A NEW GEOMETRY FOR SCHOOLS

STAGE A

STAGE B

CLEMENT V. DURELL

STAGE A and STAGE B

NEW GEOMETRY FOR SCHOOLS

DURELL

STAGE A and STAGE B

BELL
A NEW GEOMETRY
FOR SCHOOLS

This book is issued in the following
styles:—

STAGE A
STAGE B (PARTS I-III)
STAGES A & B together

The three parts of Stage B are also available
in separate form, and Parts I & II are also
issued bound together.

EXERCISES & THEOREMS
IN GEOMETRY

This is an alternative arrangement of the
material in A New Geometry, in which, after the
Stage A work, Exercises, Constructions and
Theorems respectively are for the most part
collected into separate sections.

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A NEW GEOMETRY
FOR SCHOOLS

BY
CLEMENT V. DURELL, M.A.
AUTHOR OF "GENERAL ARITHMETIC," "A NEW ALGEBRA FOR SCHOOLS,"
"TRIGONOMETRY," Etc.

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A volume of HINTS AND SOLUTIONS is issued for the use of teachers.

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PREFACE

It is now almost fourteen years since the author's Elementary Geometry was published, and, in writing this entirely new book, he has taken the opportunity to recast his treatment of the subject in the light of the experiences gained, and the suggestions received, since Elementary Geometry appeared. He has been able, also, as will be seen later, to make full use of the Second Report of the Mathematical Association on the Teaching of Geometry.

This book contains a course of geometry from the first stage up to the standard of the School Certificate and similar examinations. The object of the author has been to provide a treatment which lends itself both to class-teaching and to individual use by the pupil. The plan adopted throughout is to develop each group of geometrical facts by the following successive stages:

(i) Examples for oral discussion.

These are illustrated extensively by diagrams in order to simplify black-board work.

This oral work gives the pupil a clear understanding of the relevant facts, familiarises him with the arguments which will be used later in the formal proofs of theorems, and trains him in methods for solving riders. It includes, when appropriate, questions in which the data are numerical.

(ii) An exercise of numerical examples.

This gives practice in applying the facts deduced from oral discussion and ensures a firm grasp of these facts.

(iii) Formal proofs of the corresponding theorems.

The preliminary work makes it possible to deal with these proofs rapidly. Practice in writing out theorems is essential for examination purposes, but it will often be found sufficient
to confine this to the key-theorem of each group, regarding
the others as simple riders.

(iv) An exercise of riders.

The early examples in each exercise are direct and very
simple applications of the properties of the group. Some assis-
tance is supplied for the harder examples, but notes on method
and hints of useful constructions are included in the text.

The prominence in the text of the examples for oral
discussion is due to the author's conviction that this not
only facilitates the learning of formal proofs of theorems but,
what is far more important, is the best method of strengthening
the power of the pupil to tackle riders by showing him the
types of constructions most often required, by helping him to
assimilate the fundamental facts, and by making him familiar
with the forms of argument he must be able to employ.
Although these examples are called "oral," it is suggested
that all pupils should be required to write down the answers
to each question.

The examples in each exercise are classified under three
heads:
(A) Normal course: plain numbers.
These examples cover all essential types and have been
graded carefully. Most of them should be done by all pupils.
(B) Extra practice: numbers enclosed in brackets.
These examples provide further training if needed and are
parallel to those in A and do not extend the ground covered.
(C) Advanced course: asterisked numbers.
These are intended only for those pupils who run ahead
of the class.

For the convenience of teachers who prefer to make their
own selection, these groups are not printed in separate
sections, but the examples are arranged in order of difficulty.
NEW GEOMETRY

A volume of Hints and Solutions will be available for use with either book.

The author owes especial gratitude to Mr. K. R. Imeson for helpful advice. He tendered also cordial thanks to the several teachers who have been good enough, out of their experience, to make valuable suggestions and criticisms, which have enabled him to take into due consideration different needs and varying points of view.

C. V. D.

April, 1939.

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at end

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therefore

because

is equal to

is equivalent to

is congruent to

is not equal to

is approximately
equal to
the difference between
is greater than
is less than
is parallel to

adj. adjacent
alt. alternate
corr. corresponding
ext. exterior
int. interior
opp. opposite
quad. quadrilateral
rect. rectangle
seg. segment
sq. square
st. straight
vert. opp. vertically opposite

Ilgram parallelogram

angle
right angle
triangle
circumference
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### SQUARE ROOTS 1 TO 10

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</tbody>
</table>

### ADD |

| 1 2 3 4 5 6 7 8 9 |

### SQUARE ROOTS 10 TO 100

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</tbody>
</table>

### ADD |

| 1 2 3 4 5 6 7 8 9 |

### NATURAL SINES

| 1 2 3 4 5 6 7 8 9 |

### ADD |

| 1 2 3 4 5 6 7 8 9 |

### NATURAL COSINES

| 1 2 3 4 5 6 7 8 9 |

### SUBTRACT |

| 1 2 3 4 5 6 7 8 9 |

### NATURAL TANGENTS

| 1 2 3 4 5 6 7 8 9 |

### ADD |

| 1 2 3 4 5 6 7 8 9 |
STAGE A

FUNDAMENTAL IDEAS AND ASSUMPTIONS

Lines and Points

Lines are either straight or curved, e.g. the letter T is formed by two straight lines, the letter S is a curved line, and the letter D is partly straight and partly curved. You can make a straight line by folding a sheet of paper, the crease is a straight line. Straight lines are drawn by using a straight edge or ruler.

Two lines, straight or curved, see fig. 1, cross one another, i.e. intersect, at a point, called the point of intersection of the lines. A point has no size; it marks a position.

 Representation of a Point

<table>
<thead>
<tr>
<th>Right</th>
<th>Wrong</th>
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<tr>
<td>A</td>
<td>B C D</td>
</tr>
<tr>
<td>B C D</td>
<td>A</td>
</tr>
</tbody>
</table>

Fig. 2

Never represent a point by a blob, but always fix its position by taking the intersection of two lines; the positions of points B, C, D on the line BCD are shown by drawing short cross-lines. If a point is represented by a dot, the dot must be very small, in fact only just visible, smaller than this full stop. The dot then becomes invisible when two cross-lines are drawn through it.

Solids and Surfaces

Any object which occupies space is called a geometrical solid, e.g. a box, a sheet of paper, a tennis net or a sponge. Most solids are irregular in shape. There are some simple...
STAGE A

Kinds of solids whose names should be learnt. It is a great help to have models available; some of them should be constructed in thin cardboard or thick paper as soon as the use of the ruler, compass, and set-squares have been explained, see Exercise 23, p. 82. A photograph of the five regular solids is reproduced at the beginning of the book.

Fig. 3

A solid is bounded by a surface or portions of two or more surfaces. A surface, like a shadow, has no thickness; the surface is that which separates the space occupied by the solid from the space outside the solid. A surface may be plane or curved, e.g. the surface of a ball is curved, the surface of the top of a table is plane or as nearly plane as the carpenter can make it, and the surface of a jug is partly curved and partly plane.

The planeness of a surface is tested e.g. by a carpenter) by using a straightedge. The edge is placed against the surface: if it does not touch the surface at all points of the straightedge, the surface is not plane.

LINES, SURFACES AND SOLIDS

A plane surface is called, for short, a plane, and any portions of planes which bound a solid are called faces of the solid. A line in which two faces of a solid meet is called an edge of the solid, and a point at which three or more edges of a solid meet is called a corner or a vertex (plural, vertices).

EXERCISE 1

1. How many straight lines are used in (i) F, (ii) M, (iii) X?

2. Take a piece of paper and fold it so as to show creases like those in fig. 4. How many points are there whose positions are fixed by the intersections of two creases? What is the greatest number of other points whose positions are fixed by two creases if you make one more crease in the paper?

3. Use a straightedge to find which sets of three or more points in fig. 5 lie on a straight line (i.e. are collinear).

4. Mark on your paper points situated roughly as in fig. 6. Use a straightedge to mark in your figure a point which is collinear with (i) both A, E and C, G; (ii) both E, A and D, E; (iii) both A, B and C, E; (iv) both E, D and F, H.

[5] There are 4 huts scattered about in a field. Represent them by 4 points on your paper. How many paths may be needed if each pair of huts is connected by a straight path? Represent the paths by straight lines. How many paths may be needed if there are 5 huts? (How many paths lead to any one hut?)

6. What are the names of solids shaped like (i) a brick, (ii) an orange, (iii) a garden roller, (iv) a clown’s hat, (v) this page, (vi) a penny, (vii) a wedge, (viii) a funnel?

7. Write down the number of (i) faces, (ii) edges, (iii) corners of a brick.
[8] Write down the number of (i) faces, (ii) edges, (iii) corners of a prism with a 5-sided base, see fig. 3.

9. Draw a figure like fig. 6 (i). Then draw the lines $AA_1$, $BB_1$, $CC_1$, $DD_1$. What solid does the final figure represent? The result is improved if some of the lines are dotted. Do this.

![Fig. 6](image)

10. Repeat No. 9 for fig. 6 (ii).

11. Repeat No. 9 for fig. 6 (iii).

12. Make a sketch of (i) a triangular prism, (ii) a pyramid on a triangular base.

13. Make a table showing the number of faces, corners, and edges of various solids such as a cuboid, triangular prism, triangular pyramid, prism with 6-sided base, a beheaded 4-sided pyramid; also for a prism whose base has $n$ sides.

<table>
<thead>
<tr>
<th>Name of Solid</th>
<th>Number of Faces $F$</th>
<th>Number of Corners $C$</th>
<th>Number of Edges $E$</th>
<th>$F + C - E$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
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</tbody>
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What result is suggested by this table?

14. Point out in the room 3 lines which meet in a point but do not all lie in the same plane.

15. Point out in the room 2 planes which meet each other (i) in a single straight line, (ii) at only one point.

16. What does fig. 7 represent if you think of the point $O$ as (i) nearer to you than the plane through $A, B, C$, (ii) farther away from you than the plane through $A, B, C$?

Before Nos. 17-19 are worked, models of the octahedron, dodecahedron, and icosahedron should be shown and discussed, or reference may be made to the photograph; but the models should not remain in view while the examples are being done.

17. A solid has 8 faces, each of which is a triangle, and 4 faces meet at each corner. Use these facts to calculate the number of edges and number of corners.

18. A solid has 12 faces, each of which is a pentagon, and 3 faces meet at each corner. Use these facts to calculate the number of edges and number of corners.

19. A solid has 20 faces, each of which is a triangle, and 5 faces meet at each corner. Use these facts to calculate the number of edges and number of corners.

20. If in No. 5 there are $n$ hats, how many paths may be needed? [How many paths lead to any one hat?]

**Use of Ruler**

By using a ruler a straight line can be drawn through any two given points $A$, $B$; this is called joining $A$ and $B$. If the pencil point moves from $A$ to $B$ it is said to move in the sense $AB$, if it moves from $B$ to $A$ it is said to move in the sense $BA$. When joining two points always arrange the paper so that the line is drawn from left to right across the desk, not up or down it. The process of continuing the line $AB$ beyond $B$ in the sense $AB$ is called producing $AB$; similarly the instruction, produce $BA$, means continue the line $BA$ beyond $A$ in the sense $BA$.

Two scales are marked on the ruler: one shows inches and tenths of an inch, the other shows centimetres (cm.) and tenths of a centimetre, that is millimetres (mm.).

*Measurements of length should always be expressed in decimals, not fractions, because measurements are only approximate.*
Example. Measure in inches the line $AB$, fig. 8.

(i) Arrange the paper so that $AB$ runs across the desk, not up or down it.

(ii) Suppose $A$ is at the left end of the line $AB$. Place the ruler along $AB$ and arrange it so that one of the chief graduations is opposite $A$.

(iii) Place the ruler as close as you can to $AB$, and if it has a thick edge, stand it on its edge so as to get the graduation-marks close to $AB$. We now see that the length of $AB$ lies between 2·6 in. and 2·7 in., and we then judge by eye the number of hundredths of an inch and so estimate $AB$ as 2·64 in.

Another method of measurement consists in taking the dividers and opening them out so that one point rests on $A$ and the other point on $B$, and then reading off the answer by placing the dividers on the graduated scale; but it must be remembered that the zero graduation will get worn if it is used every time a measurement is made in this way.

Note. 2 inches is written 2 in. or $2^\prime$;
3 feet is written 3 ft. or $3^\prime$;
4 yards is written 4 yd. or $4^\prime$.

When drawing a line of given length, start by drawing a line which is too long, mark a point $A$ close to one end by drawing a short cross-line. Use the ruler to make a very small dot on the line at $B$ so that $AB$ is of the required length, and then draw a short cross-line through $B$. The dot should be so small that it becomes invisible as soon as the cross-line is drawn.

If dividers are used to mark off lines of given length, care must be taken that any holes made in the paper are very tiny; short cross-lines should be drawn as before.

Plane Figures

If part of a plane is marked off by straight or curved lines drawn in the plane, this part is called a plane figure. If it is bounded entirely by straight lines, it is called a plane rectilinear figure, but the word, plane, is usually omitted. Rectilinear figures are named according to the number of their sides:

![Fig. 10]

Also an 8-sided figure is called an octagon, and a 10-sided figure is called a decagon.

Any rectilinear figure may be called a polygon; e.g. a hexagon is a 6-sided polygon.

The symbol $\triangle ABC$ means the triangle $ABC$, and the usual abbreviation for quadrilateral is quad.

The point of intersection of two consecutive sides of a polygon is called a vertex, and a line joining two vertices which are not consecutive is called a diagonal, e.g. if in fig. 10 you join $AC$, the line $AC$ is one of the diagonals of the quadrilateral $ABCD$ or of the pentagon $ABCDE$ or of the hexagon $ABCDEF$. Polygons are named by taking the vertices in order in a clockwise or counter-clockwise direction. In fig. 10, the quadrilateral is called $ABCD$ or $ADCB$, not $ABDC$ or $ACBD$.

Perimeter of a Figure. The total length of the boundary of a figure is called its perimeter. Thus the perimeter of a polygon is the sum of the lengths of its sides.
STAGE A

Notation. The statement, $AB = 2''$, means that the distance of $B$ from $A$ is 2 inches. If $AB$, $CD$ are two given lines, the sum of the lengths of $AB$ and $CD$ is denoted by $AB + CD$. Thus the perimeter of the pentagon $ABCD$ is the value of $AB + BC + CD + DE + EA$.

If the line $PQ$ is longer than the line $XY$, the difference of their lengths is denoted by $PQ - XY$; if it is not known which line is the longer, the difference is denoted by $PQ - XY$.

**EXERCISE 2**

Nos. 1–9 refer to fig. 11. In Nos. 1–2, give the measurements both in inches and centimetres and keep them for future use.

**USE OF RULER**

1. Measure $AB$, $BC$, $CA$. Find the perimeter of $\triangle ABC$.
2. Measure $DE$, $EF$, $FD$. Find the perimeter of $\triangle DEF$.
4. Measure in cm., $LK$, $KN$, $MK$, $KP$. Find the sum of the lengths of the diagonals of quad. $LMNP$.
5. Measure the distances between the following pairs of points:
   - $A$, $F$ in cm.; $L$, $C$ in cm.; $M$, $A$ in inches.
7. Measure in cm. the lengths of $PY$, $ZN$, $PN$, $ZY$; and find in cm. the values of (i) $PY + ZN$; (ii) $PN + ZY$.
8. Find in cm. the value of $DE + EF - DF$.
9. Which of the lines $LP$, $EF$ do you think is the longer? Write down your answer, then check by measurement.

10. Which of the lines $AB$, $CD$ in fig. 12 do you think is the longer? Write down your answer, then check by measurement.

11. Draw a straight line 10 cm. long. Measure its length in inches and so express 1 cm. in inches.

12. Draw a straight line 5 in. long. Measure its length in cm. and so express 1 inch in cm.

13. Draw two straight lines $AOB$, $COD$ crossing each other and such that $AO = OB = 1.3$ and $CO = OD = 2.1$. Join $AC$, $AD$, $BC$, $BD$ and measure these four lines. What do you notice about the results?

14. Write down the number of diagonals of (i) a quadrilateral, (ii) a pentagon, (iii) a hexagon.

15. If the diagonals which pass through one corner of a hexagon are drawn, how many triangles are formed?

*16. Repeat Nos. 14, 15 for (i) a decagon, (ii) an $n$-sided polygon.

**Use of Compass**

A circle is a plane figure bounded by a curved line, all points of which are at the same distance from a point called the centre of the circle.
STAGE A

Take your compass and draw a circle. The point O where the steel point of one leg rests is the centre of the circle, and the tip of the pencil attached to the other leg traces out the curved line which is called the circumference of the circle.

Fig. 13

The line joining the centre O to any point P on the circumference is called a radius (plural, radii) of the circle; the length of any radius OP is the distance between the compass-points and therefore the pencil-point of the compass traces out a curve all points of which are at the same distance from the centre.

The straight line QR joining any two points on the circumference is called a chord, and any chord AOB through the centre O of the circle is called a diameter.

The words, radius, diameter, circumference, are also used to mean the length of the radius, diameter, circumference respectively, and the curved line is itself often called a circle.

Any portion of the circumference is called an arc, and a part of the circumference AOB cut off by a diameter AB is called a semicircular arc or for short a semicircle. In fig. 13, the arc QKR is called a minor arc because it is less than a semicircle, and the arc QLR is called a major arc because it is greater than a semicircle.

If two circles have the same centre, they are called concentric, see fig. 17, p. 12.

By using a compass it is possible to draw a circle having any given point as centre and any given length as radius. A compass may also be used to cut off from a given straight line any given length.

USE OF COMPASS

To draw a circle having a given line AB as diameter, find the middle point O of AB. This may be done by measuring AB, but later you will learn another way of finding the position of the middle point. Then draw a circle with O as centre and OA as radius.

Use of Intersecting Arcs. If A and B are given points, say 1-3" apart, and if it is required to find the position of a point P at given distances from A, B, say 1" from A and 0-65" from B, then P is somewhere on a circle, centre A, radius 1", and also somewhere on a circle, centre B, radius 0-65". Therefore P is one of the points of intersection of these two circles. To find the position of P it is unnecessary to draw the whole of each circle, but only enough of each circle to obtain their points of intersection. There are two possible positions of P, denoted by $P_1$ and $P_2$ in fig. 15.

Fig. 14

Fig. 15

Example. Construct a triangle ABC such that $AB = 3\text{ cm.}$, $BC = 1\text{ cm.}$, $CA = 4\text{ cm.}$

It is best to draw first the longest side CA.

Then B lies on a circle, centre C, radius 4 cm., and also lies on a circle, centre A, radius 3 cm.

Fig. 16, drawn on a reduced scale, shows only enough of each circle to give the two points of intersection $B_1, B_2$.

Then $\triangle AB_1C$ and $\triangle AB_2C$ satisfy the given conditions.
USE OF COMPASS

10. Draw two circles of radii 3 cm., 4 cm. so that their centres are 5 cm. apart. Draw their common chord, i.e. the line joining the points at which the circles cut, and measure its length.

In figs. 20–22, each arc is a semicircle. Draw the figures.

11. [14] Draw a straight line ABC so that AB = 1 in., BC = 1½ in.
   (i) Draw two circles passing through B and having A and C as centres.
   (ii) Draw two circles passing through C and having A and B as centres.

15. Draw a line AB of length 7 cm. Construct a point C such that CA = 5 cm., CB = 6 cm. Is there more than one possible position for C?

16. Draw a line AB of length 2 in. Construct a triangle ABC such that CA = CB = 2½ in. ABC is called an isosceles triangle because two of the sides are of equal length.

Construct figs. 23–25, the units being cm. Measure PQ in each figure. [It is best to draw HK first in each case.]

17. [19] Construct a triangle whose sides are the same lengths as those of \( \triangle ABC \) in fig. 11, p. 8. Draw the circles on AB and AC as diameters. Do the circles cut again on BC?

21. Construct a triangle whose sides are the same lengths as those of \( \triangle DEF \) in fig. 11, p. 8. Draw the circles on DE and DF as diameters. Do the circles cut again on FE produced?

22. Draw a line AB of length 2 in. Construct an equilateral triangle ABC, i.e., a triangle with its three sides of equal length.
[23] Take a point K; then construct a circle of radius 5 cm., passing through K. Construct two chords KL, KM each of length 8 cm. Draw the diameter KA and measure AL, AM.

24. Draw a line AB 3 cm. long. Construct a point C such that CA = CB = 4 cm.; hence construct a circle of radius 4 cm. which passes through A and B. What is the perimeter of \( \triangle ABC \)?

[25] Draw a line PQ 2.6 in. long. Construct a circle of radius 2 in. to pass through P and Q.

26. Draw a circle of radius 4 cm. and take any point A on its circumference. With centre A, radius 4 cm., draw part of a circle cutting the first circle at B, F. With centre B, radius 4 cm., mark off the point C, see fig. 26. With centre C, radius 4 cm., mark off D; and with centre D, radius 4 cm., mark off E. This is called "stepping the radius round the circle." Draw the hexagon ABCDEF and measure EF.

*27. In fig. 27, ABC is an equilateral triangle, and A, B, C are the centres of the three arcs, each of radius 2 cm. Draw the figure.

![Fig. 27](image)

*28. In fig. 28, HK = 1.5 in. and the radii of the arcs are 1.5 in., 1 in.; draw the crescent.

*29. Construct a quadrilateral the same size as the quadrilateral LMNP in fig. 11, p. 8. First construct \( \triangle LNP \) and then construct \( \triangle LNM \). Check your drawing by measuring PM.

*30. Draw a circle of radius 2 in. and take any point A on its circumference. Construct points C, E on the circle so that \( \triangle ACE \) is an equilateral triangle. [If fig. 26 is constructed by the method of No. 26, \( \triangle ACE \) is equilateral.]

PARALLEL LINES

Use of Set-Squares

Place your set-square \( \triangle ABC \) with one of the shorter sides AC in contact with a straightedge LM and rule a line along AB.

![Fig. 29](image)

Now slide the set-square along LM into the position \( \triangle A_1B_1C_1 \) and rule a line along \( A_1B_1 \). Slide it also along LM into other positions and so obtain other lines \( A_2B_2, A_3B_3, \) etc.

These lines \( AB, A_1B_1, \) etc., lie in a plane and will not meet however far they are produced; they are called parallel straight lines.

The symbol for parallel is \( \parallel \). In diagrams, arrows are often used to indicate that lines are given parallel, see fig. 29.

![Fig. 30](image)

Parallelogram. A quadrilateral whose opposite sides are parallel is called a parallelogram. Thus in fig. 30, \( ABCD \) is a parallelogram if \( AB \parallel DC \) and \( AD \parallel BC \).

The abbreviation for parallelogram is \( \text{p} \).
To draw a straight line through a given point \( P \) parallel to a given straight line \( HK \).

Turn the paper round so that \( HK \) runs across the desk and place the longest side \( AB \) of the set-square along \( HK \).

Place a straightedge \( LM \) against one of the shorter sides \( AC \) of the set-square. \textit{Hold the straightedge firmly} and slide the set-square along it into the position \( A_1B_1C_1 \), where \( A_1B_1 \) is close to \( P \). Then hold the set-square firmly and draw the line through \( P \) along \( A_1B_1 \). When produced, this gives the required line \( H_1K_1 \) parallel to \( HK \).

With small set-squares it may be impossible to reach the given point by one movement. Thus in fig. 31, to draw a line through \( Q \) parallel to \( HK \), after sliding the set-square to the position \( A_2B_2C_2 \), put the straightedge along \( B_2C_2 \) and then slide the set-square into the position \( A_2B_2C_2 \) where \( A_2B_2 \) is now close to \( Q \); then draw the line through \( Q \) along \( A_2B_2 \).

Exercise 4, Nos. 1–3, should now be worked.

Right Angles. The outside corners of this page are \textit{right-angled corners}; right-angled corners fit one another and two can be placed side by side along a straight edge, see fig. 32.
A similar method may be used to draw a perpendicular to HK from a given point P outside HK; this is called drawing (or dropping) a perpendicular from P to HK. If the perpendicular cuts HK at N, the point N is called the foot of the perpendicular from P to HK; to construct the foot of the perpendicular, it may be necessary to produce the straight line. In fig. 37, the foot of the perpendicular from K to BC lies on CB produced.

**Alternative Method.** The following method is not so easy to learn, but it is quicker to use and more accurate.

First place the **LONGEST** side of the set-square along the given line HK, and put a straightedge LM along one of the other sides; then turn the set-square so that the third side now rests against the straightedge. The longest side of the set-square is now perpendicular to HK.

**EXERCISE 4 †**

_In this exercise, parallels and perpendiculars should be drawn by using set-squares._

1. Rule a line AB and mark points P, Q on one side of AB and points R, S on the other side of AB, at different distances from AB. Draw lines parallel to AB through P, Q, R, S.

† Practice in the use of instruments is also supplied by the construction of surfaces of solids, see Ex. 23, p. 82. Such work may be taken at intervals during the remainder of the introductory course.
12. Draw a line AB 6 cm. long and draw through A any three other lines AP, AQ, AR. Drop perpendiculars from B to PA, QA, RA, produced if necessary, and call the feet of the perpendiculars X, Y, Z. Now draw the circle on AB as diameter; what do you notice about it?

Vertical and Horizontal Lines and Planes

If an object is suspended by a string, the line of the string would, if produced far enough, pass through the centre of the earth; the line is called a vertical line. Any plane which contains a vertical line, e.g. the surface of the wall of a room, is called a vertical plane. When a bricklayer is building a wall, he uses a plumb-line, which consists of a small lump of lead at the end of a string to test whether the surface of the wall is a vertical plane.

Any straight line which is perpendicular to a vertical line is called a horizontal line, and if all the lines that can be drawn in a plane are horizontal, the plane is called a horizontal plane, e.g. the surface of still water in a tank. A surface is called level if it is part of a horizontal plane. You can test whether the floor of this room is a horizontal plane by using an instrument called a spirit-level, in which the adjustment to the horizontal is shown by the position of a bubble in a glass tube containing alcohol.

If a line or plane is neither vertical nor horizontal, it is called oblique.

The words, perpendicular and vertical, must not be confused. Two intersecting lines are perpendicular if they form a right-angled corner; a line is vertical if it points to the centre of the earth.

EXERCISE 5 (Oral)

1. Point (i) vertically downwards, (ii) vertically upwards, (iii) horizontally to your front, (iv) horizontally to your right.
2. Name some vertical lines and vertical planes in the room.
3. Name some horizontal lines and horizontal planes in the room.

4. Hold up a box so that one of its faces is a horizontal plane. How many of its faces altogether are (i) vertical planes, (ii) horizontal planes? How many of its edges are (i) horizontal lines, (ii) vertical lines?

5. Hold up a box so that one of its edges is vertical. How many of its faces altogether are (i) vertical lines, (ii) horizontal lines? How many of its faces altogether are (i) vertical planes, (ii) horizontal planes?

6. Hold up a box so that one of its faces is a vertical plane. How many of its faces are vertical planes? How many of its edges are horizontal lines?

7. Hold up a box so that one of its edges is horizontal. How many of its edges must be horizontal lines? Must any of the edges be vertical lines? Must any of its faces be (i) horizontal planes, (ii) vertical planes?

8. Point out two intersecting vertical planes in the room; what can you say about their line of intersection?

9. Point out a vertical plane intersecting a horizontal plane; what can you say about their line of intersection?

10. Hold the cover of your book so that its surface is an oblique plane. Can you draw on it (i) a vertical line, (ii) a horizontal line?

11. Can you draw (i) two horizontal lines on an oblique plane, (ii) two oblique lines on a vertical plane?

12. Arrange your compass so that the two legs are at right angles. (i) Hold one leg vertically; must the other leg be horizontal? (ii) Hold one leg horizontally; must the other leg be vertical? (iii) Hold one leg obliquely; can the other leg be (a) horizontal, (b) vertical?

13. Hold your pencil so that it represents an oblique line. Is there (i) a vertical plane, (ii) a horizontal plane, in which this line lies?

14. Can you arrange two oblique planes (e.g. two pages of this book) so that they intersect (i) in a horizontal line, (ii) in a vertical line?

15. How many horizontal lines, passing through a given point, can be drawn (i) on the wall of a room, (ii) on the floor of a room?
Compass Directions. If the sun is shining, the shadow of a vertical pole cast on level ground at 12 o'clock (true noon) is a horizontal line pointing due North. Fig. 41 represents two perpendicular lines SON, WOE drawn on a horizontal plane. If ON points North, OE, OS, OW point East, South, West respectively. The directions North (N.), East (E.), South (S.), West (W.) are those of horizontal straight lines.

Start by pointing north and then turn round until you are pointing north again, the total amount of turning is called one revolution and is equivalent to 4 right angles; the amount of turning in half a revolution is 2 right angles, and in one-quarter of a revolution is 1 right angle.

Point in any direction OA and then turn till you are pointing along OB; the amount of turning is called the size of the angle AOB, written \( \angle AOB \), and can be measured as a fraction of a revolution or more often as a fraction of a right angle, where

\[
\text{one right angle} = \text{one-quarter of one revolution.}
\]

The abbreviation for right angle is rt. \( \angle \).

In fig. 42 (i), the direction of turning from OA to OB is indicated by the arrow on the small circular arc and is the same as that in which the hands of a clock move; it is therefore called a clockwise rotation.

In fig. 42 (ii) the direction of turning from OA to OB is reversed and is counter-clockwise.

Point north and then turn clockwise through half a right angle; the direction in which you are now pointing is midway between the directions north and east and is called North-East (N.E.); similarly for the directions North-West (N.W.), South-East (S.E.), South-West (S.W.).

EXERCISE 6

[Give sizes of angles in right angles or fractions of a right angle.]

What are the sizes of the angles turned through in Nos. 1–10?

1. Point S., turn clockwise and point W.
2. Point W., turn clockwise and point E.
3. Point E., turn counter-clockwise and point S.
4. Point N. turn counter-clockwise and point E.
5. Point S., turn clockwise and point N.W.
6. Point W., turn counter-clockwise and point N.E.
7. Point S.E., turn clockwise and point N.W.
8. Point S.W., turn counter-clockwise and point N.W.
9. Point N.E., turn clockwise and point S.
10. Point N.W., turn counter-clockwise and point W.

What is the final direction after the turns in Nos. 11–22?

1. Point S., turn clockwise through 1 rt. \( \angle \).
2. Point E., turn clockwise through 2 rt. \( \angle \).
3. Point W., turn counter-clockwise through 3 rt. \( \angle \).
4. Point N., turn counter-clockwise through 5 rt. \( \angle \).
5. Point S., turn clockwise through \( \frac{1}{2} \) rt. \( \angle \).
6. Point W., turn counter-clockwise through \( \frac{1}{2} \) rt. \( \angle \).
7. Point E., turn counter-clockwise through 3 \( \frac{1}{2} \) rt. \( \angle \).
8. Point N., turn clockwise through 4 \( \frac{1}{2} \) rt. \( \angle \).
9. Point N.E., turn clockwise through 3 rt. \( \angle \).
10. Point S.E., turn clockwise through 2 \( \frac{1}{2} \) rt. \( \angle \).
11. Point N.W., turn counter-clockwise through 1 \( \frac{1}{2} \) rt. \( \angle \).
12. Point S.W., turn counter-clockwise through 3 \( \frac{1}{2} \) rt. \( \angle \).

Through what angle does the minute-hand of a clock turn in (i) 1 hour; (ii) 30 min.; (iii) 5 min.; (iv) \( \frac{1}{2} \) hours?
Through what angle does the hour-hand of a clock turn in (i) 6 hours; (ii) 4 hours; (iii) 20 min.; (iv) 45 min.?

25. Draw a neat free-hand figure representing 4 places A, B, C, D if B is 1 mile due south of A, and C is 1 mile south-west of B, and D is 2 miles due east of C. A man walks from A to B and then from B to C and then from C to D. Mark on your figure the angles through which he turns at B and at C, using arrows and circular arcs as in fig. 42. What are the sizes of the angles?

26. A man is walking due east along a road; he then takes a road running north-west and a little later turns into another road running east. Sketch his walk and mark on your sketch the angles through which he turns. What are their sizes?

Angle Notation

If two lines OA, OB are drawn in different directions from a point O, and if we first face along OA and then turn and face along OB, the amount of turning depends on whether the rotation is clockwise or counter-clockwise; whichever of these two amounts is the smaller is denoted by $\angle AOB$, also by $\angle BOA$.

$\angle AOB$ and $\angle BOA$ are called the arms of the angle, and O is called the vertex of the angle. If an angle is denoted, as here, by 3 letters, the middle letter is the vertex of the angle and the outside letters are any points on the arms of the angle. But it is often simpler to use a single small letter to represent an angle. For example, in fig. 45, $a$ represents $\angle PKQ$, $b$ represents $\angle QKR$, $c$ represents $\angle PKR$.

**N.B.** Whenever you use small letters to represent angles, draw a large figure and mark the angles by distinct circular arcs, whenever it is not obvious what angle each small letter represents.

Angles at a Point. In fig. 47, the angles $a$, $b$ have the same vertex O; they are called angles at the point O.

In fig. 48, the angles $c$, $d$ have the same vertex O and one arm $OQ$ in common, and lie on opposite sides of that arm; they are called adjacent angles. If in fig. 49, the adjacent angles $e_1$, $e_2$ are equal, ON is said to bisect $\angle XOY$ or is called the bisector of $\angle XOY$. 
Addition and Subtraction. Take your dividers or compass and gradually open out the legs. Keep one leg along OP in fig. 50 and suppose OQ, OR are successive positions of the other leg. Then the rotation from OP to OQ followed by the rotation from OQ to OR is equivalent to a single rotation from OP to OR, and, assuming that the total rotation is less than 2 right angles, we write

\[ \angle POQ + \angle QOR = \angle POR, \]

or

\[ a + b = \angle POR. \]

Similarly we say that

\[ \angle POR - \angle POQ = \angle QOR. \]

Draw any number of lines OP, OQ, OR... from a point O, see fig. 50. Point along OP, then turn counter-clockwise and point along OQ; continue turning counter-clockwise, pointing in succession along OR, OS, OT, and finally along OP again. The total angle turned through is one revolution or 4 right angles; but the successive angles turned through are \( a, b, c, d, e; \)

\[ \ldots a + b + c + d + e = 4 \text{ right angles}. \]

This is true, whatever the number of angles at O. In words,

The sum of all the angles at a point, each being adjacent to the next, is 4 right angles.

**Angles at a Point**

Adjacent Angles on a Straight Line

Fig. 51 represents any straight line CE meeting another straight line ACB at C. Point along CA, then turn through \( \angle ACE \) and point along CE. Continue turning until you are pointing along CB. Since ACB is a straight line, the total amount of turning is half a revolution or 2 right angles. Therefore with the notation of fig. 51,

\[ a + b = 2 \text{ rt. } \angle s. \]

The angles \( a, b \) which CE makes with the straight line ACB are called adjacent angles on a straight line, and the result just obtained is stated in the form,

The sum of adjacent angles on a STRAIGHT LINE is 2 right angles.

**Abbreviation for reference:** adj. \( \angle s \) on st. line.

Supplementary Angles

Two angles are called supplementary if their sum is two right angles.

For example, \( \frac{3}{2} \text{ rt. } \angle \) and \( 1 \frac{1}{2} \text{ rt. } \angle s \) are supplementary angles. Also in fig. 51, where ACB is a straight line, \( a \) and \( b \) are supplementary angles because the sum of adjacent angles on a straight line is 2 right angles.

If the sum of the adjacent angles POR, ROQ, see fig. 52, is 2 right angles, the total amount of turning from OP to OR and then from OR to OQ is half a revolution; therefore POQ is a straight line.

This fact is stated as follows:

If the sum of two adjacent angles is 2 right angles, the exterior arms of the angles are in one straight line.

**Abbreviation for reference:** adj. \( \angle s \) supp.
**Stage A**

**Vertically Opposite Angles**

When two straight lines intersect, either pair of opposite angles are called vertically opposite. Thus in fig. 53, \( a, c \) are vertically opposite angles; so also are \( b, d \). Here the word *vertically* means *having the same vertex*.

![Diagram](image0)

**Fig. 53**

If two straight rods \( PQ \), \( RS \) cross one another at \( K \) and are hinged together at \( K \) like a pair of scissors, when \( KP \) is turned so as to lie along \( KR \), \( KQ \) is turned so as to lie along \( KS \); therefore the amounts of turning represented by \( \angle PKR \), \( \angle QKS \) are equal. Therefore with the notation of fig. 53,

\[ a = c \quad \text{and} \quad b = d. \]

This result is stated in the form,

*If two straight lines intersect, the vertically opposite angles are equal.*

**Abbreviation for reference:** vert. opp. \( \angle s \).

It may be proved as follows: with the notation of fig. 53,

\[ a + b = 2 \text{ rt. } \angle s, \quad \text{adj. } \angle s \text{ on st. line,} \]
\[ b + c = 2 \text{ rt. } \angle s, \quad \text{adj. } \angle s \text{ on st. line,} \]
\[ \therefore a + b = b + c. \]
\[ \therefore a = c. \]

Similarly it may be proved that \( b = d \).

**Exercise 7**

Nos. 1–6 refer to fig. 55.
1. If \( a = \frac{1}{2} \text{ rt. } \angle \) and \( b = \frac{1}{3} \text{ rt. } \angle \), find \( \angle POR \).
2. If \( \angle QOS = \frac{1}{4} \text{ rt. } \angle s \) and \( \angle QOR = \frac{1}{4} \text{ rt. } \angle s \), find \( c \).

![Diagram](image1)

**Fig. 55**

[3] If \( \angle POQ = \frac{1}{2} \text{ rt. } \angle \), \( \angle POR = \frac{1}{2} \text{ rt. } \angle \), \( \angle POS = \frac{1}{2} \text{ rt. } \angle s \), find \( b \) and \( c \).
4. Express in terms of small letters, (i) \( \angle POR \), (ii) \( \angle POS \), (iii) \( \angle POR + \angle QOS \), (iv) \( \angle ROS - \angle POR \).
5. Express in capital letters as simply as possible, (i) \( a + b \), (ii) \( b + c \), (iii) \( a + b + c \).
6. Express the following statements in terms of small letters:
   (i) \( OR \) bisects \( \angle QOS \); (ii) \( \angle POR = \angle QOS \);
   (iii) \( OP \) is perpendicular to \( OR \).

Nos. 7–11 refer to fig. 56, in which \( ACB \) is a straight line.
7. If \( \angle P = \frac{1}{4} \text{ rt. } \angle \), find \( f \).
8. If \( \angle f = \frac{1}{2} \text{ rt. } \angle \), find \( g \).
9. Find \( g \) if \( f \) is twice \( g \).
10. If \( CP \) bisects \( \angle ACE \) and if \( g = \frac{1}{2} \text{ rt. } \angle \), find \( \angle PCA \).

![Diagram](image2)

**Fig. 56**

[11] If \( CQ \) bisects \( \angle BCE \) and if \( \angle ECQ = \frac{1}{3} \text{ rt. } \angle \), find \( \angle ACQ \).

