illustrate your answers.

9. If \( b = 8, \quad c = 7, \quad B = 39^\circ \), find \( C \) and \( a \).

10. If \( b = 4, \quad c = 3, \quad B = 127^\circ 20' \), find \( C \) and \( a \).

11. If \( a = 9.24, \quad c = 7.48, \quad A = 37^\circ 40' \), find \( C \) and \( b \).

With the notation of Fig. 545 (i), using Pythagoras,
\[ c^2 = (a - x)^2 + h^2 = a^2 - 2ax + x^2 + h^2 \text{ and } x^2 + h^2 = b^2; \]
\[ \therefore c^2 = a^2 + h^2 - 2ax. \]

With the notation of Fig. 545 (ii), using Pythagoras,
\[ c^2 = (a + y)^2 + h^2 = a^2 + 2ay + y^2 + h^2 \text{ and } y^2 + h^2 = b^2; \]
\[ \therefore c^2 = a^2 + b^2 + 2ay. \]

Now in Fig. 545 (i), \( x = b \cos C \),
\[ \therefore c^2 = a^2 + b^2 - 2ab \cos C. \]

And in Fig. 545 (ii), \( y = b \cos (180^\circ - C) \),
\[ \therefore c^2 = a^2 + b^2 + 2ab \cos (180^\circ - C). \]

So far, nothing has been said about the meaning of the cosine of an obtuse angle. If we define the cosine of an obtuse angle to be equal to minus the cosine of its supplement, that is, \( \cos C = -\cos (180^\circ - C) \), the two formulae, just proved, will be the same.

We can then say that for any triangle \( \triangle ABC \), whether \( C \) is acute or obtuse,
\[ c^2 = a^2 + b^2 - 2ab \cos C. \]

Similarly,
\[ a^2 = b^2 + c^2 - 2bc \cos A \] and \[ b^2 = c^2 + a^2 - 2ca \cos B. \]
Oval Examples

1. Use tables to write down the values of
   (i) $\cos 120^\circ$; (ii) $\cos 140^\circ$; (iii) $\cos 170^\circ$; (iv) $\cos 96^\circ$;
   (v) $\cos 103^\circ 18'$; (vi) $\cos 112^\circ 20'$; (vii) $\cos 144^\circ 40'$.

2. Use tables to find the angles whose cosines equal
   (i) $-0.6561$; (ii) $-0.3256$; (iii) $0.9135$; (iv) $-0.7218$;
   (v) $-0.3347$; (vi) $-0.2005$; (vii) $-0.3980$; (viii) $-0.1063$.

Extensions of Pythagoras' Theorem

The proof of the cosine formula shows that (see Fig. 545)

(i) If $AN$ is an altitude of $\triangle ABC$, and if $\angle ACB$ is acute,
   \[ AB^2 = AO^2 + BC^2 - 2BC \cdot CN. \]

(ii) If $AN$ is an altitude of $\triangle ABC$, and if $\angle ACB$ is obtuse,
   \[ AB^2 = AO^2 + BC^2 + 2BC \cdot CN. \]

These two facts are called the "extensions" of Pythagoras' theorem.

Example 3.—In $\triangle ABC$,

\[ a = 3, \ b = 5, \ c = 7; \text{ find } c. \]

From the cosine formula, $c^2 = a^2 + b^2 - 2ab \cos C$,
\[ 7^2 = 3^2 + 5^2 - 2(3)(5) \cos C; \therefore 30 \cos C = 9 + 25 - 49; \]
\[ \therefore \cos C = \frac{15}{30} = 0.5. \]

From the tables, $\cos 60^\circ = 0.5$; \therefore $\cos 120^\circ = -0.5$;
\[ \therefore C = 120^\circ. \]

Exercise XVIII. b

1. Use tables to find the values of the cosines of
   (i) $110^\circ$; (ii) $150^\circ$; (iii) $165^\circ$; (iv) $96^\circ$;
   (v) $132^\circ 24'$; (vi) $147^\circ 20'$; (vii) $159^\circ 45'$; (viii) $160^\circ 52'$.

2. Use tables to find the angles whose cosines equal
   (i) $-0.6428$; (ii) $-0.2088$; (iii) $-0.8131$; (iv) $-0.9409$;
   (v) $0.2717$; (vi) $-0.5702$; (vii) $0.9001$; (viii) $0.0256$.

3. If $a = 2, \ b = 3, \ c = 5^\circ$, find $c$.
4. If $b = 2, \ c = 5, \ A = 124^\circ$, find $a$.
5. If $a = 5, \ b = 6, \ c = 7$, find $C$.
6. If $a = 5, b = 6, \ c = 9$, find $C$. 

Note. $A = 180^\circ - 35^\circ 37'$ is impossible, because this value of $A$ would make $A + B$ greater than $180^\circ$. 
FURTHER DEVELOPMENTS

7. If \( b = 10 \), \( c = 5 \), \( A = 41^\circ \), find \( a \) and \( C \).

8. If \( a = 2 \), \( c = 1 \), \( B = 164^\circ 18' \), find \( b \) and \( A \).

9. If \( a = 7 \), \( b = 6 \), \( c = 10 \), find \( A \) and \( C \).

10. If \( a = 6 \), \( c = 10 \), \( B = 15^\circ 24' \), find \( b \) and \( A \).

11. If \( a = 5 \), \( b = 8 \), \( C = 37^\circ 20' \), find \( c \) and \( B \).

12. If \( a = 97 \), \( b = 86 \), \( c = 74 \), find \( C \) and \( A \).

13. If \( b = 32 \), \( c = 44 \), \( A = 119^\circ 15' \), find \( a \) and \( B \).

14. If \( a = 4 \), \( c = 5 \), \( B = 132^\circ 10' \), find \( b \) and \( C \).

15. If \( a = 8 \), \( b = 5 \), \( c = 4 \), find \( A \) and \( C \).

Applications of the Sine and Cosine Formulae

The problems set for scale-drawing in Part I, Ex. III, \( h \), and Ex. V, \( j \), may be solved trigonometrically, if further practice is required after Ex. XVIII, \( c \) has been worked.

EXERCISE XVIII, \( c \)

1. A base line \( AB \) 1000 yards long is measured on level ground, running due south from \( A \) to \( B \). The bearings of a church \( C \) from \( A \) and \( B \) are S. 72\(^\circ\) E. and N. 55\(^\circ\) E. Find the distance of \( C \) from \( A \).

2. Three villages \( A, B, C \) are connected by straight level roads; \( B \) is 5 miles from \( A \) and 4 miles from \( C \), and \( \angle ABC = 150^\circ \). How far is \( A \) from \( C \) ?

3. A crane \( ABC \) (see Fig. 546) carries a load \( D \) as shown; \( AB \) is vertical. Find \( \angle BAC \) and the height of \( D \) above the level of \( A \).

4. Two ships leave harbour at noon in directions N. 14\(^\circ\) W., S. 37\(^\circ\) W., and steam at 10, 12 knots respectively. How many sea-miles are they apart at 12.30 P.M.? [1 knot = 1 sea-mile per hour.]

5. The elevation of the top of a tower is 32\(^\circ\) from a point \( A \), and 49\(^\circ\) from another point \( B \). 100 feet nearer the foot of the tower, which is in line with \( AB \) and at the same level. Find the height of the tower.

6. The centres \( A, B \) of two circles of radii 5 cm., 8 cm., are 10 cm. apart, and their common chord is \( CD \). Calculate \( \angle CAB \) and the length of \( CD \).

7. In the framework in Fig. 547, \( AB = AE = 20 \) ft., also \( AC = AD \). Find the lengths of \( CD \) and \( BC \).

8. In the framework in Fig. 548, \( AB = 10 \) ft.; find \( BC \) and \( BD \).

9. In \( \triangle ABC \), \( AB = 6 \) cm., \( \angle A = 38^\circ \), \( \angle B = 74^\circ \). \( O \) is the circumcentre of \( \triangle ABC \), that is, the centre of the circle which passes through \( A, B, C \). Calculate the circumradius. \([ \text{Join } O \text{ to } A, \text{ to } B, \text{ and to the mid-point of } AB \text{.} \]

10. \( A, B, C \) are 3 points on a circle, and the tangent at \( C \) meets \( AB \) produced at \( T \). If \( AB = 2 \) in., \( \angle CAB = 31^\circ \), \( \angle CBS = 72^\circ \), find \( BC \) and \( CT \).
11. In Fig. 549, find $\angle BCD$ if $BD : AC = 9 : 10$.