Nos. 12–14 refer to fig. 57.
12. If \( p = \frac{1}{2} \text{ rt. } \angle \), find \( q \).
13. If \( q = \frac{1}{3} \text{ rt. } \angle \), find \( p \).
14. If \( g \) is twice \( p \), find \( p \).
15. In fig. 53, p. 28, if \( b = \frac{1}{3} \text{ rt. } \angle s \), find \( a, d, c \).
16. In fig. 53, p. 28, if \( b \) is three times \( c \), find \( a, d \).

17. Arrange the angles \( a, b, c, d, e \), shown in fig. 58, in ascending order of magnitude and state, for each angle, whether it is acute, obtuse, or reflex.

![Diagram](image3)

**Fig. 58**

[18] What is the reflex angle between south-east and west?

[19] If in fig. 55, \( a = \frac{1}{2} \text{ rt. } \angle \), \( b = \frac{1}{2} \text{ rt. } \angle \), \( c = \frac{1}{2} \text{ rt. } \angle \), find the reflex angle (i) which \( OP \) makes with \( OS \), (ii) which \( OQ \) makes with \( OR \). Draw your own figure.
20. Find whether $\text{FON}$ is a straight line in fig. 59.
   (i) if $a = b = \frac{2}{3}$ rt. $\angle$, and $c = \frac{1}{3}$ rt. $\angle$;
   (ii) if $a = c = \frac{1}{3}$ rt. $\angle$, and $b = 1\frac{1}{3}$ rt. $\angle$;
   (iii) if $a = b = c = \frac{2}{3}$ rt. $\angle$.

Use of Protractor

In practical work it is convenient to use smaller units for measuring angles to avoid awkward fractions, although a decimal notation would be simpler and is used in Germany. One revolution is divided into 360 equal angles, each of which is called a degree. Hence

1 right angle = 90 degrees ($90^\circ$).
Further 1 degree = 60 minutes (60').
1 minute = 60 seconds (60').

The size of an angle is measured in degrees by using a protractor. Notice on your protractor that the graduations are marked with numbers, one bigger than 90° and one smaller than 90°; use your common sense to decide which of these readings you must choose in any given case.

**Exercise 8**

*Give the sizes of angles in degrees unless otherwise stated.*

1. Express in degrees: 2 rt. $\angle$s; 4 rt. $\angle$; $\frac{1}{6}$ rt. $\angle$; $\frac{1}{8}$ rt. $\angle$.
2. Express in degrees: 4 rt. $\angle$s; $\frac{1}{3}$ rt. $\angle$; $\frac{1}{2}$ rt. $\angle$s; $\frac{1}{6}$ rt. $\angle$s.
3. Express in right angles: 270°; 30°; 30°; 135°; 45°; 30°.
4. Express in right angles: 22$\frac{1}{2}$°; 60°; 150°; 210°; 315°.
5. Through what angle does the hour-hand of a clock turn in 1 hour; 10 min.; 45 min.; 2$\frac{1}{2}$ hours?
6. Through what angle does the minute-hand of a clock turn in 15 min.; 5 min.; 20 min.; 30 min.?

7. Open your compass to an angle of 100°. What is the reflex angle between the legs?
8. A wheel makes 20 revolutions per minute. Through what angle does a spoke turn in 1 second?

**Measure the following angles (Nos. 9–10) in fig. 11, p. 8**.

9. $\angle \text{MRF}, \angle \text{MRD}$.
10. $\angle \text{CYN}, \angle \text{CYP}$.

11. $\angle \text{NLP}, \angle \text{NLM}$.
12. The angles of $\triangle \text{ABC}$.
13. The angles of $\triangle \text{DEF}$.
   [Keep the results of Nos. 12, 13 for future use.]
14. The angles of the quadrilateral $\text{LMNP}$.
15. The reflex angle $\text{KRS}$.
16. The reflex angle $\text{BTN}$.

Draw the following angles, make them point different ways:
17. 55°, 125°, 200°.
18. 100°, 72°, 330°.
19. 38°, 142°, 250°.
20. 85°, 167°, 300°.
21. Draw at a point the following angles so that each is adjacent to the next: 70°, 130°, 20°, 90°. What is the size of the remaining angle at the point?
22. Open your compass to an angle of 70°. Through what extra angle must one leg be turned to bring the legs into line?
23. Without looking at your protractor, say what other number of degrees is opposite to the graduation (i) 50°, (ii) 130°.

24. If in fig. 60, $\text{OX}$ is drawn to bisect $\angle \text{POR}$, find $\angle \text{QOX}$.

25. In fig. 61, $\text{CN}$ is perpendicular to the straight line $\text{ACB}$, find the value of $\angle \text{ACP} - \angle \text{PCB}$.

Nos. 26–29 refer to fig. 62, in which $\text{ACB}$ is a straight line.
26. Find $x$ if $y = 72$.
27. Find $y$ if $x = 134$.
28. Find $y$ if $x = 2y$.
29. Find $x$ if $x - y = 42$.
STAGE A

No. 30-33 refer to fig. 63, in which L MN is a straight line.
30. Find y if x = 25, z = 62.
31. Find x if y = z = 67.
32. Find x if y = z = 2z.
33. Find y if x + z = y.
34. If, in fig. 63, it is not given that L MN is a straight line, what conclusion can you draw (i) if x = 40, y = z = 70, (ii) x = 30, y = 85, z = 75?
35. Repeat No. 34 if (i) x = 35, y = 75, z = 80; (ii) x = y = 3z = 135°.
36. In fig. 64, find (i) \( \angle AKD \); (ii) the angle between the bisectors of \( \angle BKC \) and \( \angle AKD \).
37. In fig. 65, find x.

Fig. 64

Fig. 65

Fig. 66

38. Fig. 66 represents three intersecting straight lines. (i) Find \( a \) if \( b = 70^\circ \), \( c = 35^\circ \). (ii) Find x if \( c = 2c \), \( a = 3c \), \( b = 4c \).

*39. Draw a neat, but not accurate, figure showing 6 lines in order, \( OA, OB, OC, OD, OE, OF \) such that \( \angle AOB = 43^\circ \), \( \angle BOC = 67^\circ \), \( \angle COD = 90^\circ \), \( \angle DOE = 59^\circ \), \( \angle EOF = 51^\circ \). Find whether (i) \( OA \) is in line with \( OD \), (ii) \( OB \) is in line with \( OE \), (iii) \( OC \) is in line with \( OF \).

*40. Two straight lines \( AOB, COD \) cut at \( O \); \( OP \) is drawn bisecting \( \angle BOD \) and \( PO \) is produced to \( Q \); \( OR \) is drawn between \( OA \) and \( OQ \) so that \( \angle AOR \) is twice \( \angle ROQ \). If \( \angle BOC = 90^\circ \), find \( \angle POR \).

Compass Bearings

Directions in a horizontal plane are described as follows:
In fig. 67, the direction of \( A \) from \( O \) is obtained by pointing north and then turning through \( 40^\circ \) towards the east; it is therefore described as N. 40° E. or sometimes as 40° E. of N.

Similarly the direction of \( B \) from \( O \) is written S. 64° E. or 64° E. of S.

Directions should always be measured either from the north or from the south, never from the east or from the west. For example, the direction of \( C \) from \( O \) in fig. 67 is not written W. 30° N., but is written N. 60° W. because \( \angle NOC = 90^\circ - 30^\circ = 60^\circ \).

The Army method of describing directions is to give the angle, described clockwise, which the direction makes with north line \( ON \); this is called the true bearing.

For example, in fig. 67, \( \angle NOB = 180^\circ - 64^\circ - 116^\circ \); therefore the true bearing of \( B \) from \( O \) is 116°. The true bearing of \( C \) from \( O \) is the reflex angle \( NOC \), i.e. 270° + 30° or 300°.

EXERCISE 9

Give the sizes of angles in degrees.
Write shortly the following final directions, the turning being clockwise in each case:

1. Point N., turn 70°.
2. Point S., turn 42°.
3. Point E., turn 110°.
4. Point W., turn 130°.
5. Point E., turn 212°.
6. Point N., turn 150°.
7. Point W., turn 165°.
8. Point E., turn 300°.
9. Point S.E., turn 50°.
11. Write down the true bearings of the final directions in Nos. 2, 4, 5.

Draw neat (but not accurate) figures to find the angle \( \angle AOB \) if the directions of \( A \) and \( B \) from \( O \) are as follows:

12. N. 10° E., N. 15° W.  
13. N. 40° E., S. 20° E.  
14. N. 50° E., S. 10° W.  
15. N. 20° E., S. 80° W.

16. Express more simply (i) 150° E. of N., (ii) 170° W. of S.

*17. Points of the Compass. The direction which bisects the angle between north and north-east is called north north-east and is written N.N.E.; similarly the direction which bisects the angle between north-east and north-east is called east north-east and is written E.N.E., and so on for other bisectors between directions south and south-east, etc. Draw a large circle and mark the 16 points on the circumference representing these directions from the centre of the circle.

Find the angles between the following directions:

*18.  
(i) N. and N.N.E., (ii) S.W. and E.N.E.,  
(iii) W.S.W. and W.N.W.

*19.  
(i) E. and S.S.E., (ii) N.E. and E.S.E., (iii) N.W. and E.N.E.

*20. The direction which bisects the angle between S. and N.N.E. is called S. by E., and the direction which bisects the angle between S.E. and N.N.E. is called N.E. by N. Similarly N.E. by E. means the direction midway between N.E. and E.N.E., and E. by N. means the direction midway between E. and E.N.E., and so on. Draw a larger circle concentric with the circle drawn for No. 17 and mark on it the 16 points representing these additional directions. The two figures then show the 32 points of the compass.

Find the angles between the following directions:

*21.  
(i) N. and N. by E., (ii) S. and S.E. by E.,  
(iii) W.N.W. and N.W. by N.

*22.  
(i) E. and S.E. by S., (ii) W. by S. and S. by W.,  
(iii) N.E. by E. and S.E. by S.

**Corresponding, Alternate, and Interior Angles**

A straight line such as \( L \) in fig. 68 which cuts two or more other straight lines is called a transversal.

Certain pairs of angles formed by the transversal with lines it cuts have special names.

In fig. 68, \( a_1 \) and \( a_2 \) are called corresponding angles; so also are the pairs \( b_1, b_2; c_1, c_2; d_1, d_2 \).

The pairs of interior angles on opposite sides at opposite ends of the transversal are called alternate angles, i.e. \( c_1, c_2 \) are alternate, so are \( a_1, a_2 \). \( a_1 \) and \( b_2 \) are called interior angles on the same side of the transversal, so are \( c_1 \) and \( d_2 \); they are sometimes called allied angles.

**Parallel Straight Lines**

When we slide a set-square along a straight edge and rule lines along one of the other sides of the set-square, we are drawing lines which cut a transversal so as to make corresponding angles equal, because, see fig. 69, each of the angles \( a_1, a_2 \) is equal to the same angle of the set-square. Also if \( a_1 = a_2 \), it follows that \( b_1 = b_2 \) because \( a_1 + b_1 = 2 \) rt. \( \angle s = a_2 + b_2 \), adj. \( \angle s \) on st. line.
Therefore if we assume that the lines PQ, RS drawn in this way with a set-square are parallel, we have the following test for parallel lines:

If two straight lines in a plane are cut by a third straight line and if a pair of corresponding angles are equal, then the first two straight lines are parallel.

And if we assume that through any given point one and only one straight line can be drawn parallel to a given straight line, the converse statement is true:

If a straight line cuts two parallel straight lines, the corresponding angles are equal.

Abbreviation for reference: With the notation of fig. 76, p. 37, we write

$$b = c, \text{ corr. } \angle s, \text{ PQ } \parallel \text{ RS }.$$  

K.B. The correct way of stating the reason why $b = c$ must be noted carefully. It is not sufficient to say, $b = c \text{ corr. } \angle s$; it is essential to add PQ $\parallel$ RS.

For example, in fig. 77, $b$ and $d$ are corresponding angles, but they are not equal because EF is not parallel to ST.

**EXERCISE 10**

[Arrows in the diagrams indicate that lines are given parallel.]

1. Draw a figure like fig. 70. $a_1, a_2$ and $a_3$, $a_4$ are corresponding angles. Mark all the other pairs of corresponding angles in the same way, $b_1, b_2; b_3, b_4$ etc.

2. Draw a figure like fig. 70 and mark in it all the pairs of alternate angles $p_1, p_2; q_1, q_2$ etc.

3. In fig. 71, state the names for the following pairs of angles:
   (i) $a, b;$ (ii) $a, c;$ (iii) $c, e;$ (iv) $b, d;$ (v) $b, c;$ (vi) $d, e.$

4. Make a neat, but not accurate, copy of fig. 72 and fill in the sizes of all the remaining angles.

5. Make a neat, but not accurate, copy of fig. 73 and fill in the sizes of all the remaining angles.

6. In fig. 74, find the sizes of $a, b, c, d$. Give reasons.

7. In fig. 75, find the sizes of $e, f, g, h$. Give reasons.

8. In fig. 76, prove that $a = c$. State the reasons clearly.

9. In fig. 77, find the values of (i) $a + b$; (ii) $c + d$; (iii) $p + q$. Give reasons in the proper form.

10. In fig. 76, prove that $c + d = 2 \text{ rt. } \angle s$. State the reasons clearly.
The examples in Exercise 10 illustrate the fact that if two straight lines in a plane are cut by a transversal so that a pair of corresponding angles are equal, then also the alternate angles are equal, and the sum of each pair of interior angles on the same side of the transversal is two right angles. Hence

If two straight lines in a plane are cut by a third straight line and if either (i) a pair of alternate angles are equal, or (ii) the sum of the interior angles on the same side of the cutting line is two right angles,

then the first two straight lines are parallel.

The converse statement is also true:

If a straight line cuts two parallel straight lines,

(i) the alternate angles are equal,
(ii) the sum of the interior angles on the same side of the cutting line is two right angles.

Abbreviation for reference: with the notation of fig. 78, we write

(i) $PQ \parallel RS$ if $a = b$, alt. $\angle s$.
(ii) $PQ \parallel RS$ if $b + c = 2 \text{ rt. } \angle s$, int. $\angle s$.

With the notation of fig. 79

$a = k$, corr. $\angle s$, $PQ \parallel XY$
and $b = k$, corr. $\angle s$, $RS \parallel XY$.

$\therefore a = b$.

But these are corr. $\angle s$, $\therefore PQ \parallel RS$.

This result is stated as follows:

If two straight lines in a plane are each parallel to a third straight line in the plane, then the first two straight lines are parallel to each other.

Example for Oral Discussion

With the data of fig. 80, find $p$.

Construction. Draw $CN$ parallel to $BA$.

Complete the following, giving the reasons clearly:

$CN$ is parallel to $ED$
because \[ \angle B \triangleq = 180^\circ \text{...} \]
$\therefore \angle B = \angle D$
$\therefore \angle D = \angle C$
$\therefore p = \ldots$

Note. In fig. 80, the straight line $CN$ is dotted because it is not part of the original figure, but has been added as a construction for the sake of the argument.
EXERCISE 11

Nos. 1-6 refer to fig. 81. State the reasons clearly.
1. If $b = 70^\circ$, find $r$.
2. If $s = 105^\circ$, find $a$.
3. If $c = 65^\circ$, find $p$.
4. If $q = 80^\circ$, find $d$.
5. If $a = 2q$, find $q$.
6. If $d - r = 72^\circ$, find $r$.

Nos. 7-10 refer to fig. 82. State the reasons clearly.
7. If $p = 150^\circ$, $q = 160^\circ$, find $a$, $b$.
8. If $a = 35^\circ$, $b = 25^\circ$, find $c$.
9. If $c = 50^\circ$, $p = 145^\circ$, find $a$, $b$.
10. If $p = 4a$ and $q = 5b$, find $a$, $b$, $c$.

Find the unknown marked angles in Nos. 11-19. State the reasons clearly.

11. [Diagram]
12. [Diagram]
13. [Diagram]
14. [Diagram]

Fig. 81
Fig. 82
Fig. 83
Fig. 84
Fig. 85
Fig. 86
Fig. 87
Fig. 88

ANGLES AND PARALLELS

Nos. 20-23 refer to fig. 92. State the reasons clearly.
20. If $\angle A = 55^\circ$ and $\angle B = 60^\circ$, find (i) $\angle ACD$, (ii) $\angle ACB$.
21. If $\angle A = 65^\circ$ and $\angle ACD = 100^\circ$, find (i) $\angle KCD$, (ii) $\angle B$.
22. If $\angle ACK = 48^\circ$ and $\angle KCD = 44^\circ$, find each of the angles of $\triangle ABC$.
23. If $\angle A = x^\circ$, $\angle B = y^\circ$, find $\angle ACD$ in terms of $x$, $y$. If $\angle C = z^\circ$, what can you say about $x$, $y$, $z$?

Nos. 24-27 refer to fig. 93. State whether $AB$ is parallel to $CD$ or not, with the following data.
24. $p = 70^\circ$, $r = 60^\circ$.
25. $q = 125^\circ$, $r = 55^\circ$.
26. $p = 65^\circ$, $s = 115^\circ$.
27. $t = 60^\circ$, $r = 65^\circ$.

If figs. 94, 95 contain parallel lines, name them and give reasons.
28. [Diagram]
29. [Diagram]
If figs. 96, 97 contain parallel lines, name them and give reasons.

32. In fig. 98, prove that

\[ \angle TSR = \angle NPQ. \]

State the reasons clearly.

**Angles of a Triangle**

A triangle is called **acute-angled** if ALL of its angles are acute.

A triangle is called **right-angled** if one of its angles is a right angle; the side opposite the right angle is called the hypotenuse.

A triangle is called **obtuse-angled** if one of its angles is obtuse.

**Examples for Oral Discussion**

1. In fig. 101, \( \angle ACD \) is an exterior angle of \( \triangle ABC \).
   Draw \( CM \parallel BA \) and complete the following, with reasons:
   
   \[ \angle ACM = \ldots; \quad \angle MCD = \ldots; \quad \angle ACD = \ldots \]

2. In fig. 102, find the size of the exterior angle \( CAE \), giving the reasons clearly.

**EXERCISE 12**

Arrows in the diagrams indicate that lines are given parallel.

Find the unknown marked angles (following the alphabetical order) in figs. 103, 104. State the reasons clearly.

1. \[ \text{Fig. 103} \]

2. \[ \text{Fig. 104} \]
Find the unknown marked angles in figs. 105-108. Give reasons.

3. [4]

Fig. 105

Fig. 106

5. [6]

Fig. 107

Fig. 108

7. In fig. 109, BC is produced to D. Copy and complete with full reasons,

\[ \angle ACK = \ldots \]

\[ \angle KCD = \ldots \]

\[ \therefore \text{ adding, ext. } \angle ACD = \ldots \]

The fact proved in No. 7 may be stated as follows:

If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

**Abbreviation for reference:** ext. \( \angle \) of \( \Delta \).

*Use this fact for the remaining examples in this exercise.*

Complete the equations in Nos. 8-11, which refer to fig. 110.

8. \( a + b = \ldots = \ldots \)

[9] \( b + c = \ldots = \ldots \)

10. \( q_1 = \ldots = \ldots \)

[11] \( p_1 = \ldots = \ldots \)

Fig. 110

Fig. 111

Fig. 112

Fig. 113

Fig. 114

Fig. 115

Fig. 116

Fig. 117

Fig. 118

Fig. 119

Fig. 120

Fig. 121
Sum of the Angles of a Triangle

Examples for Oral Discussion.

1. Cut out a paper triangle. Tear off two of the corners and fit the pieces so that the three corners are at the same point. What does this suggest?

2. Draw a large triangle \( \triangle ABC \). Place your pencil along \( BC \); hold it at \( C \) and turn it through \( \angle C \) so that it lies along \( AC \). Now hold it at \( A \) and turn it through \( \angle A \) so that it lies along \( AB \). Lastly, hold it at \( B \) and turn it through \( \angle B \) so that it lies along \( CB \). What is the size of the total angle through which it has turned? What does this suggest?

3. In fig. 122, find the angles \( m, n, p \). Give reasons. What is the sum of the angles of \( \triangle ABC \)?

4. In fig. 123, what other angle equals \( r \)? What other angle equals \( s \)? What do you know about \( r + s + t \)? What does this prove?

The fact illustrated by Examples 1–3 and proved in No. 4 is stated as follows:

The sum of the three angles of a triangle is two right angles.

Abbreviation for reference: \( \angle \) sum of \( \Delta \).

Hence if a triangle is right-angled, the other two angles are acute and their sum is 1 right angle.

Two angles are called complementary if their sum is one right angle. Thus the two acute angles in a right-angled triangle are complementary.

5. Measure the angles of \( \triangle ABC \) in fig. 11, p. 8; find their sum. What is the error in your answer?
Nos. 19–22 refer to fig. 130 in which \( \angle ABC \), \( \angle ACB \) and \( \angle DBC \) are the bisectors of \( \angle ABC \), \( \angle ACB \) and \( \angle DBC \). Draw your own figure for each question. You need not give reasons, but show up all the working.

19. \( \angle ABC = 30^\circ \), \( \angle ACB = 70^\circ \), find (i) \( \angle BPC \), (ii) \( \angle BKC \).

20. \( \angle ABC = 20^\circ \), \( \angle BAC = 110^\circ \), find \( \angle DAC \).

21. \( \angle ABC = 36^\circ \), \( \angle ACB = 64^\circ \), find (i) \( \angle QPR \), (ii) \( \angle FQR \).

22. \( \angle ACB = 74^\circ \), \( \angle BAC = 80^\circ \), find (i) \( \angle BHA \), (ii) \( \angle PRQ \).

23. In \( \triangle ABC \), \( \angle B = 38^\circ \), \( \angle C = 54^\circ \); \( \text{AD} \) is the perpendicular from \( \text{A} \) to \( \text{BC} \); \( \text{AE bisects} \ \angle \text{BAC} \); find \( \angle EAD \).

24. In \( \triangle ABC \), \( \angle A = 74^\circ \), \( \angle B = 28^\circ \); \( \text{BC} \) is produced to \( \text{X} \); the lines bisecting \( \angle \text{ABC} \) and \( \angle \text{ACX} \) meet at \( \text{K} \); find \( \angle \text{BKC} \).

25. In \( \triangle ABC \), \( \angle B = 110^\circ \), \( \angle C = 50^\circ \); \( \text{AD} \) is the perpendicular from \( \text{A} \) to \( \text{CB} \) produced; prove that \( \angle \text{DAB} = \angle \text{BAC} \).

26. In \( \triangle ABC \), \( \angle B = \angle C \); \( \text{BC} \) is produced to \( \text{D} \). If \( \angle \text{ACD} = x^\circ \), find \( \angle \text{A} \) in terms of \( x \).

27. In \( \triangle ABC \), the bisector of \( \angle BAC \) cuts \( \text{BC} \) at \( \text{D} \). If \( \angle B = x^\circ \) and \( \angle C = y^\circ \), find \( \angle \text{ADC} \) in terms of \( x, y \).

28. In fig. 131, find \( a \) in terms of \( b, c, d \). [\( r \) is inserted in the figure to help you.]

Fig. 131

29. In fig. 132, find \( h \) in terms of \( e, f, g \). [It is inserted in the figure to help you.]

Fig. 132

30. In fig. 133, find \( d \) in terms of \( a, b, c \).

Fig. 133

31. In fig. 134, find \( k \) in terms of \( e, f, g \).

Fig. 134

32. In fig. 135, find \( d \) in terms of \( a, b, c \). [Produce \( \text{PS and QR to meet at} \text{V} \).]

Fig. 135

33. In fig. 136, find \( e \) in terms of \( f, g, h \). [Join \( \text{AC} \) and produce it.]

Fig. 136

**Angles of a Polygon**

A polygon is called convex if each of its interior angles is less than two right angles, and is called re-entrant if one or more of its interior angles is reflex.

![Convex polygon](image1)

**Convex polygon**

![Re-entrant polygon](image2)

**Re-entrant polygon**

A polygon is called equilateral if all its sides are of equal length, and is called equiangular if all its angles are equal. The number of its sides is equal to the number of its angles.

If a polygon is both equilateral and equiangular, it is called regular. If a triangle is equilateral, it must also be equiangular; this is not true for any other polygon. If a polygon is regular, a circle can be drawn through its vertices.
Sum of the Angles of a Convex Polygon

Fig. 138 represents any 5-sided polygon ABCDE.
Take any point K inside ABCDE and join K to each vertex of the polygon, thus forming 5 triangles.
The sum of all the angles of the 5 triangles is $5 \times 2$ right angles, i.e. 10 right angles.
But this sum is made up of (i) $\angle AKB, \angle BKC, \angle CKD, \ldots$, 
i.e. all the angles at K, and (ii) $\angle ABC, \angle BCD, \angle CDE, \ldots$, 
i.e. all the angles of the polygon.
But the sum of all the angles at K is 4 right angles.
\[ \therefore \text{the sum of the angles of the 5-sided polygon is} \]
\[ (10 - 4) \text{ right angles, i.e. 6 right angles.} \]

\[ \text{Fig. 138} \]

The reader should now repeat this argument for convex polygons of 6 sides, 7 sides, 100 sides, n sides.
The fact which can be proved in this way is stated as follows:

The sum of the interior angles of a convex polygon with n sides is $(2n - 4)$ right angles.

Exterior Angles of a Convex Polygon

Fig. 139 represents any convex 5-sided polygon ABCDE with its sides AB, BC, CD, DE, EA produced in order, thus forming an exterior angle at each vertex.

Draw a large figure like this on the floor and, starting at any point N on AB, walk along NB to B, then turn and walk along BC to C and so on till you get back to N.

What are the separate angles through which you have turned? What is the total angle? What does this suggest?

The sum of the exterior angles of ABCDE can be calculated as follows:

The sum of the interior $\angle$ and exterior $\angle$ at each corner is 2 rt. $\angle$s.
\[ \therefore \text{sum of all interior} \angle \text{s and all exterior} \angle \text{s is} \]
\[ (5 \times 2) \text{ rt.} \angle \text{s.} \]
But sum of all interior $\angle$s is 6 rt. $\angle$s.
\[ \therefore \text{sum of all exterior} \angle \text{s is} (10 - 6) \text{ rt.} \angle \text{s, i.e. 4 rt.} \angle \text{s.} \]
The reader should now repeat this argument for convex polygons of 6 sides, 7 sides, 100 sides, n sides.
The fact which can be proved in this way is stated as follows:

If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is four right angles.

Exercise 14

1. What is the sum of the angles of a quadrilateral?
2. ABCD is a quadrilateral such that $\angle B = 112^\circ$, $\angle C = 75^\circ$, $\angle D = 61^\circ$, find $\angle A$.
3. In the quadrilateral ABCD, $\angle A = \angle B = \angle C$ and $\angle D = 120^\circ$, find $\angle A$.
4. ABCDE is a pentagon. If $\angle A = \angle B = \angle C = \angle D = 115^\circ$, find $\angle E$.
Find the unknown marked angles in figs. 140–142.
5. [6]
7. [6]
8. Find the size of an interior angle of a regular convex polygon with 10 sides.

10. Each exterior angle of a convex polygon is 40°; how many sides has it?

11. Each interior angle of a convex polygon is 150°; how many sides has it?

(12) Can a regular convex polygon be drawn such that each exterior angle is (i) 20°, (ii) 16°, (iii) 18°? If so, state the number of sides.

13. Can a regular convex polygon be drawn such that each interior angle is (i) 144°, (ii) 140°, (iii) 130°? If so, state the number of sides.

14. The angles of a pentagon are $2x°, 3x°, 4x°, 5x°, 6x°$; find the value of $x$.

[15] The angles of a pentagon are $x, 2x, x + 30, x - 10, x + 40$, degrees; find the value of $x$.

16. ABCDE is a regular pentagon; AB, DC are produced to meet at N; find $\angle$AND.

*17. The angles of a quadrilateral taken in order are $y°, 3y°, 5y°, 7y°$. Prove that two of its sides are parallel.

*18. Prove that the sum of the interior angles of an 8-sided polygon is twice the sum of those of a pentagon.

*19. Fig. 143 represents a crossed pentagon in which the 5 marked angles are equal. Find the size of each of these angles.

*20. The sum of the interior angles of an n-sided convex polygon is double the sum of the exterior angles. Find the value of n.

Data necessary to fix the Shape and Size of a Triangle

If you are asked to make an exact copy of the triangle ABC, fig. 11, p. 8, what measurements must you make to enable you to do so? There are 3 sides and 3 angles. Is it necessary to make six measurements? If not, what different selections of the six are sufficient?

CONGRUENCE

EXERCISE 15 (Oral)

1. In fig. 11, p. 8, measure AB, AC (in cm.) and $\angle$BAC. Are these measurements sufficient for making a copy of $\triangle$ABC? If so, draw it and measure BC in your copy.

The average of the results obtained by the class, striking out any results which differ so widely from the rest that it is evident a mistake has been made, should be taken and compared with the length of the original.

[2] Make a copy of $\triangle$DEF in fig. 11, p. 8, by measuring ED, EF (in cm.) and $\angle$E. Measure DF in your copy.

Results should be compared as in No. 1.

Hence we conclude that the shape and size of a triangle are fixed if the lengths of two sides and the size of the angle included by those two sides, called for short the included angle, are known.

Hence, if two triangles are drawn so that they agree as regards the measurements of two separate sides and the included angle, then they must agree as regards all other corresponding measurements. Such triangles are said to be equal in all respects or congruent.

The symbol for "is congruent to" is $\cong$.

Corresponding Vertices and Sides. The statement that the triangles $\triangle$ABC, QXK are congruent should mean that $\triangle$ABC can be placed on the top of $\triangle$QXK so that A is at Q, and B is at X, and C is at K; and this may be indicated more clearly by writing the statement in the form,

$\triangle$ABC $\cong$ QXK are congruent.

We call A, Q corresponding vertices, so also are B, X and C, K; we call BC, XK and CA, KQ and AB, QX corresponding sides.
When the triangles are written with the letters in the proper order, it is unnecessary to look at the figure to pick out pairs of equal sides or pairs of equal angles because the arrangement ABC shows at once what angle or side in either triangle corresponds to any given angle or side in the other. Sides which correspond are opposite to equal angles.

Note. It may be necessary to turn a triangle over before it can be fitted on to a congruent triangle. Can you fit \( \triangle ABC \) without turning it over on to \( \triangle QXX \) marked (i) in fig. 144 or on to \( \triangle QXX \) marked (ii) in fig. 144?

3. In fig. 11, p. 8, measure LN (in cm.), and \( \angle NLP, \angle LNP \). Are these measurements sufficient for making a copy of \( \triangle PNL \)? If so, draw it and measure LP in your copy.

Results should be compared as in No. 1.

4. In fig. 11, p. 8, measure LN (in cm.), and \( \angle NLM, \angle L MN \). Can you now say what \( \angle L MN \) is without measuring it? Make a copy of \( \triangle L MN \) without making any more measurements, and measure MN in your copy.

Results should be compared as in No. 1.

Hence we conclude that the shape and size of a triangle are fixed if the sizes of two angles and the length of a side, of known situation with respect to these angles, are given.

Hence if two triangles are drawn so that they agree as regards the measurements of two pairs of angles and one pair of sides correspondingly situated, then they must agree as regards all other corresponding measurements—that is, they are congruent.

5. Draw \( \triangle ABC, PQR \) with the data of fig. 145. \( \angle A = \angle P, \angle C = \angle R \) and \( AC = PQ \); but the triangles are obviously not congruent. Explain why the two equal sides \( AC, PQ \) are not corresponding sides by stating the sizes of the angles opposite to these sides.

6. In fig. 146, corresponding marks indicate that angles and sides are given equal. Are the triangles congruent? If so, state the fact in the proper way, using capital letters.

[Sketch of \( \triangle ABC \) and \( \triangle QXT \) with corresponding marks.]

7. Draw two triangles \( DEF, PZR \) and insert in your figure markings to show that \( \angle E = \angle Z, \angle D = \angle R \) and \( DE = PZ \). Are the two triangles congruent? Give reasons.

8. In fig. 11, p. 8, measure in inches, \( PM, PN, NM \). Are these measurements sufficient for making a copy of \( \triangle PMN \)? If so, draw it and measure \( \angle MPN \) in your copy.

Results should be compared as in No. 1.

9. Make a copy of \( \triangle PLM \) in fig. 11, p. 8, by measuring in inches, \( PM, ML, LP \). Measure \( \angle LMP \) in your copy.

Results should be compared as in No. 1.

Hence we conclude that if two triangles are drawn so that they agree as regards the measurements of all three sides, then they must agree as regards all other corresponding measurements—that is, they are congruent.

10. In fig. 147, corresponding marks indicate that the sides are given equal. Are the triangles congruent? If so, state the fact in the proper way.

Draw the two triangles so that \( QNKZ \) is a straight line, given that \( QB = 5 \) cm., \( BN = 6 \) cm., \( NQ = 7 \) cm., and measure the largest angle in each triangle.

11. Why is it possible to draw a triangle with angles \( 35^\circ, 62^\circ, 80^\circ \)? Draw two such triangles in which the longest sides are 4 cm., 8 cm., respectively, and measure the remaining sides. What do you notice about each pair of corresponding sides?

Can you draw a triangle with angles \( 38^\circ, 62^\circ, 90^\circ \)?
If two angles of a triangle are given, the third angle can be calculated. The statement that the angles of a triangle are 35°, 62°, 80° is equivalent to the statement that two of its angles are 35°, 62°, and therefore contains only two independent facts. It is impossible to copy a triangle unless three independent measurements are given.

If the angles of a triangle are given, the shape of the triangle is fixed, but its size is not known unless the length of one side is also given.

12. What measurements must you make to construct a triangle of the same shape as \( \triangle ABC \) in fig. 11, p. 8, but of any size? Construct two such triangles so that the length of the side corresponding to \( BC \) is (i) 4 cm., (ii) 6 cm. Measure the side which corresponds to \( AB \) in each triangle.

13. Draw a triangle \( ABC \) so that
\[ AB = 7 \text{ cm., } AC = 5.5 \text{ cm., } \angle B = 35°. \]
Measure \( \angle ACB \).
First draw \( \angle PBQ = 35° \), see fig. 148, and cut off \( BA \) from \( BP \) so that \( BA = 7 \text{ cm.} \); then use your compass to find the position of \( C \) on \( BQ \) so that \( AC = 5.5 \text{ cm.} \).

![Fig. 148](image)

There are two possible positions, \( C_1 \) and \( C_2 \) of \( C \) and therefore there are two triangles, \( \triangle ABC_1 \) and \( \triangle ABC_2 \) of different shapes and sizes, which satisfy the data.

Thus three independent measurements, the lengths of two sides and the size of a not-included angle, do not fix the shape and size of the triangle.

14. Draw a triangle \( ABC \) so that
\[ AB = 5 \text{ cm., } AC = 7 \text{ cm., } \angle B = 90°. \]
Measure \( \angle ACB \).
Proceed as in No. 13, see fig. 149.
There are two possible positions, \( C_1 \) and \( C_2 \) for \( C \), but if we fold the paper so that \( AB \) is the crease, \( \triangle ABC \) will coincide with \( \triangle ABC_2 \); therefore in this case there are not two triangles of different sizes which satisfy the data.

Hence we conclude that the shape and size of a triangle are not always fixed uniquely when the measurements of two sides and a not-included angle are given.

If however the given not-included angle is a right-angle, the measurements of the hypotenuse and one other side are sufficient to fix the shape and size of the triangle.

Hence if two right-angled triangles are drawn so that they agree as regards the lengths of their hypotenuses and one other pair of sides, then they must agree as regards all other corresponding measurements; that is, they are congruent.

15. Draw a triangle \( ABC \) so that
\[ AB = 7 \text{ cm., } AC = 8 \text{ cm., } \angle B = 35°. \]
Measure \( \angle ACB \). Is it possible to construct triangles of different sizes to fit the given measurements?

Note. This example illustrates the more general statement that the measurement of two sides and a not-included angle fixes the size of the triangle uniquely if the given not-included angle is opposite to the longer of the two given sides, but not otherwise.

16. Draw an obtuse-angled triangle \( ABC \) so that \( AB = 12 \text{ cm., } AC = 9 \text{ cm., } \angle B = 40° \). Measure \( BC \).

The conclusions of this Exercise are as follows:
(i) the shape and size of a triangle are fixed by
(ii) one side and two angles, the situation of the side with respect to the angles being given;
(iii) three sides.

Also in the special case when the triangle is right-angled:
(iv) the hypotenuse and one side.

But an ambiguous case may arise if two sides and a not-included angle are measured. Two triangles can be drawn to fit the measurements, see fig. 148, if the measured not-included angle is opposite to the shorter of the two measured sides.
These results provide the following tests for congruence:

1. Two triangles are congruent if ANY TWO SIDES of the first are equal to any two sides of the second, and if the INCLUDED ANGLES are also equal.
   Abbreviation for reference: SAS or 2 sides, inc. ∠.

2. Two triangles are congruent if ANY TWO ANGLES of the first are equal to any two angles of the second, and if ANY SIDE of the first is equal to the CORRESPONDING SIDE of the second.
   Abbreviation for reference: ASA; AAS or 2 ∠s, corr. side.

3. Two triangles are congruent if the THREE SIDES of the first are equal to the three sides of the second.
   Abbreviation for reference: SSS or 3 sides.

4. Two RIGHT-ANGLED triangles are congruent if their HYPOTENUSES are equal and if ONE OTHER SIDE of the first is equal to one other side of the second.
   Abbreviation for reference: RHS or rt. ∠, hyp., side.

EXERCISE 16

The two triangles in fig. 150 are given congruent. Express this in the form \(\triangle ABC\) are congruent for the data in Nos. 1–2, and mark on your own figure the dimensions of \(\triangle MPT\).

1. (i) \(\angle P = 64°, \angle T = 59°\);
   (ii) \(\angle H = 67°, \angle P = 69°\);
   (iii) \(PH = 4.3\ cm, PT = 4.2\ cm\).

2. (i) \(\angle T = 59°, \angle P = 57°\);
   (ii) \(\angle H = 59°, HT = 4.2\ cm\);
   (iii) \(PT = 4.3\ cm, HT = 4.3\ cm\).

Are the triangles in figs. 151–155 congruent? If so, state the test used and express the fact in the form \(\triangle ABC\) are congruent.
Mark on your own figure the remaining dimensions, the unit of length being 1 cm.

In fig. 155, \(AB = 6.3\ cm, \angle C = 64\frac{1}{2}°\).
STAGE A

In Nos. 8–11, show the data for the two triangles $\triangle ABC$, $\triangle XYZ$ on a figure by corresponding marks (cf. fig. 144, p. 53). Say whether the triangles must be congruent. If they are congruent, state the test used and express the fact in the proper way.

8. (i) $AB = YZ$, $AC = XZ$, $\angle A = \angle Z$.
   (ii) $AB = XY$, $\angle A = \angle Z$, $\angle B = \angle Z$.

9. (i) $AC = XZ$, $AB = XY$, $\angle C = \angle Z$.
   (ii) $AB = YZ$, $\angle A = \angle Z$, $\angle C = \angle X$.

10. (i) $BC = ZX$, $AC = YX$, $BA = YZ$.
     (ii) $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.

     (ii) $AC = YZ$, $BC = XY$, $\angle C = \angle Y$.
     (iii) $AB = AC$, $XY = XZ$, $\angle A = \angle Y$.

State clearly the reasons which prove that figs. 156–161 contain congruent triangles and express the fact properly.

12. [13]

14. [15]

16. [17]

CONSTRUCTIONS

*18. Draw a triangle $\triangle ABC$ in which $AB = AC$, and draw the line which bisects $\angle BAC$ and cuts $BC$ at $N$. Prove that (i) $\angle B = \angle C$, (ii) $\angle ANB = 1$ right angle.

*19. Draw two lines $\triangle AKB$, $\triangle CKD$ cutting each other at $K$ so that $AK = KB$ and $CK = KD$. Join $AC$, $CB$, $BD$, $DA$. (i) Prove that $AC = BD$. (ii) What angle is equal to $\angle ACK$ and what follows from this fact? (iii) What angle is equal to $\angle BCK$? Give reasons. What follows from this fact?

*20. Draw a line $AB$ and bisect it at $N$; draw through $N$ the line perpendicular to $AB$ and take any two points, $P$ and $Q$, on it. Prove that (i) $PA = PB$, (ii) $\angle PQA = \angle PBQ$.

Constructions with Ruler and Compass

The bisection of lines and angles and the drawing of perpendiculars and parallels to given lines can be performed without the use of either a graduated ruler or protractor or set-square. The methods are indicated in Exercise 17, Nos. 1–6, and should be worked through by all. But in general in practical work, set-squares should be used for drawing parallels and perpendiculars unless otherwise stated.

EXERCISE 17

[Set-squares and protractors must not be used in this Exercise.]

1. (i) Bisect the given angle $\angle ACB$.
   In fig. 162 (i), $P$, $Q$ lie on a circle, centre $C$; two equal circles, centres $P$, $Q$, cut at $R$.

(ii) Prove that the construction is correct by using the triangles in fig. 162 (ii).
2. (i) C is a given point on a given line AB; construct the
perpendicular at C to AB.

In fig. 163 (i), X, Y lie on a circle, centre C; two
equal circles, centres X, Y, cut at K.

(i)

(ii) Prove that the construction is correct by using the
triangles in fig. 163 (ii).

3. (i) Construct the perpendicular bisector of a given line
AB, i.e. the line which bisects AB at right angles.

In fig. 164 (i), two equal circles, centres A, B, cut at P, Q.

(i)

(ii) Prove that the construction is correct. First prove,
see fig. 164 (ii), that \( \triangle APQ, BPQ \) are congruent;
this shows that \( \angle APQ = \angle BPQ \); then prove that
\( \triangle APN, BPN \) are congruent.

4. (i) C is a given point outside a given line AB; construct
the perpendicular from C to AB.

In fig. 165 (i), X, Y lie on a circle, centre C; two
equal circles, centres X, Y, cut at R.

(ii) Prove that the construction is correct. First prove,
see fig. 165 (ii), that \( \triangle CXR, CYR \) are congruent,
and so prove that \( \triangle CXN, CYN \) are congruent.

5. (i) Given an angle ACB and a line RP, construct a line
RQ so that \( \angle RQ = \angle C \).

In fig. 166, X, Y lie on a circle, centre C; \( RX' = CX \);
the circle, centre R, radius equal to CY, cuts the
circle, centre X', radius equal to XY, at Y'.

(ii) Prove that with this construction \( \angle RQ = \angle C \).
by showing that \( \triangle RX'Y', CXY \) are congruent.

6. (i) C is a given point outside a given line AB; construct
the line through C parallel to AB.

In fig. 167, P, Q are any points on AB; the circle,
centre P, radius equal to QC, cuts the circle, centre C,
radius equal to PQ, at R.

(ii) Prove that the construction is correct. In fig. 167 (ii),
use congruent triangles to prove that \( \angle QPC = \angle RCP \).
What follows?
STAGE A

Do not use a protractor or set-square for any of the following constructions. Show all the construction lines, but do not state the method and do not prove it is correct.

7. Construct an equilateral triangle. Exercise 16, No. 18, shows that all its angles are equal; hence construct an angle equal to 30°.

8. Draw a triangle with sides of lengths 5 cm., 6 cm., 7 cm., and construct the bisector of each angle of the triangle.

9. Draw a triangle with sides of lengths 5 cm., 6 cm., 7 cm.; construct the perpendicular from each vertex to the opposite side.

10. Draw an acute-angled triangle; make all its sides unequal. Construct the perpendicular bisectors of the three sides.

[11] (i) Draw any obtuse angle. Construct the lines which divide it into four equal angles.

(ii) Construct an angle equal to 22°30′

[12] Draw a circle and take three points A, B, C on the circumference, so that AB is longer than AC. Construct the bisector of ∠BAC and the perpendicular bisector of BC. Do they meet, when produced, on the circumference?

[13] Draw an obtuse-angled triangle. Construct the perpendiculars from each vertex to the opposite side. Do these perpendiculars, when produced, meet one another at the same point?

14. Draw any triangle ABC. Use a construction to complete the parallelogram ABCD.

Plains and Maps

If you look at a drawing through a magnifying glass, you appear to see a figure of the same shape but of larger size. In the same way, the picture projected on the screen at a cinema is the same shape as the picture on the photographic film, but greatly enlarged.

The fact that it is possible to have drawings of different size but the same shape is assumed by an architect when he draws the plans of a building, and by a land-surveyor when he makes a map. The shape of the figure formed by the lines in a map is the same as that of the figure formed by the roads, railways, etc., which they represent, but all lengths are reduced in a fixed ratio, called the scale of the map.

SCALE DIAGRAMS

Example for Oral Discussion

1. Fig. 168 represents four straight roads AB, BC, CD, DA enclosing an estate; the plan is drawn on a scale in which 1 cm. represents 200 yd. A is due north of B, and C is due east of B.

(i) What are the lengths in yards of the sides AB, BC, CD?

(ii) Draw the plan and, by measuring AC, find the distance of A from C.

(iii) Measure ∠BAD in your figure. What is the bearing of D from A?

2. If a plan is drawn on the scale of 1 in. to 500 yd., the ratio of 1 in. to 500 yd. is called the Representative Fraction (or R.F.) of this plan. What is its value?

When drawing a plan to scale, start by making a rough sketch containing all the given measurements, unless one is already provided. Then choose a convenient scale and state what your scale is.

EXERCISE 18

[Use your set-squares whenever it is simpler to do so.]

1. A map is made on a scale of 5 miles to the inch.

(i) What distance is represented by a line 2·4 in. long on the map?

(ii) What is the length of a line on the map which represents a road 8 miles long?

(iii) What is the R.F. of the map?

2. A plan is drawn on a scale of 10 ft. to the inch.

(i) What distance is represented by a line 7·2 in. long on the plan?

(ii) What is the length of a line on the plan which represents a fence 15 yd. long?

(iii) What is the R.F. of the plan?
3. Fig. 169 represents a rectangular courtyard ABCD. Draw a plan of it on the scale, 20 ft. to the inch, and find the length of the diagonal AC of the courtyard.

4. Fig. 170 represents three church towers, H, O, K; H is due north of O. Draw a map on the scale, 1 mile to the cm., and find (i) the distance of K from H, (ii) the bearing of K from H.

5. A, B are points 500 yd. apart on a straight road, see fig. 171. Draw a map on the scale, 1 mile to the cm., and find from a plan (i) the distance of C from A, (ii) the shortest distance of the hut from the road.

6. Fig. 172 represents three church towers P, Q, R; Q is due east of P. Find (i) the bearing of R from P, (ii) the bearing of R from Q, (iii) the shortest distance of R from the straight road PQ.

7. A, B, C are three towns; A is 15 miles due north of B and is 25 miles due west of C. Find the distance and bearing of C from B.

8. E, F, G are three church towers; F is 4 miles due south of E; G is 5 miles due south-west of E. Find the distance and bearing of G from F.

9. Three straight roads form a triangle LMN; LM = 600 yd., MN = 450 yd., NL = 350 yd. Find the shortest distance of N from the road LM.

10. A, B are points 350 yd. apart, on a straight level road; C is a flagstaff 200 yd. from A and 250 yd. from B. Find the shortest distance of C from the road AB.

11. B is 250 yd. due north of A; a tree T bears N. 42° E. from A and bears S. 38° E. from B. Find the distance of T from A.

12. P, Q, R are three towers; Q is 3½ miles due north of P; R is 7½ miles from P and bears N. 35° E. from P. Find the distance and bearing of R from Q.

13. A straight passage runs from A to B, 100 yd., and then turns through an angle of 70° and runs on to C, 80 yd. from B. What distance would be saved by having a passage direct from A to C?

14. X is 500 yd. due north of Y; the bearing of Z from X is S. 70° W., and the bearing of Z from Y is N. 35° W. Find the distance of Z from Y.

15. A gun whose range is 8000 yd. is placed at a point 6000 yd. from a straight railway line. What length of the railway is within range of the gun?

16. Southampton is 12 miles S. 22° W. from Winchester; Romsey is 10 miles S. 58° W. from Winchester. Find the distance and bearing of Romsey from Southampton.

17. Andover is 12 miles from Winchester and 15 miles from Salisbury; Salisbury is 20 miles due west of Winchester. Andover is on the north side of the road from Winchester to Salisbury. Find the bearing of Andover from Salisbury.

18. A man rows north at 4 miles an hour and the current carries him north-east at 5 miles an hour. How far is he from his starting-point after half an hour?

19. An aeroplane points north and flies at 120 miles an hour; the wind carries it south-west at 30 miles an hour. What is its distance and bearing from the starting-point after 20 minutes?

20. Bristol is 55 miles due north of Dorchester; Exeter is 42 miles from Dorchester and 64 miles from Bristol and is on the west side of the road from Dorchester to Bristol. If a wireless station is located 33 miles north-east of Exeter, find its distance and bearing from Dorchester.
STAGE A

Heights and Distances

Suppose you are looking through a telescope \( T \) which is pointed horizontally at a flagstaff. If you now tilt the telescope in a vertical plane so that it points upwards at the top of the flagstaff, the angle through which you have tilted it is called the angle of elevation of the top of the flagstaff from \( T \). In fig. 173, if \( TA \) is horizontal and if \( A TP \) is a vertical plane, \( \angle A TP \) is called the angle of elevation of \( P \) as seen from \( T \).

Suppose you are standing on the top of a cliff and looking through a telescope \( T \) which is pointed horizontally out to sea. If you now turn the telescope in a vertical plane so that it points downwards at a boat on the sea, the angle through which you have turned the telescope is called the angle of depression of the boat from \( T \).

In fig. 174, if \( TA \) is horizontal and if \( A TQ \) is a vertical plane, \( \angle A TQ \) is called the angle of depression of \( Q \) as seen from \( T \).

EXERCISE 19

[Use your set-squares whenever it is simpler to do so.]

1. The angle of elevation of the top of a tower \( AB \) from a point \( C \) on the ground, 120 ft. from the foot of the tower, is 35°. See fig. 175. Find the height of the tower.

2. The angle of depression of a boat \( B \) from the top \( K \) of a cliff \( HK \), 300 ft. high, is 40°, see fig. 176. Find the distance of the boat from the foot \( H \) of the cliff.

3. At a distance of 150 ft. from the foot of a tower, the angle of elevation of the top of the tower is 28°. Find the height of the tower.

4. A kite is flown at the end of a string 400 ft. long which makes an angle of 65° with the horizontal. Find the height of the kite above the ground.

5. Find the angle of elevation of the sun at a time when a vertical pole 12 ft. high casts a shadow 20 ft. long on the ground.

6. Find the angle of depression from the top of a tower 140 ft. high of an object on the ground at a distance of 200 ft. from the foot of the tower.

7. A ladder 15 ft. long is resting against a vertical wall; the foot of the ladder being 8 ft. away from the wall. Find (i) the angle which the ladder makes with the horizontal, (ii) the height of the top of the ladder above the ground.

8. Fig. 177 represents two shears servings \( PR, QR\), whose ends \( P, Q \) rest on level ground. Find the height of \( R \) above the ground.

9. From the top \( A \) of a cliff, 250 ft. above sea-level, the angles of depression of two boats \( B, C \) are 64°, 48°; and \( ABC \) is a vertical plane. Find the distance of \( B \) from \( C \).

10. The angle of elevation of the top of a tower from a point on the ground 150 yd. from the foot of the tower is 28°. What will it be from a point on the ground 100 yd. from the foot of the tower?

11. One end of a string 5 ft. long is fastened to a peg and a stone is attached to the other end. The stone sways to and fro so that the string in its extreme positions makes 25° with the vertical on each side. Find the distance between the two extreme positions of the stone.

12. From a point \( P \) on the ground the angle of elevation of the top \( B \) of a tower \( AB \) is 30°. From a point \( Q \) on the ground, 100 ft. nearer to the foot \( A \) of the tower, the angle of elevation of \( B \) is 50°. Find the height of the tower.
Similar Triangles

The method for solving problems by drawing to scale depends on the assumption that it is possible to increase or decrease the sides of a triangle in any given ratio without changing its shape, i.e., without altering the size of its angles. Two such triangles are called similar.

The measurements given in fig. 178, 2 angles and 1 side, fix the shape and size of \( \triangle ABC \).

Draw accurately the triangle in fig. 178, measure \( AB, AC \), and calculate the size of \( \angle A \). You should obtain the results shown in fig. 179.

4 cm. 6 cm. 5 cm. 6 cm.

\( \angle A = 63^\circ \) \( \angle B = 56^\circ \) \( \angle C = 41^\circ \)

\( \triangle B'C' \) = 4.5 cm.

The shape and size of a triangle are fixed if we are given the lengths of two sides and the size of the included angle. If \( A'B'C' \) is any triangle similar to the triangle \( ABC \) whose dimensions are shown in fig. 179, \( \angle A' = \angle A = 83^\circ \); \( A'B' \) can have any length whatever, but the triangles are similar, therefore when the length of \( A'B' \) has been chosen, the length of \( A'C' \) must be taken so that \( \frac{A'C'}{AB} = \frac{A'B'}{BC} \), and we then have sufficient data for drawing \( \triangle A'B'C' \).

For example, if \( A'B' \) is taken to be 3 cm., \( A'B' = \frac{3}{5} \), therefore \( A'C' = \frac{3}{5} \times 5 \) cm. = 3.75 cm.

6. Draw \( \triangle A'B'C' \) with \( \angle A' = \angle A = 83^\circ \), \( A'B' = \frac{3}{5} \) cm., \( A'C' = \frac{3}{5} \times 4 \) cm. Measure \( \angle B' \), \( \angle C' \) and \( B'C' \). Calculate the ratio \( \frac{B'C'}{BC} \).

Results obtained by the class should be compared as in No. 1.

The general statement is expressed as follows:

If two triangles \( ABC \), \( A'B'C' \) are such that

\[ \frac{A'B'}{AB} = \frac{A'C'}{AC} \]

then \( \triangle A'B'C' \) are similar, that is

\[ \frac{B'C'}{BC} = \frac{C'A'}{CA} = \frac{A'B'}{AB} \] and \[ \frac{\angle A}{\angle A'} = \frac{\angle B}{\angle B'} = \frac{\angle C}{\angle C'} \]
STAGE A

The shape and size of a triangle are fixed if we are given the lengths of three sides. If $A'B'C'$ is any triangle similar to the triangle $ABC$ whose dimensions are shown in fig. 179, $B'C'$ can have any length whatever, but the triangles are similar, therefore when the length of $B'C'$ has been chosen, the lengths of $C'A'$, $A'B'$ must be taken so that

$$\frac{C'A'}{BC} = \frac{B'C'}{AB}$$

and we then have sufficient data for drawing $\triangle A'B'C'$.

7. With the data for $\triangle ABC$ in fig. 179, p. 70, draw $\triangle A'B'C'$ with $B'C' = 4.5$ cm., $C'A' = 3.75$ cm., $A'B' = 3$ cm. Measure $\angle A'$, $\angle B'$, $\angle C'$. What do you notice?

The general statement is expressed as follows:—

If two triangles $ABC$, $A'B'C'$ are such that

$$\frac{BC}{CA} = \frac{CA}{AB},$$

$$\frac{B'C'}{A'B'} = \frac{C'A'}{C'B'},$$

then $\triangle A'B'C'$ are similar, that is

$$\angle A = \angle A', \angle B = \angle B', \angle C = \angle C'.$$

Thus to each of the three general tests for congruence of two triangles corresponds a test for similarity of two triangles.

EXERCISE 20

1. Are the triangles in fig. 180 similar? If so, give the reason and express the fact in the form $\triangle ABC \cdots$ are similar. Find also the lengths of the remaining sides.

2. $\triangle XTS$ is a triangle such that, with the data of fig. 180, $\triangle ABC$ are similar,

   (i) find $TX$, $TS$ and $\angle T$ if $XS = 12$ cm.
   (ii) find $TS$, $XS$ and $\angle S$ if $TX = 8$ in.

3. Are the triangles in fig. 181 similar? If so, give the reason and express the fact in the form $\triangle ABC \cdots$ are similar. Find also the sizes of the unmarked sides and angles.

4. $\triangle FZD$ is a triangle such that, with the data of fig. 181, $\triangle ABC \cdots$ are similar,

   (i) find $FZ$, $ZD$ and $\angle F$ if $FD = 7$ cm.,
   (ii) find $FZ$, $FD$ and $\angle Z$ if $ZD = 12$ in.

5. Are the triangles in fig. 182 similar? If so, give the reason and express the fact in the proper form. Find also the sizes of the remaining angles and side.

6. In fig. 183, where PQ is parallel to BC, explain why $\triangle APQ$ and $\triangle ABC$ are similar. Find the lengths of $AB$ and $QC$. 

Fig. 180

Fig. 181

Fig. 182

Fig. 183
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[7] In fig. 184, AC is parallel to PQ, and ABQ, PBC are straight lines.
   (i) Complete $\Delta ABC$ are similar because ...
   (ii) Find the lengths of BP, BQ.

Fig. 184

8. In fig. 185, AC = 10 and $\angle GKC = \angle B$.
   (i) Complete $\Delta ABC$ are similar because ...
   (ii) Find the lengths of KG, KA.

Fig. 185

[9] On a map showing three towns A, B, C, AB = 4.5, BC = 3.5, CA = 3. If the distance of A from B is 27 miles, find the distances of C from A and of C from B.

10. A pole 10 ft. high casts a shadow 3 ft. long on level ground at the same time as the shadow of a tower near the pole is 42 ft. long. Find the height of the tower.

11. In a photograph of a chest, the height measures 3.2" and the breadth 6". If the chest is 7/2 ft. broad, find its height.

*12. Explain why fig. 186 contains two similar triangles. Express this fact in the proper form and find the length of BC.

*13. In fig. 187, arrows indicate that lines are given parallel. Find p, q, r.

*14. In fig. 188, arrows indicate that lines are given parallel. Find a, b, c, d.

Fig. 188

TANGENT OF AN ANGLE 75

Trigonometrical Ratios

The following section can be postponed without interfering with the remainder of the geometry course; but some theorems which form part of the course may with advantage be expressed in trigonometrical form.

The Tangent of an Angle. Draw two angles A, B each 38° and from points P, Q on one arm of each angle draw the perpendiculars PM, QN to the other arm.

Fig. 189

Then $\angle A = \angle B$ and $\angle M = \angle N$.

$\therefore \Delta APM$ and $\Delta BQN$ are similar.

$\therefore \frac{PM}{AM} = \frac{QN}{BN}$

Thus the value of the ratio $\frac{PM}{AM}$ does not depend on the distance of P from the point A, but only on the size of $\angle A$.

The ratio $\frac{PM}{AM}$ is called the tangent of the angle $\angle PAM$, and is written $\tan \angle PAM$ or $\tan 38°$, since $\angle PAM = 38°$.

In words, see fig. 190, if a perpendicular PM is let fall from any point P in one arm of an angle $\angle x$ to the other arm, opposite side (PM)

$\tan \angle x = \frac{PM}{AM}$

adjacent side (AM)
An approximate value of the tangent of an angle can be found by measurement and computation.

If fig. 188 is drawn so that \(AM = 5\) cm., measurement shows that \(PM = 3.9\) cm., approximately.

Hence \(\tan 38^\circ = \frac{3.9}{5} = 0.78\), approximately.

It should be noted that in fig. 189, \(\angle P = 90^\circ - 38^\circ - 52^\circ\);

\[
\therefore \tan 52^\circ = \frac{AM}{PM}
\]

Thus \(\tan 52^\circ = \frac{5}{3.9} = 1.28\), approximately.

The values of the tangents of angles have been calculated, once for all, and entered in a table from which they can be obtained when required, see p. xv.

**Example 1.** From a point on the ground 200 ft. away from the foot of a tower the angle of elevation of the top of the tower is 32°. Find the height of the tower.

In fig. 191, \(AB\) represents the tower.

If \(AB = h\) ft.,

\[
\frac{h}{200} = \tan 32^\circ = 0.625,
\]

\[
\therefore h = 0.625 \times 200 = 125\text{ ft.}
\]

The height of the tower is 125 ft.

**Example 2.** In \(\triangle ABC\), \(CA = 3\) ft., \(CB = 4\) ft., and \(\angle C = 90^\circ\), find \(\angle A\).

If \(\angle A = x^\circ\),

\[
\tan x^\circ = \frac{CB}{CA} = \frac{4}{3} = 1.333.
\]

Hence from the tables

\(x^\circ = 53.1^\circ\), approximately.

**Exercise 21**

[It saves time to use squared paper for Nos. 1-8.]

Find by drawing and measurement approximate values of

1. \(\tan 25^\circ\).
2. \(\tan 72^\circ\).
3. \(\tan 45^\circ\).
4. \(\tan 60^\circ\).

Find by drawing and measurement the angles whose tangents have the values in Nos. 5-8.

5. 0.7
6. 1.6
7. 0.6
8. 1.8

Nos. 9-16 refer to a triangle \(ABC\), right-angled at \(C\).

9. If \(\angle A = 40^\circ\) and \(AC = 4\) cm., find \(BC\).
10. If \(\angle B = 65^\circ\) and \(BC = 3\) cm., find \(AC\).
11. If \(\angle A = 52^\circ\) and \(BC = 10\) cm., find \(AC\).
12. If \(\angle B = 65^\circ\) and \(AC = 4\) cm., find \(BC\).
13. If \(BC = 4\) cm., \(AC = 10\) cm., find \(\angle A\).
14. If \(BC = 5\) cm., \(AC = 5\) cm., find \(\angle A\).
15. If \(BC = 6\) cm., \(AC = 8\) cm., find \(\angle B\).
16. If \(BC = 5\) cm., \(AC = 13\) cm., find \(\angle B\).

17. From a point on the ground 100 ft. from the foot of a tower, the angle of elevation of the top of the tower is 52°. Find the height of the tower.

18. From the top of a tower 180 ft. high the angle of depression of a milestone is 27°. How far is the milestone from the foot of the tower?

19. A ladder leaning against a vertical wall makes an angle of 21° with the wall; the foot of the ladder is 5 ft. from the wall. How high up the wall does the ladder reach?

20. A man starts from \(O\) and walks 320 yd. East and then 140 yd. North. What is now his bearing from \(O\)?

21. A man starts from \(O\) and walks \(\frac{1}{2}\) mile due South and then \(\frac{1}{2}\) mile due East. What is now his bearing from \(O\)?

22. A rectangular sheet of paper is 9 in. long, 7 in. wide. What angle does a diagonal of the sheet make with the longer side?

23. What is the angle of elevation of the sun if a stick 6 ft. long casts a shadow 3 ft. 9 in. long on level ground?
24. Find the angles of a set-square if the lengths of the sides containing the right angle are 5·3" and 3·7".

25. A man stands at a distance of 85 ft. from the foot of a tower and observes that the angles of elevation of the top and bottom of a flagstaff on the top of the tower are 56° and 34° respectively. Find the length of the flagstaff.

26. A man standing on a cliff 160 ft. high observes two boats in a vertical plane with him. The angles of depression of the boats are 32°, 49°. Find the distance between the boats.

27. The shadow of a telegraph pole is 13 ft. long when the angle of elevation of the sun is 59°. Find the length of the shadow when the angle of elevation of the sun is 35°.

The Sine and Cosine of an Angle. Draw two angles A, B each 37°, and from points P, Q on one arm of each angle draw the perpendiculars PM, QN to the other arm.

Then \( \angle A = \angle B \) and \( \angle M = \angle N \),

\[ \therefore \triangle APM \sim \triangle BQN \]

\[ \therefore \text{the ratio } \frac{PM}{AP} = \frac{QN}{BQ} \]

Thus the value of the ratio \( \frac{PM}{AP} \) does not depend on the distance of P from the point A, but only on the size of \( \angle A \).

The ratio \( \frac{PM}{AP} \) is called the sine of the angle \( \angle PAM \), and is written \( \sin PAM \) or \( \sin 37° \), since \( \angle PAM = 37° \).

\[ \sin 37° = \frac{PM}{AP} = \frac{3\cdot0}{5} = 0\cdot60, \text{ approximately.} \]

And \[ \cos 37° = \frac{AP}{PM} = \frac{4\cdot0}{5} = 0\cdot80, \text{ approximately.} \]

A table of values of sines and cosines of angles is given on p. xv.
Example. A ladder 10 ft. long leans against a wall and makes an angle of 63° with the horizontal.

(i) How high up the wall does the ladder reach?
(ii) How far from the wall is the foot of the ladder?
In fig. 195, AB represents the ladder and CB represents the wall.
Let \( CB = h \) ft. and \( AC = d \) ft.

\[
\begin{align*}
(i) & \quad 10 = \sin 63^\circ \\
& \quad = 0.891 \\
& \quad h = 0.891 \times 10 = 8.91 \\
& \quad CB = 8.91 \text{ ft.}
\end{align*}
\]

\[
\begin{align*}
(ii) & \quad 10 = \cos 63^\circ \\
& \quad = 0.454 \\
& \quad d = 0.454 \times 10 = 4.54 \\
& \quad AC = 4.54 \text{ ft.}
\end{align*}
\]

**EXERCISE 22**

*It saves time to use squared paper in Nos. 1-7.*

Find by drawing and measurement approximate values of
1. \( \sin 67^\circ \)
2. \( \cos 67^\circ \)
3. \( \sin 32^\circ \)
4. \( \cos 32^\circ \)

Find by drawing and measurement the value of \( x^\circ \) in Nos. 5-7.
5. \( \sin x^\circ = 0.5 \)
6. \( \cos x^\circ = 0.9 \)
7. \( \sin x^\circ = \frac{4}{3} \)

Nos. 8-14 refer to a triangle ABC, right-angled at C.
8. If \( \angle A = 57^\circ \) and \( AB = 5 \) cm., find BC and AC.
9. If \( \angle B = 35^\circ \) and \( AB = 3 \) in., find BC and AC.
10. If \( \angle A = 71^\circ \) and \( AB = 10 \) cm., find BC and AC.
11. If \( AB = 10 \) cm. and \( BC = 9 \) cm., find \( \angle A \).
12. If \( AB = 4 \) in. and \( AC = 2 \) in., find \( \angle A \).
13. If \( AB = 5 \) in. and \( AC = 3 \) in., find \( \angle B \).
14. If \( AB = 8 \) cm. and \( BC = 7 \) cm., find \( \angle B \).

15. A ladder 20 ft. long leans against a wall and makes an angle of 72° with the ground. How high up does the wall does the ladder reach? How far from the wall is the foot of the ladder?
16. A ladder 25 ft. long stands against a vertical wall and its foot is 7 ft. from the wall. What angle does the ladder make with the horizontal?
17. A man starts at O and walks 300 yd. in a direction N. 12° E. How far east of O is he? How far north of O is he?
18. A man starts at O and walks 5 miles in a direction S. 33° W. How far west of O is he? How far south of O is he?
19. P is 3 \( \frac{1}{2} \) miles east of Q; R is due north of P and is 4 \( \frac{1}{2} \) miles from Q. What is the bearing of R from Q?
20. The string of a kite is 750 ft. long and makes an angle of 68° with the horizontal. Find the height of the kite above the ground.
21. The diagonal of a rectangle is 6 cm. long and makes an angle of 32° with one of the sides. Find the lengths of the sides of the rectangle.
22. A hill slopes upwards at an angle of 18° with the horizontal. What height does a man rise when he walks 100 yd. up the hill?
23. A picture 45 in. high rests with its lower edge against a wall and makes an angle of 7° with the wall. Find the distance of the top of the picture from the wall.
24. A pendulum 3 ft. long swings to and fro through an angle of 20° each side of the vertical. How high does the lower end of the pendulum rise?

*The examples in Exercise 19 (omitting No. 8), p. 68, may also be solved trigonometrically if further practice is required.*
Construction of Surfaces of Solids

The open box which holds the matches in a Brymav match-box is 2·3 in. long, 1·5 in. wide, and 0·7 in. high. If you cut down the edges and fold the sides of the box down level with the base, you will obtain a figure of the shape represented in fig. 196, and this is called the net of the open box.

What are the lengths of the different parts of the boundary of fig. 196, if the dimensions of the open match-box are those given above?

Which lines in the net must be equal?

Draw out the net on stiff paper or thin cardboard; make creases along the dotted lines and then fold so as to obtain an open box. Use gummed paper to fasten the edges together.

EXERCISE 23

1. Make a sketch showing the net of a closed box, 5 cm. long, 3 cm. wide, 2 cm. high. Show the dimensions of the net on your sketch. Draw the figure accurately and construct the box.

2. Fig. 197 represents the net of a triangular prism. Show on your own figure the dimensions of the net if the prism is 5 cm. high and if each edge of the base is 3 cm. Construct the prism.

3. Fig. 198 represents the net of a regular tetrahedron, see the photograph opposite p. 1. Show on your own figure the dimensions of the net if each edge of the solid is 2 in. long. Construct the solid.

4. Fig. 199 represents the net of a pyramid on a square base, see fig. 3, p. 2. Show on your own figure the dimensions of the net if each edge of the base is 3 cm. and each of the slant edges is 4 cm. Construct the pyramid.

5. Fig. 200 represents the net of a regular octahedron, each face of which is an equilateral triangle, see the photograph opposite p. 1. Show on your own figure the dimensions of the net if each edge of the solid is 4 cm. long. Construct the solid.

6. Draw on stiff paper or thin cardboard the net of a prism, 4 cm. long, whose base is a regular pentagon, see fig. 3, p. 2. To draw the pentagon, prick through the points marked 1, 2, 3, 4, 5 in fig. 11, p. 8. Construct the solid.
7. Fig. 201 represents the net of a regular dodecahedron, see the photograph opposite p. 1. Each part of the net is a regular pentagon. To draw a central pentagon, prick through the points marked 1, 2, 3, 4, 5 in fig. 11, p. 8. Then fit equal pentagons round the edges of this pentagon. Construct the solid.

8. Fig. 202 represents the net of a regular icosahedron, see the photograph opposite p. 1. Each part of the net is an equilateral triangle. Construct the solid.

9. A circular cylinder, see fig. 3, p. 2, is made of thin paper and has both ends closed. Its height is 6 cm. and its girth (i.e. the circumference of the base) is 11 cm. Draw the net from which it could be constructed. Use the fact that the circumference of a circle is \( \pi \) times the diameter, approximately, to find the diameter of each circular end of the cylinder. Construct the cylinder.

10. Draw a semicircle of radius 6 cm. on stiff paper, not cardboard, and cut it out. Coil it so as to obtain the curved surface of a circular cone, see fig. 3, p. 2. Cut out also a circle of radius 2 cm.; this will form the base of the circular cone. Construct the solid.

PARALLELS, CONGRUENCE AND INEQUALITIES

The fundamental facts about

(i) the measurement of length and angle,
(ii) the relations between angles formed by a transversal and parallel lines,
(iii) the tests for congruence and similarity

have been discussed and illustrated in Stage A. These facts form a group of assumptions on which the course of elementary geometry is based; it is therefore convenient to repeat here the statements made there and give additional examples of their use.

Angles at a Point

Definition. If a straight line \( CD \) meets another straight line \( ACB \) so as to make the two adjacent angles equal, each angle is called a right angle.

It is assumed that all right angles are equal.

\[ \text{Fig. 203} \]

\( CD \) is then said to be at right angles to \( ACB \) or perpendicular to \( ACB \), and \( C \) is called the foot of the perpendicular from \( D \) to \( AB \). Never speak of drawing a perpendicular from a line to a point.

If a straight line \( OA \) is perpendicular to each of two different straight lines \( OB \) and \( OC \), it can be proved that \( OA \) is also perpendicular to every straight line through \( O \) in the plane \( OBC \), and \( OA \) is then said to be perpendicular to the plane \( OBC \).
THEOREM 1

If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.

With the notation of fig. 204, if $\overline{ACB}$ is a straight line, then $a + b = 2 \text{ rt. } \angle s$.

Abbreviation for reference: adj. $\angle s$ on st. line.

Corollary. If any number of straight lines are drawn from a given point, the sum of the successive adjacent angles so formed is equal to four right angles.

With the notation of fig. 205,

$a + b + c + d + e = 4 \text{ rt. } \angle s$.

Abbreviation for reference: $\angle s$ at a point.

Formal proofs of Theorems 1, 2 are given in the Appendix, pp. 540-1.

Definition. Two angles are called supplementary if their sum is equal to two right angles and either is called the supplement of the other.

In fig. 204, in which $\overline{ACB}$ is a straight line, $a$ and $b$ are supplementary angles.

THEOREM 2

If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.

With the notation of fig. 206,

if $a + b = 2 \text{ rt. } \angle s$,
then $\overline{AOB}$ is a straight line.

Abbreviation for reference: adj. $\angle s$. supp.

Definition. Three or more points are said to be collinear if they lie on the same straight line.

In fig. 206, if $a + b = 2 \text{ rt. } \angle s$, then $A, O, B$ are collinear.

THEOREM 3

If two straight lines intersect, the vertically opposite angles are equal.

With the notation of fig. 207, which represents two intersecting straight lines,

$a = b$ and $p = q$.

Abbreviation for reference: vert. opp. $\angle s$.

A formal proof was given on p. 28.
Examples for Oral Discussion

1. Draw a figure to show how fig. 205, p. 86, must be constructed
   (i) if \( e + a + b = 2 \text{ rt. } \angle s; \)
   (ii) if \( b + c + d = 2 \text{ rt. } \angle s; \)
   (iii) if \( a + b + c = 2 \text{ rt. } \angle s. \)

   Draw four lines \( AH, AK, AL, AM \) as in fig. 217, p. 90, without
   the numerical data and use it for Nos. 2-4:

2. If \( \angle KAL = \angle MAH \), must \( LAH \) and \( KAM \) be straight
   lines?

3. If \( \angle KAL = \angle MAH \) and \( \angle HAK = \angle LAM \), must \( LAH \)
   and \( KAM \) be straight lines?

4. If it is given that \( LAH \) is a straight line and that
   \( \angle KAL = \angle MAH \), must \( KAM \) be a straight line?

NUMERICAL EXAMPLES

EXERCISE 24

1. Write down the supplements of \( 20^\circ, 160^\circ, 92^\circ. \)
2. Find \( x \) if \( 2x, 3x \) are supplementary angles.
3. Find \( y \) if \( 4y^2 + 30^\circ, y^2 + 40^\circ \) are supplementary angles.
4. Find the size of an angle \( x^\circ \), if it is 4 times its supplement.
5. What is the least number of times you must turn through
   \( 17^\circ \) in order to turn through (i) an obtuse angle, (ii) a reflex angle,
   (iii) more than one revolution?
6. Through what angle does the minute-hand of a clock turn
   (i) half an hour, (ii) 1 minute?
7. Through what angle does the hour-hand of a clock turn in
   (i) 9 hours, (ii) 20 minutes?
8. Find the angle between the hands of a clock (i) at 7 o'clock,
   (ii) at 20 minutes past seven.
9. A wheel has 6 spokes equally spaced; what is the angle
   between 2 adjacent spokes?
10. A wheel makes 40 revolutions per minute; through what
    angle does a spoke turn in 1 second?

ADJACENT ANGLES

Draw a figure representing the compass directions, north,
west, east, south, etc. (see fig. 43, p. 23), and use it to find
the angles between the following pairs of directions:

1. N.E. and E.
2. E. and W.S.W.
3. S.W. and N.N.W. (S.E. and W.)
4. S. and N.E.
5. E.S.E. and W.S.W.
6. 10° E. of N. and 30° E. of S.
7. 72° E. of N. and 65° W. of N.
8. 20° W. of S. and 80° E. of N.
9. 70° W. of N. and 10° E. of S.

11. Find the reflex angle between S. 80° E., N. 50° W.

*22. Two wheels A, B are geared so that A makes 6 revolutions
    when B makes 1 revolution. Through how many degrees does B
    make \( 1 \) of a revolution?

23. In fig. 208, \( \triangle ABC \) is a straight line, find \( x. \)

24. In fig. 209, \( \triangle ABC \) is a straight line. Find \( y. \)

25. In fig. 210, find \( x. \)

26. In fig. 211, find \( x \) if \( y = 40. \)

27. In fig. 211, find \( y \) if \( x = 90. \)

28. In fig. 212, find \( d \) if \( c \) is twice \( d. \) [Suppose \( d = x^\circ. \)]

29. In fig. 212, find \( d \) if \( c \) exceeds \( d \) by 1 right angle.
30. In fig. 213, ACB is a straight line. Find $a$ if $a$ is three times $b$. [Suppose $b$ is $\pi$.]

[31] In fig. 213, ACB is a straight line. Find $a$ if $a$ exceeds $b$ by $\frac{1}{4}$ of a right angle.

32. Fig. 214 represents three straight lines intersecting at a point. Find $a$ and $b$.

[33] Fig. 215 represents three straight lines intersecting at a point. Find an equation connecting $x$, $y$, $z$.

34. In fig. 216, not drawn accurately, (i) find $\angle PAT$; (ii) find the angle between the bisectors of $\angle RAQ$, $\angle RAS$.

35. In fig. 216, not drawn accurately, find three points which are collinear. Is there more than one answer? Give reasons.

36. In fig. 217, not drawn accurately, (i) find the value of $x$; (ii) find three points which are collinear. Is there more than one answer? Give reasons.

Use of Small Letters for Angles. Proofs can often be written down more shortly by using small letters to represent angles. It is also a help to use a notation for two angles which are given equal or can easily be proved to be equal which suggests this equality, e.g. $a$ and $a_1$, or $a_2$ and $a_3$, etc.

Whenever you use small letters for angles, draw a large diagram. Unless you do so, the notation will not be clear.

Example for Oral Discussion

If a straight line $EC$ meets another straight line $ACB$ at $C$, and if $CH$, $CK$ are the bisectors of $\angle AEC$, $\angle BCE$, prove that $\angle HCK$ is a right angle.

Since $\angle s HCE$, $HCA$ are given equal, we denote them by $a_1$, $a_2$; similarly for $\angle s KCE$, $KCB$.

(i) Express with small letters the fact that has to be proved.
(ii) What do you know about $\angle ECA$ and $\angle ECB$? Give the reason. Express this fact with small letters.
(iii) What do you get by putting $a_2 = a_1$ and $b_2 = b_1$? Complete the proof.

In proving a rider, always state all the necessary reasons.