![Fig. 549.](image)

12. The sides $AB$, $BC$, $CD$, $DA$ of a cyclic quadrilateral are respectively 5, 4, 7, 6 in.; calculate $\angle ABC$. [Let $AC = x$ in, $\angle ABC = \theta$; what is $\angle ADC$ in terms of $\theta$? Form two equations connecting $x$ and $\cos \theta$.]

13. $BC$ and $AD$ are the parallel sides of a trapezium $ABCD$; $AB = 8$ cm., $BC = 4$ cm., $CD = 6$ cm., $DA = 11$ cm.; calculate $\angle BAD$ and the area of $ABCD$. [Draw from $C$ a line parallel to $BA$ to cut $AD$ at $E$.]

14. The sides of a triangle are 8 cm., 9 cm., 10 cm.; find its smallest angle and its area.

15. The mechanism in Fig. 550 consists of three rods; $AB$ and $DC$ can turn about their ends $A$ and $D$, which are fixed, and there are hinges at $B$ and $C$. Calculate the total angle through which $AB$ can turn.

![Fig. 550.](image)

16. $ABC$ is a triangle such that $BC = 6$ cm., $\angle B = 38^\circ$, $\angle C = 74^\circ$. $I$ is its in-centre, that is, the centre of the circle touching $BC$, $CA$, $AB$, and inside $\triangle ABC$. Calculate the length of $IB$ and the radius of the inscribed circle. [Join $IB$, $IC$ and draw $IX$ perpendicular to $BC$.]

### ANSWERS TO PART I

**Exercise I. c (p. 4)**

3. 9. 4. 4. 5. $ACD$, $ABE$, $HBD$, $GSC$, $GFD$, $CFE$.

8. 6, 10.

**Exercise I. d (p. 6)**

13. 6. 14. 12. 15. 8. 16. 8, 18, 12. 17. 3.

18. Cuboid 6, 8, 12; $\triangle$ prism 5, 6, 9; $\triangle$ pyramid 4, 4, 6; 5-sided prism 7, 10, 15; 4-sided pyramid 5, 5, 8; $L$ 8, 12, 18; beheaded pyramid 6, 8, 12; $F + C - E = 2$.

**Exercise II. a (p. 10)**

2. 6. 4. $STV$ and $OSTV$ are st. lines.

5. $ABZ$ is st. line.

**Exercise II. b (p. 12)**

1. 15 in.; 8 cm. 7. All meet at a second point.

**Exercise II. c (p. 14)**

1. 4, 5, 6, 10. 2. 2, 5. 3. Quad., $\Delta$. 4. 2 quads. or $\triangle$ and pentagon.

5. (i) 4, 5, 8; (ii) 4, 6, 9; (iii) 3, 10, 12; (iv) 4, 15, 18; $C + F = S + 1$.

6. 5; 8. 7; 3; 9. 8. (i) 5, 20; (ii) 7, 35. 9. 2 sectors.

In Fig. 21, $AB = 2.37$ in. approx.
SIMPLIFIED GEOMETRY

Exercise II. d (p. 17)

1. PQ SR PS QR AB BC CA
   In. 2:00 1:50 3:50 3:44 3:83 3:11 3:65
   cm. 5:09 3:82 8:88 8:75 9:72 7:90 7:75

2. DE EF FD MP KN
   In. 4:47 1:98 5:00 1:53 1:22
   cm. 11:35 5:02 12:69 3:88 3:10

3. (i) 1:41 in., 3:57 cm.; (ii) 2:39 in., 6:06 cm.;
   (iii) 2:26 in., 5:75 cm.; (iv) 3:36 ln., 8:52 cm.;
   (v) 4:76 in., 12:06 cm.

4. 25:4 cm. 5. 10:4 in. 6. D, P, A, Q, F, E, B.


10. 3:64 in.; 0:304 in.

11. HK = 3:0 cm., VZ = 2:9 cm.

12. SM = 5:03 cm., PQ = 5:09 cm.

Exercise II. e (p. 21)

1. 3 in., 2 in., 1 in. 3. 12 in. 4. 9 cm.

Exercise II. f (p. 23)

[Answers are given frequently to a higher degree of accuracy than can
   be obtained by drawing, to assist in deciding what margin of error is
   permissible.]

1. 3, 4, 5 cm. 2. 1, 2, 2:5 in. 3. 4:27 cm. 4. 6:71 cm.

5. 4:99 cm. 6. 7:60 cm. 7. 5:31 cm. 12. Six.

Exercise III. b (p. 29)

3. 4. 4. E, F, H, L, T. 5. III, IX; I, VII; II, VIII.

6. (i), (ii), (iii) Yes; (iv) PA, PD, PQ; (v) No; (vi) Contains
   PQ.

Exercise III. c (p. 31)

6. (a) Third corner is (i) IV or XII; (ii) VI or X;
   (iii) C; (iv) C; (v) VI; (vi) XI; (d) 6; (c) II, VIII; I, VII;
   (d) V, XI; I, VII.

Exercise III. d (p. 33)

1. 3. 5 cm. 2. 2, 1:5 in.; 2:5, 2:5 in.


Exercise III. e (p. 36)

1. 12. 2. 28. 4. 56. 5. 1 sq. yd., 9 sq. ft.

6. (i) square, side 1 in.; 4; 4 sq. in.; (ii) square, side 1:5 in.;
   2:5; 2:5 sq. in.

7. (i) square, side 3 in.; y sq. ft.; 6 sq. in.; (ii) square,
   side 6 in.; 4 sq. ft.; 36 sq. in.; (iii) square, side 9 in.;
   y sq. ft.; 81 sq. in.

8. (i) 100, 7; (ii) 216, 2:16 sq. in. 9. 216 sq. ft.; 5 ft.

Exercise III. f (p. 39)

1. 40 ft., 20 ft. 2. 40 ft., 60 ft. 3. 20 ft.

4. 42:4 ft. 5. 44:7, 28:3 ft. 6. 80 ft.

7. 2:5 cm.; 25 ft. 8. 1:5 cm.; 32 ft.

Exercise III. g (p. 40)

4. 7 sq. cm. 5. 11 sq. cm.

Exercise III. h (p. 41)

1. 13:2 ft. 2. 5, 9 ft. 3. 83:8 ft.; 53:9 ft.

4. 11:5 ft. 5. 7:42 mi.

6. (i) 36:5 yd.; (ii) 40:7 yd.; (iii) 29:3 yd.; (iv) 21:75 yd.;
   40:7 yd.

7. (i) 16:2 ft.; (ii) 13:0 ft. 8. (i) 18:3 in.; (ii) 17:3 in.;
   14:3 in.

9. 262 yd. 10. 18:3 ft.; 1:0 ft.
SIMPLIFIED GEOMETRY

Exercise IV. a (p. 45)
7. 4, 8, 4, 2. 8. (i) Yes; (ii) 4, 8; (iii) 2, 4; (iv) Yes.

Exercise IV. b (p. 48)
[The unit of angle measurement is a right angle.]
1. 1. 2. 1. 3. 3. 4. 2. 5. 3. 6. 3.
7. $\frac{1}{4}$. 8. 1. 9. $2\frac{1}{4}$. 10. 2. 11. $2\frac{1}{4}$. 12. 3.
13. 1. 14. $1\frac{1}{4}$. 15. 2. 16. $1\frac{1}{4}$. 17. $1\frac{1}{4}$. 18. 2.
25. N.E. 26. W. 27. N.E. 28. (i) 1; (ii) 2; (iii) $\frac{1}{2}$; (iv) 0.
29. (i) 2; (ii) 1; (iii) 3; (iv) 4; (v) $\frac{1}{4}$; (vi) $1\frac{1}{4}$; (vii) 6; (viii) 10.
30. (i) 4; (ii) 1; (iii) $\frac{1}{2}$; (iv) $\frac{1}{2}$; (v) $1\frac{1}{2}$; (vi) $2\frac{1}{2}$; (vii) 8; (viii) 50.

Exercise IV. c (p. 50)
[The unit of angle measurement is a right angle.]
1. 1, 2, 3, $1\frac{1}{2}$, 2$\frac{1}{2}$. 7. (i) 2 c.; (ii) 3 cc.; (iii) 5 cc.; (iv) $3\frac{1}{2}$ c.
9. 3; 2$\frac{1}{2}$, 3$\frac{1}{2}$.

Exercise IV. d (p. 52)
[The unit of angle measurement is a right angle.]
1. $\angle AOB$, 1; $\angle BOC$, $\frac{1}{3}$; $\angle AOC$, $1\frac{1}{3}$.
2. $\angle BCR$, 1; $\angle RCS$, $\frac{1}{3}$; $\angle PGS$, $1\frac{1}{3}$; $\angle OCR$, $\frac{1}{3}$; $\angle QCS$, $\frac{1}{3}$.
3. $\frac{1}{4}$, 1, $1\frac{1}{4}$; 3. 5. $\frac{1}{4}$ cc., $\frac{1}{4}$ c., 1 c.; 1 c.
6. 1; $1\frac{1}{4}$; $1\frac{1}{2}$; $\frac{1}{4}$; $\frac{1}{2}$, $\frac{1}{2}$.
7. $\angle QOS$, $b + c$; $\angle PQR$, $a + b$; $\angle ROS$, $c$; $\angle POS$, $a + b + c$.
8. $\frac{1}{2}$; $\frac{1}{2}$; $1\frac{1}{2}$; 2; $\frac{1}{2}$. 9. $2\frac{1}{2}$; $1\frac{1}{2}$; 4.
10. (i) $1\frac{1}{2}$; (ii) $\frac{1}{2}$, $1\frac{1}{2}$; (iii) $\frac{1}{2}$; (iv) $1\frac{1}{2}$.

Exercise V. a (p. 57)
1. $270^\circ$, $135^\circ$, $225^\circ$, $22\frac{1}{4}^\circ$, $18^\circ$, $15^\circ$, $72^\circ$, $150^\circ$, $720^\circ$.
2. $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{4}$, $1\frac{1}{4}$, $1\frac{1}{2}$, $2\frac{1}{4}$, $3\frac{1}{4}$, 5.
3. $\angle$ obtuse, $\angle$ acute, $\angle$ reflex, $\angle$ obtuse, $\angle$ reflex.
5. $290^\circ$. 6. $100^\circ$. 7. $360^\circ$, $30^\circ$, $120^\circ$, $45^\circ$.
8. $180^\circ$, $30^\circ$, $150^\circ$, $240^\circ$. 9. $50^\circ$, $230^\circ$, $140^\circ$.
10. $130^\circ$, $310^\circ$, $40^\circ$.
11. Sum is (i) $180^\circ$, (ii) $360^\circ$.
12. Sum is (i) $180^\circ$, (ii) $90^\circ$, (iii) $360^\circ$; (iv) Difference is $180^\circ$.
13. (i) $c = 45^\circ$, $b = d = 140^\circ$; (ii) $d = 130^\circ$, $a = c = 50^\circ$; (iii) $a = 45^\circ$, $b = d = 135^\circ$.
14. (i) $h = 80^\circ$; (ii) $g = 70^\circ$; (iii) Each $= 60^\circ$; (iv) $f = h = 50^\circ$.

Exercise V. b (p. 62)
3. $51^\circ$, $76^\circ$, $52^\circ$. 4. $93\frac{1}{4}^\circ$, $63\frac{1}{4}^\circ$, $231^\circ$. 5. $102^\circ$, $86\frac{1}{2}^\circ$, $93\frac{1}{4}^\circ$, $78\frac{1}{2}^\circ$.
6. $57^\circ$, $21^\circ$, $29\frac{1}{2}^\circ$, $57^\circ$. 7. $126\frac{1}{4}^\circ$, $53\frac{1}{4}^\circ$, $126\frac{1}{4}^\circ$, $53\frac{1}{4}^\circ$.
8. $155^\circ$, $59\frac{3}{4}^\circ$, $25^\circ$, $129\frac{1}{4}^\circ$. 9. $54\frac{1}{4}^\circ$. 10. $134^\circ$.
11. $144\frac{1}{4}^\circ$. 12. $76^\circ$. 16. $130^\circ$, $70^\circ$.

Exercise V. c (p. 63)
1. 4-62 cm. 2. 3-09 cm. 3. 4-51 cm. 4. 4-02 cm. 5. 3-21 in.
6. 5-83 cm. 7. 2-69 in. 8. 3-05 in. 9. 5-88 cm. 10. 5-12 cm. 11. 5-88 cm.

Exercise V. d (p. 64)
2. $45^\circ$, $67\frac{1}{4}^\circ$, $67\frac{1}{2}^\circ$. 3. $67\frac{1}{2}^\circ$, $157\frac{1}{4}^\circ$, $112\frac{1}{4}^\circ$. 4. $45^\circ$, $135^\circ$, $135^\circ$.
SIMPLIFIED GEOMETRY

9. N. by W.; NW. by N.; NW. by W.; W. by N.

10. Midway between (i) E. and E.S.E.; (ii) SE. and S.E.; (iii) SE. and E.S.E.; (iv) S. and S.S.W.; (v) SW. and S.W.; (vi) W. and W.S.W.

11. 11°. 12. 22°.

Exercise V.e (p. 66)

1. N. 30° E.; N. 80° W.; S. 70° W.; N. 80° E.; S. 80° E.; N. 65° W.; S. 35° E.; N. 75° W.; N. 35° E., S. 15° E., S. 60° W., N. 20° W.

2. 30°, 260°, 250°, 80°, 100°, 295°, 145°, 285°.

3. 45°, 110°, 55°, 100°, 55°, 170°.

4. 45°, 125°, 160°, 50°, 100°, 180°.

5. N. 70° E.; S. 60° E.; 120°. 6. 50°. 7. 40°. 8. 55°.

9. 35°.

10. S. 20° E.; N. 70° W.; S. 40° W.

Exercise V.f (p. 67)

1. 64-9 ft. 2. 94-3, 26-0 yd.; 24-4 yd.

3. 5-48 mi., N. 50° E.

4. 38-2 ft. 5. 272 ft. 6. 2-83 mi., N. 48° W.

7. 748 yd.; 465 yd.; N. 67° W.; 8. 82-1 ft.

8. 26-9, 20-0 yd.

10. 40, 47-0 mi.; S. 54° E.; 40 mi. 11. 46-0, 38-6 yd.

12. 44°, 30°.

Exercise V.g (p. 69)

1. 180°; 120°; 50°; x + y = 180. 2. 70°.

3. 45°; 135°. 4. 125°; 140°; N. 65° W.; 135°, 44°; 84°, 62°.

5. 60°. 6. 20°. 8. 40°.

Exercise V.h (p. 72)

1. 38°; 265°; 90°. 2. 80°; x = 120; 148°, 32°.

3. 130°, 40°; 130; y = 35. 4. t = 40; 45°; z = 15.

5. 145°; 25°. 6. 55°; 15°. 8. 60°. 10. 130°.

11. 144°. 12. 75°.

Exercise V.j (p. 73)

1. 78°; x° + y°. 2. 90° in each case. 3. 25°; y°.

4. 112°; 180° - z°.

Exercise VI.a (p. 77)

3. Yes; no. 4. (i) Yes; yes. (ii) Yes; no.

5. Yes; yes. 6. Yes. 7. Yes; yes. 8. No; yes.

9. Yes; no. 10. (i) HD, GC, FB; (ii) BC, AD, AE, BF.


15. Yes; no. 16. Yes, if desk is level; no.

Exercise VI.b (p. 79)

2. Yes. 3. East and West. 4. Yes, no, yes.

9. 90°; rectangle. 10. 5, 5 cm.

Exercise VI.c (p. 80)

1. (i) 55° r; 55° l. 2. x°, l; y°, r; x = y.

3. 180° - y°, r; 180° - x°, l; x = y. 6. ∠BCD = 50°.

Exercise VI.d (p. 82)

1. Each 65°. 2. Each 70°. 3. (i) Each 120°; (ii) Each 115°.

6. 7. ∠A = 108°. 8. (i) p = 70°, q = 110°.
SIMPLIFIED GEOMETRY

9. (i) 70°, 110°, 110°; 180°, 180°. (ii) 115°, 65°, 65°; 180°, 180°.
10. (i) 115°, 65°, 180°; (ii) 80°, 100°, 180°; (iii) 50°, 50°, 180°.
11. \[x^2, 180^\circ - x^2, 180^\circ; \text{sum} = 180^\circ.\]
12. 145°; 85°.