The proof of this rider may be set out as follows:

$$(a_1 + a_2) + (b_1 + b_2) = 2 \text{ rt. } \angle s \text{ adj. } \angle s \text{ on st. line},$$

but $a_1 = a_2$ and $b_1 = b_2$ given,

$\therefore 2a_1 + 2b_1 = 2 \text{ rt. } \angle s,$

$\therefore a_1 + b_1 = 1 \text{ rt. } \angle,$

$\therefore \angle HCK = 1 \text{ rt. } \angle.$
**EXERCISE 25**

Nos. 1–12 refer to fig. 219.

1. Express with small letters, (i) \( \angle POR \), (ii) \( \angle QOS + \angle ROT \).

[2] Express with small letters, (i) \( \angle QOT \), (ii) \( \angle POS - \angle POR \).

3. Express with capital letters as simply as possible, (i) \( c+d \), (ii) \( a+b+c \).

4. Express the following statements with small letters:
   (i) \( OS \) bisects \( \angle ROT \); (ii) \( TOP \) is a straight line.

5. Express the following statements with small letters:
   (i) \( \angle QOS = \angle ROT \); (ii) \( OS \) is perpendicular to \( OQ \).

6. \( \angle POQ = \angle ROS \), prove that \( \angle POR = \angle QOS \).

7. \( \angle QOS = \angle ROT \), prove that \( \angle QOR = \angle SOT \).

8. Prove that \( \angle POR + \angle QOS = \angle POS + \angle QOR \).

9. If \( OR \) is perpendicular to \( OP \), and if \( OS \) is perpendicular to \( OQ \), prove that
   (i) \( \angle POQ = \angle ROS \), (ii) \( \angle POS + \angle QOR = 2 \text{ rt. } \angle s \).

10. If \( OS \) bisects \( \angle ROT \), prove that \( \angle QOS - \angle TOS = \angle QOR \).

*11. If \( OR \) bisects \( \angle QOT \), prove that \( \angle ROS = \frac{1}{2} \angle QOS - \angle TOS \).

*12. If \( \angle POT = 2 \angle QOS \) and if \( OR \) bisects \( \angle POT \), prove that \( \angle POQ = \angle ROS \).

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**ADJACENT ANGLES**

Nos. 13–15 refer to fig. 220, in which \( OM, ON \) are the bisectors of \( \angle EOH, \angle EOK \).

13. Prove that \( \angle MON = \frac{1}{2} \angle HOK \).

[14] Prove that \( \angle MOE + \angle MOK = 2 \angle MON \).

15. Prove that \( \angle HON + \angle KOM = 3 \angle MON \).

---

Nos. 16–18 refer to fig. 221, which represents three intersecting straight lines. *State the reasons clearly.*

16. If \( p = q \), prove that \( a = b \).

17. If \( b = c \), prove that \( s = t \).

18. If \( a = c \), prove that \( q + t = 2 \text{ rt. } \angle s \).

19. In fig. 222, \( HCK \) is a straight line and \( CK \) bisects \( \angle ACB \). Prove that \( s = t \).

---

20. In fig. 223, not drawn accurately, \( ABC \) is a straight line and \( p_1 = p_2 \). Prove that \( MCN \) is a straight line.

*21. If in fig. 222 it is not given that \( HCK \) is a straight line, but it is given that \( s = t \) and \( \angle KCB = \angle KCA \), prove that \( HCK \) is a straight line.

*22. If in fig. 223 it is not given that \( ABC \) is a straight line, but it is given that \( \angle BCM = \angle ACN \) and that \( \angle ACM = \angle BCN \), prove that \( ABC \) and \( MCN \) are straight lines.
New Geometry

Parallels and Transversals

Definition. If two straight lines are coplanar, i.e. lie in the same plane, and if they never meet however far they are produced either way, they are called parallel straight lines.

The following assumption about parallel straight lines is called Playfair’s Axiom.

Through a given point, one and only one straight line can be drawn parallel to any given straight line which does not pass through the given point.

Definition. If a straight line cuts two or more other straight lines, it is called a transversal.

The meanings of the words, alternate, corresponding, interior, applied to angles formed by a transversal with the lines it cuts, have been given on p. 35.

In fig. 224,

a and b are called alternate angles,
c and d are called corresponding angles,
d and b are called interior angles on the same side of the transversal, or shortly interior angles; they are also called allied angles.

The use of pairs of these angles in providing tests for lines to be parallel, and the relations between pairs of these angles formed by a transversal cutting two parallel lines have been discussed in detail in Stage A, see pp. 55–55. These results are expressed by the following theorems.

Theorem 5

Two straight lines are parallel if a transversal makes

(i) a pair of alternate angles equal,
or (ii) a pair of corresponding angles equal,
or (iii) a pair of interior angles on the same side of the transversal supplementary.

With the notation of fig. 224, PQ || RS, if

(i) a = b,
or if (ii) c = b,
or if (iii) b + d = 2 rt. Ls.

Abbreviations for reference: (i) alt. Ls equal,
(ii) corr. Ls equal,
(iii) int. Ls supp.

A formal proof of Theorem 5 (i) is given in the Appendix, p. 544.

Theorems 5 (ii), 5 (iii) can be deduced from Theorem 5 (i).

1. If c = b, then a = b.

   a = c vert. opp. Ls,
   but c = b given.
   \[ a = b \]

2. If b + d = 2 rt. Ls, then a = b.

   a + d = 2 rt. Ls adj. Ls on st. line,
   but b + d = 2 rt. Ls given.
   \[ a + d = b + d \]
   \[ a = b \]
THEOREM 6

If a transversal cuts two parallel straight lines,
i) alternate angles are equal,
ii) corresponding angles are equal,
iii) interior angles on the same side of the transversal are supplementary.

With the notation of fig. 225, in which $PQ \parallel RS$,

(i) $a = b$,
and
(ii) $c = b$,
and
(iii) $b + d = 2 \text{rt. } \angle s$.

Abbreviations for reference: (i) alt. $\angle s$, $PQ \parallel RS$;
(ii) corr. $\angle s$, $PQ \parallel RS$;
(iii) int. $\angle s$, $PQ \parallel RS$.

A formal proof is given in the Appendix, p. 545.

Note. Markings in the diagrams for Theorems refer only to what is known from the data and construction, and not to any new facts discovered in the course of the proof.

THEOREM 7

Coplanar straight lines which are parallel to the same straight line are parallel to one another.

This follows from Playfair's Axiom, because two intersecting straight lines cannot both be parallel to a third straight line.
NEW GEOMETRY

Find the unknown marked angles in figs. 231–234. Draw your own figure and insert another parallel line in each case.

6. \[ \triangle ABC \]

7. \[ \triangle DEF \]

8. \[ \triangle GHI \]

9. \[ \triangle JKL \]

10. In fig. 235, prove that CD is parallel to BE.


*12. In fig. 237, ABCD is a straight line; prove BK \parallel CP.

*13. In fig. 238, prove that AB is parallel to DE.

EXERCISE 27

[Arses indicate that lines are given parallel.]

In the following examples draw your own figure and mark on it any necessary construction. State the reasons clearly.

1. In fig. 239, prove that \( p = q \). [No construction.]

2. In fig. 240, prove that \( p + q = 2 \) rt. \( s \). [No construction.]

3. In fig. 241, prove that \( m = n \). [Produce \( DE \) to meet \( BC \) at \( K \).]

4. In fig. 242, prove that \( h = k \). [Produce \( BC \) to \( E \).]

5. Draw any triangle ABC and produce BC to D. Prove that \( \angle ACB = \angle A + \angle B \). [Draw CK parallel to BA.]

6. Draw a triangle ABC in which \( \angle B = \angle C \). Produce BA to E. Draw AP parallel to BC. Prove that AP bisects \( \angle CAE \). [No construction.]

7. In fig. 243, C\( BK \) is a straight line and BQ bisects \( \angle ABK \); prove that \( \angle A = \angle C \). [No construction.]

8. In fig. 244, prove that \( p + q = g - s \). [No construction.]
NEW GEOMETRY

[9] In fig. 245, prove that \( \angle BCD = p + q \). [Draw \( CN \) parallel to \( ED \).]

*10. In fig. 246, prove that \( r + s - t = 2 \text{ rt. } \angle s \).

*11. If in fig. 245, when \( BD \) is joined, \( BC \) and \( DC \) are the bisectors of \( \angle ABD \) and \( \angle EDB \), use the fact proved in No. 9 to show that \( \angle BCD \) is a right angle.

*12. Draw a figure like fig. 238, p. 98, but omitting the numerical measurements. If \( AB \) is parallel to \( DE \), prove that \( \angle ABC + \angle CDE - \angle BCD = 2 \text{ rt. } \angle s \).

**Formal Proofs**

In Stage A, various fundamental geometrical facts, called *theorems*, have been discussed and illustrated practically, but *proofs* have not been given. The proofs of some of these facts are given in an appendix to which reference may be made at a later stage. But for the present the reader should *assume* the truth of the fundamental facts given about—

(i) angles at a point,
(ii) angles made by a transversal with parallel lines,
(iii) angle-tests for lines to be parallel,
(iv) tests for triangles to be congruent.

By making use of these assumptions, it is then possible to give systematic proofs of other important theorems.

A general statement of the fact which it is required to prove is called the *general enunciation* of the theorem. If this fact is stated in terms of the letters of a particular diagram, it is called the *particular enunciation*.

---

WRITING OUT THE PROOF OF A THEOREM

If what is required to be proved is expressed in the form of a *general enunciation*, set out the work in the following order:

1. State what is given:
2. State what it is required to prove:
3. State the construction, if any is necessary:
4. State the proof; this must include all necessary reasons, using suitable abbreviated references.

*Never give the number of a theorem as a reference.*

For example, the general enunciation of Theorem 8 is printed at the top of p. 102, and the argument which follows the diagram is expressed in terms of the letters of the diagram and contains the following stages:

1. **Given.** (2) **To Prove.** (3) **Construction.** (4) **Proof.**

   It is necessary to distinguish carefully between what is *given in the figure* and what is *added to the figure* for purposes of proof. Thus in Theorem 8, the fact that \( BC \) is produced to \( D \) is part of what is given and must be included in the particular enunciation because it is needed for stating what has to be proved; it is not part of the "construction," i.e., it is not an addition to the figure which is made to help in the proof.

   If the particular enunciation is given, the letters used in the figure must be those in the given enunciation, but it is then unnecessary to repeat over again the statements of what is given and what has to be proved, because this is merely equivalent to copying the printed question.

**Examination Requirements.** In Theorem 8, the enunciation contains two separate statements, and the proof of the second depends on that of the first. If in an examination a candidate is asked to prove only the second statement, full marks are usually not given unless the proof of the first statement is also included. Throughout this book, the theorems are so arranged that this remark applies to all enunciations which contain more than one part.
THEOREM 8

(1) If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the interior opposite angles.

(2) The sum of the angles of a triangle is equal to two right angles.

Given a triangle ABC with BC produced to D.

To prove that

(i) \( \angle ACD = \angle A + \angle B \);

(ii) \( \angle A + \angle B + \angle ACB = 2 \) right angles.

Construction. Through C draw CE \parallel BA.

Proof. (i) With the notation in the figure,

\[ p_3 = p_1 \text{ alt. } \angle s, \text{ } CE \parallel BA, \]

\[ q_3 = q_1 \text{ corr. } \angle s, \text{ } CE \parallel BA, \]

\[ .\text{ } by \text{ addition, } p_3 + q_3 = p_1 + q_1, \]

\[ \therefore \angle ACD = \angle A + \angle B. \]

(ii) Add \( \angle ACB \) to each side,

\[ \therefore \angle ACD + \angle ACB = \angle A + \angle B + \angle ACB. \]

But \( \angle ACD + \angle ACB = 2 \) rt. \( \angle s \).

adj. \( \angle s \) on \( \angle s \) line,

\[ \therefore \angle A + \angle B + \angle ACB = 2 \) rt. \( \angle s. \]

Corollary 1. If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.

ANGLES OF A TRIANGLE

Corollary 2. In a right-angled triangle, (i) the right angle is the greatest angle, (ii) the sum of the two remaining angles is \( 1 \) right angle, and each of these angles is acute.

Corollary 3. At least two of the angles in every triangle are acute.

Corollary 4. From a given point outside a given line, not more than one straight line can be drawn perpendicular to the given line.

Abbreviations for reference: For Theorem 8 (1), ext. \( \angle \) of \( \triangle \).

For Theorem 8 (2), \( \angle \) sum of \( \triangle \).

Note: The corollaries form suitable examples for oral discussion. Exercise 29, Nos. 4-8 illustrate the usefulness of Corollary 1.

Corollary 4 is used on p. 171.

Examples for Oral Discussion

1. ABC is a triangle in which \( \angle A + \angle B = \angle C \), prove that \( \angle C \) is a right angle.

   First Method. Produce BC to D.

   Explain why \( \angle ACD = \angle ACB \), and complete the proof.

   Second Method. What do you know about \( \angle A + \angle B + \angle C \) ?

2. Prove that from a given point P outside a given straight line BC, not more than one straight line can be drawn perpendicular to BC.

   If possible, suppose that two different lines PM, PN can be drawn both perpendicular to BC.

   Explain why PMN and PMN cannot both be right angles.

N.B. The tendency to avoid using the exterior angle property of a triangle and to rely only on the angle-sum property must be checked from the start. It is often advisable to give extra credit to those pupils who use the exterior angle property when it is shorter to do so.
NUMERICAL EXAMPLES

EXERCISE 28

[Use the property of the exterior angle instead of the property of the angle-sum of a triangle whenever it is shorter to do so.]

Find the value of \( z \) and the angle \( a \) in figs. 250–252:

1. \[ \text{Fig. 250} \]

2. \[ \text{Fig. 251} \]

3. \[ \text{Fig. 252} \]

4. In a right-angled triangle, one angle is 43°; find the other acute angle.

Nos. 5–12 refer to the angles \( A, B, C \) of a triangle \( ABC: \)

5. Find \( C \) if \( A = 40^\circ \), \( B = 105^\circ \); also if \( A = 90^\circ \), \( B = x^\circ \).

6. Find \( B \) if \( A = 15^\circ \), \( C = 18^\circ \).

7. Find \( A \) if \( B = C = 52^\circ \).

8. Find \( A \) if \( A = B \) and \( C = 48^\circ \).

9. Find \( C \) if \( A + B = 3C \).

*10. Find \( A \) if \( 2A = 15^\circ \) and \( B + C = 30^\circ \).

*11. Express \( \frac{1}{2}B + \frac{1}{2}B \) in terms of \( A \).

*12. Simplify \( \frac{1}{2}(B + C - A) \).

13. Can a triangle be drawn having its angles equal to (i) 45°, 60°, 80°; (ii) 30°, 60°, 75°; (iii) 100°, 110°, \( x^\circ \)?

14. The angles of a triangle are \( 2x^\circ \), \( 3x^\circ \), \( 4x^\circ \). Find \( x \).

15. The angles of a triangle are \( y^\circ \), \( 2y^\circ - 20^\circ \), \( 3y^\circ - 40^\circ \). Find \( y \).

16. Two exterior angles of a triangle are 120°, 130°. Find the third unequal exterior angle.

Find the unknown marked angles in figs. 253–257. Arrows indicate that lines are given parallel.

17. \[ \text{Fig. 253} \]

18. \[ \text{Fig. 254} \]

19. \[ \text{Fig. 255} \]

20. \[ \text{Fig. 256} \]

21. \[ \text{Fig. 257} \]

22. In \( \triangle ABC \), \( AX \) bisects \( \angle BAC \) and \( AD \) is perpendicular to \( BC \). If \( \angle A = 60^\circ \) and \( \angle B = 70^\circ \), find \( \angle DAX \).

23. In \( \triangle ABC \), \( \angle B = 100^\circ \), \( \angle C = 50^\circ \); \( AD \) is the perpendicular from \( A \) to \( BC \) produced. Prove that \( \angle DAC = 2\angle DAB \).

24. In \( \triangle ABC \), \( \angle B = 90^\circ \), \( \angle A = 2\angle C \), and the bisector of \( \angle A \) cuts \( BC \) at \( K \). Prove that \( \angle AKC = 4\angle C \).

Nos. 25–29 refer to fig. 238, in which \( BH, CK \) are the bisectors of \( \angle ABC, \angle ACB \).

25. If \( \angle A = 80^\circ \), \( \angle ABC = 30^\circ \), find \( \angle BIC \).

26. If \( \angle ABC = 44^\circ \), \( \angle BIC = 125^\circ \), find \( \angle BIC \).

*27. If \( \angle A = 74^\circ \), \( \angle ACB = 66^\circ \), and if the perpendicular \( AD \) from \( A \) to \( BC \) cuts \( BH, CK \) at \( P, Q \), find the angles of \( \triangle IPQ \).

*28. If \( \angle BKC = 95^\circ \), \( \angle BHC = 82^\circ \), find \( \angle A \).

*29. If \( \angle BIC = 132^\circ \), find \( \angle A \).
1. If, in fig. 259, \( p = 2c \), prove that \( b = c \).

2. If, in fig. 259, \( b = 2c \), prove that \( c = \frac{1}{2} p \).

3. In fig. 260, BA is parallel to CD, prove that \( r = d + e \).

4. ABCD is a quadrilateral such that the diagonal AC bisects \( \angle DAB \) and bisects \( \angle DCB \). Prove that \( \angle ABC = \angle ADC \).

5. In fig. 261, BE and CF are perpendicular to AC and AB. Prove that \( \angle BEF = \angle FCE \).

6. In fig. 261, BE and CF are perpendicular to AC and AB. Prove that (i) \( \angle FHB = \angle BAC \); (ii) \( \angle BHC \), \( \angle BAC \) are supplementary.

7. In fig. 262, if \( b = r \), prove that \( c = a \).

8. In fig. 262, find \( r \) in terms of \( b, c, a \).

9. In fig. 263, if \( n_1 = n_2 \), prove that \( q = r \).

10. In fig. 263, if \( n_1 = n_2 \), prove that \( \angle ACD = \angle BAD \). [Put in a small letter for \( \angle BAC \).]

11. In fig. 264, if \( a = b \), prove that \( p = q \).

12. In fig. 264, find \( q \) in terms of \( a, b, p \).

13. In fig. 265, find \( t \) in terms of \( a, b, c \).

14. In fig. 266, find \( p \) in terms of \( d, e, f \).

15. Draw a triangle ABC right-angled at A and draw the perpendicular AD from A to BC. What angle in your figure is equal to \( \angle DAC \)? Give reasons.

16. In fig. 267, prove that \( p + q + r = 4 \) right angles.

17. In fig. 267, \( p + q = 3r \), prove that the triangle is right-angled.

18. ABCD is a parallelogram; the bisectors of \( \angle BAD \), \( \angle ABC \) meet at P; prove that \( \angle APB \) is a right angle.

19. In the triangle ABC, the bisectors of the angles \( \angle ABC \), \( \angle ACB \) meet at I, prove that \( \angle BIC = 90^\circ + \frac{1}{2} \angle BAC \).

20. The side BC of \( \triangle ABC \) is produced to D; the bisector of \( \angle BAC \) cuts BC at K; prove that \( \angle ABK = \frac{1}{2} \angle AKD \).

21. The side BC of \( \triangle ABC \) is produced to D; the bisectors of the angles \( \angle ABC \), \( \angle ACB \) meet at Q; prove that \( \angle BQC = \frac{1}{2} \angle BAC \).

22. ABCD is a tetrahedron (i.e., a pyramid on a triangular base). If the plane angles at each of the corners A, B, C add up to two right angles, prove that the plane angles at the corner D add up to two right angles. Sketch the net of ABCD.
NEW GEOMETRY

Sum of the Angles of a Polygon

If we wish to find the sum of the angles of a quadrilateral $ABCD$, the simplest method is to draw a diagonal $AC$, thus forming the two triangles $ABC$, $ACD$.

With the notation in fig. 268, from $\Delta ABC$, $m + n + p = 2$ rt. $\angle s$, $\angle sum of \Delta$,
from $\Delta ACD$, $r + s + t = 2$ rt. $\angle s$, $\angle sum of \Delta$;
but the angles of $ABCD$ are $r + m, n, p + s, t$.

$\therefore$ by addition, the sum of the angles of the quadrilateral $ABCD$ is $4$ right angles.

Similarly, if we wish to find the sum of the angles of a pentagon $ABCDE$, the simplest method is to draw the two diagonals $AC, AD$, thus forming the three triangles $ABC, ACD, ADE$.

Hence it follows that the sum of the angles of a pentagon is $5$ right angles.

If, however, the polygon has a large number of sides it is better to use a different method.

Examples for Oral Discussion

1. Find the sum of the interior angles of the hexagon $ABCD$.

   Take any point $O$ inside the hexagon; join it to each vertex.

   Copy and complete the following:

   The total sum of all the angles of the 6 triangles $OAB, OBC, OCD, ODE, OEF, OFA$, is . . . ;
   but the sum of the 6 angles at $O$ is . . . ;

   $\therefore$ the sum of the interior angles of $ABCD$ is . . .

2. Repeat No. 1 for a 7-sided polygon.

3. Work out the sum of the interior angles of (a) a 10-sided polygon, (b) a 100-sided polygon, (c) an $n$-sided polygon.

4. If the sides of a convex pentagon $ABCDE$ are produced in order, i.e. as shown in fig. 271, find the sum of the exterior angles so formed.

   Copy and complete the following:

   The interior $\angle +$ the exterior $\angle$ at each vertex $= 2$ rt. $\angle s$;
   $\therefore$ the sum of the 5 int. $\angle s +$ the sum of the 5 ext. $\angle s = . . .$;

   but the sum of the 5 int. $\angle s = . . .$;
   $\therefore$ the sum of the 5 ext. $\angle s = . . .$.

5. Repeat No. 4 for a convex 7-sided polygon.

6. Work out the sum of the 100 exterior angles of a convex 100-sided polygon. Repeat for an $n$-sided polygon.

7. Draw a large convex pentagon $ABCDE$ on the ground.

   Start at any point $K$ on $AB$ and walk along $KB$ to $B$, then turn and walk along $BC$ to $C$, and so on until you have arrived back at $K$. What are the separate angles through which you have turned? What is the total angle?

8. Apply the argument of No. 7 to a pentagon $ABCDE$ which is not convex, see fig. 272.

   Copy and complete the following:

   The angles turned through are $p, q, r, t$ counter-clockwise, and $s$ clockwise.

   But the total angle turned through is . . .

   $\therefore p + q + . . . = . . .$

   These examples illustrate the fact that the sum of the exterior angles of a polygon formed by producing the sides in order is $4$ right angles if the polygon is convex, but not otherwise.
NUMERICAL EXAMPLES

EXERCISE 30

1. Three of the angles of a quadrilateral are 80°, 100°, 110°; find the other angle.

2. Three of the angles of a quadrilateral are 112°, 75°, 51°; find the other angle.

3. ABCD is a quadrilateral in which \( \angle A = \angle B = \angle C \), and \( \angle D = 120° \); find \( \angle A \).

4. The angles of a quadrilateral taken in order are \( 2x°, 3x°, 7x°, 8x° \). Find the value of \( x \) and prove that two of the opposite sides are parallel.

5. The angles of a pentagon are \( x°, 2x°, 2x°, 2x°, 3x° \); find the value of \( x \).

6. The angles of a pentagon are \( x°, 2x°, 2x°, 2x°, 3x° \); find the value of \( x \).

7. ABCDE is a regular pentagon; AB, DC are produced to meet at P; find \( \angle BPC \).

8. Find the sum of the interior angles of a polygon which has (i) 30 sides, (ii) 40 sides.

9. Prove that the sum of the interior angles of an octagon (8 sides) is twice the sum of the interior angles of a pentagon.

10. Find the size of an exterior angle of (i) a regular octagon (8 sides), (ii) a regular decagon (10 sides).

11. Find the size of an exterior angle of a regular polygon having (i) 15 sides, (ii) \( n \) sides.

12. Find the number of sides of a polygon (i) if each exterior angle is 40°, (ii) if each interior angle is 144°.

13. Find the number of sides of a polygon (i) if each exterior angle is 15°, (ii) if each interior angle is 160°.

14. If the sum of the interior angles of a polygon is 30 right angles, find the number of its sides.

15. Five of the angles of a hexagon are equal to each other and each is greater than the remaining angle by 72°. Find the size of the smallest angle.
Theorem 9

1. The sum of the interior angles of a convex polygon of \(n\) sides is \((2n - 4)\) right angles.

2. If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is four right angles.

![Fig. 275](image)

Given a convex polygon \(A_1B_1C_1D_1E_1\ldots\) of \(n\) sides \(A_1B_1, B_1C_1, C_1D_1, D_1E_1, \ldots\) produced to \(L, M, N, P, \ldots\)

To prove that:

1. The sum of the interior angles \(A_1B_1C_1, B_1C_1D_1, \ldots\) is \((2n - 4)\) right angles.
2. The sum of the exterior angles \(L_1A_1, M_1B_1, N_1C_1, \ldots\) is four right angles.

Construction. Take any point \(O\) inside the polygon and join it to each vertex.

Proof. (1) The polygon has \(n\) sides and therefore has been divided into \(n\) triangles by the construction.

The sum of the angles of each triangle is 2 rt. \(\angle\)s.

\[\therefore\text{the sum of the angles of the } n \text{ triangles is } 2n \text{ rt. } \angle\text{s,}\]

\[\therefore\text{sum of } \angle\text{s of polygon } + \text{ sum of } \angle\text{s at } O = 2n \text{ rt. } \angle\text{s;}
\]

\[\text{but sum of } \angle\text{s at } O = 4 \text{ rt. } \angle\text{s,}\]

\[\therefore\text{sum of } \angle\text{s of polygon } + 4 \text{ rt. } \angle\text{s } = 2n \text{ rt. } \angle\text{s.}\]

\[\therefore\text{sum of } \angle\text{s of polygon } = (2n - 4) \text{ rt. } \angle\text{s.}\]
EXERCISE 31

1. **ABCD** is a quadrilateral in which \( \angle A = \angle C \) and \( \angle B = \angle D \). Prove that (i) \( \angle A + \angle B = 2 \) rt. \( \angle s \); (ii) \( ABCD \) is a parallelogram.

2. **ABCD** is a quadrilateral in which \( \angle A \) + \( \angle C \) = 2 rt. \( \angle s \). If \( AB \) is produced to \( E \), prove that \( \angle CBE = \angle D \).

3. In fig. 277, \( NA, NB, NC, ND \) are the bisectors of the angles of the quadrilateral \( ABCD \). Prove \( \angle ANB + \angle CND = 2 \) rt. \( \angle s \).

4. If in fig. 277, \( NA \) and \( NB \) are the bisectors of \( \angle DAB, \angle CBA \), prove that \( \angle ADC + \angle BCD = 2 \angle ANB \).

5. **ABCD** is a quadrilateral in which \( \angle A \) and \( \angle C \) are each right angles. Prove that the bisectors of \( \angle B \) and \( \angle D \) are parallel.

6. In fig. 278, prove that \( m = a + b + d \).

7. If, in fig. 278, \( m = 2a \) and \( a = \frac{b}{2} \), prove that \( b = d \).

8. If, in fig. 278, the bisectors of \( \angle ABC, \angle ADC \) meet at \( K \), prove that \( a + m = 2 \angle BKD \).

9. Find the sum of the marked angles in fig. 279.

*10. If, in fig. 280, \( p \) and \( q \) are supplementary angles, prove that \( p = 90^{\circ} + \frac{1}{2}(x + z) \).

*11. The sides of any convex hexagon are produced and the bisectors of the exterior angles so formed are drawn and produced to form another convex hexagon \( LMPQR \). Prove that \( \angle L + \angle M + \angle N + \angle Q = 4 \) rt. \( \angle s \).

CONGRUENCE

Tests for Congruence

Three general tests and one special test for the congruence of two triangles have been discussed in Stage A. The first of these, repeated here, is numbered Theorem 4, although printed after Theorem 9, because if at a later stage it is desired to reduce the number of assumptions made, this test for congruence is used to establish the test for parallel lines given in Theorem 5, p. 95. The method is given in the Appendix.

THEOREM 4

If two triangles have TWO SIDES of the one equal to TWO SIDES of the other, each to each, and also the angles INCLUDED by those sides equal, the triangles are congruent.

In \( \triangle ABC, PQR \),

if \( AB = PQ, AC = PR \) and \( \angle A = \angle P \),

then \( \triangle ABC \equiv \triangle PQR \), are congruent.

Abbreviation for reference: SAS or 2 sides, inc. \( \angle \).

A formal proof of Theorem 4 is in the Appendix, p. 542.
THEOREM 10

If two triangles have TWO ANGLES of the one equal to TWO ANGLES of the other, each to each, and also A SIDE of one equal to the CORRESPONDING SIDE of the other, the triangles are congruent.

In $\triangle ABC$, $PQR$,

if $\angle B = \angle Q$ and $\angle C = \angle R$ and if either (i) $BC = QR$ or (ii) $AB = PQ$ or (iii) $AC = PR$,

then $\triangle ABC$ are congruent.

**Abbreviation for reference**: ASA, AAS or $2\angle s$, corr. side. A formal proof of Theorem 10 is given in the Appendix, p. 546.

**Note**: Since the sum of the angles of a triangle is two right angles, if two angles of one triangle are equal to two angles of another triangle, then the third angles are also equal.

The two sides which are given equal must be corresponding sides, i.e., they must be opposite to angles which are known to be equal.

**EXERCISE 32**

Nos. 1–8 are suitable for oral discussion.

The data in Nos. 1–7 refer to two triangles $ABC$, $DEF$. In each case show the data on your own figure by suitable marking. Do the data show that the triangles must be congruent? If so, state the test used and express the fact in the proper way.

1. $AB = EF$, $AC = DF$, $\angle A = \angle F$.
2. $CA = FD$, $CB = FE$, $\angle B = \angle E$.
3. $AC = EF$, $\angle A = \angle E$, $\angle C = \angle F$.
4. $AC = DF$, $\angle B = \angle D$, $\angle C = \angle F$.
5. $BC = DF$, $\angle B = \angle D$, $\angle C = \angle E$.
6. $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.
7. $AB = AC$, $DE = DF$, $\angle A = \angle D$.

8. If $\triangle ABC$ and $\triangle XYZ$ are congruent, which angle is equal to $\angle C$?

Which side is equal to $XY$?

9. If, in fig. 283, $PR$ bisects $\angle QPS$ and $PS = PQ$, prove that $RS = RQ$.

10. If, in fig. 284, $AB$ and $CD$ bisect each other, prove that

(i) $AD = BC$, (ii) $AD$ is parallel to $CB$.

11. If, in fig. 284, $CB$ is parallel to $AD$ and if $N$ is the mid-point of $AB$, prove that $CN = ND$.

12. If, in fig. 284, $AD$ is equal and parallel to $CB$, prove that

(i) $N$ is the mid-point of $AB$, (ii) $AC = DB$, (iii) $AC$ is parallel to $DB$.

13. $ABCD$ is a quadrilateral in which $AB = CD$ and $\angle B = \angle C$. Prove that $AC = BD$.

14. A line $AP$ is drawn bisecting the angle $BAC$; $PX$, $PY$ are the perpendiculars from $P$ to $AB$, $AC$. Prove that $PX = PY$.

15. If the straight line $XOY$ bisects at right angles the straight line $AOB$, prove that $XA = XB$.

16. $ABCD$ is a quadrilateral in which $AB$ is parallel to $DC$, and $AD$ is parallel to $BC$. Prove that $AB = DC$ and that $AD = BC$. [Join $AC$.]

17. $O$ is the centre of each of the circles in fig. 285; $AOB$, $POQ$ are straight lines. Prove that $A Q = PB$.

18. In fig. 286, $AB = AC$, $AP = AQ$, and $\angle BAC = \angle PAQ$ Which two triangles in the figure are congruent? Give reasons.

19. $D$ is the mid-point of the base $BC$ of $\triangle ABC$; $BX$, $CY$ are perpendiculars from $B$, $C$ to $AD$, produced if necessary. Prove that $BX = CY$. 

**Fig. 283**

**Fig. 284**

**Fig. 285**

**Fig. 286**
20. X is the centre of each semicircle AQ, BCP in fig. 287; XPQ, AXCDB are straight lines.

(i) Which triangle in the figure is congruent to ΔAXP? Give reasons.

(ii) Name other two congruent triangles. Which angle is equal to ∠XPB? Give reasons.

(iii) Prove that

∠APB + ∠QCD = 2 rt. ∠s.

21. In fig. 288, AB = AC and APQR is a straight line. If ∠BAC = a, prove that AP = CQ.

*22. The perpendicular bisectors of the sides AB, AC of ΔABC meet at O. Prove that OB = OC. [Join OA and prove that OB = OA.]

*23. The bisectors of ∠B, ∠C of ΔABC meet at I; IQ, IR are the perpendiculars from I to AB, AC. Prove that IQ = IR. [Draw the perpendicular IP from I to BC.]

*24. AOB is a diameter of a circle, centre O; OP, OQ are two perpendicular radii; PH, QK are the perpendiculars from P, Q to AB. Prove that PH = OK.

Draw two figures, one in which P and Q are on the same side of AB and the other in which P and Q are on opposite sides of AB. Prove the fact in each case.

Isosceles Triangles

If two sides of a triangle are equal, the triangle is called isosceles; the point of intersection of the equal sides is called the vertex, the angle included by the equal sides is called the vertical angle and the third side is called the base of the isosceles triangle.

If the three sides of a triangle are equal, the triangle is called equilateral.

If a triangle has three unequal sides, it is called scalene.

**THEOREM 11**

If two sides of a triangle are equal, then the angles opposite to those sides are equal.

Given a triangle ABC in which AB = AC.
To prove that ∠B = ∠C.

**Construction.** Let AD be the bisector of ∠BAC and let it meet BC at D.

**Proof.** In the Δs ABD, ACD,

\[ \angle B = \angle C. \]

\[ \triangle ABD \text{ and } \triangle ACD \text{ are congruent, } S.A.S. \]

\[ \therefore \angle B = \angle C. \]

**Abbreviation for reference:** base ∠s, isos. ∆.

Theorem 11 is often stated in the form:

The angles at the base of an isosceles triangle are equal.
THEOREM 12

If two angles of a triangle are equal, then the sides opposite to those angles are equal.

Given a triangle ABC in which $\angle B = \angle C$.

To prove that $AB = AC$.

Construction. Let AD be the bisector of $\angle BAC$ and let it meet BC at D.

Proof. In the $\triangle$s ABD, ACD,

\[
\angle B = \angle C \quad \text{given},
\]
\[
a_1 = a_2 \quad \text{constr.},
\]
\[
AD = AD.
\]

$\therefore \triangle$s $\triangle ABD, ACD$ are congruent AAS.

$\therefore \triangle ABD = \triangle ACD$.

Abbreviation for reference: sides opp. equal $\angle s$.

Important Note. Theorem 11 is used to prove the SSS test for congruence (Theorem 13, p. 129) and the RHS test for congruence (Theorem 14, p. 130), and therefore neither of these tests can be used in proving Theorem 11. The construction used for the proof of Theorem 11 must therefore be noted carefully, because any different way of drawing AD would make the proof worthless. To avoid confusion, it is best to use the same construction for Theorem 12 as for Theorem 11.

The following deductions from Theorems 11, 12 are suggested as examples for oral discussion because they are often of use in rider-work.
NEW GEOMETRY

The fact that a theorem is true does not imply that the converse theorem is also true, although it often may be so. For example, the statement,

"If two angles are right angles, then they must be equal,"

is true.

But the converse statement,

"If two angles are equal, then they must be right angles."

is not true.

NUMERICAL EXAMPLES

EXERCISE 33

[Arrows indicate that lines are given parallel.]

1. The vertical angle of an isosceles triangle is 110°; find the base angles.

2. One base angle of an isosceles triangle is 62°; find the vertical angle.

3. One of the angles of an isosceles triangle is $x$°; find the other angles in terms of $x$. [Two sets of answers.]

4. Find the angles of an isosceles triangle if a base angle is double the vertical angle.

5. Find the angles of an isosceles triangle if the vertical angle is three times a base angle.

6. In fig. 292, $AB = AC$ and $CX = CB$, find $\angle ACX$.

7. In fig. 293, $OA = OB = OC$, find $\angle OCB$.

8. In fig. 294, $O$ is the centre of the circular arc PQR. Find the angles of $\triangle PQR$.

9. In fig. 295, $AN = AC$, find $\angle NAB$.

10. In fig. 296, $AB = AC$ and $BPC$ is a straight line; prove that $BP \parallel OA$ and $BQ \parallel PA$.

11. In $\triangle ABC$, the bisector of $\angle A$ cuts $BC$ at $D$. If $AD = DB$ and if $\angle C = 66°$, find $\angle B$.

12. In fig. 297, $ABCD$ is a straight line and $BE = BC$, prove that $CE = CD$.

13. In fig. 298, prove that $DA = DC$.

14. The side $BC$ of $\triangle ABC$ is produced to $D$; $\angle BAC = 40°$, $\angle ACD = 75°$. If $P$ is a point on $AB$ such that $AP = AC$, prove $PB = PC$.

15. In $\triangle ABC$, $AB = AC$; $D$ is a point on $AC$ produced such that $BD = BA$. If $\angle CBD = 36°$, prove that $BC = CD$.

16. In Nos. 16–20, work throughout in terms of the small letter which is to appear in the answer.

17. In fig. 299, if $AB = AC$, find $x$ in terms of $y$.

18. In fig. 300, $AB = AC$ and $BK$ bisects $\angle ABC$. Find $\angle AKB$ in terms of $x$. 
18. In fig. 301, \( AB = AC = BD \). Find \( y \) in terms of \( z \).

19. In \( \triangle ABC \), \( AB = AC \); \( P, Q \) are points on \( AB, AC \) respectively such that \( AP = PQ \). If \( \angle ABC = z \), find \( \angle BPQ \) in terms of \( x \).

20. \( ABCDE \) is a regular pentagon. Prove that the bisector of \( \angle BAC \) is perpendicular to \( AE \).

21. In \( \triangle ABC \), \( \angle B = 40^\circ \), \( \angle C = 120^\circ \). The bisector of \( \angle B \) cuts \( AC \) at \( X \) and cuts the perpendicular from \( C \) to \( AB \) at \( N \). Prove that \( XN = XC \).

22. In fig. 302, the sides of \( \triangle ABC \) are produced as shown. If \( AB = AC, BG = BH, \) and \( AK = KG \), find \( \angle BAC \) and prove that \( HC = HK \).

**Exercise 34**

[Arrows indicate that lines are given parallel.]

1. In fig. 303, \( AP \) bisects \( \angle BAC \). Prove that \( AQ = QP \).

2. In fig. 304, \( O \) is the centre of the circumcircle of \( \triangle ABC \). Prove that \( \angle OAB + \angle OCB = \angle ABC \). [Join \( OB \).]

3. The side \( BC \) of \( \triangle ABC \) is produced to \( D \). If \( AB = AC \), prove that \( \angle ABD + \angle ACD = 2\angle BAC \).

4. The side \( BC \) of \( \triangle ABC \) is produced to \( D \). If \( \angle ACD = 2\angle BAC \), prove that \( CA = CB \).

5. In \( \triangle ABC \), \( AB = AC \). If the bisector of \( \angle B \) meets \( AC \) at \( P \), prove that \( \angle APB = 3\angle PBC \).

6. In fig. 305, \( IB, IC \) are the bisectors of \( \angle ABC, \angle ACB \). If \( AB = AC \), prove that \( IB = IC \).

7. \( ABCD \) is a quadrilateral in which \( AB = AD \) and \( \angle ABC = \angle ADC \). Prove that \( CB = CD \). [Join \( BD \).]

8. In \( \triangle ABC \), \( AB = AC \). If a line parallel to \( BC \) cuts \( AB, AC \) at \( H, K \) respectively, prove that \( BH = CK \).

9. In fig. 306, \( AB = AC, BP = QC \). \( BPQC \) is a straight line. Prove that \( \angle APQ = \angle AQP \). [Prove two triangles are congruent.]

10. \( ABCD \) is a quadrilateral in which \( \angle B \) and \( \angle C \) are equal acute angles. If \( AB = CD \), prove that \( \angle A = \angle D \). [Produce \( BA \) and \( CD \) to meet at \( K \).

11. In \( \triangle ABC \), \( AB = AC \). If \( AC \) is produced to any point \( X \) and if \( BX \) is joined, prove that \( \angle ABX + \angle AXB = 2\angle ACB \).

12. \( IB, IC \) are the bisectors of \( \angle ABC, \angle ACB \) of \( \triangle ABC \), see fig. 305. If \( AB = AC \) and if \( BC \) is produced to \( D \), prove that \( \angle ACD = \angle BIC \).

13. \( D \) is the mid-point of the side \( BC \) of \( \triangle ABC \). If \( AD = BD \), prove that \( \angle BAC \) is a right angle.

14. In \( \triangle ABC \), \( AB = AC \). \( BA \) is produced to \( E \) and \( AX \) is drawn bisecting \( \angle CAE \). Prove that \( \angle AX \) is parallel to \( BC \).

15. In \( \triangle ABC \), \( AB = AC \). If \( CN \) is the perpendicular from \( C \) to \( AB \), prove that \( \angle NCB = \frac{1}{2} \angle A \).

16. In fig. 307, \( P, Q, R \) are points on the sides of \( \triangle ABC \) such that \( BP = PC \) and \( QC = PB \). If \( AB = AC \), prove that \( i) \ PQ = PR \), \( ii) \angle RPQ = \angle B \).

17. In fig. 308, \( AP = AB, CB = CQ \), and \( ABC \) is a straight line. Prove that \( \angle PBQ \) is a right angle.
18. In fig. 309, \( \angle ABK = \angle C \) and BN bisects \( \angle KBC \). Prove that \( AN = AB \).

![Fig. 309]

19. In fig. 310, \( \angle XAC = \angle B \) and \( \angle YAB = \angle C \). Prove that \( AX = AY \).

![Fig. 310]

20. In \( \triangle ABC \), \( AB = AC \); \( BC \) is produced to any point \( K \), and \( AK \) is joined. If a triangle is drawn in which two of the angles are equal to \( \angle CAK \), \( \angle BAK \), respectively, prove that the third angle is equal to \( 2 \angle ACK \).

21. In fig. 311, not drawn accurately, \( AX = AY \) and the straight line \( CYX \) bisects \( \angle ABC \). Find an angle in the figure equal to \( \angle ABC \).

![Fig. 311]

22. In fig. 312, \( O \) is the centre of the circle; \( PBCA \) and \( PQR \) are straight lines. If \( PQ \) is equal to the radius of the circle, prove that \( \angle AOR = 3 \angle BOQ \).

![Fig. 312]

23. \( ABC \) is an acute-angled triangle; \( ABP \), \( ACQ \) are equilateral triangles outside \( \triangle ABC \). Prove that \( \triangle PAC = \triangle BAQ \).

If \( CP \) cuts \( BQ \) at \( R \), prove that \( \angle BRC = 120^\circ \).

24. \( Y \) is any point on the side \( BC \) of the equilateral triangle \( ABC \); \( BYK \) is an equilateral triangle outside \( \triangle ABC \). Prove that (i) \( AY = KC \); (ii) \( \angle BAY = \angle KCY \); (iii) \( \angle YAC = \angle YKC \).

25. In fig. 313, \( AB = AC \) and \( PQ = PC \). Prove that \( \angle BPC \) is a right angle.

![Fig. 313]

26. The side \( BA \) of \( \triangle ABC \) is produced to \( E \) and the bisectors \( AP \), \( AQ \) of \( \angle CAB \), \( \angle CAE \) cut \( BC \), \( BC \) produced, at \( P \), \( Q \), see fig. 314. If \( AP = AQ \), prove that (i) \( \angle APQ = 45^\circ \), (ii) \( \angle ACB = \angle ABC = 90^\circ \).

27. In \( \triangle ABC \), \( \angle A \) is a right angle; \( AD \) is the perpendicular from \( A \) to \( BC \). If \( P \) is a point on \( BC \) such that \( CP = CA \), prove that \( AP \) bisects \( \angle BAC \).

28. In fig. 315, \( PBC \), \( PHK \) are straight lines. If \( AH = AK \), prove that \( p = 4(0 - c) \).

29. In \( \triangle ABC \), \( \angle B = \angle C = 40^\circ \); the bisector of \( \angle ABC \) meets \( AC \) at \( D \). If \( Q \) is a point on \( BC \) such that \( BQ = BD \). Prove that \( CQ = AD \).

[Take \( R \) on \( BC \) so that \( BR = BA \); join \( DR \).]

30. The side \( BC \) of \( \triangle ABC \) is produced to \( D \). The bisectors of \( \angle BAC \), \( \angle ACD \) meet at \( P \). If the line through \( P \) parallel to \( BC \) cuts \( BA \), \( CA \), produced if necessary, at \( Q \), \( R \) respectively, prove that \( QR \) is equal to the difference of \( BQ \) and \( CR \).

[Prove \( PQ = QB \); what line equals \( QR \).]

31. In fig. 316, \( AB = BC \), \( CE = ED \), and \( ACD \), \( ABP \) are straight lines. Prove that (i) \( \triangle DPE = \triangle EBC \); (ii) \( \angle BPE = PE \).

32. A line \( AD \) meets \( BC \) at \( D \) and divides \( \triangle ABC \) into two isosceles triangles. Prove that in \( \angle ABC \) either one angle is a right angle or one angle is twice another angle or one angle is three times another angle.

33. If, in fig. 312, \( K \) is a point on the circle such that the centre \( O \) lies inside \( \angle KQR \). Prove that (i) \( \angle QKR = 2 \angle QKR \), (ii) \( \angle QOR = 2 \angle QOR \). [Join \( KO \) and produce \( K \).]
†Theorem 13

If two triangles have the THREE SIDES of one equal to the THREE SIDES of the other, each to each, the triangles are congruent.

Given two $\triangle ABC$, $XYZ$, in which $AB = XY$, $BC = YZ$, $CA = ZX$.

To prove that $\triangle ABC$, $XYZ$ are congruent.

Construction. On the opposite side of $YZ$ from $X$, draw $YP$ so that $\angle PYZ = \angle B$ and $YP = BA$. Join $PZ$, $PX$.

Proof. In $\triangle PYZ$, $ABC$,

$PY = AB$ given,
$YZ = BC$ given,
$\angle PYZ = \angle B$ constr.,

$\therefore \triangle PYZ$ are congruent $\triangle ABC$ SAS.

$\therefore ZP = CA$ and $\angle YPZ = \angle A$.

But $CA = ZX$ given,
$\therefore ZP = ZX$.

Also $YP = BA$ constr., and $BA = YX$ given,
$\therefore YP = YX$.

$\therefore YP$ and $ZP$ are isosceles triangles,

$\therefore \angle YPX = \angle YXP$ and $\angle ZPX = \angle XZP$ base $\angle s$,

$\therefore$ adding in fig. (i) and subtracting in fig. (ii),
$\angle YPZ = \angle YXZ$.

This is also true in fig. (iii) where $XZP$ is a straight line.

But $\angle YPZ = \angle A$ proved,
$\therefore \angle XYZ = \angle A$.

$\therefore$ in $\triangle ABC$, $XYZ$,
$AB = XY$ given,
$AC = XZ$ given,
$\angle A = \angle XYZ$ proved,

$\therefore \triangle ABC$ are congruent $\triangle XYZ$ SAS.

Abbreviation for reference: SSS or 3 sides.

Note. It may appear at first sight that cases (ii), (iii) can be avoided by choosing $YZ$ and $BC$ as two equal sides which are not shorter than the other pairs of equal sides. But actually the proof that in fact this leads only to case (i) is not easy and is certainly much longer than the consideration of the three cases.

†The proofs of Theorems 13, 14 are included in the text because it may be considered useful to take them as examples for oral discussion. Pupils should not be required to reproduce these proofs at a first reading.
THEOREM 14

If two triangles have TWO SIDES of the one equal to TWO SIDES of the other, each to each, and the angles opposite to one pair of equal sides are RIGHT ANGLES, the triangles are congruent.

Given two \( \triangle ABC, XYZ \), in which
\[ AC = XZ, \quad AB = XY, \quad \angle B = \angle Y = 1 \text{ rt. } \angle, \]

To prove that \( \triangle ABC, XYZ \) are congruent.

Construction. Produce \( ZY \) to \( Q \), making \( YQ = BC \). Join \( XQ \).

Proof. Since \( \angle XYZ = 1 \text{ rt. } \angle \) and \( ZYQ \) is a straight line, \( \angle XYQ = 1 \text{ rt. } \angle \).

\[ \because \quad \triangle ABC, XYZ, \text{ are congruent } \triangle XYZ, \]
\[ \because \quad \angle B = \angle Y = 1 \text{ rt. } \angle, \]
\[ \therefore \quad \triangle ABC, XYZ, \text{ are congruent } \triangle XYZ, \]
\[ \because \quad \angle C = \angle Q \text{ and } AC = XQ; \]
\[ \because \quad \angle Z = \angle Q \quad \text{base } \angle s, \text{ isos. } \triangle; \]
\[ \therefore \quad \angle C = \angle Q \quad \text{proved}, \quad \angle C = \angle Z. \]
EXERCISE 35

1. AB and CD are two equal chords of a circle, centre O; prove that $\angle AOB = \angle COD$.

[2] The sides of the quadrilateral $ABCD$ are all equal; prove that $AC$ bisects $\angle BAD$.

3. In fig. 320, $BA = BC$ and $KA = KC$; prove that $\angle BAK = \angle BCK$.

4. In fig. 321, the perpendiculars from P to $AB$, $AC$ are equal; prove that $AP$ bisects $\angle BAC$.

5. In fig. 322, if $ON$ is the perpendicular from the centre $O$ of the circle to a chord $AB$, prove that $AN = NB$.

[6] In fig. 322, if $N$ is the mid-point of a chord $AB$ of a circle, centre $O$, prove that $ON$ is perpendicular to $AB$.

7. $ABCD$ is a quadrilateral such that $AB = CD$ and $AD = BC$. Prove that $AD$ is parallel to $BC$. [Join $AC$.]

[8] Two circles cut at $A$, $B$; $O$ is the centre of one circle; prove that the centre of the other circle lies on the bisector of $\angle AOB$.

9. $N$ is the mid-point of the side $BC$ of $\triangle ABC$; $NX$, NY are the perpendiculars from $N$ to $AB$, $AC$. If $NX = NY$, prove that $AB = AC$.

[10] $ABCD$ is a quadrilateral such that $AD = BC$ and $AC = BD$. If $AC$ cuts $BD$ at $K$, prove that (i) $KC = KD$, (ii) $AB \parallel DC$.

11. $AB$, $CD$ are two equal lines not in the same plane. If $AC = BD$, prove that $\angle ABC = \angle BDC$. If the triangle $ABC$ is rotated about $BC$, must $A$ pass through $D$?

12. In $\triangle ABC$, $AB = AC$ and $\angle A = 1$ rt. $\angle$; $XAY$ is any straight line through $A$; $BH$, $CK$ are the perpendiculars from $B$, $C$ to $XAY$; prove that $AH = CK$.

13. In fig. 323, $AB = BH$, $BC = BK$, $AK = CH$, and $ABC$, $BHK$ are straight lines. Prove that (i) $\angle ABK = 1$ rt. $\angle$, (ii) $CH$ is perpendicular to $AK$.


15. In fig. 325, the bisector of $\angle BAC$ meets at $P$ the line which bisects $BC$ at right angles; $PX$, $PY$ are the perpendiculars from $P$ to $AB$, $AC$.

Find in the figure three pairs of congruent triangles and prove the congruence. Prove also that $AX = \frac{1}{2}(AB + AC)$.

16. In fig. 326, $AB$ is the diameter of the circle, centre $O$; the chord $PQ$ makes $45^\circ$ with $AB$; $PM$, $QN$ are the perpendiculars from $P$, $Q$ to $AB$. Prove $ME = ON$. [Prove $\triangle PMO = \triangle ONQ$.]

17. $P$ is any point on the side $AB$ of a triangle $ABC$. The triangle is rotated about $BC$; $A'$, $P'$ are the new positions of $A$, $P$. Prove that $\angle P'AC = \angle PAC$.

18. $P$, $Q$ are two points on the perpendicular bisector of $AB$. At $P$ a perpendicular $PR$ is erected to the plane of $PQ$ and $AB$. Prove that $\angle ARQ = \angle BRQ$. 
Formal Constructions. In theoretical constructions the instruments allowed are restricted to—

(i) a straightedge (ungraduated ruler) for
   (a) joining two given points by a straight line,
   (b) producing a given straight line;

(ii) a compass for
   (a) drawing a circle with a given centre and radius,
   (b) cutting off from a straight line a length equal to
       that of a given straight line.

When performing a construction, the figure must be drawn accurately and all construction-lines must be shown; none of them must be rubbed out.

To secure a high standard of accuracy in constructions,
(i) use a hard pencil and see that it is properly sharpened.
   A line drawn on paper has width, but the less the width, the more accurate your drawing will be. It is impossible to draw a fine line with a soft pencil.

(ii) Draw circles with long radii, so that points which have
     to be joined are as far apart as possible.
     There is less error in the position of a line which is
     obtained by joining two points far apart than two
     points close together. You may see why this is so if
     you try to hold a long rod at two points close to one
     end when someone else is pushing at the other end at
     right angles to the rod.

(iii) Avoid intersections of lines and circles in which the
     angles at which they cut are small.
     Think how an intersection would appear under a
     microscope by regarding lines as strips, say 1 cm.
     wide. If the angle formed by the strips is small, the
     area common to the two strips is large.

(iv) Take care of your compass. You will spoil it if you use
     it for ordinary writing or for ruling lines.
CONSTRUCTION 2

Bisect a given finite straight line.

Given a finite line AB.

To construct the mid-point of AB.

Construction. With centres A, B and any sufficient radius, the same for each, draw arcs of circles cutting at P, Q.
Join PQ and let it cut AB at C.
Then C is the mid-point of AB.

Proof. Join PA, PB, QA, QB.

In Δs PAQ, PBQ,

\[ AP = BP \]
\[ AQ = BQ \]
\[ PQ = PQ. \]

\[ \therefore \Delta PQS \] are congruent \[ \text{SSS}. \]

\[ \therefore \angle APQ = \angle BPQ. \]

\[ \therefore \text{in } \Deltas PAC, PBC, \]

\[ AP = BP \]
\[ PC = PC \]
\[ \angle APC = \angle BPC \] proved.

\[ \therefore \Deltas PAC, PBC \] are congruent \[ \text{SAS}. \]

\[ \therefore AC = BC. \]

Note. Since also \[ \angle ACP = \angle BCP \] and since these are adjacent angles on a straight line, each of these angles is a right angle. Therefore PQ is the perpendicular bisector of AB.

CONSTRUCTION 3

Draw a straight line at right angles to a given straight line from a given point in the line.

Given a point C in a line AB.

To construct a line from C perpendicular to AB.

Construction. With centre C and any radius, draw an arc of a circle cutting AB at P, Q.
With centres P, Q and any sufficient radius, the same for each, draw arcs of circles cutting at R.
Join CR.
Then CR is the required line perpendicular to AB.

Proof. Join PR, QR.

In Δs PCR, QCR,

\[ CP = CQ \]
\[ PR = QR \]
\[ CR = CR. \]

\[ \therefore \Deltas PCR \] are congruent \[ \text{SSS}. \]

\[ \therefore \angle PCR = \angle QCR, \]

but these are adjacent angles on a straight line,

\[ \therefore \angle PCR \] is a right angle.

Note. This construction is a special case of the construction for bisecting a given angle. In fig. 329, CR bisects the straight angle AOB.
Two alternative methods for Construction 3 are indicated in figs. 330, 331. It is a useful exercise for the reader to perform each of these constructions and prove that they are correct.

Given a point C on AB, to draw the line through C perpendicular to AB.

1. With centre C and any radius, draw the arc PQR cutting CB at P.
   With centre P and the same radius, draw an arc cutting the arc PQR at Q.
   With centre Q and the same radius, draw an arc RS cutting the arc PQR at R.
   With centre R and the same radius, draw an arc cutting the arc RS at S.
   Join CS.
   Then CS is perpendicular to AB.
   [For the proof, note that \( \triangle s \) PCQ, CQR, SQR are equilateral.]

2. Take any point O outside AB.
   Draw a circle, centre O, radius OC, and let it cut AB again at D.
   Join DO and produce it to meet the circle at E.
   Join CE.
   Then CE is perpendicular to AB.
   [For the proof, note that \( \triangle s \) ODC, OEC are isosceles.]
Two alternative methods for Construction 4 are indicated in figs. 333, 334. It is a useful exercise for the reader to perform each of these constructions and prove that they are correct.

Given a line \( AB \) and a point \( C \) outside \( AB \), to draw the line from \( C \) perpendicular to \( AB \).

1. Take any two points \( P \), \( Q \) on \( AB \).
2. Draw an arc of a circle, centre \( P \), radius \( PC \).
3. Draw an arc of a circle, centre \( Q \), radius \( QC \), cutting the first arc at \( R \).
4. Join \( CR \), cutting \( AB \) at \( N \).

Then \( CN \) is the perpendicular from \( C \) to \( AB \).

[For the proof, prove \( \triangle QPC \cong \triangle QPR \) and then prove \( \triangle NPC \cong \triangle NPR \).]

Note. The points \( P \) and \( Q \) should be taken as far apart as possible because this makes it easier to fix the position of \( R \) accurately.

2. Take any point \( D \) on \( AB \).

Join \( CD \) and construct the mid-point \( O \) of \( CD \).

With centre \( O \), radius \( OC \), draw an arc of a circle cutting \( AB \) at \( N \).

Join \( CN \).

Then \( CN \) is the perpendicular from \( C \) to \( AB \).

[For the proof, note that \( \triangle ODN \), \( OCN \) are isosceles.]

The next construction shows how an angle can be copied. The method consists in constructing a triangle one of whose angles is the given angle and then constructing a triangle the lengths of whose sides are equal to those of the first triangle.

**Construction 5**

From a given point in a given straight line, draw a straight line making with the given line an angle equal to a given angle.

![Figure 335](image_url)

Given a point \( A \) on a given line \( AB \) and an angle \( \angle YXZ \).

To construct a line \( AC \) such that \( \angle CAB = \angle ZXY \).

**Construction.** With centre \( X \) and any radius, draw an arc of a circle cutting \( XY \), \( XZ \) at \( P \), \( Q \).

With centre \( A \) and the same radius, draw an arc \( EF \) of a circle cutting \( AB \) at \( E \).

With centre \( E \) and radius equal to \( PQ \), draw an arc of a circle cutting the arc \( EF \) at \( F \).

Join \( AF \) and produce it to \( C \).

Then \( AC \) is the required line.

**Proof.** Join \( PQ \), \( EF \).

In \( \triangle PXQ \), \( EAF \),

\[
\begin{align*}
\angle ODN + \angle OCN &= \angle ODN + \angle OCN \\
\angle ODN &= \angle OCN \\
\triangle ODN \cong \triangle OCN
\end{align*}
\]

:. \( \triangle PXQ \) are congruent SSS,

:. \( \angle YXZ = \angle BAC \).

The most important use made of Construction 5 is for constructing a line passing through a given point parallel to a given straight line. See also Ex. 36, No. 14.

In practical work, parallels are always drawn by using set-squares. But if a formal construction is required, set-squares cannot be used.
CONSTRUCTION 6

Through a given point, draw a straight line parallel to a given straight line not passing through the given point.

![Diagram of construction 6]

Given a point C outside a given line AB.

To construct a line through C parallel to AB.

Construction. Take any point P on AB. Join PC.

From the point C on CP, construct the line CD so that \( \angle PCD \) is equal to \( \angle CPB \) and alternate to it. Then CD is the required line.

Proof. Since the transversal PC cuts the lines AB, DC so as to make the alternate angles \( \angle CPB, \angle CDB \), AB is parallel to DC.

EXERCISE 36

[Use only a compass and straightedge for these constructions.]

1. Construct an equilateral triangle, given one side.

2. Construct an isosceles right-angled triangle, given one of the shorter sides.

3. Take two points A and B. Construct a point C on AB produced such that BC = 3AB.

Construct the following angles:


[9] Construct a right angle and construct two lines dividing the right angle into three equal angles.

10. Take two points A and D. Construct an equilateral triangle ABC such that AD is the perpendicular from A to BC.

11. Draw a triangle ABC and construct the bisector of each angle of the triangle. Do the bisectors meet at a point?

12. Draw an obtuse angle. Divide it into four equal angles.

13. Take two points A and B. Construct C and D so that \( \angle ABC \) and \( \angle BCD \) are alternate angles and are right angles and so that \( AB = BC = CD \). Do AD and BC bisect each other?

14. Draw any triangle ABC. Construct D so that CD = SA and AD = BC. Explain why this is a construction for drawing a line through A parallel to BC.

15. Draw any triangle and construct the perpendicular bisector of each side. Do they meet at a point?

16. Draw any circle, centre O, and draw any chord PQ which is not a diameter. Construct the perpendicular bisector of PQ. Does it pass through O?

17. Draw a circle and take three points A, B, C on the circumference. Construct the bisector of \( \angle BAC \) and the perpendicular from BC; do they meet, when produced, on the circumference?

18. Draw any acute-angled triangle and construct the perpendicular from each vertex to the opposite side.

19. Draw a triangle ABC so that \( \angle C \) is obtuse. Construct the perpendiculars from A to AB produced, from B to AC produced, and from C to AB. Do these lines, when produced, meet at a point?

20. Take three points P, Q, R so that \( \angle PQR \) is obtuse. Join PQ and QR. Construct a line through P perpendicular to PQ without producing PQ.

21. Draw a triangle ABC and take a point O inside the triangle. Construct a point P on BC such that \( \angle BPC = \angle BAC \).

22. Draw a triangle ABC and a straight line DE. Construct a line DF such that \( \angle EDF = \angle EBC \).

23. Draw a triangle ABC in which AB is greater than AC. Produce AB, AC to P, Q. Construct the bisectors of \( \angle ABC \), \( \angle ACB \) and let them meet at H. Construct the bisector of \( \angle BPC \), \( \angle QCB \) and let them meet at K. Does the bisector of \( \angle BAC \) pass through H and K?

24. Draw a triangle ABC in which AB is greater than AC. Construct a point P on BC such that the perpendiculars from P to AB and AC are equal. State your method shortly.
Parallelograms

Definition. A quadrilateral which has both pairs of opposite sides parallel, see fig. 337, is called a parallelogram.

Abbreviation: Parag or parm.

It is suggested that the properties of a parallelogram in Theorems 15, 16 and the tests for a quadrilateral to be a parallelogram in A Theorems 17-20 should be taken orally as riders, as follows:

Examples for Oral Discussion

Properties of a Parallelogram

1. $ABCD$ is a parallelogram. Prove that
   
   (i) $AB = DC$, $AD = BC$.
   
   (ii) $\angle A = \angle C$, $\angle B = \angle D$.

   Draw your own figure. Show on it by suitable markings what is given. Join $BD$.

   Explain with full reasons why $\triangle ABD \equiv \triangle CDB$.

   Note. Your proof also shows that the area of a parallelogram is bisected by a diagonal.

2. If the diagonals of a parallelogram $ABCD$ cut at $K$, prove that $AK = KC$ and $BK = KD$.

   Prove that $\triangle AKB \equiv \triangle CKD$.

Tests for a Parallelogram

3. $ABCD$ is a quadrilateral in which $AB = DC$ and $AB$ is parallel to $DC$. Prove that $ABCD$ is a parallelogram.

   Draw your own figure and show on it by suitable markings what is given. Join $BD$.

   Why is it sufficient to prove that $AD$ is parallel to $BC$?

   Explain with full reasons why $\triangle ABD \equiv \triangle CDB$.

4. $ABCD$ is a quadrilateral in which $\angle A = \angle C$ and $\angle B = \angle D$. Prove that $ABCD$ is a parallelogram.

   Draw your own figure and show on it by suitable markings what is given. No construction.

   What is the sum of the angles of any quadrilateral?

   Use the data to prove that $AB$ is parallel to $DC$ and that $AD$ is parallel to $BC$.

5. $ABCD$ is a quadrilateral in which $AB = DC$ and $AD = BC$. Prove that $ABCD$ is a parallelogram.

   Draw your own figure and show on it by suitable markings what is given. Join $BD$.

   Explain with full reasons why $\triangle ABD \equiv \triangle CDB$.

   Complete the proof, i.e., prove that $AB$ is parallel to $DC$, and $AD$ is parallel to $BC$.
6. \(ABCD\) is a quadrilateral in which the diagonals bisect each other at \(K\), i.e. \(AK = KC\) and \(BK = KD\). Prove that \(ABCD\) is a parallelogram.

Use \(\triangle AKB\), \(\triangle CKD\) to prove that \(AB\) is parallel to \(DC\).

How can you prove that \(AD\) is parallel to \(BC\)?

7 (A discovery example.) Use the results of the preceding examples to discover as many properties as you can of \(ABCD\) and \(ABEF\) in which \(ABCD\) and \(ABEF\) are given parallelograms. Give reasons.

Answer the same question if \(ABCD\), \(ABEF\) are parallelograms in different planes. [Assume that two lines which are parallel to the same line are parallel to one another, even if all three lines do not lie in one plane.]

**Rectangle, Square, and Rhombus**

A parallelogram in which ONE angle is a right angle is called a rectangle (abbreviation, rect.).

A rectangle which has TWO ADJACENT sides equal is called a square (abbreviation, sq.).

The square \(ABCD\) is often called the \(square\) on \(AB\), or \(sq.\) on \(AB\).
I. Properties of a Rectangle

(i) All the angles of a rectangle are right angles.

If $\angle A = 1 \text{ rt. } \angle$, then $\angle B = \angle C = \angle D = 1 \text{ rt. } \angle$.

[No construction: use parallels.]

(ii) The diagonals of a rectangle are equal.

[Prove that $\triangle DAB \cong \triangle CBA$.]

(iii) If the diagonals of a parallelogram are equal, the parallelogram is a rectangle.

[Prove that $\triangle DAB \cong \triangle CBA$; why does $\angle A + \angle B = 2 \text{ rt. } \angle$?]

II. Properties of a Square

(i) All the sides of a square are equal.

If $AB = AD$, then $AB = BC = CD$.

[No construction: opp. sides ||gram are equal.]

(ii) The diagonals of a square are equal and cut at right angles.

(iii) The angle which a diagonal makes with a side of the square is $45^\circ$.

(iv) If the diagonals of a parallelogram are equal and cut at right angles, the parallelogram is a square.

III. Properties of a Rhombus

(i) All the sides of a rhombus are equal.

If $AB = AD$, then $AB = BC = CD$.

(ii) The diagonals of a rhombus are bisected by the diagonals.

(iii) The diagonals of a rhombus cut at right angles.

(iv) If the diagonals of a parallelogram cut at right angles, the parallelogram is a rhombus.

PARALLELOGRAMS

NUMERICAL EXAMPLES

EXERCISE 37

1. $ABCD$ is a rectangle. If $\angle BAC = 32^\circ$, find $\angle DBC$.

2. $ABCD$ is a rectangle. If $\angle ACD = 67^\circ$, find $\angle ADB$.

3. $ABCD$ is a rhombus. If $\angle ABC = 56^\circ$, find $\angle ACD$.

4. $ABCD$ is a rhombus. If $\angle BAC = 35^\circ$, find $\angle ADC$.

5. The diagonals of a rectangle $ABCD$ cut at $K$.

If $\angle AKB = 110^\circ$, find $\angle ACK$ and $\angle ACD$.

6. The diagonals of a rectangle $ABCD$ cut at $N$.

If $\angle ANB = 74^\circ$, find $\angle DNC$.

7. $ABCD$ is a square; a straight line $CPQ$ cuts $BD$ at $P$ and cuts $BA$ at $Q$. If $\angle CPD = 80^\circ$, find $\angle CQA$.

8. $ABCD$ is a regular pentagon; $ABPQ$ is a square inside the pentagon. Find $\angle CBP$ and $\angle DBQ$.

9. $ABC$ is an equilateral triangle; $BCPQ$ and $BCHK$ are the two squares on $BC$. Find $\angle APB$ and $\angle AHB$.

10. The diagonals of a square $ABCD$ cut at $K$. From $AB$ a part $AQ$ is cut off equal to $AK$. Prove that $\angle AKQ = 3 \angle BKQ$.

11. The side $AD$ of the square $ABCD$ is produced to $E$, and the bisector of $\angle EDB$ meets $AC$ produced at $X$. Find $\angle AXD$ and prove that $CD = CX$.

12. $ABCD$ is a rhombus in which $\angle B = 108^\circ$; $CAPQ$ is another rhombus such that $P$ lies on $AB$ produced. Find the acute angle which $AQ$ makes with $BC$.

13. $ABCD$ is a regular pentagon; $ABP$ is an equilateral triangle inside the pentagon. Find the angles of $\triangle PEB$.

14. The diagonals of a rectangle $ABCD$ cut at $K$; $KAP$ is an equilateral triangle drawn so that $B$ and $P$ are on the same side of $AC$. If $\angle ACD = 25^\circ$, find the angles of $\triangle ABP$.

15. $ABCD$ is a square; $ABX$ is an equilateral triangle inside the square. Find $\angle DXC$.

16. $ABCD$ is a regular pentagon and $EDCP$ is a parallelogram inside the pentagon. Find $\angle FPQ$ and prove that $APQ$ is a straight line.

17. The diagonals of the rectangle $ABCD$ cut at $K$, and $AK$ is greater than $AB$. The circle, centre $A$, radius $AK$, cuts $AB$ produced at $E$. If $\angle AKB = 4 \angle BKE$, find $\angle BAC$.

18. $ABCD$ is a square; $P$ is a point on $CA$ produced such that the parallelogram $DAPQ$ is a rhombus. If $QC$ cuts $PD$ at $R$, find the angles of $\triangle DRQ$ and prove that $RP = RC$. 

19. $ABCD$ is a square; $P$ is a point on $CA$ produced such that the parallelogram $DAPQ$ is a rhombus. If $QC$ cuts $PD$ at $R$, find the angles of $\triangle DRQ$ and prove that $RP = RQ$.
THEOREM 15

(1) The opposite sides and angles of a parallelogram are equal.

(2) Each diagonal bisects the area of the parallelogram.

\[ \text{Fig. 349} \]

Given a parallelogram \(ABCD\) and a diagonal \(BD\),

To prove that

1. \(AB = DC\), \(AD = BC\)
2. \(\angle A = \angle C\), \(\angle B = \angle D\).

And

Area of \(\triangle ABD\) = Area of \(\triangle CDB\).

Proof. (1) With the notation in the figure,

- In \(\triangle ABD\) and \(\triangle CDB\),
  - \(p_1 = p_2\) alt. \(\angle s\), \(AB \parallel DC\),
  - \(q_1 = q_2\) alt. \(\angle s\), \(AD \parallel BC\),
  - \(BD = DB\).

\[ \therefore \text{\(\triangle ABD\) and \(\triangle CDB\) are congruent \(\text{ASA}.\)} \]

\[ \therefore AD = CB, AB = CD, \]
\[ \angle A = \angle C. \]

(2) Since \(\triangle ABD = \triangle CDB\), the triangles are equal in area.

Similarly, by joining \(AC\) it may be proved that

\[ \triangle ABC = \triangle CDA. \]

\[ \therefore \angle B = \angle D. \]

And \(AC\) bisects the area of the parallelogram.

Abbreviations for reference:

1. opp. sides \(\parallel\) gram.
2. opp. \(\angle s\) \(\parallel\) gram.
THEOREM 17

If one pair of opposite sides of a quadrilateral are equal and parallel, the other pair of opposite sides are also equal and parallel.

Given a quadrilateral $ABCD$ in which

$AB = DC$ and $AB \parallel DC$.

To prove that $AD = BC$ and $AD \parallel BC$.

Construction. Join $BD$.

Proof. With the notation in the figure, in $\triangle ABD, CDB$,

$AB = CD$ given,

$BD = DB$,

$P_1 = P_2$ alt. $\angle s, AB \parallel DC$.

$\therefore \triangle ABD \cong \triangle CDB$ SAS.

$\therefore AD = CB$,

and $\angle q = \angle r$,

but these are alt. $\angle s$,

$\therefore AD$ is parallel to $BC$.

Abbreviation for reference: 2 sides equal and $\parallel$.

This theorem is also stated in the form:

A quadrilateral, which has one pair of equal and parallel sides, is a parallelogram.

Fig. 351

THEOREM 18

If the opposite angles of a quadrilateral are equal, the quadrilateral is a parallelogram.

Given a quadrilateral $ABCD$ in which

$\angle A = \angle C$ and $\angle B = \angle D$.

To prove that $ABCD$ is a parallelogram.

Proof. The sum of the angles of a quadrilateral is $4$ rt. $\angle s$,

$\therefore \angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle s$,

but $\angle A = \angle C$ and $\angle B = \angle D$ given,

$\therefore 2\angle A + 2\angle B = 4$ rt. $\angle s$,

$\therefore \angle A + \angle B = 2$ rt. $\angle s$;

but these are interior angles on the same side of the transversal $AB$,

$\therefore AD$ is parallel to $BC$.

In the same way it may be proved that

$\angle A + \angle D = 2$ rt. $\angle s$;

but these are interior angles on the same side of the transversal $AD$,

$\therefore AB$ is parallel to $DC$.

$\therefore$ both pairs of opposite sides of $ABCD$ are parallel;

$\therefore ABCD$ is a parallelogram.

Abbreviation for reference: opp. $\angle s$ equal.
THEOREM 19

If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.

Given a quadrilateral \(ABCD\) in which \(AB = DC\) and \(AD = BC\).

To prove that \(ABCD\) is a parallelogram.

Construction. Join BD.

Proof. In \(\triangle BAD, DCB\),

\[
\begin{align*}
AB &= CD \text{ given}, \\
AD &= CB \text{ given}, \\
BD &= DB.
\end{align*}
\]

\(\therefore \triangle BAD \cong \triangle DCB\) SSS.

\(\therefore\) with the notation in the figure,

\(p = r;\)

but these are alt. \(\angle s\),

\(\therefore\) \(AB\) is parallel to \(DC\);

and

\(q = s;\)

but these are alt. \(\angle s\),

\(\therefore\) \(AD\) is parallel to \(BC\).

\(\therefore\) both pairs of opposite sides of \(ABCD\) are parallel;

\(\therefore\) \(ABCD\) is a parallelogram.

Abbreviation for reference: opp. sides equal.

THEOREM 20

If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

Given a quadrilateral \(ABCD\) in which the diagonals \(AC, BD\) bisect each other at \(K\), so that

\(AK = KC\) and \(BK = KD\).

To prove that \(ABCD\) is a parallelogram.

Proof. With the notation in the figure, in \(\triangle AKB, CKD\),

\(AK = CK \text{ given},

BK = DK \text{ given},

\(p_1 = p_2\) vert. opp. \(\angle s\).

\(\therefore \triangle AKB \cong \triangle CKD\) SAS.

\(\therefore q = r;\)

but these are alt. \(\angle s\),

\(\therefore\) \(AB\) is parallel to \(DC\).

Similarly, from the \(\triangle AKD, CKB\) it may be proved that the alt. \(\angle s\), \(t\) are equal,

\(\therefore\) \(AD\) is parallel to \(BC\).

\(\therefore\) both pairs of opposite sides of \(ABCD\) are parallel;

\(\therefore\) \(ABCD\) is a parallelogram.

Abbreviation for reference: diags. bisect each other.
EXERCISE 38

1. State, without proof, what you know about the parallelogram $ABCD$ (i) if $AC$ bisects $\angle BAD$, (ii) if $AC = BD$, (iii) if $AC$ is perpendicular to $BD$.

2. In fig. 355, $ABCD$ and $ABXY$ are parallelograms such that $DCYX$ is a straight line. Use the $SAA$ test to prove that $\triangle ADY \cong \triangle BCO$.

3. In fig. 356, $ABCD$ is a parallelogram and $A$ is the centre of the circle. Prove that $\angle KAD = \angle CDA$. [No construction.]

4. The diagonals of the parallelogram $ABCD$ cut at $K$; any line through $K$ cuts $AB$, $CD$ at $X$, $Y$. Prove that $KX = KY$.

5. $P$ is the mid-point of the side $BC$ of a parallelogram $ABCD$; $DP$ and $AP$ meet, when produced, at $Q$. Prove that $AB = BQ$.

6. $ABCD$ is a parallelogram. Prove that the perpendiculars from $B$ and $D$ to $AC$ are equal.

7. In fig. 357, $ABCD$ and $ABQP$ are parallelograms. Prove that $CDQP$ is a parallelogram.

8. In fig. 358, $\triangle PQR$ is formed by drawing lines through $A$, $B$, $C$ parallel to $BC$, $CA$, $AB$ respectively. Prove that $A$, $B$, $C$ are the mid-points of the sides of $\triangle PQR$.

9. Two equal circles, centres $A$, $B$, cut at $C$, $D$. Prove that $AC$ is a rhombus.

10. $ABCD$ is a rhombus. If the bisector of $\angle DAC$ cuts $CD$ at $P$, prove that $\angle DPA = \angle DAP$.

11. $ABCD$ is a parallelogram such that the bisectors of $\angle A$ and $\angle B$ meet on $CD$. Prove that $AB = 2BC$.

12. In $\triangle ABC$, $\angle A$ is a right angle; $ABPQ$ and $ACXY$ are squares lying outside $\triangle ABC$. Prove that $PAX$ is a straight line.

13. In fig. 359, $ABCD$ and $APQR$ are squares. Prove that $BP = DR$.

14. $ABCD$ is a parallelogram; $BLM$ and $BCK$ are squares outside the parallelogram. Prove that (i) $AE = BK = \angle BCD$, (ii) $KL = BD$.

15. The side $AB$ of the parallelogram $ABCD$ is produced to $X$, and the bisector of $\angle CBA$ meets $DA$ produced and $DC$ produced at $E$ and $F$. Prove that $DE = DF = BA + BC$.