Exercise VI.e (p. 84)
1. 110°. 2. 50°. 3. 115°. 4. 108°. 5. 60°. 6. 115°.
7. 40°, 60°. 8. 145°, 125°. 9. 70°. 10. 40°, 70°. 11. 108°.
12. 12, 75°. 13. 85°. 14. \(x = 36\). 15. \(y = 25\).
16. \(z = 36\). 17. 110°, 70°; 180°. 18. 36°, 70°, 74°; 180°.

Exercise VI.f (p. 87)
4. \(b + c > 180°\). 5. 26°. 6. 15°; 85°, 120°. 7. No.
8. Yes. 9. Yes. 10. No.

ANSWERS TO PART II

Oral Work (p. 101)
1. (i) 75°, 43°, 118°. (ii) 46°, 32°, 98°. (iii) B, C.
2. QPR, QRP; QRL; QPR, PQR. 3. 3; 6.

Exercise VII.a (p. 102)
1. 52°, 84°, 136°; 42°, 25°, 70°. 2. 70°, 60°, 130°; 34°.
3. 41°, 54°, 54°; 42°, 36°, 42°.

Exercise VII.b (p. 103)
1. 60°; 55°, 65°. 2. 65°, 35°, 70°, 70°, 40°; 120°, 65°, 55°.
3. 70°; 120°. 4. 55°; 75°, 75°. 5. 72°, 144°; 70°, 80°.

Oral Work (p. 104)
1. (i) 81°, 28°, 109°; 180°, 71°; (ii) B; A; \(\text{sum} = 180°\).

Exercise VII.c (p. 105)
1. 75°; 14°; 154°. 2. 40°; 50°. 3. 60°. 4. 37°.
5. 20°, 6. 72°, 72°, 36°. 7. \text{sum} = 90°. 8. Yes, no; no.
9. 52°, 59°, 65°; 10. 25°, 25°, 63°.
11. 30°, 100°; 53°, 110°, 82°. 12. 36°, 72°; 72°, 72°, 74°.
13. 145°; 45°, 125°. 14. 44°; \(x = 36, y = 72\). 15. 25°.
16. 130°; 132°. 17. 87°. 18. 36°, 63°.

Exercise VII.d (p. 107)
1. 6, 2, 8, 3, 10. 4, 36, 196, 2a - 4. 6, \(p + q + r = 360°\).
7. Equals 360°. 9, \(p + q + r - s = 360°\). 10. 8 rt, \(\angle a\); 2 rt, \(\angle a\).

Exercise VII.e (p. 109)
1. 12, 14, 20. 2. 120°; 110°. 3. \(x = 72\), \(y = 108\); 125°.
4. 60°, 120°. 5. 144°. 6. 89°. 7. 9. 18. 9. 12. 10. 60°.

Exercise VII.f (p. 111)
2. \(\angle ANT, \angle ACB\). 3. \(\angle NAB, \angle ACB\).
5. \(t = p + q + r + s\); \(t = d - a = b + c\). 6. \(\angle ABQ\).
Exercise VIII: b (p. 118)
2. (i) About LM and about HK; (iv) Rhombus.

Exercise VIII: c (p. 119)
1. 6-65 cm.; 5-28 cm.; 76°. 5. Yes. 6. ∠A = 57°.
7. Yes. 8. BC = 7-90 cm.; CA = 7-75 cm.; AB = 9-72 cm.;
   ∠A = 52°, ∠B = 51°, ∠C = 76°. 9. PF = 10-37 cm.; RS = 8-52 cm., SP = 8-85 cm., ∠P = 21°, ∠R = 57°, ∠S = 102°.
10. EF = 5-02 cm., FD = 12-69 cm., DE = 11-35 cm., ∠E = 93°.

Exercise VIII: d (p. 122)
1. (i) ΔKNQ; KN = 2-7 in., NQ = 2-6 in., QK = 2-5 in.,
   ∠Q = 64°. (ii) ΔQNK; QN = 2-7 in., NK = 2-6 in.,
   KQ = 2-5 in., ∠K = 64°. (iii) ΔQNK; QN = 2-7 in.,
   KN = 2-7 in., ∠N = 60°, ∠Q = 56°, ∠K = 64°.
2. (i) Same as 1 (ii); (ii) ΔKNQ; NK = 2-7 in., KQ = 2-6 in.,
   QN = 2-5 in., ∠N = 60°; (iii) Same as 1 (i).
3. (i) Same as 1 (ii); (ii) Same as 2 (ii); (iii) Same as 1 (i).
4. No; yes; no. 5. ΔABC = ΔKCP, ∠P = 78°, ∠K = 45°,
   ∠Z = 62°. 6. ΔDEF = ΔTRL, TL = 9-8, ∠L = 57°, ∠T = 64°.
7. No. 8. ΔLKG = ΔKRG, KRG = 7-7, QK = 8-5, ∠Q = 72°,
   ∠R = 45°; FG = 7-1 or 9-6. 10. ∠X = 37°; XY = 6 or 9.
11. ΔABC = ΔAPK, ∠PAK = 48°, ∠P = 64°, ∠C = ∠K = 68°.
12. ΔABD = ΔCDB, CB = 7-9, CD = 7-7, ∠C = ∠A = 78°.
13. ΔPQS = ΔRSQ, ∠SQR = 57°, ∠R = 63°, ∠RSQ = ∠SQP = 60°.
14. ΔABD = ΔADC, ∠ADB = ∠ADC = 90°.
15. ΔAPQ = ΔBPQ, ΔAPN = ΔBPN, ΔAQN = ΔBQN; angles at
   N are 90°. ∠AQN = 45° = ∠QAN = ∠QBN; ∠PAN = ∠PBN
   = 58°; QN = BN = 4-25.

Exercise VIII: e (p. 124)
1. No, angle sum not 180°. 2. DE > DF + FE.
3. (i) No, ZK > ZY + YX; (ii) 104°; (iii) XZ is st. line.

Exercise VIII: f (p. 127)
1. (i) ΔABC = ΔZYX; S.A.S.; (ii) No.
2. (i) ΔABC = ΔXYZ; A.A.S.; (ii) Ambiguous.
3. (i) ΔABC = ΔXYZ; A.S.A.; (ii) No.
4. (i) No; (ii) ΔABC = ΔXYZ; S.S.S.
5. (i) No; (ii) ΔABC = ΔXYZ; 90°, H.S.
6. (i) ΔABC = ΔXYZ; A.A.S.; (ii) No.
7. AB = FD or BC = DE or CA = EF.
8. ΔA = ΔF or BC = DE. 9. ΔABD = ΔCBD; A.S.A.
10. ΔAPN = ΔPB; S.A.S. 11. ΔABD = ΔCDB; A.S.A.
12. ΔEOF = ΔHOG; A.A.S. 13. ΔABN = ΔACN; S.A.S.
14. ΔONA = ΔONB; 90°, H.S.

Exercise VIII: g (p. 129)
1. 24, 2 in.; 3; 10, 8 cm.; 5; 1-61 in., 0-644, 53°; 6-44 cm.,
   0-644, 53°. 2. 9, 7-5, 6, 4-6 cm.; 1; 2, 12-5 in.
3. 28°, 1-54. 4. 1-16. 5. 1-54. 6. 104°.
7. DE = 11-35 cm., EF = 5-02 cm., FD = 12-69 cm.;
   ∠E = 93°. 8. 60°. 9. 3; 10. 62° or 118°.

Exercise VIII: h (p. 131)
1. ΔABC, ΔQRP; test (i); QP = 9-4, ∠Q = 46°, ∠R = 73°.
2. (i) 3-6, 8-4 cm. or 4-8, 3-6 cm.; (ii) 46°, 61°.
3. ΔDEF, ΔZYX; test (iii); ∠X = 122°, ∠Y = 23°, ∠Z = 35°.
4. ΔGHK, ΔVST; test (ii); SV = 2, TV = 3-27, ∠S = 79°.
5. (i) 18, 19-6 in.; (ii) 79°, 64°.
6. ΔABC, ΔAKH; test (ii); HK = 2-2, KC = 2-7.
7. ΔABC, ΔMBP; test (ii); AC = 6, MC = 6.
SIMPLIFIED GEOMETRY

8. \( \triangle PSN, \triangle ORQ \); test (ii); \( PN = 2\frac{3}{4}, PS = 4\frac{1}{2} \).