16. In fig. 360, $H$, $K$ are the mid-points of $AB$, $AC$, and $HKP$ is a straight line. Prove that (i) $CP = AH$, (ii) $CPhB$ is a parallelogram, (iii) $HK = \frac{1}{2} BC$.

17. In fig. 361, $ABCD$ and $APCQ$ are parallelograms. Prove that (i) $AC$, $BD$, $PQ$ are concurrent, i.e. pass through the same point; (ii) $PB$ is parallel to $DQ$.

18. $ABCD$ is a rhombus; $PABQ$ is a straight line such that $PA = AB = BQ$. Prove that $PD$ and $QC$ when produced out at right angles.
19. In fig. 362, Q is any point on the diagonal AC of a parallelogram ABCD. Prove that the areas of XQRD and PQYB are equal. [The area of a parallelogram is bisected by a diagonal.]

20. If fig. 363 represents a cube, prove that $\angle EBG = 60^\circ$.

21. If fig. 363 represents a cuboid, prove that $\angle EBG = \angle DGB$.

22. N is the mid-point of the side AB of the rectangle ABCD; NP is drawn perpendicular to the plane of ABCD; P is joined to A, B, C, D. Point cut, with reasons, pairs of congruent triangles.

23. Fig. 364 represents a framework of 5 rods AB, BC, CD, BK, CK, jointed where they meet. The rods AB, DC can turn round the ends A, D which are fixed and the framework remains in one plane. P is a mark on the rod BC. Prove that P moves along the circumference of a circle whose centre lies on AD and find the path traced out by K. [Take $K = 4^\circ$, $DE = 7^\circ$.]

24. Fig. 365 represents a right prism whose ends ABC, XYZ are equilateral triangles; P is the mid-point of the edge BY. Prove that PC = PX.

25. The height of a right pyramid on a square base is equal to half the diagonal of the base. Prove that the faces are equilateral triangles.

26. ABCD is a parallelogram; P is any point not in its plane; O is the mid-point of PB; AQ is produced to R so that AQ = QR. Prove that RO bisects PC.

27. ABCD is a square; E is a point on AC such that AE = AB; the line through E perpendicular to AC cuts BC, DC at F, G. Prove that $\angle FAG = 45^\circ$.

28. In $\triangle ABC$, $\angle C$ is a right angle; ABPQ is a square outside $\triangle ABC$; PN is the perpendicular from P to AC. Prove that $PN = CA + CB$.

29. In fig. 366, ABCD is a square, $\angle APB$ is a right angle, and CXQ, DQ are drawn parallel to AP, BPX respectively. Prove that A

(i) $PX = AP - PB$, (ii) $\angle APQ = 45^\circ$.

30. Three lines AOB, COD, EOF, not in the same plane, meet at O which is the mid-point of each. The parallelograms AECF, SFDO are completed. Prove that (i) PQ passes through O, (ii) PF = EQ.

31. BDG is the base of a tetrahedron and E is its vertex. Through each edge of the tetrahedron a plane is drawn parallel to the opposite edge, thus forming the parallelepiped in fig. 363. Use this construction to prove that the lines joining the mid-points of the opposite edges of a tetrahedron are concurrent and bisect each other.

What can you say about the opposite edges of the tetrahedron if the edges of the parallelepiped are all equal?

Construction of Triangles and Quadrilaterals. Examples of the construction of triangles and quadrilaterals from simple sufficient data have been given in Stage A. The sole purpose of Exercise 39 is to give material for the revision of this work; it may therefore be omitted if such revision is unnecessary.
REVISION EXERCISE

EXERCISE 39

[Always make a neat sketch of the required figure and mark the data on it before you start to construct the figure.]

Construct when possible \( \triangle ABC \) from the given measurements in Nos. 1-9, choosing your own unit of length. If there are two different solutions, construct both. If there is no solution, say so.

1. (i) \( a=3, b=4, c=5 \); measure \( A \).
   (ii) \( a=3, b=4, c=8 \); measure \( A \).

2. \( a=5, B=30^\circ, C=45^\circ \); measure \( b \).

3. \( a=4, A=48^\circ, B=33^\circ \); measure \( b \).

4. (i) \( a=7, A=110^\circ, B=40^\circ \);
   measure \( b \).
   (ii) \( a=5, B=125^\circ, C=70^\circ \);
   measure \( b \).

5. \( b=7, c=5, A=125^\circ \); measure \( a \).

6. (i) \( b=5, c=7, C=72^\circ \); measure \( a \).
   (ii) \( b=6, c=4, C=40^\circ \); measure \( a \).
   (iii) \( b=8, c=6, C=65^\circ \); measure \( a \).

7. \( a=b=6, A=50^\circ \); measure \( c \).

8. \( a=11, b=7, A=90^\circ \); measure \( c \).

9. \( a:b:c=4:3:2 \); measure \( A \).

Construct the quadrilateral \( ABCD \) from the given measurements in Nos. 10-15, choosing your own unit of length.

10. \( AB=4, BC=4, CD=3, CA=110^\circ \); measure \( AD \).

11. \( AB=5, AC=6, AD=4, BD=7, CD=3 \); measure \( BC \).

12. \( AB=5, BC=6, CD=3, DA=4, \angle D=100^\circ \); measure \( \angle B \).

13. \( \angle B=70^\circ, \angle C=95^\circ, \angle D=105^\circ, AB=5, AD=4 \); measure \( BC \).

14. \( AB=5, \angle CAB=35^\circ, \angle ABD=47^\circ, \angle ACB=65^\circ, \angle ABD=54^\circ \); measure \( CD \).

15. \( AB=BC=3, AD=DC=5, \angle B=120^\circ \); measure \( \angle D \).

CONSTRUCTIONS

16. Construct an isosceles triangle with a base of 6 cm and a vertical angle of 70\(^\circ\); measure its sides.

17. Construct an isosceles triangle with a base of 4 cm and a height of 5 cm; measure its vertical angle.

18. Draw a circle of radius 5 cm; construct a triangle \( ABC \) such that \( A, B, C \) lie on the circumference, and \( AB=8 \) cm, \( AC=7 \) cm; measure \( \angle BAC \). [Two answers.]

Construction of Triangles and Parallelograms. For the construction of parallelograms, rectangles, etc., from sufficient data, familiarity with the properties discussed in pp. 148-155 is essential. These may be summarised as follows:

(1) In any parallelogram,
   (a) opposite angles are equal;
   (b) opposite sides are equal;
   (c) the diagonals bisect each other;
   (d) each diagonal bisects the area.

(2) A quadrilateral is a parallelogram, if
   (a) both pairs of opposite angles are equal;
   (b) both pairs of opposite sides are equal;
   (c) one pair of opposite sides are equal and parallel;
   (d) the diagonals bisect each other.

(3) In any rectangle,
   (a) all the angles are right angles;
   (b) the diagonals are equal.

(4) In any square,
   (a) the diagonals are equal;
   (b) the diagonals cut at right angles;
   (c) the angle which each diagonal makes with each side of the square is 45\(^\circ\).

(5) In any rhombus,
   (a) all the sides are equal;
   (b) the diagonals cut at right angles;
   (c) the angles are bisected by the diagonals.
CONSTRUCTION 7

Describe a square on a given straight line.

Given a straight line AB.
To construct a square having AB as one side.

Construction. From A draw a line AC perpendicular to AB.
From AC cut off AP equal to AB.
Through P draw PQ parallel to AB.
Through B draw a line parallel to AP cutting PQ at R.
Then ABPR is the required square.

Proof. By construction, ABPR is a parallelogram.
Since \( \angle BAP = 1 \text{ rt. } \angle \), ABPR is a rectangle.
Since AB = AP, the rectangle ABPR is a square.

When constructing a figure from numerical data,
(1) make a neat sketch of the required figure;
(2) mark on your sketch the given measurements;
(3) try to find or draw some triangle in the figure which can be constructed from the data, or by deductions from the data, see Examples 1–3.

Examples for Oral Discussion
1. Construct a parallelogram ABCD such that
\[ AB = 6 \text{ cm.}, \ AC = 10 \text{ cm.}, \ BD = 8 \text{ cm.} \]
Measure AD.

Draw a neat sketch of ABCD, showing the data on it, and let the diagonals cut at K.
(i) Mark on your sketch the lengths of AK and BK.
(ii) What part of the figure can now be constructed?
(iii) Complete the construction and measure AD.

2. Construct a trapezium ABCD in which AB is parallel to DC, and
\[ AB = 8.5 \text{ cm.}, \ BC = 3.5 \text{ cm.}, \ CD = 4.5 \text{ cm.}, \ DA = 3 \text{ cm.} \]
Measure BD.

Draw a neat sketch of ABCD, showing the data on it. In your sketch, draw DP parallel to CB to meet AB at P.
(i) Mark on your sketch the lengths of DP and AP.
(ii) What part of the figure can now be constructed?
(iii) Complete the construction and measure BD.

3. Construct a triangle ABC in which
\[ \angle B = 60^\circ, \ \angle C = 40^\circ, \ \text{perimeter} = 9 \text{ cm.} \]
Measure BC.

Draw a neat sketch of \( \triangle ABC \), showing on it the sizes of \( \angle B, \angle C \), and complete fig. 370 in the way indicated by the markings, see p. 164.
In your sketch, produce CB to P so that BP = BA, and produce BC to Q so that CQ = CA. Join AP, AQ.

(i) Mark on your sketch the length of PQ and the sizes of ∠P, ∠Q.
(ii) What part of the figure can now be constructed?
(iii) B lies on the perpendicular bisector of AP. Why is this?
(iv) Complete the construction and measure BC.

EXERCISE 40

[In this exercise set-squares may be used for drawing parallels and perpendiculars.]

1. Construct the rectangle ABCD, given that AB = 4 cm. and AC = 6 cm.; measure AD.

2. Construct the parallelogram ABCD, given that AB = 4 cm., AD = 5 cm., AC = 6 cm.; measure ∠BAD.

3. Construct the rhombus ABCD, given that BD = 7 cm., ∠B = 40°; measure AC.

4. Construct the square ABCD, given that AC = 5 cm.; measure AB.

5. Construct the rectangle ABCD, given that the diagonals intersect at an angle of 54° and that BD = 8 cm.; measure the sides.

6. Construct a rhombus ABCD, given that AB = 5 cm., AC = 6 cm.; measure ∠BAD.

7. Construct a parallelogram ABCD, given that AB = 7 cm., AC = 10 cm., BD = 8 cm.; measure BC.

8. Construct a parallelogram ABCD, given that AC = 4 in., BD = 5 in., and that the diagonals intersect at an angle of 50°, measure the longer side.

9. Construct the rhombus ABCD, given that AC = 6 cm., BD = 9 cm.; measure AB.

10. Construct △ABC, given that ∠A = 70°, ∠C = 35°, and the length of the perpendicular from A to BC is 2 in.; measure BC.

11. Construct △ABC, given that ∠C = 68°, AB = 6 cm., and the length of the perpendicular from A to BC is 4 cm.; measure BC.

In Nos. 12–16, construct △ABC, choosing your own unit of length.

12. a + b = 11, b + c = 16, c + a = 13; measure ∠A.

13. A = 25°, C = 65°, c = 7; measure a.

14. b = c, a = 4, B = 24°; measure b.

15. A + B = 118°, B + C = 96°, a = 7; measure c.

16. A : B : C = 1 : 2 : 3, a : c = 5; measure c.

17. D is a point on the side BC of an equilateral triangle ABC. Given that BD = 3 cm. and ∠DAC = 40°; construct △ABC and measure BC.

18. Draw two parallel lines AB, CD, 6 cm. apart, and take any point O between AB and CD. Construct a straight line POQ cutting AB, CD at P, Q so that PQ = 6 cm.

In Nos. 19–24, construct a trapezium ABCD in which AB is parallel to DC.

19. AB = 8 cm., BC = 4 cm., CD = 3 cm., DA = 3.6 cm.; measure ∠A.

20. AB = 5 cm., BC = 6 cm., CD = 2 cm., DA = 4 cm.; measure ∠A.

21. AB = 8 cm., CD = 5 cm., ∠A = 72°, ∠B = 40°; measure BC.

22. AB = 4 cm., CD = 7 cm., ∠A = 130°, ∠B = 70°; measure AD.

23. AB = 6.5 cm., CD = 3 cm., AC = 7 cm., BD = 5 cm. Describe shortly your method. [In your sketch, complete the parallelogram CDBP.]

24. AB = 4.5 cm., CD = 2.5 cm., AC = 4 cm., BD = 5 cm. Describe shortly your method.

25. The perpendicular distance between the opposite sides of a parallelogram are 5 cm., 4 cm., and one angle is 70°. Construct the parallelogram and measure one of its longer sides.

26. The perpendicular distance between one pair of opposite sides of a parallelogram is 4 cm. and the lengths of the diagonals are 8 cm. and 6 cm. Construct the parallelogram. Describe shortly your method.
NEW GEOMETRY

In Nos. 27–33, construct the triangle ABC, choosing your own unit of length.

*27. \( \angle A = 60^\circ \), \( \angle B = 70^\circ \), \( a + b + c = 12 \); measure \( a \)

*28. \( \angle A = 90^\circ \), \( a = 10 \), \( b + c = 13 \); measure \( b \) [In your sketch produce CA to P so that AP = AB; join BP; what is \( \angle CPB \)]

*29. \( \angle B = 80^\circ \), \( b = 10 \), \( a + c = 13 \); measure \( c \).

*30. \( \angle A = 70^\circ \), \( c = 7 \), \( a + b = 14 \); measure \( a \) [Produce AC to P so that CP = CB. C is on the perpendicular bisector of BP; why?]

*31. \( \angle B = 35^\circ \), \( a = 8 \), \( b + c = 10 \); measure \( b \).

*32. \( \angle B = 25^\circ \), \( a = 9 \), \( c - b = 4 \); measure \( c \) [From AB cut off AP equal to AC; join CP.]

*33. \( \angle A = 70^\circ \), \( a = 9 \), \( b - c = 2 \); measure \( b \) [From AC cut off AP equal to AB; join BP. Calculate \( \angle CPB \).]

*34. Construct an isosceles triangle of height 5 cm. and perimeter 18 cm.; measure its base.

*35. Construct a triangle ABC, given that \( BC = 1 \text{ in.} \), the length of the perpendicular from A to BC is 1-2 in. and that the length of the line joining B to the mid-point of AC is 0-9 in. Describe shortly your method.

*36. Construct a convex quadrilateral PQRS given that \( PQ = 8 \text{ cm.}, RS = 6 \text{ cm.}, \angle QPR = 65^\circ \); and that S is 3-5 cm. from PQ and that the diagonals cut at right angles. Describe shortly your method.

Inequalities

The fundamental theorem on inequalities is as follows:—

An exterior angle of a triangle is greater than either of the interior opposite angles.

This can be regarded as a corollary of Theorem 8, p. 102, but can be proved without the use of parallels, see Appendix, p. 543; this proof provides an instructive study on congruent triangles.

The symbol \( > \) means is greater than.

The symbol \( < \) means is less than.

Thus 5 in. \( > \) 4 in. and \( 70^\circ < 90^\circ \).
THEOREM 22 (Proof by Exhaustion)

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Given a triangle ABC in which \( \angle B > \angle C \).
To prove that \( AC > AB \).

**Proof.** Either \( AC < AB \) or \( AC = AB \) or \( AC > AB \).

If \( AC < AB \), the angle opposite the greater side AB is greater than the angle opposite the smaller side AC,
\[ \therefore \angle C > \angle B, \] contrary to what is given;
\[ \therefore AC \text{ is not less than } AB. \]

If \( AC = AB \), \( \angle C = \angle B \), base \( \angle s \), isos. \( \triangle \),
which is contrary to what is given;
\[ \therefore AC \text{ is not equal to } AB. \]

\[ \therefore AC \text{ is neither less than } AB \text{ nor equal to } AB, \]
\[ \therefore AC > AB. \]

**Corollary.** In an obtuse-angled triangle, the greatest side is opposite to the obtuse angle; in a right-angled triangle, the hypotenuse is the greatest side.

This method of proof is called a proof by exhaustion.
It consists in taking in turn each possible supposition and proving that all except one of them are untrue. It then follows that the remaining supposition must be true. An alternative direct proof follows.

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THEOREM 22 (Direct Proof)

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Given a triangle ABC in which \( \angle B > \angle C \).
To prove that \( AC > AB \).

**Construction and Proof.** Since \( \angle B > \angle C \), there is a point X on AC between A and C such that \( \angle ABX = \angle C \).
Let the bisector of \( \angle XBC \) cut XC at K.

With the notation in the figure,
\[ \angle ABK = m_1 + p_1, \]
\[ \angle AKB = m_2 + p_2 \]
ext. \( \angle \) of \( \triangle \).

But \( m_1 = m_2 \) and \( p_1 = p_2 \) constr.,
\[ \therefore \angle ABK = \angle AKB; \]
but these are angles of \( \triangle AKB \),
\[ \therefore AK = AB. \]

But K lies between C and X and therefore between C and A,
\[ \therefore AC > AK; \]
\[ \therefore AC > AB. \]
THEOREM 23

Of all straight lines which can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

Given a straight line $AB$, a point $C$ outside $AB$, the perpendicular $CN$ from $C$ to $AB$, and any other point $P$ on $AB$.

To prove that $CN < CP$.

Proof. In $\triangle CNP$,
$$\angle N = 1 \text{ rt. } \angle \text{ given},$$

$$\therefore \angle P + \angle C = 1 \text{ rt. } \angle \text{ sum of } \triangle = 2 \text{ rt. } \angle s,$$

$$\therefore \angle P < 1 \text{ rt. } \angle,$$

$$\therefore \angle P < \angle N.$$

$$\therefore \angle CNP < \angle N.$$

$$\therefore \angle NP < \angle N.$$

$$\therefore \angle CN < CP.$$

Corollary. Conversely, if $CK$ is the shortest straight line from $C$ to any point of $AB$, then $CK$ is perpendicular to $AB$.

For if $CK$ is not perpendicular to $AB$, it is greater than the perpendicular from $C$ to $AB$.

Distance of a Point from a Straight Line

From a given point $P$ outside a given straight line $AB$, one, and only one, straight line can be drawn perpendicular to $AB$, produced if necessary.

A construction has been given showing how one perpendicular can be drawn, and if two perpendiculars could be drawn they would form with the given line a triangle in which two of the angles are right angles; this is impossible, because the sum of the three angles of any triangle is two right angles (see p. 102).

Definition. The length of the perpendicular $PX$ from any point $P$ to a straight line $AB$ is called the distance of $P$ from $AB$. It is the shortest distance of $P$ from any point on $AB$.

A point $P$ is said to be equidistant from two straight lines $AB$ and $CD$ if the perpendiculars $PX$, $PY$ from $P$ to $AB$, $CD$ are equal.

Note. In fig. 376, the perpendicular from $P$ to $CD$ meets $CD$ at a point on $DC$ produced.

Distance between Two Points. The distance between two points $A$, $B$ is the length of the straight line $AB$, and it may reasonably be taken as true that this is the shortest distance from $A$ to $B$ because in fact it is as obvious as other assumptions that have previously been made (explicitly or implicitly). A "proof" is, however, added here because

(i) the construction and method of proof are useful in rider-work and in other constructions;

(ii) it is still set as a theorem in some examinations.
THEOREM 24

Any two sides of a triangle are together greater than the third side.

Given a triangle ABC.

To prove that \( BA + AC > BC \).

Construction. Produce BA to P and cut off AX from AP equal to AC.

Join CX.

Proof. With the notation in the figure, since \( AC = AX \) \( \text{const} \), \( m_1 = m_2 \) base \( \angle s \), isos. \( \triangle \).

But \( \angle BCX > \angle m_1 \); \( \therefore \angle BCX > m_2 \);

\( \therefore \) in \( \triangle BCX \), the side opposite \( \angle BCX \) is greater than the side opposite \( m_2 \).

\( \therefore \) BX > BC.

But \( BX = BA + AX = BA + AC \) \( \text{const} \),

\( \therefore \) BA + AC > BC.

Corollary. The difference between any two sides of a triangle is less than the third side.

NUMERICAL EXAMPLES

EXERCISE 41

1. In \( \triangle ABC \), \( AB = AC \) and \( \angle B = 62^\circ \). Which is the longer, \( AB \) or \( BC \)?

2. The sides \( CA, CB \) of \( \triangle ABC \) are produced to \( H, K \); the bisectors of \( \angle ABC, \angle ACB \) meet at \( I \); \( \angle BAH = 126^\circ \), \( \angle ABK = 118^\circ \). Which is the longer, IB or IC?

3. In fig. 378, \( \triangle ABCD \) is a straight line; \( BQ, CQ \) are the bisectors of \( \angle PBD, \angle PCD \). Which is the longer, (i) PB or PC, (ii) BC or QC?

4. In fig. 379, \( \triangle ABCD \) is a straight line. Which is the longer, (i) RB or RC, (ii) PB or PR?

5. In \( \triangle ABC \), \( \angle A = 90^\circ , \angle B = 58^\circ \); \( AB, AC \) are produced to \( H, K \); PB, PC are the bisectors of \( \angle HBC, \angle KCB \). Which is the longer, (i) AB or AC, (ii) PB or PC?

6. In \( \triangle ABC \), \( \angle B = 30^\circ , \angle C = 60^\circ \); the bisector of \( \angle BAC \) cuts BC at X. Arrange in order of length, the shortest first, AX, BX, CX.

7. \( P \) is a point between B and C on the side BC of \( \triangle ABC \), such that \( PA = PC \); \( \angle C = 58^\circ \), \( \angle PAB = 35^\circ \). Which is the longer, PB or PC?

8. \( P \) is a point between B and C on the side BC of the equilateral triangle \( ABC \). Arrange in order of length, the shortest first, the sides of \( \triangle ABP \).

9. Is it possible to draw a triangle whose sides are of lengths (i) 2 1\( \frac{1}{4} \) cm, 3 6 cm, 6 5 cm; (ii) 2 in., 3 in., 4 in.; (iii) 1 in., 2 in., 3 in.?
NEW GEOMETRY

[10] How many unequal triangles can be drawn such that the lengths of two sides are 4 ft., 7 ft., and such that the length of the third side is a whole number of feet?

11. ABCD is a convex quadrilateral in which AB = 7 cm., BC = 2 cm., CD = 3 cm., DA = 4 cm. (i) Between what limits must the length of AC lie? (ii) Prove that \( \angle DCB > \angle DBC \) and that \( \angle ADB > \angle DAB \).

12. ABCD is a trapezium in which AB, DC are the parallel sides; AC cuts BD at K. If \( \angle CAB = 41^\circ \) and \( \angle AKB = 160^\circ \), find which is the greater, AC or BD.

*13. In fig. 380, RQ is parallel to BC. Prove that \( (i) AR > PR > QC; \) \( (ii) BP > PQ. \)

*14. In \( \triangle ABC \), \( \angle B = 90^\circ \), \( \angle C = 29^\circ \); prove that \( AB < \frac{1}{2} AC \).

EXERCISE 42

1. In fig. 381, AB = AC and BAP is a straight line. Prove that PB > PC.

2. In fig. 382, AB = AC and BQR is a straight line. Prove that AQ < AC < AR.

[3] In \( \triangle ABC \), BC > CA > AB. Prove that \( \angle A > 60^\circ > \angle C \).

4. In \( \triangle ABC \), \( \angle B = 90^\circ \), \( \angle C = 45^\circ \). Prove that \( BC > \frac{1}{2} AC \).

5. In \( \triangle ABC \), the bisector of \( \angle BAC \) cuts BC at D. Prove that BA > BD.

[6] In \( \triangle ABC \), AB > AC and the bisector of \( \angle BAC \) cuts BC at D. Prove that \( \angle BDA > \angle CAD \).

7. In \( \triangle ABC \), AB > AC. If the bisectors of \( \angle ABC \), \( \angle ACB \) meet at I, prove that IB > IC.

INEQUALITIES

[8] ABCD is a convex quadrilateral in which \( \angle B = \angle D \) and \( AB > AD \). Prove that \( \angle ABD < \angle A \).

[9] In fig. 383, AB > AD and \( \angle B = \angle D \). Which is the greater, \( \angle C \) or \( \angle D \)? Give reasons.

[10] ABCD is a quadrilateral. Prove that \( AB + BC + CD > AD \). [Join AC.]

[11] Prove that any side of a triangle is less than half the perimeter of the triangle.

12. In fig. 384, prove that, if a circle is drawn with C as centre and CN as radius, it does not meet AB at any point except N.

13. O is any point inside the triangle ABC. Prove that \( \angle BOC > \angle BAC \). [Produce BO to meet AC at K.]

[14] D is any point on the side BC of \( \triangle ABC \). Prove that \( AD < \frac{1}{2}(BC + CA + AB) \).

15. P is any point inside the convex quadrilateral ABCD. Prove that \( PA + PB + PC + PD \) is not less than \( AC + BD \). What is the position of P if \( PA + PB + PC + PD = AC + BD ? \)

16. In the convex quadrilateral ABCD, AB is the greatest side and CD is the least side. Prove that \( \angle C > \angle D \). [Join BD.]

17. In fig. 385, \( \angle B \) is acute and \( \angle B = 2 \angle C \). Prove that AC < \( \frac{1}{2} AC \).

18. The sides AB, AC of \( \triangle ABC \) are produced to P, Q; the bisectors of \( \angle PBC \), \( \angle QCB \) meet at X. If \( AB > AC \), find which is the greater, \( XB \) or \( XC \). Give reasons.
19. If in fig. 386, $BE = EC$, prove that $ED > EA$.

20. O is any point inside the triangle $ABC$. Prove that $BA + AC > BO + OC$. [Produce $BO$ to meet $AC$ at $K$.]

21. $ABCD$ is a trapeziium in which $AB$, $DC$ are parallel; $AC$ cuts $BD$ at $K$. If $AC > BD$, prove that $AK > KB$. [Draw $CP$ parallel to $DB$ to meet $AB$ produced at $P$.]

22. D is the mid-point of the side $BC$ of $\triangle ABC$. If $AB > AC$, prove that $\angle BAD < \angle DAC$. [Produce $AD$ to $P$ so that $AD = DP$. Join $BP$.]

23. D is the mid-point of the side $BC$ of $\triangle ABC$. Prove that $AD < \frac{1}{2}(AB + AC)$.

24. In $\triangle ABC$, $AB > AC$. If the bisector of $\angle BAC$ cuts $BC$ at $D$, prove that $BD > DC$. [Produce $AC$ to $P$ so that $AP = AB$. Join $DP$.]

25. In fig. 387, $A'$ is the image of $A$ in the straight line $CD$, i.e. $CD$ bisects $AA'$ at right angles; $A'EB$ is a straight line. Prove that (i) $\angle AEC = \angle BED$; (ii) $AP + PB > AE + EB$. [Join $A'P$.]

This is called the light-path theorem. If a ray of light from a source $A$ is reflected in a mirror $CD$ so as to travel to $B$, it follows the shortest path $AB$, and for this path the angle of incidence, $\angle AEC$, is equal to the angle of reflection, $\angle BED$.

26. O is any point inside the triangle $ABC$. Prove that $OA + OB + OC > \frac{1}{2}(2AC + CA + AB)$. [No construction.]

27. Prove that the sum of the lengths of the diagonals of a convex quadrilateral is greater than the semi-perimeter.
NEW GEOMETRY

4. In fig. 391, \( AH = HB \) and \( AK = KC \). Prove that
(i) \( HK \) is parallel to \( BC \), (ii) \( HK = \frac{1}{2} BC \).

Draw \( CP \) parallel to \( BA \) to meet \( HK \) produced at \( P \).
(i) Explain why \( \triangle CKP \cong \triangle AKH \).
(ii) Prove that \( CP = BH \). What then follows from the fact that \( CP \) is equal and parallel to \( BH \)?

![Fig. 391]

5. In fig. 392, the side \( AB \) of \( \triangle ABC \) is divided into 5 equal parts by the lines \( EK, FL, GM, HN \) drawn parallel to \( BC \) to cut \( AC \) at \( K, L, M, N \).

Prove that \( EK = \frac{1}{5} BC \), \( FL = \frac{2}{5} BC \), \( GM = \frac{3}{5} BC \), \( HN = \frac{4}{5} BC \).

Draw \( KP, LQ, MR, NS \) parallel to \( AB \) to cut \( BC \) at \( P, Q, R, S \).

(i) What can you say about the points \( K, L, M, N \)?
(ii) What can you say about the points \( P, Q, R, S \)?

6. If, in fig. 392, \( AH = \frac{1}{2} AB \), what can you say about the length of \( HN \)?

NUMERICAL EXAMPLES

EXERCISE 43

1. In fig. 393, if \( AP = PB = 2 \) in., \( BC = 3 \) in., and \( AC = 3.6 \) in., find the lengths of \( AQ \) and \( PQ \).

[2] \( X, Y, Z \) are the mid-points of the sides \( BC, CA, AB \) of \( \triangle ABC \). If \( BC = 5 \) in., \( CA = 6 \) in., \( AB = 7 \) in., find the lengths of \( XY, YZ \) and \( ZX \).

INTERCEPT THEOREMS

3. In fig. 393, if \( AP = 2 \) in., \( PB = 3 \) in., \( BC = 4.5 \) in., and \( AC = 4.6 \) in., find the lengths of \( AQ \) and \( PQ \). [Look at fig. 392.]

4. In fig. 393, if \( AP \parallel AB \), what can you say about \( AQ \) and \( PQ \)? Give reasons.

5. In fig. 394, find the values of \( a \) and \( b \).

6. In fig. 394, find \( x \) if \( y = 12 \).

7. In fig. 394, find \( z \) if \( y = 12 \).

8. If, in fig. 393, \( QR \) is drawn parallel to \( AB \) to meet \( BC \) at \( R \), and if \( SP = 10 \) cm., \( AQ = 12 \) cm., \( BR = 15 \) cm., and \( RC = 20 \) cm., find \( QC \) and \( AP \).

9. In fig. 395, if \( AC = CB = 4^\circ \), find \( CR \). [Find \( CK \) and \( KR \).]

10. In fig. 395, if \( AC = 3^\circ \) and \( CB = 6^\circ \), find the length of \( CR \).

Fig. 395

11. In fig. 396, if \( AC = CB = 10^\circ \), find \( CR \). [Find \( CK \) and \( KR \).]

12. In fig. 396, if \( AC = 5^\circ \), \( CB = 4^\circ \), find the length of \( CR \).

[13] The tops \( A, C \) of two vertical poles \( AB, CD \) which stand on level ground are joined by a rod. If \( AB = 5 \) ft. and \( CD = 12 \) ft., find the height of the mid-point of \( AC \) above the ground.

14. At the corners of a rectangular horizontal court, vertical poles of heights 12, 10, 6, 7 ft. are erected in order round the court. The tops are joined diagonally by straight wires. Will the wires intersect? If not, what alteration in the height of the shortest pole is necessary to make them do so?
**THEOREM 25**

The straight line joining the mid-points of two sides of a triangle is parallel to the third side and equal to one-half of it.

Given the mid-points $H$, $K$ of the sides $AB$, $AC$ of $\triangle ABC$.

To prove that:
(i) $HK \parallel BC$,
(ii) $HK = \frac{1}{2} BC$.

**Construction.** Through $C$ draw $CP$ parallel to $BA$ to meet $HK$ produced at $P$.

**Proof.**
(i) With the notation in the figure,
in $\triangle CKP$, $\triangle AKH$,

$\begin{align*}
    m_1 &= m_2, \\
    n_1 &= n_2, \\
    CK &= AK & \text{given},
\end{align*}$

$\therefore \triangle CKP$ are congruent $\triangle AKH$.

$\therefore CP = AH$ and $PK = HK$.

but $AH = BH$ given, $\therefore CP = BH$.

Also $CP$ is drawn parallel to $BH$.

$\therefore$ the lines $CP$, $BH$ are equal and parallel,

$\therefore BCPH$ is a parallelogram.

$\therefore HP \parallel BC$, i.e. $HK \parallel BC$.

(ii) Also $BC = HP$ opp. sides $\parallel$gram,

but $HK = KP$ proved,

$\therefore HK = \frac{1}{2} BC$.

*Abbreviation for reference:* mid-point theorem.

**THEOREM 26**

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

Given the mid-point $H$ of the side $AB$ of $\triangle ABC$, and a line through $H$ parallel to $BC$ cutting $AC$ at $K$.

To prove that

$AK = KC$.

**Construction.** Through $C$ draw $CP$ parallel to $BA$ to meet $HK$ produced at $P$.

**Proof.** Since $HP \parallel BC$ given, and $CP \parallel BH$ constr.,

$BCPH$ is a parallelogram,

$\therefore CP = BH$ opp. sides $\parallel$gram,

but $BH = HA$ given, $\therefore CP = HA$.

In $\triangle CPK$, $\triangle AKH$, with the notation in the figure,

$\begin{align*}
    m_1 &= m_2, \\
    n_1 &= n_2, \\
    CP &= AH & \text{given},
\end{align*}$

$\therefore \triangle CPK$, $\triangle AKH$ are congruent $\triangle AKH$.

$\therefore CK = AK$.

*Abbreviation for reference:* Intercept theorem.
THEOREM 27

If three or more parallel straight lines make equal intercepts on a given transversal, they make equal intercepts on any other transversal.

Given the parallel lines $BQ$, $CR$, $DS$, $ET$, ... cutting a transversal at $B$, $C$, $D$, $E$, ... so that $BC = CD = DE = ...$ and cutting another transversal at $Q$, $R$, $S$, $T$, ...

To prove that $QR = RS = ST = ...$

Construction. Through $R$, $S$, $T$, ... draw lines parallel to $BCDE$ to meet $BQ$, $CR$, $DS$, ... produced, if necessary, at $H$, $K$, $L$, ...

Proof. Since $BH || CR$ given,

and $RH || CB$ constr.,

$CRHB$ is a parallelogram,

$\therefore RH = CB$ opp. sides //gram.

Similarly it may be proved that $SK = DC$,

but $BC = CD$ given, $\therefore RH = SK$.

$\therefore$ in $\Delta$s $RHQ$, $SKR$, with the notation in the figure,

$m_1 = m_2$ corr. $\angle$s, $QH \parallel RK$,

$n_1 = n_2$ corr. $\angle$s, $RH \parallel SK$,

$RH = SK$ proved.

INTERCEPT THEOREMS

$\therefore \frac{\Delta RHQ}{\Delta SKR}$ are congruent $\therefore AAS$,

$\therefore QR = RS$.

Similarly it may be proved that $RS = ST$, etc.

Abbreviation for reference: Intercept theorem.

Note. Theorem 26 is a special case of Theorem 27, and could be regarded as a corollary of Theorem 27; separate proofs are given for examination purposes.

The construction used for Theorem 25 is the same as is used for Theorem 26 and Theorem 27; there are other constructions which could be used, but it is less confusing if the same construction is used for each of this group of theorems.

Examples for Oral Discussion

1. In fig. 400, $AC = CB$ and the parallel lines $AP$, $CR$, $BQ$ meet another transversal at $P$, $R$, $Q$. Prove that $CR = \frac{1}{2}(AP + BQ)$.

Join $AQ$ and let it cut $CR$ at $K$.

(i) Explain why $AK = KQ$.

(ii) What do you know about the length of $CK$?

(iii) What do you know about the point $R$ and the length of $KR$?

2. In fig. 401, $P$, $Q$, $R$, $S$ are the mid-points of the sides of any quadrilateral $ABCD$. Prove that $PQRS$ is a parallelogram.

Join $AC$.

(i) Explain why $PQ$ is parallel to $AC$.

(ii) What do you know about the length of $PQ$?

(iii) What do you know about $SR$?
NEW GEOMETRY

3. Find out whether the statement in No. 2 applies to a
quadrilateral ABCD which is not convex, see fig. 383, p. 175.
Does it apply to a quadrilateral ABCD if ABC, ADC are
in different planes?

4. In fig. 402, F is the mid-point of the hypotenuse AB of
the right-angled triangle ABC. Prove that CF = 1/2 AB.

Draw FN parallel to BC to meet AC at N.

(i) What do you know about the
length of AN?

(ii) Explain why \( \triangle FNC \cong \triangle FNA \).

Complete the proof.

5. What can you say about the circle
whose diameter is the hypotenuse of a
right-angled triangle?

EXERCISE 44

1. In fig. 403, P, Q, R are the mid-points of BC, CA, AB;
prove that PQAR is a parallelogram.

2. In fig. 404, ABCD is a parallelogram and H is the
mid-point of AB. Prove that DH = BC.

3. P, Q are the mid-points of the sides AB, DC respectively
of the parallelogram ABCD; PD, BQ cut AC at H, K respectively,
(i) Explain why PBQD is a parallelogram. (ii) Prove that
AH = HK = KC.

[4] In fig. 398, p. 179, if AP = 2PB, prove that AQ = 2QC.
[Draw another parallel through the mid-point of AP.]

INTERCEPT THEOREMS

5. In fig. 405, H, K, X, Y are the mid-points of PB, PC, QB,
QC. Prove that HK = XY.

6. In fig. 406, AD = DB. Prove that AH = BK.

[7] Four parallel lines cut one transversal at A, B, C, D and
cut another transversal at P, Q, R, S. If AB = CD, prove that
PQ = RS. [Through Q, S draw QH, SK parallel to DA to meet
AP, CR, produced if necessary, at H, K.]

8. In fig. 407, AH = HK = KB; prove that (i) HP = 1/2 BC;
(ii) KQ = 1/2 BC. [Draw PX and QY parallel to AB.]

9. In fig. 408, H is the mid-point of AB and of PQ; K is the
mid-point of AC and of PR. Prove that QR = BC.

[10] P is the mid-point of the side BC of \( \triangle ABC \); Q is the
mid-point of AP; BQ produced meets AC at R. Prove that
AC = 2AR. [Draw PK parallel to BR to meet AC at K.]

[11] In \( \triangle ABC \), \( \angle B = 1 \) right angle; BCX is an equilateral
triangle. Prove that the line from X parallel to AB bisects AC.
[There are two cases.]

12. In fig. 396, p. 179, if AC = CB, prove that for any lengths
of AP and BQ, CR = 1/2(BQ - AP).
13. \( AB \) is a given straight line and \( O \) is a given point outside \( AB \); \( P \) is a variable point on \( AB \). Prove that the midpoint of \( OP \) lies on a fixed straight line.

14. \( A \) and \( B \) are given points; \( P \) is a variable point on a given circle, centre \( B \). Prove that the midpoint of \( AP \) lies on a fixed circle whose centre is the midpoint of \( AB \).

15. If the diagonals of a quadrilateral cut at right angles, prove that the midpoints of the four sides are the vertices of a rectangle.

16. If the diagonals of a quadrilateral are equal, prove that the midpoints of the four sides are the vertices of a rhombus.

17. \( ABCD \) is a rectangle; \( V \) is any point not in its plane; \( L, M, N, P \) are the mid-points of \( VA, VB, VC, VD \). Prove that \( L, M, N, P \) are coplanar and are the vertices of a rectangle.

18. \( Q, R \) are the mid-points of the sides \( AB, BC \) of \( \triangle ABC \); the perpendiculars \( AD, BE \) from \( A, B \) to \( BC, AC \) intersect at \( H \); \( P \) is the mid-point of \( AH \). Prove that \( \angle PQR \) is a right angle.

19. In fig. 409, \( O \) is the centre of the circle; \( AOBD \) is a straight line such that \( AB = BD \). Prove that \( PR = RD \). [Join \( AP \)].

20. In fig. 410, \( BN \) is the perpendicular from \( B \) to the bisector of \( \angle ABC \); \( D \) is the mid-point of \( BC \). Prove that \( DN = \frac{1}{2} (AB - AC) \). [Produce \( BN \) and \( AC \) to cut at \( K \)].

21. \( D \) is the mid-point of the side \( BC \) of \( \triangle ABC \); \( CA \) is produced to \( E \). If \( BR \) is the perpendicular from \( B \) to the bisector of \( \angle BAE \), prove that \( DR = \frac{1}{2} (AB + AC) \).

22. \( D \) is the mid-point of the side \( BC \) of \( \triangle ABC \); \( BR, CS \) are the perpendiculars from \( B, C \) to any straight line passing through \( A \). Prove that \( DR = DS \). First take the case when the line through \( A \) lies outside the angle \( \angle BAC \) and then the case when it lies inside \( \angle BAC \). [Draw the perpendicular \( DK \) from \( D \) to \( \triangle ABC \)].

23. \( Q, R \) are the mid-points of the sides \( AC, AB \) of \( \triangle ABC \); \( \angle BAC = \angle BQC \). [Look at Example 4, p. 184.]

24. \( A \), \( B \), \( C \), \( D \) are the vertices of \( \triangle ABC \); \( E, F, G \) are the perpendiculars from \( A, B, C \) to \( HK \). Prove that \( AP + CR = BQ + DS \). [Let \( AC \) cut \( BD \) at \( O \); draw \( \triangle ON \) parallel to \( \triangle HK \). Look at Example 1, p. 183.]

25. \( B, C \) are the mid-points of \( \angle B \), \( \angle C \) of \( \triangle ABC \); \( AH, AK \) are the perpendiculars from \( A \) to \( BP, CQ \); prove that \( KH \) is parallel to \( BC \). [Produce \( AH, AK \) to cut \( BC \) at \( X, Y \).]

26. A square box \( ABCD \) rests in the rack of a railway carriage with one edge against the wall, see fig. 411. The point of contact \( E \) is 11 in. from the wall; \( E \) is the mid-point of \( AB \), and \( A \) is 5 in. from the wall. Find the distances of \( B \) and \( C \) from the wall. [Draw \( \triangle ABC \) and \( \triangle ABE \) and \( \triangle CDE \).]

27. In fig. 412, \( D \) is the mid-point of \( BC \); \( PQQ \) is a straight line such that \( AP = AQ \). Prove that \( AP = \frac{1}{3} (AB + AC) \). [Hint: \( D \) is the mid-point of \( BC \); therefore with \( BC \) as one side, make a triangle for which the mid-point theorem can be used.]
CONSTRUCTION 8

Divide a given straight line into any given number of equal parts.

Given a line AB.

To construct points dividing AB into any given number of equal parts, say 5.

Construction. From A draw any line AC making any convenient angle with AB, and from AC cut off equal lengths AF, FG, GH, ... the number of such lengths being the required number of equal parts, in this case 5. Let the equal lengths be AF, FG, GH, HK, KL. Join LB, and through F, G, H, K draw lines parallel to LB to meet AB at P, Q, R, S. Then AP, PQ, QR, RS, SB are the required equal parts.

Proof. Since the parallel lines FP, GQ, HR, KS, LB, together with a parallel through A, make equal intercepts on AC, they also make equal intercepts on AB.

\[ \therefore AP = PQ = QR = RS = SB. \]

Examples for Oral Discussion

Plain Scales. The meaning of a scale has been explained on p. 64.

1. Draw a scale of \( \frac{1}{4} \) to read feet and inches and long enough to measure 4 ft.

   1 ft. is represented by \( \frac{1}{4} \) ft.
   \( \therefore \) 4 ft. is represented by \( 0\cdot4 \) ft., \( = 4\cdot8 \) in.

   Draw a line AB, 4-8 in. long, and use a construction to divide it into 4 equal parts, each of which then represents 1 ft. Use a construction to divide the first of these parts AC into 12 equal parts, each of which then represents 1 in.

2. Draw a scale, 8 in. to 1 mile, to read to 10 yd. and long enough to measure 1000 yd.

   First Method. 1760 yd. is represented by 8 in.

   \[ \therefore 1000 \text{ yd. is represented by } \frac{8 \times 1000}{1760} \approx 4\cdot55 \text{ in.} \]

   Draw a line AB, 4-55 in. long, and use a construction to divide it into 10 equal parts, each of which then represents 100 yd. Use a construction to divide the first of these parts AC into 10 equal parts, each of which then represents 10 yd.

   Second Method. Draw a line AD, 8 in. long; this represents 1760 yd.

   \[ 1000 : 1760 = 25 : 44 \]

   Use a construction to find a point C on AD such that

   \[ AC : AD = 25 : 44 \]

   Then AC represents 1000 yd. Proceed as before.

3. Draw a scale of \( \frac{1}{4} \) to read feet and inches and long enough to measure 6 ft.

4. Draw a scale, 1\( \frac{1}{2} \) in. to 1 ft., to read feet and inches and long enough to measure 4 ft.

5. Draw a scale of 1: 20,000 to read to 100 metres and long enough to measure 3 km.
NEW GEOMETRY

6. Draw a scale, 1 ft. to 1 mile, to read to 10 yd. and long enough to measure 1000 yd.

7. Draw a scale of 1: 2500 to read to half-chains and long enough to measure 2 furlongs.

8. The Diagonal Scale. With a ruler graduated in tenths of an inch, it is possible to measure a length in inches correct to one place of decimals and to make an estimate to two places of decimals. More accurate measurements can be made by using a diagonal scale, see fig. 414.

9. Construct a diagonal scale to show eighths and sixty-fourths of an inch.

10. Construct a diagonal scale, 1 in. to 1 mile, to read miles, furlongs and chains up to 4 miles.

Definitions. (1) Three or more straight lines are said to be concurrent if they pass through the same point.

(2) The straight line joining any vertex of a triangle to the mid-point of the opposite side is called a median of the triangle.

Examples for Oral Discussion

Nos. 1-7 refer to fig. 415, in which E, F, G are the mid-points of AC, AB, AH. Copy this figure.

1. What two facts do you know about EG?
2. Explain why BGCH is a parallelogram.
3. Let AGH cut BC at D, show this on your figure and explain why BD = DC.

This proves that the three medians of any triangle are concurrent. The point of intersection of the medians is called the centroid of the triangle.

4. Prove that DG = ½ GA and that DG = ½ DA.
5. What are the corresponding facts about EG and FG?

6. If the lengths of the medians AD, BE, CF of ΔABC are 3 cm., 4½ cm., 5 cm., find the lengths of the sides of ΔABC, and then construct ΔABC.

7. Construct fig. 415, given that BE = 7½ cm., CF = 6 cm., BC = 7 cm.
THEOREM 28

(1) The three medians of a triangle are concurrent.

(2) The point at which the medians intersect is one-third of the way along each median, measured towards the vertex.

\[ \text{Given the mid-points } E, F \text{ of the sides } AC, AB \text{ of } \triangle ABC \]
\[ \text{and that } BE, CF \text{ cut at } G. \]

To prove that if AG produced cuts BC at D,

(1) \( BD = DC. \)
(2) \( DG = \frac{1}{3} DA, \quad EG = \frac{1}{3} EB, \quad FG = \frac{1}{3} FC. \)

Construction. Produce AGD to H so that AG = GH.
Join BH, CH.

Proof. (1) Since AF = FB and AG = GH,
\[ \text{FG is parallel to BH} \quad \text{mid-point theorem.} \]
Since AE = EC and AG = GH,
\[ \text{EG is parallel to CH} \quad \text{mid-point theorem.} \]
Since FGC \parallel BH and EGB \parallel CH,
\[ \text{GCHB is a parallelogram,} \]
\[ \therefore \text{the diagonals GH, BC bisect each other,} \]
\[ \therefore BD = DC. \]

(2) For the same reason, GD = DH,
\[ \therefore GH = 2 GD. \]

Similarly, it may be proved that \( EK = 2 KE\) and \( FG = \frac{1}{3} FC. \)

Abbreviation for reference: centroid theorem.

EXERCISE 45

1. If the medians AD, BE, CF of the triangle ABC meet at G, prove that G is the centroid of \( \triangle DEF. \)

2. \( ABCD \) is a parallelogram; \( P \) is the mid-point of \( AB; \)
\( CP \) cuts \( BD \) at \( Q; \) prove \( AQ \) produced bisects \( BC. \) \[ \text{[Join AC.]} \]

3. \( AD \) is a median of \( \triangle ABC; \) \( DA \) is produced to \( K \) so that \( AK = 2 DA. \)
Prove that \( BA \) produced bisects \( CK. \)

4. \( ABCD \) is a parallelogram; \( BD \) is produced to \( P \) so that \( \sqrt{3} = DP. \)
Prove \( CD \) produced bisects \( AP, \) and \( AD \) produced bisects \( CP. \)

5. If \( G \) is the centroid of \( \triangle ABC \) and if \( AG = BC, \) prove that \( \triangle BGC \) is a right angle.

In Nos. 6–12, \( AD, BE, CF \) are the medians of \( \triangle ABC. \)

6. Construct \( \triangle ABC, \) given that \( AB = 5 \text{ cm}, AC = 6 \text{ cm}, \)
\( AD = 5 \text{ cm}; \) measure \( BC. \) \[ \text{[Can you draw } \parallel \text{gram } CABC?] \]

7. Construct \( \triangle ABC, \) given that \( BE = 6 \text{ cm}, CF = 5 \text{ cm}, \)
\( BC = 8 \text{ cm}; \) measure \( AD. \)

8. Construct \( \triangle ABC, \) given that \( AD = 6 \text{ cm}, BE = 7 \text{ cm}, \)
\( CF = 9 \text{ cm}; \) measure \( BC. \)

9. Prove that \( 2 BE + 2 CF > 3 BC. \)

10. Prove that \( 2AD + 3BC > 4BE. \) \[ \text{[Consider } \triangle BDG. \]

11. Prove that \( 4(AD + BE + CF) > 3(BC + CA + AB). \)

12. Prove that \( BE + CF > AD. \) \[ \text{[See fig. 416.]} \]

*13. \( ABCD \) is a tetrahedron; \( P, \) \( Q \) are the centroids of the faces \( BCD, ACD. \)
Prove that the straight lines \( AP, BQ \)
intersect, and if \( G \) is their point of intersection, prove that \( (i) \) \( PQ \) is parallel to \( AB \) and equal to \( \frac{1}{3} AB, \) \( (ii) \) \( PG = \frac{1}{3} PA. \) \[ \text{[Join } A \text{ and } B \text{ to the mid-point of } CD.] \]

*14. \( ABCD \) is a tetrahedron; \( P, \) \( Q, \) \( R, \) \( S \) are the centroids of the faces opposite \( A, B, C, D \) respectively. \Prove that \( AP, \)
\( BQ, CR, DS \) are concurrent.
LOCI

If we look at the tip of the seconds-hand of a watch we see that it occupies a series of different positions in the course of each minute, and if we combine together all these different positions we obtain the circumference of a circle which the tip of the seconds-hand traces out each minute. This aggregate of all possible positions of the tip is called its locus. The reader is no doubt familiar with the word "aggregate" in connection with cricket scores; here we use it to mean the resulting curve obtained by combining all the different positions of a small object moving according to some given law.

When it is stated that the locus of a small object which moves about subject to some given law is a certain curve, two complementary ideas are involved:

1. The position of every point on the curve satisfies the given law.
2. Every position of the object which satisfies the given law lies on the curve.

It often happens that the conditions of the problem prevent the object from describing the whole of a curve; in this case it must be stated what part of the curve forms the locus.

Suppose, for example, a door is capable of being opened through an angle of 110°, but no more, what is the locus of a small mark on the handle?

The locus is an arc $AB$ of a circle such that, if $O$ is the centre of the circle, $\angle AOB = 110^\circ$; it is not a complete circle.

In trying to discover what the locus is in any given problem it may be possible to visualise the path along which the object moves, as in the case of the seconds-hand considered above, but it is often better to start by marking in a figure a number of possible positions of the object and then try to guess what straight lines or curves form the locus. In either case it is necessary to prove that the guess is correct; but it is often harder to discover what the locus is than to prove the result when discovered. Oral practice in guessing loci should therefore be given before proceeding to theoretical proofs.

Example for Oral Discussion

A, B are two given points in a given plane and $P$ is a point in the plane such that $\angle APB = 60^\circ$. Find experimentally the aggregate of all the possible positions of $P$.

Stick two pins into the paper at $A$ and $B$, perpendicular to the paper, and slide your set-square between the pins so that the arms of the angle $60^\circ$ of the set-square pass through $A$ and $B$. Prick in the paper a number of possible positions of the vertex of the angle.

We see that the locus consists of two arcs of two distinct circles $AP, B$ and $AP, B$, not the whole circumference of one circle.

Note. The formal proof of this locus depends on a later theorem, Theorem 55, p. 332, for the present it must be regarded as an experimental result.

In this example, if the restriction that $P$ lies in a given plane is removed, the aggregate of all its possible positions forms a surface which can be obtained by revolving the arc $AP, B$ about the line $AB$ as axis through 4 right angles.

Definition. The aggregate of all points whose positions are determined by a given law is called the locus of points subject to that law.

When there are an unlimited number of possible positions of a point $P$, all subject to some given law, we call any possible position of $P$ a variable point, and it is often said that the variable point $P$ moves in accordance with the given law and traces out the resulting locus. This form of words is not strictly accurate, because a point marks a position in a plane or in space and cannot move; its use is justified by regarding the phrase, "variable point," as meaning some small object, such as the tip of a pencil, which is free to move.
EXERCISE 46 (Oral)

What are the loci described in Nos. 1–14?
1. A mark on the platform of a merry-go-round.
2. A small lump of lead dropped from your hand.
3. The tip of the pendulum of a clock.
4. The top of your head if you slide downstairs on a tea-tray.
5. The tip of your nose in a swing.
6. The centre of the wheel of an engine running along (i) a straight railway line, (ii) a circular railway line.
7. A mark on a see-saw.
8. The end of the chain by which a donkey is tethered to a post if the chain is kept fully stretched.
9. A mark on the top of a trap-door in the floor when the trap-door is opened to its full extent.
10. The tip of a man's nose on a moving staircase if he stands still from the time he steps on till the time he steps off.
11. The centre of the top of a box when the base of the box is made to slide about on the top of a table.
12. The centre of a marble which rolls about inside a spherical bowl.
13. The highest point of the shade of an electric light hanging from the ceiling when the light swings about.
14. The right-angled corner of your set-square if you rotate it round the hypotenuse as axis.
15. A variable point \( P \) is due north of a fixed point \( A \). Describe precisely the locus of \( P \).
16. A variable point \( P \) is at a given distance from a given point \( A \). What is the locus of \( P \) (i) if \( P \) lies in a plane through \( A \), (ii) without this restriction?
17. A variable point \( P \) is at a given distance from a given line \( AB \). What is the locus of \( P \) (i) if \( P \) lies in a plane through \( AB \), (ii) without this restriction?
18. A variable circle, centre \( P \), of given radius, passes through a fixed point \( A \). What is the locus of \( P \) (i) if \( P \) lies in a plane through \( A \), (ii) without this restriction?
29. A thin straight rod is 3 ft. long; P is a variable point such that its distance from the nearest point of the rod is always 2 ft. What is the complete locus of P (i) if P lies in a given plane through the rod, (ii) without this restriction?

*30. AB is a long string with a weight attached to B; the end A is free to slide on the rim of a fixed vertical circular ring and AB itself remains vertical. What is the locus of B?

*31. A and B are given points such that AB = 5 cm.; P is a variable point such that PA < 4 cm. and PB < 3 cm. What is the complete locus of P (i) if P lies in a given plane through AB, (ii) without this restriction?

*32. A long string is wound round a prism whose section is a regular hexagon. Describe the locus of the free end of the string when the string is unwound, being kept taut and in a plane perpendicular to the axis of the prism.

*33. A sphere rolls on the inside surface of a hollow circular cone. What is the locus of its centre?

*34. A variable point P lies on a variable line AQ which passes through a given point A and makes a given angle with a given plane. What is the locus of P?

*35. ABCD is a fixed rhombus; P is a variable point in the plane of ABCD such that \( \angle APB = \angle APD \). Find out whether the diagonals and the diagonals produced form part of the locus of P. There is also a circular arc which belongs to the locus of P; sketch this arc in your figure by using experimental methods.

*36. ABCD is a square; P is a variable point inside the square such that the sum of its distances from AB and AD is equal to a side of the square. Use experimental methods to find the locus of P and then prove your result is correct. [Use squared paper.]

*37. A ladder 10 ft. long rests with one end against a vertical wall and the other end on a horizontal floor. If the ladder slips down, remaining in a plane perpendicular to the wall, find the locus of its mid-point.

*38. A circular cone rolls on a plane. What is the locus of the centre of the base of the cone?

*39. A solid, consisting of two unequal spheres glued together, rolls on a plane. What are the loci of the centres of the spheres?

In solving problems on loci, there are in general three stages:

1. The locus must be discovered.
   Sometimes this can be done experimentally. But when this is impracticable, it is necessary to reduce the problem to one of the standard locus theorems.

2. It must be proved that the position of every point on the specified locus satisfies the given law.

3. It must be proved that every point whose position satisfies the given law lies on the specified locus.

From theorems which will be discussed at a later stage dealing with areas, angle properties of a circle and similar figures, useful standard locus-theorems can be deduced.

At this stage, we consider two locus-theorems which have many important practical and theoretical applications.

I. The locus of points which are equidistant from two given points is the perpendicular bisector of the straight line joining the given points.


II. The locus of points which are equidistant from two given intersecting straight lines is the pair of lines which bisect the angles between the given lines.


The enunciation of each theorem specifies the locus; the complete proof of each theorem involves the proving of two distinct statements, either of which is the converse of the other, corresponding to stages (2) and (3) enumerated above.

The first of these two locus theorems is established by the two theorems, 29 (1), 29 (2), pp. 200, 201, which deal with stage (2) and stage (3) respectively. The second of these locus theorems is established by the two theorems, 32 (1), 32 (2), pp. 208, 209.
THEOREM 29 (1)

Any point on the perpendicular bisector of the line joining two given points is equidistant from the given points.

Given two points A, B and any point Q on the perpendicular bisector HK of AB.

To prove that QA = QB.

Construction. Let HK cut AB at N.

Proof. In \( \triangle ANQ, BNQ \),

\[
\begin{align*}
AN &= BN & \text{given}, \\
QN &= QN, \\
\angle ANQ &= \angle BNQ & \text{rt. \( \angle \)}, \text{given.}
\end{align*}
\]

\( \therefore \triangle ANQ \) and \( \triangle BNQ \) are congruent \( \text{SAS} \).

\( \therefore QA = QB. \)

---

THEOREM 29 (2)

A point which is equidistant from two given points lies on the perpendicular bisector of the straight line joining the given points.

Given two points A, B and a point P such that \( PA = PB \).

To prove that P lies on the perpendicular bisector of AB.

Construction.

Let N be the mid-point of AB. Join PN.

Proof. In \( \triangle ANP, BNP \),

\[
\begin{align*}
AN &= BN & \text{constr.}, \\
PA &= PB & \text{given}, \\
PN &= PN.
\end{align*}
\]

\( \therefore \triangle ANP \) and \( \triangle BNP \) are congruent \( \text{SSS} \).

\( \therefore \angle ANP = \angle BNP, \)

but these are adjacent \( \angle \)s on a straight line, therefore each is a right angle.

\( \therefore \) PN is perpendicular to AB and bisects it,

\( \therefore \) P lies on the perpendicular bisector of AB.

Examples for Oral Discussion

1. The perpendicular bisectors of the sides $AB$, $AC$ of a triangle $ABC$ meet at $O$. Prove that
   
   (1) $OA = OB = OC$;
   
   (2) the perpendicular bisector of $BC$ passes through $O$.

   ![Fig. 421]

(i) Explain why $\triangle OZA = \triangle OZB$; then complete the proof of (1).

(ii) If $X$ is the mid-point of $BC$, explain why $\triangle OXB = \triangle OXC$; then complete the proof of (2).

This argument provides a proof of the theorem that the perpendicular bisectors of a triangle are concurrent. The proof may be abbreviated, see Theorem 30, by using the locus theorem just established.

Since $OA = OB = OC$, the circle, centre $O$, radius $OA$ passes through $A, B, C$. It is called the circumcircle of $\triangle ABC$, and $O$ is called the circumcentre; the radius of this circle is called the circumradius.

2. If $A, B, C$ are any three given points which do not lie on a straight line, one and only one circle can be drawn to pass through them.

Perform the necessary construction and prove that the statement is correct.

3. If $AD$, $BE$, $CF$ are the altitudes of any triangle $ABC$, i.e., the perpendiculars from the vertices to the opposite sides, prove that $AD$, $BE$, $CF$ are concurrent.

   Through $A, B, C$ draw lines parallel to $BC, CA, AB$ respectively to form the triangle $PQR$.

   (i) Explain why $BC = AR$ and prove that $AQ = AR$.
   
   (ii) Explain why $AD$ is the perpendicular bisector of $QR$.
   
   (iii) What can you say about $BE$ and $CF$?

   Complete the proof.

   The point of intersection of the altitudes is called the orthocentre of the triangle.

   In fig. 423, the triangle $DEF$, whose vertices are the feet of the altitudes, is called the pedal triangle of the triangle $ABC$.

   If you draw an obtuse-angled triangle, you will see that its orthocentre lies outside the triangle.

   **Note.** The circumcentre of $\triangle ABC$ is the centre of the circle which passes through $ABC$, but the "orthocentre" is not the centre of any important circle associated with $\triangle ABC$. The word "centre" is often used to denote the common point of intersection of three or more concurrent straight lines. Thus the common point of intersection of the medians of a triangle (see p. 192) might be called the "median-centre" of the triangle, although actually it is called the centroid.

   **Notation.** If corresponding points are taken on the sides of a triangle $ABC$, e.g. mid-points, feet of altitudes, etc., they should be denoted by consecutive letters of the alphabet, e.g. $X, Y, Z$ or $D, E, F$, etc., see figs. 421, 423, with $X$ on the side opposite $A$, $Y$ opposite $B$, $Z$ opposite $C$, and similarly for $D, E, F$. It is easier both to write out an argument and to follow it if this is done.
THEOREM 30

The perpendicular bisectors of the three sides of a triangle are concurrent.

Given a triangle ABC and the perpendicular bisectors PX, QY, RZ of BC, CA, AB.

To prove that PX, QY, RZ are concurrent.

Construction.
Let QY meet RZ at O.
Join OA, OB, OC.

Proof. Since O lies on the perpendicular bisector of AB,

\[ OA = OB. \]

Since O lies on the perpendicular bisector of AC,

\[ OA = OC, \]
\[ OB = OC. \]

\[ \therefore \ O \text{ is equidistant from the points } B, C; \]
\[ \therefore \ O \text{ lies on the perpendicular bisector } PX \text{ of } BC. \]
\[ \therefore \ PX, QY, RZ \text{ meet at } O. \]

Abbreviation for reference: Circumcentre theorem.

THEOREM 31

The altitudes of a triangle are concurrent.

Given a triangle ABC, and the perpendiculars AD, BE, CF from A, B, C to BC, CA, AB.

To prove that AD, BE, CF are concurrent.

Construction. Through A, B, C draw straight lines parallel to BC, CA, AB respectively, to form the triangle PQR.

Proof. BC is parallel to RA \[ \text{constr.}, \]
CA is parallel to BR \[ \text{constr.}, \]
\[ \therefore \ BC \parallel RA \text{ opp. sides lgram.} \]

Similarly, since BCQA is a parallelogram,
\[ BC = AQ. \]
\[ \therefore \ RA = AQ. \]

Since AD is perpendicular to BC \[ \text{given}, \]
and since BC is parallel to RQ \[ \text{constr.}, \]
\[ \therefore \ AD \text{ is the perpendicular bisector of } RQ. \]

Similarly, BE, CF are perpendicular bisectors of RP, PQ.
But the perpendicular bisectors of the sides of \( \triangle PQR \)
are concurrent,
\[ \therefore \ AD, BE, CF \text{ are concurrent.} \]

Abbreviation for reference: Orthocentre theorem.
EXERCISE 47

1. Draw a triangle $ABC$ in which $BC = 7$ cm, $CA = 6$ cm, $AB = 5$ cm. Construct the circle which passes through $A$, $B$, $C$ and measure its radius.

2. Given an arc of a circle, show how to construct the centre of the circle.

3. Draw a triangle $ABC$ in which $\angle B > \angle C$; construct in the simplest way a point $P$ on $AC$ such that $\angle PBC = \angle C$.

4. Draw a triangle $ABC$ in which $\angle A$ is obtuse. Construct a point $P$ on $BA$ produced such that $PB - PC = AB$.

5. Draw a quadrilateral $ABCD$ in which $AB$ is not parallel to $DC$. Construct a point $P$ such that $PA = PB$ and $PC = PD$.

6. Draw a triangle $XYZ$ and construct a point $P$ such that $PX = PY$ and $XP$ is perpendicular to $YZ$.

7. Draw a triangle $ABC$ in which $AB = 2$ in., $BC = 3$ in., $\angle B = 60^\circ$. Find a point $P$ on $CB$ produced such that $PC = PA = 1\frac{1}{2}$ in. Measure $PC$. [Find two points from which $P$ is equidistant.]

8. Given two points $A$, $B$ and a line $CD$, construct a circle to pass through $A$ and $B$ and have its centre on $CD$. Is this always possible?

9. Given a circle and two points $A$, $B$ inside it. Construct a circle to pass through $A$, $B$ and have its centre on the circumference of the given circle. Is there more than one solution?

10. In $\triangle ABC$, $AB = AC$. If the perpendicular bisector of $AB$ cuts $BC$, or $BC$ produced, at $X$, prove that $\angle AXB = \angle SAC$.

11. Two circles, centres $A$, $B$, cut at $H$ and $K$. Prove that $AB$ bisects $HK$ at right angles.

12. $ABC$ is a triangle right-angled at $C$; the perpendicular bisector of $AC$ cuts $AB$ at $K$. Prove that $KA = KB = KC$.

Where is the circumcentre of a right-angled triangle?

13. $A$, $B$ are given points; $APBQ$ is a variable rhombus. Find the locus of $P$.

14. $A$, $B$ are given points; $AQ$ is a variable line; $P$ is the image of $B$ in $AQ$, i.e. $AQ$ is the perpendicular bisector of $BP$. Find the locus of $P$.

15. $A$, $B$, $C$, $D$ are four points on the circumference of a circle. Prove that the perpendicular bisectors of $AB$, $AC$, $AD$, $BC$, $BD$, $CD$ are concurrent.

16. $AB$ is the longer of the two parallel sides $AB$, $DC$ of the trapezium $ABCD$. If $P$ is a point on $AB$ such that $DP = DA$ and $CP = CB$, prove that $AB = 2CD$.

17. In fig. 426, $RP$, $RQ$ are the bisectors of the equal angles $APB$, $AQB$. If $RP = RQ$, prove that $A$, $R$, $B$ lie on a straight line. [Join $PQ$.]

18. The angle $BAC$ of $\triangle ABC$ is obtuse, and the perpendicular bisectors of $AB$, $AC$ cut $BC$ at $H$, $K$. Prove that $\angle HAK = 2\angle BAC - 180^\circ$.

- The diagonals of the quadrilateral $ABCD$ cut at $K$. Circles are drawn through $A$, $K$, $B$; $B$, $K$, $C$; $C$, $K$, $D$; $D$, $K$, $A$. Prove that their centres are the vertices of a parallelogram.

20. In fig. 427, $AB$ and $AC$ are given straight lines; $APQR$ is a variable parallelogram of given perimeter. Prove that the locus of $Q$ is part of a straight line. What is the complete locus if $P$, $R$ can also lie on $BA$, $AC$ produced? [Take $K$ on $AB$ so that $PK = PQ$ and prove that $KQ$ is in a fixed direction.]

The following examples depend on the orthocentre property.

21. Where is the orthocentre of a right-angled triangle?

22. If $D$ is the orthocentre of $\triangle ABC$, prove that $A$ is the orthocentre of $\triangle BDC$.

23. $X$, $Y$, $Z$ are the mid-points of the sides of $\triangle ABC$; prove that the orthocentre of $\triangle XYZ$ is the circumcentre of $\triangle ABC$.

24. If $H$ is the orthocentre of $\triangle ABC$, prove that the angles $BHC$, $BAC$ are equal or supplementary. [$\triangle ABC$ may be acute-angled or obtuse-angled.]

25. In $\triangle ABC$, $\angle A = 45^\circ$; $H$ is the orthocentre of $\triangle ABC$ and $CH$ cuts $AB$ at $F$; prove that $BF = FH$.

26. $Q$ is a point inside the parallelogram $ABCD$ such that $\angle QBC = \angle QCD$ are right angles; prove that $AQ$ is perpendicular to $BD$. 
THEOREM 32 (1)

A point which lies on the bisector of a given angle is equidistant from the arms of that angle.

Given an angle $\angle AOB$, a point $Q$ on the bisector $ON$ of $\angle AOB$, and the perpendiculars $QX, QY$ from $Q$ to $OA, OB$.

To prove that $QX = QY$.

Proof. In $\triangle QXO, QYO$,

\[
\begin{align*}
\angle QOX &= \angle QOY \text{ \ given}, \\
\angle QXO &= \angle QYO \text{ \ rt. } \angle s, \text{ \ given}, \\
QO &= QO.
\end{align*}
\]

$\therefore \triangle QXO \cong \triangle QYO \ \text{AAS}.$

$\therefore QX = QY.$

---

THEOREM 32 (2)

A point which is equidistant from two intersecting straight lines lies on one of the lines which bisect the angles between the given lines.

Given two straight lines $AOB, COD$ and a point $P$ such that the perpendiculars $PH, PK$ from $P$ to $AB, CD$ are equal.

To prove that $P$ lies on the bisector of one of the angles between $AOB, COD$.

Construction. Join $OP$.

Proof. Suppose that $P$ lies within the angle $BOD$.

In $\triangle PHO, PKO$,

\[
\begin{align*}
\angle PHO &= \angle PKO \text{ \ rt. } \angle s, \text{ \ given}, \\
PH &= PK \text{ \ given}, \\
PO &= PO.
\end{align*}
\]

$\therefore \triangle PHO \cong \triangle PKO \ \text{RHS}.$

$\therefore \angle PHO = \angle PKO.$

$\therefore P$ lies on the bisector of $\angle BOD$.

In the same way, if $P$ lies within any one of the angles $BOC, COA, AOD$, it lies on the bisector of that angle.

Abbreviation for reference: \(\angle\) bisector locus.

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Fig. 429
Examples for Oral Discussion

1. The bisectors of $\angle B$, $\angle C$ of $\triangle ABC$ meet at $I$; $IP$, $IQ$, $IR$ are the perpendiculars from $I$ to $BC$, $CA$, $AB$. Prove that

   (i) $IP = IQ = IR$.
   (ii) $IA$ bisects $\angle BAC$.

   (i) Explain why $\triangle IPB \equiv \triangle IRB$; then complete the proof of (i).
   (ii) Join $IA$ and explain why $\triangle IRA \equiv \triangle IQA$; then complete the proof of (ii).

This argument provides a proof of the theorem that the bisectors of the three angles of any triangle are concurrent. The proof may be abbreviated, see Theorem 33, by using the locus theorem just established.

Since $IP = IQ = IR$, the circle, centre $I$, radius $IP$, passes through $P$, $Q$, $R$.

Also since the least distance of a point from a straight line is the perpendicular distance, the circle, centre $I$, radius $IP$, does not meet $BC$ at any point except $P$, and similarly does not meet $CA$, $AB$ at any points except $Q$, $R$.

This circle is said to touch $BC$, $CA$, $AB$ and is called the inscribed circle of the triangle $ABC$; its centre $I$ is called the in-centre, and the radius of the circle is called the in-radius.

External Bisector of an Angle. If one arm $AB$ of an angle $ABC$ is produced to $D$, the bisector of the angle $DBC$ is called the external bisector of the angle $ABC$.

2. The external bisectors of $\angle B$, $\angle C$ of $\triangle ABC$ meet at $I_1$.

   (i) $I_1P_1 = I_1Q_1 = I_1R_1$;
   (ii) $I_1A$ bisects $\angle BAC$;
   (iii) the circle, centre $I_1$, radius $I_1P_1$, touches $BC$, $AB$ produced, and $AC$ produced.

   (i) Explain why $\triangle I_1P_1B \equiv \triangle I_1R_1B$; then complete the proof of (i).
   (ii) Join $I_1A$ and explain why $\triangle I_1R_1A \equiv \triangle I_1Q_1A$; complete the proof of (ii).
   (iii) Explain why the circle, centre $I_1$, radius $I_1P_1$, meets $BC$ at no point except $P_1$, and complete the proof.

The circle which touches $BC$, $AB$ produced, $AC$ produced, see fig. 433, is called an escribed circle of $\triangle ABC$, and its centre $I_1$ is called an ex-centre.
There are three escribed circles of any triangle; the circle in fig. 433 is said to be escribed to BC. Similarly, two other circles can be constructed, escribed to AB and escribed to AC respectively.

Note. The proof of example 2 shows that the external bisectors of any two angles of a triangle and the internal bisector of the third angle are concurrent.

Intersection of Loci

If the position of a point is given by two distinct conditions, it may be possible to construct the two corresponding loci and so fix the position, or the several possible positions, of the point by taking the intersection of these lines or curves.

For example, we have just constructed the possible positions of points which are equidistant from the three sides of the triangle ABC:

(i) the locus of points equidistant from BA, BC consists of the internal and external bisector of \( \angle ABC \);
(ii) the locus of points equidistant from CA, CB consists of the internal and external bisector of \( \angle ACB \).

These two loci, two pairs of straight lines, intersect at four points, the In-centre and the three ex-centres of \( \triangle ABC \). Each of these points is equidistant from the lines which form the sides of \( \triangle ABC \).

The use of the intersection of loci was illustrated by the proof of Theorem 30 and is again illustrated by the proof of Theorem 33. The reader should employ the same method to prove that the external bisectors of two angles of a triangle and the internal bisector of the third angle are concurrent.

THEOREM 33

The internal bisectors of the three angles of a triangle are concurrent.

Given a triangle \( \triangle ABC \) and the internal bisectors \( AF, BG, CH \) of \( \angle A, \angle B, \angle C \).

To prove that \( AF, BG, CH \) are concurrent.

Construction. Let \( BG, CH \) meet at \( I \). From \( I \) draw the perpendiculars \( IP, IQ, IR \) to \( BC, CA, AB \).

Proof. Since \( I \) lies on the bisector of \( \angle ABC \),
\[ \text{IP is equidistant from the straight lines BC, BA}, \]
\[ \therefore IP = IR. \]

Since \( I \) lies on the bisector of \( \angle ACB \),
\[ \text{I is equidistant from the straight lines CB, CA}, \]
\[ \therefore IP = IQ, \]
\[ \therefore IQ = IR. \]

\[ \therefore I \text{ is equidistant from the straight lines AC, AB and lies within the angle BAC}, \]
\[ \therefore I \text{ lies on the internal bisector } AF \text{ of } \angle BAC; \]
\[ \therefore AF, BG, CH \text{ meet at I}. \]

Abbreviation for reference: In-centre theorem.
EXERCISE 48

(Give all possible answers of the required constructions)

1. ABC is an equilateral triangle of side 3 in. What is the locus of (i) points distant 2 in. from A, (ii) points distant 1 in. from BC? Construct a point P 2 in. from A and 1 in. from BC.

2. ABC is an equilateral triangle of side 4 cm. Find points on AC and BC produced which are 2 cm. from BC.

3. ABC is an equilateral triangle of side 4 cm. Find points on AC and BC produced which are 2 cm. from BC.

4. ABC is an equilateral triangle of side 4 cm. Construct a point P, 2 cm. from AB and 3 cm. from AC.

5. Draw \( \triangle ABC \) such that \( AB = 4 \text{ cm}, \ BC = 5 \text{ cm}, \ CA = 4.5 \text{ cm} \). Construct a point P which is equidistant from CA, CB and is 3 cm. from B.

6. Draw two straight lines \( AO, BO, CO \) cutting at an angle of 60°. Sketch the locus of a variable point P whose distance is always 2 cm. from one of the given lines.

7. ABCD is a rectangle such that \( AB = 5 \text{ cm}, \ BC = 4 \text{ cm} \). Sketch the locus of points whose distances are always 1 cm. from some side of the rectangle, either inside or outside ABCD. What part of this locus must be omitted if they are never less than 1 cm. from any side of ABCD? Rub it out.

8. ABC is a triangle such that \( AB = 4 \text{ cm}, \ BC = 5 \text{ cm}, \ CA = 6 \text{ cm} \). Find the locus of points whose distances are always 1 cm. from some side of the triangle and never less than 1 cm. from any side, either inside or outside ABC. [Use the "draw and rub out" method indicated in Nos. 6, 7.]

9. A, B are two given points such that \( AB = 5 \text{ cm} \). Find the locus of points which are always 4 cm. from one of the points A, B and never less than 4 cm. from the other.

10. Draw a circle, radius 4 cm., and draw a straight line ABCD cutting the circle at B, C such that BC = 4 cm. Find the locus of points which lie on the line AD and are always more than 1 cm. from the nearest point of the circumference of the given circle.

11. ABC is an equilateral triangle of side 1 in. Sketch the locus of points in the plane of ABC whose distances are always 1 in. from some one of the three points A, B, C, and never less than 1 in. from the other two.

12. ABCD is a square of side 6 cm. Sketch the locus of points which are equidistant from the lines AB, AD, and distant not less than 2 cm. from any side of the square.

13. Draw any triangle ABC. Construct points on BC and AD produced which are equidistant from AB and AC.

14. Draw any triangle ABC. Construct a point P which is equidistant from AB and AC, and also equidistant from B and C. Construct also the circumcircle of \( \triangle ABC \). Does P lie on its circumference?

15. Draw a triangle ABC such that \( BC = 5 \text{ cm}, \ CA = 4 \text{ cm}, \ AB = 6 \text{ cm} \). Construct the four points which are equidistant from the three sides of the triangle. Construct also the in-circle and as much of the three escribed circles as there is room for on your paper.

16. Draw a parallelogram ABCD. Construct the two points which are equidistant from the three sides AB, BC, CD, and construct the two circles which touch these three lines.

17. The diagonals of the parallelogram ABCD cut at K. If K is equidistant from AB and AD, prove that ABCD is a rhombus.

18. Construct a triangle ABC such that \( BC = 5 \text{ cm}, \ the \ median AA' = 4 \text{ cm}, \ and \ the \ perpendicular AD from A to BC is 3 \text{ cm} \). State shortly your method.