9. \( \triangle ANB, \triangle PMB \) in Fig. 279, test (iii), hence \( \angle ANB = 90^\circ \);
\( \triangle ANB, \triangle CAB, \triangle CNA \), test (ii); \( AC = 7\frac{1}{2}, CN = 4\frac{1}{2} \).

Exercise VIII. j (p. 136)
10. 45°, 60°, 150°, 15°.

Oral Example (p. 137)
Length = 15 units, breadth = 8 units; area = 120 units of area.

Exercise IX. a (p. 138)
1. 26 sq. ft. 2. 33 sq. ft. 3. 55 sq. ft. 4. 44 sq. ft.
5. 44 sq. ft. 6. 18 sq. ft. 7. (63 - 30) sq. ft.
8. (90 - 35) sq. ft. 9. (144 - 4 \times 25) sq. ft. 10. 104 sq. ft.
11. 748 sq. ft. 13. \((a + b)(c + d) = ac + ad + bc + bd \).
14. \((x + 2)(x + 3) = x^2 + 5x + 6 \);
\((a + b)(x + y + z) = ax + ay + az + bx + by + bz \).
16. \((a + b)^2 = a^2 + 2ab + b^2 \).
17. 23 by 19; 437 sq. mm., 4-37 sq. cm.
18. Unit, \(\frac{1}{2} \) in.; 10 by 7; 70; 4\frac{1}{2} sq. in.
19. 30 units of area.
20. 25 sq. mi.; 4 sq. mi.; between 24 and 25 sq. mi.
21. 100 sq. mi.; 16 sq. mi.; between 96 and 100 sq. mi.

Exercise IX. b (p. 141)
1. 48, 24 sq. in.
2. 12 sq. cm., 1\frac{1}{2} sq. ft., \(\frac{1}{2} \) sq. yd.; \(\frac{1}{2} \) bh sq. in.
3. 42-5 sq. in.; 17, 6 in.
4. 458 sq. ft. 5. 18 sq. ft.
6. 68 sq. cm. 7. 12-5 sq. in. 8. 49 sq. ft. 9. 8 in.
10. 5 in.; 5 cm. 11. 33 sq. in. 12. 14 sq. in.

Exercise IX. c (p. 143)
1. (i) 21, 21 sq. in.; (ii) 23 sq. in.; (iii) 49 sq. in.; (iv) 28 sq. in.
2. (i) 0-9 cm., 1-2 cm.; 1-1 sq. cm.; (ii) 2-0 cm., 0-53 cm.; 1-1 sq. cm.
3. 1-81; 2-72 in.; 5-43 or 5-44 sq. in. 4. 72 sq. in.; 9 in.

Exercise IX. d (p. 146)
1. In Fig. 317, (i) \( AD = 4 \) cm.; (ii) \( BE = 3 \) cm.; (iii) if \( CF = 2 \) cm., \( AB = 12 \) cm.
2. 28, 10, 4 sq. in.; 14 sq. in.
3. 48 sq. cm.; 6 cm.; 6 cm.; 6\frac{1}{4} cm.
4. 36 sq. cm., 9 cm., 9 cm.; 3 cm.
5. 2\frac{1}{2} cm. 6. 5-33 cm or 11-31 cm.

Oral Example (p. 149)
Area of \( ABCD \) is 24 sq. in.

Exercise IX. e (p. 150)
1. 29-8 (5) sq. cm. 2. 38-8 sq. cm.
3. \( 8X = 7-58 \) cm., \( BY = 10-11 \) cm.; area = 15-2 sq. cm.
4. 27, 15 sq. cm.; 42 sq. cm.
5. 42 sq. cm.
6. 9, 6, 15 sq. in. 7. 9 sq. in.; 3-6 in. 8. 41-6 sq. cm.

Exercise X. a (p. 154)
1. \( \angle t, \angle r; \angle f, \angle g \).
2. \( \angle f \).
3. \( BA, BC \).
4. \( CA, CB \).
5. 114°, 6. 54°.
6. 58°, 7. 86°.
8. 76°.
9. 67°.
10. 68°, 11. 132°, 12. 50°; \( AC = AB \).
13. \( AB = BC \).
14. 15. \( \angle h + \angle r = 180° \).
16. (i) \( AB = AC \); (ii) \( BA = BC \).
17. (i) \( \angle - \angle t, AC = AB \); (ii) \( \angle r = 90° \).
18. \( \angle r = \angle v = 60° \).
19. Same as No. 18.
20. \( AB = AC \).

Oral Examples (p. 155)
1. \( x^2 + x^2 = 148°; \angle ABC = 74° \).
2. \( 4x^2 + 4x^2 + x^2 = 180°; \angle BAC = 20° \).
SIMPLIFIED GEOMETRY

Exercise X.b (p. 155)
1. 104°, 2. 68°, 3. 80°, 20°, or 50°, 50°.
2. (i) 25°, 25°; (ii) 53°, 53°, or 70°, 40°; (iii) 60°, 60°.
3. 70°, 70°, 40°, or 40°, 10°, 100°.
4. (i) 25°, 25°, 130°; (ii) 54°, 54°, 72°, or 72°, 72°, 36°; (iii) 60°, 60°, 60°.
5. (i) 50°; (ii) 35°; (iii) 50°; (iv) x = 2y.
6. 90°, 45°, 45°; 108°, 36°, 36°. 9. 72°, 72°, 36°; 80°, 80°, 20°.
7. (i) 100°; (ii) 110°; (iii) 144°; (iv) 2x + y = 360°; (v) 30°.
8. 30°, 18°; (ii) 34°, 26°. 12. x = 2x; \angle OBP, \angle OBP.
9. 13. 68°, 112°. 14. 15°, 40°. 15. \angle SOC, \angle SOD, \angle COD.

Exercise X.c (p. 157)
1. 50°; 23°; 2x°. 2. 140°, 63°, 29°.
3. (i) 62°, 46°, 108°, 54°; (ii) 2x°, 2y°.
4. (i) 42°, 108°; 60°, 33°; (iii) 2x°, 2y°.
5. (ii) 130°; 25°, 65°; sum 90°; (iii) 2x°, 2y°; x + y = 90.

Exercise X.d (p. 160)
1. Each 70°. 2. 120°, 60°. 3. 55°. 4. 220°, 110°.
5. 120°, 120°. 6. 160°, 200°, 100°. 7. 75°, 105°. 8. 115°, 9. 75°. 10. 180°. 11. 180°. 12. HSTK; \angle p.
13. HLMK; \angle q. 14. \angle MON, \angle MKN.
15. \angle THS, \angle TKS (or \angle TLS, TMS, TNS).

Oral Examples (p. 163)
1. (i) \angle p, \angle q; (ii) 220°; 70°, 110°; 180°; (iii) 260°, 100°, 50°, sum 180°. 2. (i) 50°, 190°; (ii) 70°, 110°.

Exercise X.e (p. 164)
5. \angle ADD = 70° = \angle ACB; \angle DAB = 80°; \angle ABD = 30°; \angle KAD = 100°; \angle DBE = 150°.
6. \angle ACB = 50°, \angle BDC = 55°, \angle ADG = 75°, \angle ABC = 75°, \angle CBE = 105°.
7. \angle BAC = 50°, \angle DAC = \angle DBC = 50°, \angle DCB = 100°, \angle BCF = 80°.
8. \angle DAC = \angle DBC = 35°; \angle BAC = \angle BDC = 45°; \angle ABD = = \angle ACD = 45°, \angle ADB = 60°, \angle DNC = 95°, \angle BNC = \angle AND = 85°, \angle KAB = 105°, \angle EBC = 100°, \angle FCB = 75°.

ANSWERS TO PART II

9. \angle CBD = \angle ABD = \angle CAD = 35°, \angle ADB = \angle ACB = 60°, \angle ADG = 70°, \angle BAC = \angle BDC = 50°, \angle FCB = \angle BNC = \angle AND = 85°, \angle ANS = \angle DNC = 95°.
10. 90°.

Exercise X.f (p. 165)
1. 110°, 55°, 125°, 65°. 2. 55°. 3. 107°, 146°, 70°.
4. 47°, 62°. 5. 65°. 6. 50°, 110°, x = 2y, 7. 105°.
8. 50°, 35°, 80°. 9. 72°, 36°, 72°. 10. 72°, 144°.

Oral Examples (p. 173)
(i) 16, 9, 16, 9 sq. in.; (ii) 25 sq. in.; 5 in.; (iii) EX = 3.2 in., CX = 1.8 in.

Exercise XI.a (p. 174)
1. 64, 36, 64, 36, 100 sq. cm.; 10, 6-4, 3-6 cm.
2. 144, 169, 144, 25, 25 sq. cm.; 5, 11\sqrt{5}, 51\frac{1}{4} \text{ cm}.
3. 49, 625, 49, 576, 376 sq. in.; 24, 1-9-6, 23-04 in.
4. 4, 9, 3-6 sq. in.; 3-6 in. 5. 225, 289, 64 sq. cm.; 8 cm.
6. 225, 144, 81 sq. cm.; 15, 12, 9 cm.

Oral Examples (p. 174)
1. x = 15. 2. BN = 5 in.; AN = 12 in.; area = 60 sq. in.
3. HE = 8.

Exercise XI.b (p. 175)
1. 13 in. 2. 6 cm. 3. 6-4 cm. 4. 2-9 in.
5. 3-2 cm. 6. 2 in.; 3 sq. in. 7. 5 in. 8. 7-2 cm.
9. 18\frac{2}{3} ft. 10. 5-7 in. 11. 2-8 in. 12. 5-2 cm.

Exercise XI.c (p. 176)
1. \sqrt{3-3} - 61. 2. 3, 16, 3-74 in. 3. 2-24. 4. 6 in.; 3-6 cm.
5. 1-5 in. 6. 21 in. 7. 8 in. 8. 9 in. 9. 12 in.
10. 19-2 in. 11. 15, 13, 15-8, 13, 15-8 in. 12. 60, 65 ft.

Exercise XII.a (p. 180)
1. (i) 9 cm. by 6 cm. (or 9 cm. by 13-5 cm.); (ii) 6 cm. by 4 cm.; (iii) 12 in. by 8 in.; (iv) 1-8 cm. by 1-2 cm.; (v) 15 cm. by 10 cm.; (v) 45 cm. by 30 cm.
Simplified Geometry

2. (i), (iii), (iv). 3. $2x = 5y$ (or $2y = 5x$). 4. A, B; no.
5. All; no. 6. 25 ft. by 10 ft. or 10 ft. by 4 ft. 7. All.
8. 15, 12, 9 in.; 2, 2$\frac{3}{4}$, 3$\frac{1}{2}$ in. 9. (i), (ii), (iv), (vi).

Exercise XII. b (p. 182)

1. 72; (i) 2, 5, 6 cm.; (ii) 1-2, 3, 3-6 in.
2. 21-4 mi.; 6. 82$\frac{1}{2}$° E.
3. All similar; (i) PR = 2 in., PQ = 3$\frac{3}{4}$ in.; (ii) TK = 2-4 in.,

\[KX = 4-8 \text{ in.};\] (iii) $4\frac{1}{2}, 2\frac{1}{2} \text{ cm.}$. 4. All. 5. 11$\frac{1}{2}$, 5 in.
6. The smaller part. 7. No. 8. 1 in. by 2 in.; 2 in. by 4 in. 9.

\[AX = 1\frac{1}{2}XY, AB = 1\frac{1}{2}BC; AQ = 3QY = YC.\]
10. PQYX, PQCB, XYCB; none.

Exercise XII. c (p. 184)

1. 3 ft. by 2 ft. by 1-2 ft.; 20 cm. by 13$\frac{1}{4}$ cm. by 8 cm.
2. Yes. 3. 2$\frac{1}{2}$, 2 cm. or 6, 3 cm. or 8, 5$\frac{1}{2}$ cm.
3. (i) Height 40 in.; (ii) Diameter 14-4 in.
4. No; yes, the largest. 7. No, yes.
5. No. 8. Smallest face of (i) and the middle-sized face of (ii).

Exercise XII. d (p. 185)

1. (i) 1:2, 1:4; (ii) 1:3, 1:9; (iii) 2:3, 4:9; (iv) $p : q, p^2 : q^2$.
2. 1:9, 1:25, 1:100, 1:9. 3. 9:16.
6. Yes; 16:1:4; 1:16. 8. (ii) 8 cm., 1:4; (iii) 5-6 cm., 25:49; (iv) $\frac{1}{3}$, 1:9.
10. Area of largest = sum of areas of other two.

Exercise XII. e (p. 187)

1. 1:8, 1:27, $x^3 : y^3$; 1:4, 1:9, $x^2 : y^2$.
2. $x^8$, $x^7$, $x^6$, $x^5$, $x^4$, $x^3$, $x^2$. 3. 27:125.
4. 8. 5. 3 in., 2 in.; 27:8, 9:4.

Exercise XII. f (p. 188)

1. 1:3600; 9-6 sq. in. 2. 24 sq. ft. 3. 1000.
4. 44, 216, 612 lb. 5. 9. 6. 3$\frac{3}{8}$ qts.; 5$\frac{1}{2}$ oz. 7. 64,000.
8. 2s. 3d. 9. 6 in. 10. 192 sq. in.; 212$\frac{1}{2}$ F.

Answers to Part III

Exercise XIII. a (p. 192)

7. 70°; 150°. 8. 45°. 9. 40°, 110°. 10. 6-14 cm.
11. 5-5, 3-16 cm. 12. 1-80 in. 13. No; no; between 8-5 cm. and 1-5 cm.; 6-10 cm. 14. 106$\frac{1}{2}$; 8 cm. 15. 1-56 in.; 1-56 sq. in.

Exercise XIII. b (p. 194)

6. At rt. $\angle$ s; 30°, 120°. 11. CP || AB.

Exercise XIII. c (p. 196)

3. 2-30, 2-5, 3-35 cm. 4. 3-75 cm.
6. 6-65, 1-35 cm.

Oral Examples (p. 197)

1. $r = 0$-5. 2. $r^2 = (12 - r)^2 + 4^2$; $r = 6\frac{1}{2}$.

Exercise XIII. d (p. 197)

1. 3 cm. 2. 9-8 cm. 3. 13 cm. 4. 2$\frac{1}{2}$ in. 5. 12 cm.
6. 5-3 cm. 7. 7 in. or 1 in. 8. 5 in. 9. 7-25 cm.
10. 2-8 cm. 11. 16-1 in. 12. 8 cm.

Exercise XIV. a (p. 203)

4. (iii) $\frac{1}{2}$. 7. $\frac{3}{2}, \frac{1}{2}$. 8. $\frac{3}{2}, \frac{1}{2}$. 1
SIMPLIFIED GEOMETRY

Exercise XIV. b (p. 205)
1. \(\frac{1}{2}, 2, 3, \frac{1}{3}, \frac{3}{4}, \frac{4}{3}, 3\).
2. 2 : 1, 2 : 3, 2 : 3, 1 : 1.
3. P, Q, R, G.
4. 3 : 2 ; 3 : 1 ; 4 : 7 ; 3 : 5.
5. P, R, T ; A.

Exercise XIV. c (p. 205)
1. \(\frac{3}{2}, \frac{2}{3}\).
2. \(\frac{4}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}\).
3. \(\frac{7}{10}, \frac{2}{3}\).
4. 4:8, 14, 4:8 in.

Oral Examples, Diagonal Scale (p. 209)
(i) 0:2, 0:04, 0:12, 0:16 in. ; (ii) 0:24, 0:68, 0:72 in. ;
(iii) 2:24, 1:68, 2:72 in. ; (v) \(\frac{1}{2} + \frac{1}{8}, \frac{1}{4} + \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\);
(vi) 2\(\frac{1}{8}\) ; (vii) 3\(\frac{1}{8}\).

Exercise XIV. d (p. 210)
1. 4:24, 3:39, 1:84.
2. 1:57, 2:76 in. ; 0:39 (4) in.
3. 1\(\frac{1}{4}\) in.
4. 7 in. ; 1:9 in.
5. 13 cm. ; 4:2 in.
6. 5:5 cm. ; 7:2 cm. ; 8:2 cm.
7. 8:1 : 2.
9. A ; BA produced.
10. \(\frac{1}{2}\), \(\frac{2}{3}\). ; 6, 15 in.
11. \(\frac{3}{8}\).

Oral Examples (p. 212)
1. (i) 10\(\frac{3}{4}\), 13\(\frac{1}{2}\) cm. ; (ii) \(\frac{7}{8}\), \(\frac{9}{8}\).
2. (ii) \(\frac{9}{8}\).

Exercise XIV. e (p. 213)
1. 13\(\frac{3}{4}\), 10\(\frac{3}{8}\) cm. ; 0:8, 0:8.
2. (i) 2\(\frac{1}{4}\), 3 in. ; 1\(\frac{1}{8}\) ;
(ii) 2\(\frac{1}{4}\), 3:6 in. ; \(\frac{3}{8}\), \(\frac{5}{8}\).
3. (i) 2\(\frac{1}{2}\) in. ; (ii) 2\(\frac{1}{4}\), 3:2 in. ;
(iii) \(\frac{4}{3}\), \(\frac{5}{3}\). ; 4. (i) \(\frac{5}{8}\) ; 5\(\frac{1}{4}\) cm. ; 2\(\frac{1}{2}\) cm. ; 2\(\frac{1}{2}\) cm.
(ii) 5\(\frac{3}{8}\), 3:6 cm. ; (iii) 5\(\frac{3}{4}\), 3:2 cm. ; 5. 9\(x\) + 8\(y\) = 72.
6. 6\(\frac{2}{3}\) in. ; 3:2.
8. 14:4 in. 10. 2:4 in. ; \(\frac{3}{4}\).
11. 2 : 5 ; 5 : 1.
12. 3:6 in.

ANSWERS TO PART III

Exercise XV. a (p. 220)
1. 30\(^\circ\), 100\(^\circ\), 50\(^\circ\).
2. 55\(^\circ\), 85\(^\circ\), 40\(^\circ\).
4. Rhombus ; 35\(^\circ\), 145\(^\circ\).
5. 4 cm. ; 6:4 cm. ; 7:6 cm. ; 8:1 cm.
9. 4:48, 2:02 cm. ; 6:5 cm.

Oral Examples (p. 222)
1. \(x^2\) = 20 ; 4:5 cm.
2. \(y^2\) = 41 ; 6:4 cm.
3. \(\angle PTQ = 50^\circ\), \(\angle OTQ = 25^\circ\).

Exercise XV. b (p. 222)
1. 55\(^\circ\), 70\(^\circ\), 40\(^\circ\), 80\(^\circ\), 35\(^\circ\), 55\(^\circ\).
2. 25\(^\circ\), 140\(^\circ\).
3. 12 cm. ; 17 in. ; 4:6 in. ; 18, 8 cm. ; 5, 36\(^\circ\), 54\(^\circ\) ; 22\(^\circ\), 68\(^\circ\).
6. 50\(^\circ\). ; 7. 16 cm. ; 8. 3 cm.
9. 1:4, 0:8 in.
10. 2 cm. ; 3:5 in. ; 11. 34 in.
12. rect. ; 5, 7 cm. ; 24, 24 cm.

Oral Examples (p. 225)
1. 5, 6, 7 cm.
2. (i) Q, R ; (ii) 4:5, 3:5, 2 cm.
3. (9 - x) + (8 - x) = 6 ; 5:5 cm. ; 3:5 cm. ; 2:5 cm.

Exercise XV. c (p. 226)
1. 10 cm. ; 8, 4 cm. ; 2. 6 cm. ; 2:5 in. ; 1 in. or 5 in.
4. 2, 10 cm. ; 5. 4:5, 5:5, 6 cm. ; 6. 7 cm. ; 3 cm.
8. 2:6 cm. ; 10. 2:6, 2, 1:8 cm.
10. 4:6, 3:8, 1:6 cm.
11. Two circles, \(AC = 3\) cm., \(BC = 3:4\) cm., and two with centres on \(AB\) produced at distances 1:4 cm., 3:4 cm.
12. 5, 2 cm.

Oral Examples (p. 228)
1. 35\(^\circ\), 90\(^\circ\), 55\(^\circ\), 55\(^\circ\), 125\(^\circ\), 125\(^\circ\).
2. 18\(^\circ\), 90\(^\circ\), 72\(^\circ\), 72\(^\circ\), 108\(^\circ\), 108\(^\circ\).
SIMPLIFIED GEOMETRY

Exercise XV. d (p. 230)
1. 50°, 68°, 62°. 2. 70°. 3. 85°, 45°. 4. 70°, 26°.
5. 62°, 64°. 6. 117°; 44°. 7. 94°; 8°. 8. 54°, 36°.
9. 115°. 11. 2-44 cm. 12. 2-92 cm.

Exercise XV. f (p. 235)
6. 7-02 cm. 7. 2-60 cm. 8. 2 cm.; 5 cm.
9. 1-91 in. 10. 1-83 cm.

Exercise XVI. a (p. 240)
1. $E Q = 4\frac{1}{4}$ in., $K Q = 3\frac{3}{4}$ in., $M S = 6\frac{3}{8}$ in., $G S = 5\frac{1}{2}$ in.
2. (ii) $3\frac{1}{3}$, 2 cm.; (iii) $3-6$, 2-4 cm.; (iv) $4-5$, 5 cm.
3. (i) $6\frac{3}{8}$, 9 in.; (ii) $2\frac{1}{2}$, 2 in.; (iii) $6-4$, 9-6 in.
4. (i) $16$, 18 in.; (ii) $3\frac{1}{4}$ in. 5. 48 ft. 6. 60 ft.
7. $\frac{7}{8}$, $\frac{3}{8}$; $12$ ft. 8. 4 cm.; 4-5 cm. 9. 6 ft. 8 in.
10. 5 ft. 11. 6 in. 12. 6 in.

Exercise XVI. b (p. 243)
4. 5-32; 42-1; 3-93. 5. 3, 4, 5; right-angled.
6. (i) Yes; (ii) No; (iii) Yes; (iv) Yes. 7. $P Q = \perp B C$.
8. 4-13; 2-52; 0-83. 9. 5-88 cm. 10. 12-8.
11. $\Delta ABC$, $\Delta ARP$; 2 in. 12. $4\frac{1}{4}$ in. 13. $\Delta APC$, $\Delta DPB$; cyclic quad.
14. $\Delta APD$, $\Delta CPB$. 15. $\Delta APR$, $\Delta ASQ$; $\Delta APS$, $\Delta ARQ$; cyclic quad.

Exercise XVI. c (p. 245)
1. (i) $\Delta APD$, $\Delta CPB$; (iii) $2$ cm.; (iv) 2-4, 1-6 in.
2. (i) $\Delta AQD$, $\Delta QCB$; (iii) $8$ cm.; (iv) 14-4, 9-6 cm.
3. (i) $\Delta DQB$; (iii) 3, 2-5 cm.
4. $\Delta CAB$; 10-5 in.; 12-2 in.; $\frac{5}{2}$.

ANSWERS TO PART III
5. (ii) $\triangle CNA$; (iii) $\triangle NBA$, $\triangle NAC$; (v) 4-8, 6-4 cm.;
(vi) $3\frac{1}{4}$, 2$\frac{1}{4}$ in.
6. (i) $\triangle PAT$; (ii) $\frac{PB}{PT}$
 (iii) 6-75, 5 cm.; (iv) $\sqrt{(im)}$ in.;
$\frac{PT}{PA}$
7. $\Delta APB$, $\Delta PNB$; 6, 10, 7-5 cm. 8. 2-5 in.
9. 4 in. 10. $\Delta ABC$, $\Delta BCP$; 4 cm.: $\frac{m^2}{l}$ in.

Exercise XVI. e (p. 252)
1. 12, 20, 15 cm. 2. 0-6, 1-56 in.
3. 3-45; $AN = 1-86$ (5) cm.; 2-9, 1-2 cm. 4. 1-67; 1-95.
5. 12; 2-45; 2-65; 3-46. 6. 6-36. 7. 4-68 cm.
8. 3-29 cm. 9. 4 in.; 10 cm.; 12 in. 10. 2 in.; 1$\frac{1}{2}$ in.
11. 6 in.; 7 in.; 3-5 cm.; 6-25 cm. 12. 6, 2-5 cm.
13. 5-66, 6 cm. 14. 40 ft. 15. 16$\frac{2}{3}$ cm.

Exercise XVI. f (p. 255)
11. 3 cm., 15 cm. 12. 1-5 cm.; 3-4 cm.

Oral Examples (p. 257)
1. 0-4663, 0-8391, 1-1918, 2-1445.
2. 30°, 58°, 60°, 57°. 3. 0-9; 42°; 1-6, 58°; 2-05, 64°.
4. 7-00 cm.; 9-81 in.; 3-60 in.

Exercise XVII. a (p. 258)
1. 0-8391, 1-1918, 2-1445, 7-115, 0-4557, 0-7646, 1-1875, 1-9797, 2-0534, 0-5467, 1-3738, 1-0532.
2. 18°, 49°, 68°, 45°, 17°, 36°, 38°, 48°, 50°, 18°, 73°, 42°, 20° 2°,
20° 33°, 20° 28°, 20° 14°, 45° 25°, 55° 44°, 63° 48°, 68° 38°.
3. 26°, 34°, 56° 10°, 70° 1°, 84° 18°, 14° 2°, 18° 26°, 33° 41°,
23° 12°, 57° 32°, 60°.
Simplified Geometry

Exercise XVII.b (p. 259)

1. 0°8, 38° 39½'; 1°2, 50° 12'; 1°6, 58°; 0°7, 35°.
2. 3° 84'; 290°; 14°3; 8°11. 3. 81° 10'; 14° 28'; 126° 8'.
4. 272°; 1°37'; 9°44'. 5. 187° 5 ft. 6. 38° 39½'.
7. 15° 6 ft. 8. 8° 20 cm. 13. 5° cm. 9. 30° 58'. 10. 1° 40 in.
11. 8° 58 cm. 12. 48° 49'. 13. 24×73 sq. cm.
14. 73° 44', 106° 16'.

Oral Examples (p. 262)

1. 0° 42° 26', 0° 9063; 0° 6293, 0° 7771; 0° 7771, 0° 6293; 1° 0; 0, 0, 1.
2. 36° 52'; 41° 24'. 3. 0° 47, 28°; 0° 85, 58°; 3°, 42°.
4. 1° 36 in., 1° 46 in.; 4° 85 cm., 3° 53 cm.; 3° 09 cm., 9° 51 cm.

Exercise XVII.c (p. 263)

1. 0° 37° 46', 0° 5, 0° 9272, 0° 9994, 0° 4289, 0° 7157, 0° 0622, 0° 4184, 0° 8200, 0° 5102.
2. 0° 9272, 0° 6660, 0° 3746, 0° 0349, 0° 8231, 0° 4431, 0° 4426, 0° 8809, 0° 5605, 0° 6691.
3. 31° 56', 41° 36', 46° 48', 18° 28', 69° 22', 64° 95', 26° 46'.
4. 76°, 55°, 37° 18', 25° 42', 68° 10', 56° 8', 34° 9', 18° 7', 61° 18', 51° 44', 34° 33', 60° 39'.

Exercise XVII.d (p. 263)

1. t, 1, 1, 1; 0, 0, 0; a, 1, 1, 1; 0, 0, 0; β, 1, 1, 1; 0, 0, 0; γ, 1, 1, 1; 0, 0, 0.
2. sin B, cos C; cos P, sin R; sin E, cos F; tan Z; sin Ρ, cos R.
3. tan B; sin X, cos Z; tan F; sin C, cos B; tan P.
4. 4° 23, 9° 06; 0° 727, 1° 86. 5. 1° 43, 4° 79.
6. 4° 45; 10° 95. 7. 2350, 4415 yd. 8. 503 ft.
9. 33° 22'; 53° 8'; 63° 26'; 51° 19'. 10. 14° 29'.
11. 49° 15'. 12. 6° 82 cm. 13. 5° 88 cm. 14. 77° 22'.
15. 15° 5 sq. in. 16. 41° 24', 70° 22'.

Answers to Part III

Exercise XVII.e (p. 266)

1. (i) 1° 6616, 1° 6243, 1° 6464, 1° 6452; (ii) 1° 4396, 1° 4663, 1° 4474, 1° 4492; (iii) 0° 8998, 0° 7813, 0° 7926, 0° 7912.
2. (i) 66° 24', 66° 26', 29° 40'; (ii) 56° 18', 56° 22', 72° 15'; (iii) 31° 48', 31° 46', 65° 14'.
3. 31° 45', 39° 43', 34° 3', 45° 35'.
4. 12° 35, 15° 89'; 3° 698, 4° 125; 4° 284, 5° 230.
5. 500° 8 ft. 6. 3° 916 cm. 7. 4° 498 in. 8. 11° 16 in.
9. 0° 27 in. 10. 5° 390, 6° 426 cm. 11. 6° 34; 6° 734 in.; 12. 4° 447 in.; 13. 5° 385 in.; 41° 12', 52° 54', 85° 54'.
12. 2° 86, 4° 98, 6° 08 in. 13. 2° 01 cm. 14. 133 ft. per sec.
15. 9° 33 cm.

Exercise XVII.f (p. 268)

1. (i) 8° 485, 11 cm.; (ii) 39° 31'; (iii) 49° 24'; (iv) 66° 7' or 66° 8'; (v) 81° 12'.
2. (i) 15, 17 in.; (ii) 28° 4'; (iii) 44° 54'; (iv) 40° 8'; (v) 41° 38'; (vi) 89° 48'. 3. 10° 39'; 12° 6'.
4. 36° 52', 40° 54'. 5. 3° 86 in.; 22° 41'; 6° 62 in., 35° 38'; 3° 03 (5 in.). 6. 2° 32'. 7. 57° 48'.
8. 6° 39'; 8° 77 cm.; 61° 4'.
9. (i) 3° 46 cm.; (ii) 54° 45'; 4° 90 cm., 70° 32'.
10. (i) 4° 25 in.; (ii) 37° 53', 6° 77 (5 in.), 63° 5'.

Oral Examples (p. 272)

1. 9° 40 sq. in.; 9° 40 sq. in.; 2° 52 sq. cm.; 2° 52 sq. cm.
2. 0° 7660, 0° 4226, 0° 8878, 0° 5334.
3. 63°, 117°; 17° 48', 162° 12'; 39° 26', 140° 34'; 35° 28', 144° 32'.

Exercise XVIII.a (p. 274)

1. 0° 3429, 0° 9397, 0° 9998, 0° 6820, 0° 1219, 0° 9767, 0° 8118, 0° 9077.
Simplified Geometry

2. 24°, 156°; 49° 18', 130° 42'; 28° 40', 151° 20'; 68° 45', 111° 15'.
3. 4-37.  4. 5-16.  5. 37° 54'.  6. 44° 49½'.
7. c = 10-4, a = 8-19.  8. 61° 1', 8-95, or 118° 59', 3-13.
33° 25', 12-1.  10. 30° 36', 1-39.  11. 29° 39', 13-9 (5).
12. 9-37, 7-10.  13. 56° 11', 7-99.
14. 52° 47', 10-10, or 127° 13', 235.  15. 11-8, 9-16.

Oral Examples (p. 278)
2. 131°, 109°, 156°, 136° 12', 109° 33', 101° 34', 113° 27', 95° 45'.

Exercise XVIII b (p. 277)
2. 130°, 105°, 144° 24', 160° 12', 105° 46', 124° 46', 154° 10', 91° 28'.
3. 3-19.  4. 6-34.  5. 78° 28'.  6. 109° 28'.
7. 7-04, 27° 47'.  8. 2-97 (5), 10° 29'.  9. 43° 32', 100° 7'.
10. 4-01, 20° 42'.  11. 5-04, 105° 40' (42').  12. 47° 14', 74° 12½'.

Exercise XVIII c (p. 278)
1. 1030 yd.  2. 8-79 mi.  3. 26° 23', 10-9 ft.
4. 9-94 sea-miles.  5. 137 ft.  6. 52° 24', 7-92 cm.
7. 12-7, 10-6 ft.  8. 12-35, 24-35 ft.  9. 3-23 (5) cm.
10. 1-06, 1-53 in.  11. 136° 25'.  12. 110° 47'.
13. 46° 34'; 43-8 sq. cm.  14. 49° 27'; 34-2 sq. cm.
15. 50° 29' on each side of AD.  16. 4-36, 1-42 cm.