19. Draw on squared paper, with 1 inch as unit on each axis, the locus of points whose co-ordinates \((x, y)\) are subject to the following laws:

   (i) \( y = 2x; \)  (ii) \( y = \frac{1}{2}x + 1; \)
   (iii) \( y = -x; \)  (iv) \( 4y + 3x = 18; \)
   (v) \( y = \frac{3}{2}x; \)  (vi) \( 5y = 2x^2 - 7x + 7. \)
Nos. 20-23 should be worked on squared paper.

20. Take a point S 2 in. from the lowest main line XK on the squared paper and on a main central line XS up the paper. A variable point P is such that its distance from S is equal to its distance from the line XK. Construct a number of possible positions of P (on both sides of SX) and then draw a free-hand curve to represent the locus of P. This curve is called a parabola.

21. Take a point S 2 in. from the lowest main line XK on the squared paper and on a main central line XS up the paper. A variable point P is such that its distance from S is three times its distance from XK. Construct a number of possible positions of P (on both sides of SX) and then draw a free-hand curve to represent the locus of P. This curve is called an ellipse. How high up, how low down can the curve go?

22. In the figure for No. 21, mark a point S' on XS produced such that SX = 4:25 in. A variable point P is such that PS = PS' = 3-75 in. Draw the curve which represents the locus of P and compare the result with the locus in No. 21. [If you have a loop of thread 6 in. long and put pins in the paper at S and S' you can trace this locus very quickly.]

23. Take a point S 1 in. below the highest main line on the squared paper and on a main central line SX. Call the line, which is 2½ in. below the highest main line, XK. A variable point P is such that its distance from XK is equal to its distance from XK. Construct a number of possible positions of P (on both sides of SX) and then draw free-hand the locus of P. There are two distinct branches of the curve, one on each side of XK. The curve is called a hyperbola.

24. Take two points A, B, 6 cm. apart. A variable point P is such that PA = PB. Draw the curve which represents the locus of P. On AB produced take a point C such that BC = 2 cm. and add to your figure the circle, centre C, radius 4 cm.

For Nos. 25-27, tracing paper should be used.

25. Draw two lines OA, OB cutting at right angles. P, Q are variable points on OA, OB such that PQ = 8 cm.; R, S are points on PQ such that R is the mid-point of PQ, and PS = 3 cm. Draw a line PSRQ on the tracing paper so that PS = 3 cm., PR = 4 cm., PQ = 8 cm. and, by pricking through, obtain a number of possible positions of R, S. Then sketch the locus (i) of R, (ii) of S.

26. Mark on your paper two points A, B, 5 cm. apart. Draw on tracing paper two lines cutting at an angle of 50°. By pricking through, find the locus of a point P such that AB = 4 cm. A variable line through A cuts the circle again at P and is produced to Q so that PQ = 4 cm.; also PA is produced to Q' so that PQ' = 4 cm. Draw on tracing paper a straight line LO'PQ' such that Q'P = PQ = 4 cm. and, by pricking through, obtain a number of possible positions of Q, Q', and sketch the locus. The curve is called a cardioid.

27. Draw two perpendicular lines AB, AC. P is a variable point which is 2 cm. nearer to AB than to AC. Find the locus of P. [Draw a line DE parallel to AB such that P is equidistant from AC and DE.]

28. ABC is an equilateral triangle of side 6 cm. Construct a point P on the line AB which is 2 cm. nearer to BC than to AC. Is there more than one possible position?

29. ABC is a triangle in which AB > AC. The perpendicular bisector of BC meets the bisector of ∠BAC at P; PX, PY are the perpendiculars from P to AB, AC produced. Prove that SX = CY. [Join PB, PC.]

30. ABC is a triangle in which AB > AC. The perpendicular bisector of BC meets the bisector of ∠BAC at P; PX, PY are the perpendiculars from P to AB, AC produced. Prove that SX = CY. [Join PB, PC.]

31. ABCD is a given quadrilateral. Construct a point P which is equidistant from the lines AB, BC, and also equidistant from the lines AB, CD. What is the greatest number of possible positions of P? How many positions of P are there if ABCD is a parallelogram? Prove that there are not more than two possible positions of P if ∠A + ∠C is equal to two right angles.

32. A variable line cuts two given lines AB, CD at P, Q. If the bisectors of ∠BPQ, ∠DQP meet at R, find the locus of R.

33. AB and CD are two given lines which when produced meet off the paper. Construct that portion of the line bisecting the angle between AB and CD which lies on the paper. [Compare No. 32.]

34. In fig. 433, p. 211, if CB is produced to H and if BC is produced to K so that HS = HA and KC = CA, prove that HA = HI = HK.

35. If I is the incentre and if I1, I2, I3 are the ex-centres of a triangle ABC, prove that the line joining any two of the points I, I1, I2, I3 is perpendicular to the line joining the other two.
REVISION PAPERS 1–8

[Angle properties of parallels, triangles, polygons.]

1. What angle is equal to (i) \( \frac{1}{2} \) of its supplement? (ii) \( \frac{1}{3} \) of its complement?

2. ABCD is a straight line such that \( AB = \frac{1}{2} AC = \frac{1}{2} AD \). If \( CD = 2 \) in., calculate the distance between the mid-points of \( AB \) and \( AD \).

3. In fig. 435, arrows indicate lines are given parallel. Calculate the angles \( a \) and \( b \). Give reasons.

4. In \( \triangle ABC \), \( \angle A = 74^\circ \), \( \angle B = 52^\circ \); \( AB \) is produced to \( D \). \( BC \) is produced to \( E \). Calculate the acute angle between the bisectors of \( \angle BDC \) and \( \angle ECA \).

2. (i) Find the obtuse angle between the directions N. 37° E., S. 54° W.

(ii) Find the reflex angle between the directions N.W., W.

2. A line \( AB \), 3 in. long, is produced to points \( P \) and \( Q \) such that \( AP = 4BP \) and \( AQ = 3BQ \). Find the length of \( PQ \).

3. Calculate the angle \( c \) in fig. 436.

4. \( D \) is a point on the side \( BC \) of \( \triangle ABC \) such that \( \angle CAD = \angle ABC \). Prove that \( \angle ADC = \angle BAC \).

1. \( \triangle ABC \) and \( \triangle CDE \) are two straight lines.

(i) If \( \angle ACE = \frac{4}{5} \angle BCE \), calculate \( \angle BCD \).

(ii) If \( \angle ACE \) exceeds \( \angle BCE \) by \( 90^\circ \), calculate \( \angle BCD \).

2. \( \triangle ABC \) is a straight line; \( D \) is the mid-point of \( BC \). Prove that \( AD + AC = 2AD \).

3. \( AD \) is the perpendicular from \( A \) to the side \( BC \) of \( \triangle ABC \). Given that \( AD = 4 \) cm., \( \angle B = 55^\circ \), \( \angle C = 65^\circ \), draw \( \triangle ABC \) and measure \( BC \).

4. \( X \) is a point inside \( \triangle ABC \) such that \( \angle XAB = \angle XCA \). Prove that \( \angle AXC + \angle BAC \) equals 2 right angles.

4. \( A \) is due east of \( B \); \( P \) is N. 17° W. of \( A \) and N. 29° E. of \( B \). Calculate \( \angle APB \).

The bisector of \( \angle A \) in \( \triangle ABC \) meets \( BC \) at \( P \). Given that \( AP = 5 \) cm., \( \angle B = 28^\circ \), \( \angle C = 68^\circ \), draw \( \triangle ABC \) and measure \( BC \).

3. Fig. 437 represents a "Pentagram," that is, the inner figure is a regular pentagon. Calculate the angle \( a \).

4. In fig. 438, find \( e \) in terms of \( b, c, d \).

5. In fig. 439, not drawn accurately,

(i) find three points which are collinear;

(ii) find the angle \( b \) if \( R, A, N \) are collinear;

(iii) find what points are collinear if \( b = 2x^2 + 5^\circ \).
2. **ABCD** is a quadrilateral with its opposite sides parallel, and **AC** bisects \( \angle BAD \). If \( \angle BAD = 2y^\circ \) and \( \angle ABC = 3y^\circ \), calculate \( \angle ACB \).

3. The sum of the angles of a polygon is 12 right angles. Find the number of sides of the polygon.

4. In fig. 440, \( \angle ACD = \angle ABC \), and \( CP \) bisects \( \angle BCD \). Prove that \( \angle APC = \angle ACP \).

6

1. It requires 4 complete turns of the handle to wind up a bucket from the bottom of a well 24 ft. deep. Through what angle must the handle be turned to raise the bucket 10 ft.?

2. In fig. 441, **AB** is parallel to **EF**. Calculate the value of \( x \). Give reasons.

3. In \( \triangle ABC \), **AD** is the perpendicular from **A** to **BC**, and the bisector of \( \angle BAC \) meets **BC** at **P**. If \( \angle B = 2x^\circ \) and \( \angle C = 2y^\circ \), find in terms of \( x \) and \( y \):
   (i) \( \angle BAP \),
   (ii) \( \angle PAD \).

4. The sides \( AD \), \( BC \) of the quadrilateral **ABCD** are produced to \( P, Q \) respectively. If \( \angle DCQ = \angle A \), prove that \( \angle DPC = \angle B \).

7

1. In fig. 442, **ACB** is a straight line. If \( \angle ACQ = 2 \angle QCT \) and if \( \angle BCR = 2 \angle RCT \), find the size of \( \angle QCR \).

2. In \( \triangle ABC \), \( \angle A \) is greater than \( \angle B \). If **AN** is the perpendicular from **A** to the line bisecting \( \angle ACB \), prove that \( \angle BAN = \frac{1}{2}(\angle BAC - \angle ABC) \).

3. The angles of a pentagon taken in order are \( x + 10, 2x - 10, 2x, 2x + 10, x + 50 \), degrees. Find the value of \( x \) and prove that two pairs of sides of the pentagon are parallel.

4. In fig. 443, **PA**, **PB**, **RC**, **RD** are the bisectors of the angles of the quadrilateral **ABCD**. Prove that \( \angle QPS \) and \( \angle QRS \) are supplementary.

8*

1. If the direction N. 22° W. is the same as the direction S. 22° W., find the value of \( x \) and the true bearing of the direction (i.e., the angle which the direction makes with due north, measured clockwise).

2. In \( \triangle ABC \), the bisector of \( \angle BAC \) meets **BC** at **D** and the bisector of \( \angle ABC \) meets **AC** at **E**. If \( \angle ADC = 79^\circ \), \( \angle BEC = 83^\circ \), find \( \angle ABC \) and \( \angle ACB \).

3. Each angle of a regular polygon of \( n \) sides is \( \frac{1}{n} \) of each angle of a regular polygon of \( y \) sides. Express \( y \) in terms of \( x \) and find any values of \( x \) and \( y \) which will fit.

4. In fig. 444, the angles **DAB**, **DCB** are supplementary and \( \angle PAB = \angle ADB \). Prove that \( \angle APB = \angle BDC \).

9

[Angle properties of triangles and polygons, congruence, isosceles triangles.]

1. In \( \triangle ABC \), \( \angle A = 44^\circ \), \( \angle B = 112^\circ \). Calculate the acute angle between the bisectors of \( \angle B \) and \( \angle C \).

2. In \( \triangle ABC \), **AB** = **AC** and \( \angle A = 20^\circ \). **D** is a point on **AC** such that \( \angle BDC = 60^\circ \). Prove that \( \angle AD = DB \).

3. In fig. 445, prove that \( \angle ABC = \angle ADC \).

4. **AB**, **DC** are the parallel sides of the trapezium **ABCD**. If \( AD = DC \), prove that **AC** bisects \( \angle BAD \).
10

1. If the reflex angle $AOB$ is four times the acute angle $AOB$, find the acute angle $AOB$.

2. In fig. 446, find $d$ in terms of $a$, $b$, $c$.

3. In $\triangle ABC$, $AB = AC$; $D$ is a point on $AC$ such that $AD = BD = BC$. Calculate $\angle BAC$.

4. $ABCD$ is a straight line such that $AB = CD$; $K$ is a point outside the line such that $KB = KC$. Prove that $KA = KD$.

11

1. In $\triangle ABC$, $\angle B = 35^\circ$, $\angle C = 75^\circ$; the perpendiculars from $B$, $C$ to $AC$, $AB$ respectively cut at $Q$. Find $\angle BQC$.

2. $ABC$ is an equilateral triangle; $BC$ is produced to $D$ so that $BC = CD$. Prove that $\angle BAD$ is a right angle.

3. The bisector of $\angle A$ of $\triangle ABC$ cuts $BC$ at $D$. Through $C$ a line is drawn parallel to $DA$ to meet $BA$ produced at $P$. Prove that $AP = AC$.

4. If two circles have the same centre $O$, and if a straight line $XY$ is drawn cutting the inner circle at $A$, $B$ and the outer circle at $X$, $Y$, prove that $BX = AY$.

12

1. The sum of one pair of angles of a triangle is $100^\circ$, and the difference of another pair is $60^\circ$. Prove that the triangle is isosceles.

2. If all the marked angles in fig. 447 are equal, find the size of each. Does the figure contain any pairs of parallel lines? Give reasons.

3. In $\triangle ABC$, $AB = AC$; $BC$ is produced to $D$ so that $CD = AB$. Prove that $\angle ABD = 2\angle ADB$.

4. The diagonals of the quadrilateral $ABCD$ cut at $K$. If $AB = BC = CD$ and if $\angle ABC = \angle BCD$, prove that (i) $\angle KBC = \angle KCB$, (ii) $\angle AKD = \angle ABC$.

13

1. $O$ is a point outside a straight line $ABCD$ such that $OA = AB$, $OB = BC$, $OC = CD$; $\angle BOC = x^\circ$. Find $\angle ODA$ in terms of $x$.

2. In fig. 448, find $x$ in terms of $a$, $b$, $c$.

3. In $\triangle ABC$, $AB = AC$; $AB$ is produced to $D$ so that $BD = BC$. Prove that $\angle ACD = 3\angle ADC$.

4. $ACB$ is a straight line; $ABX$, $ACY$ are equilateral triangles on opposite sides of $AB$. Prove that $CX = BY$.

14

1. In $\triangle ABC$, $AB = AC$; $BC$ is produced to $D$ so that $BD = BA$. If $\angle BAC = 2\angle CAD$, find $\angle ABC$.

2. Find the sum of the interior angles of a 15-sided polygon.

3. In fig. 449, $AB = AC$ and $KAD$ is a straight line. Prove that $r - s = 2p$.

4. In $\triangle ABC$, $AB = AC$ and $\angle A$ is a right angle. $D$ is any point on $BC$; $BH$, $CK$ are the perpendiculars from $B$, $C$ to $AD$, produced if necessary. Prove that (i) $BH = AK$; (ii) the difference between $BH$ and $CK$ is equal to $HK$.

15

1. In $\triangle ABC$, $\angle A = x^\circ$, $\angle B = 3x^\circ$, $\angle C = 5x^\circ$. Find the value of $x$, and prove that a line $BP$ can be drawn cutting $AC$ at $P$ such that $\triangle APB$ and $\triangle PCB$ are both isosceles.

2. The bisectors of $\angle B$, $\angle C$ of $\triangle ABC$ meet at $I$. If $\angle BIC = 135^\circ$, prove that $\angle A$ is a right angle.

3. In fig. 450, arrows indicate that lines are given parallel. Prove that $a - b = c$.

4. $O$ is a point inside an equilateral triangle $ABC$; $OAP$ is an equilateral triangle such that $O$ and $P$ are on opposite sides of $AB$. Prove that $BP = OC$. 

Fig. 446

Fig. 447

Fig. 448

Fig. 449

Fig. 450
16*

1. In \( \triangle ABC \), \( \angle A = 115^\circ \), \( \angle C = 20^\circ \); \( AD \) is the perpendicular from \( A \) to \( BC \). Prove that \( AD = DB \).

2. ABCDE is a regular pentagon; \( AC \) cuts \( BE \) at \( K \). Find \( \angle EKC \).

![Figure 451]

3. In fig. 451, \( \angle ABD = \angle C \), and \( APQ \) is the bisector of \( \angle A \). Prove that \( BP = BQ \).

4. In \( \triangle ABC \), \( \angle B = 90^\circ \), \( \angle C = 30^\circ \). Prove that \( AC = 2AB \).

REVISION PAPERS 17-24 (Theorems 1-20)

[Angles of polygon, congruence, isosceles triangles, parallelograms.]

17

1. In \( \triangle ABC \), \( \angle A = 2 \angle B \) and \( \angle C - \angle B = 36^\circ \). Prove that \( \triangle ABC \) is isosceles.

2. ABCDE is a pentagon such that \( AB \) and \( DC \) when produced cut at right angles. If \( \angle A = \angle D = 2 \angle E \), find \( \angle E \).

3. In \( \triangle ABC \), \( AB = AC \), and \( BC \) is less than \( AB \). \( D \) is a point on \( BC \) produced such that \( DB = BA \). Prove that \( \angle ACD = 2 \angle ADB \).

4. The diagonals of the parallelogram \( ABCD \) cut at \( K \). Any line through \( K \) cuts \( AD, BC \) at \( P, Q \). Prove that \( DPBQ \) is a parallelogram.

REVIEW PAPERS 18

1. The angles of a quadrilateral, taken in order, are \( x, x + 20, x + 30, x + 40 \) degrees. Find the value of \( x \) and prove that the quadrilateral is a trapezium.

2. Draw the parallelogram \( ABCD \) in which \( AC = 3 \) in., \( BD = 4 \) in., \( BC = 2 \) in. Measure \( AB \).

3. In \( \triangle ABC \), \( AB = AC \). \( D \) is a point either on \( AC \) or on \( AB \) produced such that \( DB = BC \). Prove that \( \angle BDC = \angle BAC \). [Draw two figures, one in which \( \angle A \) is less than \( 60^\circ \), the other in which \( \angle A \) is greater than \( 60^\circ \).]

4. In fig. 452, \( BAC \) and \( PDQ \) are parallel lines. If \( AP, AQ \) are the bisectors of \( \angle BAC, \angle CAD \), prove that \( PD = DQ \).

19

1. Two equilateral triangles \( ABC, AYZ \) lie outside each other. If \( \angle CAY = 35^\circ \), find the acute angle at which \( BC \) and \( ZY \) intersect when produced.

2. In \( \triangle ABC \), \( \angle C = 3 \angle B \). From \( A \) a part \( AD \) is cut off equal to \( AC \). Prove that \( CD = DB \).

3. ABCD is a square; \( AXB \) is an equilateral triangle outside the square. Prove that \( \angle ACX = \angle ABX \).

4. ABCD is a parallelogram; \( BP, DQ \) are two parallel lines cutting \( AC \) at \( P, Q \). Prove that \( BP \) is parallel to \( PD \).

20

1. State (without proof) the name of a quadrilateral in which

(i) the diagonals bisect each other;
(ii) the diagonals bisect each other at right angles;
(iii) the diagonals are equal and bisect each other;
(iv) all four sides are equal;
(v) all four angles are equal.

2. \( AB, BC, CD, DE \) are four consecutive sides of a regular 20-sided polygon. Find the acute angle at which \( AB \) and \( ED \) intersect when produced.
3. **ABCD** is a parallelogram. If the bisector of \( \angle BCD \) cuts \( AD \) at its mid-point \( K \), prove that \( KB \) bisects \( \angle ABC \).

4. \( P \) is any point on the side \( AB \) of the square \( ABCD \). The line from \( A \) perpendicular to \( DP \) cuts \( BC \) at \( Q \). Prove that \( DP = AQ \).

**21**

1. **ABCDEFGH** is a regular octagon. Find the acute angle at which \( AD \) cuts \( BF \).

2. Draw a trapezium \( ABCD \) in which \( AB \) and \( DC \) are the parallel sides, given that \( AB = 7.5 \text{ cm}, \ BC = 3 \text{ cm}, \ CD = 3.5 \text{ cm}, \ DA = 3.8 \text{ cm} \). Measure \( \angle BAD \).

3. In \( \triangle ABC, AB = AC \). \( P \) is any point on \( BC \), and \( H, K \) are points on \( AB, AC \) such that \( AHKP \) is a parallelogram. Prove that \( PH + PK = AB \).

4. **ABCD** is a square. The bisector of \( \angle BCA \) cuts \( AB \) at \( P \); \( PQ \) is the perpendicular from \( P \) to \( AC \). Prove that \( PB, PQ, AQ \) are all equal.

**22**

1. In \( \triangle ABC, \angle A = 55^\circ, \angle C = 35^\circ \); \( K \) is a point on \( AC \) such that \( KB = KA \). Prove that \( KB = \frac{1}{2} AC \).

2. The diagonals of the rectangle \( ABCD \) cut at \( X \). If \( AC = 5 \text{ cm}, \angle AXB = 110^\circ \), draw the rectangle and measure \( AB \).

3. **ABCD** is a quadrilateral. If the bisectors of \( \angle ABC, \angle ADC \) are parallel, prove that \( \angle A = \angle C \).

**Fig. 453**

4. In fig. 453, \( A, B \) are the centres of two equal circles. If \( \angle PAB = \angle QBA \), prove that (i) \( \angle APQ = \angle BQF \), (ii) \( QM = AB \).

**Fig. 454**

1. In \( \triangle ABC, \angle A = 2 \angle B; \ P \) is a point on \( BC \), and \( Q \) is a point on \( AC \), such that \( AB = AP = PQ = QC \). Find \( \angle C \).

2. In fig. 454, \( AB = AC, BP = BQ \) and \( PA = PR \). Prove that \( \angle B = 3 \angle A \).

3. The bisectors of \( \angle B, \angle D \) of the quadrilateral \( ABCD \) meet at a point \( K \) inside \( ABCD; BK \) is produced to \( N \). Prove that the difference of \( \angle A, \angle C \) is equal to \( 2 \angle NKD \).

Can you prove a corresponding result in the case when \( K \) lies outside \( ABCD \)?

4. **ABCD** is a square whose diagonals cut at \( K \); \( P \) is a point on \( DK \) between \( D \) and \( K \). The perpendicular from \( B \) to \( AP \) cuts \( AK \) at \( Q \). Prove that \( \triangle ABP \equiv \triangle BCQ \).

**Fig. 455**

1. Construct a triangle \( ABC \), given that \( \angle A = 44^\circ, \angle B = 56^\circ \), and that the perimeter of \( \triangle ABC \) is 4 in. Measure \( AB \).

2. \( D \) is any point on the bisector of \( \angle BAC \); \( DP, DQ \) are drawn parallel to \( AB, AC \) to meet \( AC, AB \) respectively at \( P, Q \). Prove that \( DP = DQ \).

3. In fig. 455, prove that the sum of the angles \( a, b, c, d \) is 6 right angles.

4. **ABCD** is a parallelogram; \( ABHK \) is a square on the same side of \( AB \) as \( CD \); \( BCPQ \) is a square on the same side of \( BC \) as \( AB \). Prove that (i) \( \angle QBH = \angle BAC \); (ii) \( QH = BD \).
25

1. In \( \triangle ABC \), \( AB = AC \), and \( AB \) is produced to \( D \). Prove that \( \angle ACD - \angle ADC = 2 \angle BCD \).

2. Given a triangle \( ABC \) in which \( \angle A > \angle B \), construct in the simplest way a point \( P \) on \( BC \) such that \( AP + PC = BC \). State shortly your method.

3. In \( \triangle PQR \), \( PQ = PR \); \( S \) is a point on \( QR \) produced; \( \angle Q = 73^\circ \) and \( \angle RPS = 38^\circ \). Which is the greater, (i) \( RS \) or \( RP \), (ii) \( SQ \) or \( SP \)? Give reasons.

4. \( X, Y \) are the mid-points of the sides \( AB, AC \) of \( \triangle ABC \). \( BY \) is joined and produced to \( Q \) so that \( BY = YQ \); \( CX \) is joined and produced to \( P \) so that \( CX = XP \). Prove that (i) \( AP = AQ \), (ii) \( PAQ \) is a straight line.

26

1. \( PQRS \) is a parallelogram in which \( PQ \) is less than \( QR \). The bisector \( PX \) of \( \angle P \) cuts \( QR \) at \( X \); the bisector \( QY \) of \( \angle Q \) cuts \( PS \) at \( Y \). Prove that \( PQX \) is a rhombus.

2. Draw a quadrilateral \( ABCD \). Construct a point \( P \) equidistant from the lines \( AB, AD \) and such that \( PA = PC \). Is there more than one position of \( P \)? State shortly your method.

3. \( ABCD \) is a parallelogram. If \( \angle BAC > \angle DAC \), prove that \( BC > CD \).

4. \( P \) is a point on the side \( AB \) of \( \triangle ABC \) such that \( AP = \frac{1}{2} AB \); \( PQ \) is drawn parallel to \( BC \) to meet \( AC \) at \( Q \). By drawing sets of parallel lines, prove that \( PQ = \frac{1}{2} BC \).

27

1. In \( \triangle ABC \), \( AB = AC \); \( BA \) is produced to \( E \). If the bisector of \( \angle ABC \) meets \( AB \) at \( D \), prove that \( \angle CDE = \frac{1}{2} \angle CAE \).

2. In \( \triangle ABC \), \( D \) is the mid-point of \( BC \). Draw \( \triangle ABC \), given that \( AB = 0 \) cm, \( AC = 6 \) cm, \( AD = 4 \) cm. Measure \( BC \). [In your sketch, produce \( AD \) to \( K \) so that \( AD = BK \); join \( CK \).]

28

3. In fig. 456, \( AB = AC = CP \) and \( BQ = BP \). Which is the greater, \( QA \) or \( QP \)? Give reasons.

4. \( AB, DC \) are the parallel sides of the trapezium \( ABCD \); \( P, Q \) are the mid-points of \( AD, BC \). Prove that (i) \( PQ \) is parallel to \( AB \), (ii) \( PQ = \frac{1}{2} (AB + DC) \). [Join \( P \) and \( Q \) to the mid-point \( X \) of \( BD \).]

3. In fig. 457, prove that the sum of the angles \( a, b, c, d, e \) is two right angles. [Join \( CE \).]

4. \( ABCD \) is a quadrilateral in which \( AB = AD \) and \( \angle ABC \) is a right angle. Prove that the line through \( A \) parallel to \( BC \) bisects \( CD \).
29

1. ABCD is a square; AB and BC are produced to P and Q respectively so that BP = CQ. Prove that PD and AQ are equal and cut at right angles.

2. In fig. 468, PQN is the perpendicular bisector of AB. If BQ bisects \( \angle AQP \), find \( \angle QAB \) and prove that \( QN = \frac{1}{2} QA \).

3. AD, BC are the parallel sides of the trapezium ABCD; AP is the perpendicular from A to the bisector of \( \angle ABC \). Prove that AP bisects \( \angle BAD \). If PN is the perpendicular from P to BC, what can you say about the circle, centre P, radius PN?

4. In the convex quadrilateral ABCD, the sides AB and CD are equal but not parallel. P, Q, R, S are the mid-points of AD, AC, BD, BC, respectively. Prove that PQRS is a rhombus.

30

1. Draw a triangle ABC in which AB = 5.5 cm, BC = 4.5 cm, CA = 4 cm. Construct a point P equidistant from the lines AB, AC and at a distance of 2 cm from BC. Is there more than one position of P? State shortly your method.

2. In \( \triangle ABC \), AB = AC; the bisector of \( \angle ABC \) meets AC at D; P is a point on AC produced such that \( \angle ABP = \angle ADB \). Prove that (i) \( \angle ABD = \angle APB \), (ii) CP = CP.

3. The side BC of the equilateral triangle ABC is produced to D so that BC = CD. P is any point on BA produced. Prove that (i) \( \angle BAD \) is a right angle, (ii) PD + DC > PB.

4. Given two points A, B on opposite sides of a given line CD, construct a point P on CD such that \( \angle APC = \angle BPC \). State your method and prove that it is correct. (See p. 176, No. 25.)
33*

1. A, B, C are three points on a minor arc of a circle, centre O (i.e., an arc less than a semicircle). Prove that \( \angle ABC \) is equal to the sum or difference of \( \angle OAB \) and \( \angle OCB \). [Two cases: (i) for are \( ABC \), (ii) for are \( ACB \).]

2. In fig. 461, P, Q, R are the mid-points of the sides \( AB \), \( BC \), \( CD \) of the quadrilateral \( ABCD \) and also the mid-points of the sides \( A'B', B'C', C'D' \) of the quadrilateral \( A'B'C'D' \). Prove that \( S \) is the mid-point of \( AD \) and of \( A'D' \).

3. The sides \( AQ \), \( AR \) of the rectangle \( AQPR \) lie along fixed lines \( AB \), \( AC \). \( P \) is a variable point such that \( PQ = PR = 1 \) in. Find precisely the locus of \( P \). [Find two fixed lines from which \( P \) is equidistant.] How is the locus affected if \( Q \), \( R \) may lie on \( CA \) produced, \( BA \) produced?

4. \( P \) is the mid-point of the side \( AB \) of the parallelogram \( ABCD \); \( BD \) cuts \( CP \) at \( Q \) and cuts \( CA \) at \( K \). Prove that \( BQ = 2QK \).

34*

1. In fig. 462, \( ABCDEFGH \) is a regular octagon cut out of the square \( PQRS \). Prove that \( QD = QO \). [You may assume that \( O \) is the centre of the octagon.]

2. In \( \triangle ABC \), \( AB = AC \) and \( \angle B = 70^\circ \). Given that the perimeter of \( \triangle ABC \) is 10 cm., draw the triangle and measure \( AB \). [In your sketch draw the straight line \( PBCQ \) such that \( PB = BA = CA = CQ \); join \( AP, AQ \).]

3. In \( \triangle ABC \), \( D \) is the mid-point of \( BC \), and \( E \) is a point on \( AC \) such that \( AE = \frac{1}{2} AC \); \( AD \) cuts \( BE \) at \( K \). Prove that \( KE = \frac{2}{3} BE \). [Through \( E \) draw \( EP \) parallel to \( AD \) to cut \( BC \) at \( P \).]

4. In \( \triangle ABC \), \( \angle A \) is a right angle; \( O \) is the centre of the square \( BPQC \) outside \( \triangle ABC \). Prove that \( AO \) bisects \( \angle BAC \). [Prove that \( O \) is equidistant from the lines \( AB, AC \).]

**PART II**

(Section 1)

**AREAS**

Measurement of Area. Just as the length of a straight line is measured by comparing it with some standard length, such as 1 in., 1 yd., 1 cm., etc., so the area of a plane figure—that is, the amount of plane surface enclosed by the figure—is measured by comparing it with the area of some standard figure. Any given polygon could be taken as the standard, but the most convenient figure to select as the standard is a square.

If each side of a square is 1 in., we say that the area of the square is 1 square inch, written 1 sq. in.; if each side of the square is 1 cm., the area of the square is 1 sq. cm., and so on.

Definition. A unit of area is the area of a square whose side is of unit length.

If one of the small squares of a squared blackboard is taken as the unit of area (its side need not be an exact number of inches or centimetres), and if various figures are drawn on this blackboard, their areas may be expressed in terms of the unit of area by counting up the number of squares they contain. A fair approximation is obtained by counting more than half a square as a whole, and ignoring less than half a square. In this way the areas of the various figures can be compared with one another.

If the lengths of two adjacent sides of a rectangle are 7 units and 4 units, the rectangle can be divided by lines parallel to the sides into \( 7 \times 4 \) compartments, each of which is 4 units a square of unit area. Since the rectangle contains 28 squares of unit area, we say that its area is 28 units of area.
Consider a rectangle $2\frac{1}{4}$ in. long, $1\frac{1}{2}$ in. wide. This rectangle cannot be divided up into a whole number of squares of side 1 in. But if we choose $\frac{1}{2}$ inch as the unit of length, length of rectangle = 15 units, width of rectangle = 8 units.

\[ \therefore \text{area of rectangle} = 15 \times 8 \text{ units of area}. \]

But in this case a square of side 1 in. is 6 units long, 6 units wide, and therefore contains $6 \times 6$ units of area,

\[ i.e. \text{ 1 sq. in.} = 6 \times 6 \text{ units of area}. \]

\[ \therefore \text{area of rectangle} = \frac{15 \times 8}{6 \times 6} \text{ sq. in.} = \left( \frac{15}{6} \times \frac{8}{6} \right) \text{ sq. in.} = (2\frac{1}{4}) \text{ sq. in.} \]

Hence we see that, whether the lengths of the sides of a rectangle are measured by whole numbers or fractions, the area of the rectangle is measured by the product of the measures of its sides.

**Examples for Oral Discussion**

1. Find a unit of length for which the sides of the following rectangles are measured by whole numbers. Express the lengths in terms of this unit; state the unit of area and find the area of each rectangle in sq. in.
   (i) $2\frac{1}{4}$ in. long, $1\frac{1}{2}$ in. wide; (ii) 4-2 in. long, 1-5 in. wide; (iii) $1\frac{1}{2}$ in. long, $2\frac{1}{4}$ in. wide; (iv) $1\frac{3}{4}$ in. long, $1\frac{1}{4}$ in. wide.

2. The scale of the plan of an estate is 4 in. to the mile. What area is represented by
   (i) a square of side 1 in. on the plan?
   (ii) a square of side 0-8 in. on the plan?
   (iii) a rectangle whose area is 3-2 sq. in.?

What area on the map represents $\frac{1}{2}$ sq. mile?

---

**Theorem 34**

The area of a rectangle is measured by the product of the measures of two adjacent sides.

![Diagram](Fig. 484)

Given a rectangle $ABCD$ in which $AB$ is $x$ units long, $AD$ is $y$ units long, and the unit is chosen so that $x$ and $y$ are integers.

To prove that the area of $ABCD$ is $xy$ units of area.

**Construction.** Divide $AB$ into $x$ equal parts and through each point of division draw a line parallel to $AD$ to meet $DC$. Divide $AD$ into $y$ equal parts and through each point of division draw a line parallel to $AB$ to meet $BC$.

**Proof.** The lines parallel to $AD$ divide the rectangle $ABCD$ into $x$ equal rectangles, the lengths of whose adjacent sides are 1 unit and $y$ units.

The lines parallel to $AB$ divide each of these rectangles into $y$ equal squares, the length of each side being 1 unit.

\[ \therefore ABCD \text{ contains } xy \text{ squares, each of unit area}; \]

\[ \therefore \text{ the area of } ABCD \text{ is } xy \text{ units of area}. \]

**Note.** This proof fails if the lengths of $AB$ and $AD$ cannot be measured by integers or fractions. In this case they are called incommensurable.
NEW GEOMETRY

Notation. The rectangle $ABCD$ is often called rect. $AC$ or rect. $BD$, if this does not cause any ambiguity.

The rectangle $ABCD$ is said to be contained by $AB$, $AD$ or by $AB$, $BC$, i.e. by any pair of adjacent sides.

For brevity, we say that

area of rect. $ABCD = AB \times AD$ or $AB \cdot AD$,

but it must be clearly understood that this is merely an abbreviation for the statement that the number of units of area of the rectangle $ABCD$ is the product of the numbers of units of length of $AB$ and $AD$.

If $ABCD$ is a square, it is often spoken of as the square on $AB$ or more shortly, sq. on $AB$.

For brevity, we say that

area of sq. on $AB = AB^2$,

but this is merely an abbreviation for the statement that the number of units of area of the square $ABCD$ is the square of the number of units of length of $AB$.

If two figures are equal in area, they are called equivalent.

The symbol for "is equivalent to" is \( \equiv \). The statement \( \Delta ABC \equiv \Delta XYZ \) means that the triangles $ABC$, $XYZ$ are equal in area; it does not mean that they are congruent.

Examples for Oral Discussion

1. Find the area of the right-angled triangle $ABC$, given that $AB = 7$ in., $BC = 3$ in., $\angle ABC = 90^\circ$.

Complete the parallelogram $ABCD$.

(i) Explain why $ABCD$ is a rectangle.

(ii) What is the area of rect. $DABC$?

(iii) What is the area of $\Delta ABC$?

Give reasons.

2. What expression (using capital letters) represents the area of $\triangle PQR$ if $\angle QPR$ is a right angle?

AREAS OF RECTANGLES

EXERCISE 49

[The unit of length in each diagram is 1 in.]

1. Draw on squared paper a rectangle, 2 3 in. long, 1 4 in. wide. Calculate its area and verify by counting squares.

2. Sketch a figure to illustrate that 1 sq. yd. = 9 sq. ft.

Find the areas of figs. 466–468 in which all the corners are right-angled:

3. [Fig. 466]

4. [Fig. 467]

5. [Fig. 468]

6. A rectangle is equal in area to a square, side 4 in. Find its breadth if its length is 10 in.

7. Find the length of the side of a square which is equal in area to a rectangle 4 ft. long, 3 in. wide.

8. Copy fig. 469 and insert in each rectangular compartment its area. Complete \((a+b)^2 = \ldots\)

9. Copy fig. 470 and insert in each rectangular compartment its area. Complete \((a+b)(x+y+z) = \ldots\)

10. Draw a figure like fig. 469 to illustrate that \((2x)^2 = 4x^2\). Complete the statement: If a straight line $AB$ is bisected at $C$, then $AC^2 + \ldots AB^2$.

11. Draw a figure to illustrate that \((3y)^2 = 9y^2\). State the corresponding geometrical theorem.
12. A map is drawn on a scale of 4 miles to the inch.
   (i) What area is represented by 1 sq. in. on the map?
   (ii) What area on the map represents 8 sq. miles?

13. A map is drawn on a scale of 1 mile to the inch.
   (i) What area is represented by 10 sq. in. on the map?
   (ii) What area on the map represents 0.8 sq. mile?

14. On a map in which 6 in. represents a mile, a field is a square measuring 4 in. each way. Find the area of the field in acres, correct to nearest acre.

   Make a sketch of fig. 460 and find the areas of the following triangles obtained by joining the necessary points:
   15. (i) \( \triangle ABC \); (ii) \( \triangle AEF \).
   16. (i) \( \triangle BCD \); (ii) \( \triangle DEF \).

17. In fig. 471, \( BCQP \) is a rectangle. Find the area of \( \triangle ABC \).

18. In fig. 472, find the area of the quadrilateral \( ABCD \).

19. In fig. 473, \( EFN \) is a straight line. Find the area of \( \triangle DEF \).

20. In fig. 474, find the area of
   (i) \( \triangle BCQ \);
   (ii) quad. \( APBC \);
   (iii) \( \triangle ABC \).

21. B is 100 yd. east and 50 yd. north of A; C is 40 yd. east and 90 yd. north of A. Find the area of \( \triangle ABC \). [Draw a figure like fig. 474.]

22. Find the area of the triangle whose vertices are the points \((2, 1); (2, 5); (4, 7)\).

23. Repeat No. 22 for the points \((3, 2); (5, 4); (4, 8)\).

24. Repeat No. 22 for the points \((1, 1); (5, 2); (6, 5)\).

25. Find the area of the quadrilateral whose vertices are the points \((0, 0); (3, 2); (1, 5); (0, 7)\).

26. Repeat No. 25 for the points \((1, 3); (3, 2); (5, 5); (2, 7)\).

27. In \( \triangle ABC \), \( \angle A = 90^\circ \) and \( AB = 6 \text{ cm} \). If the area of \( \triangle ABC \) is 15 sq. cm, find the length of \( AC \).

28. \( AD \) is the perpendicular from \( A \) to the side \( BC \) of the acute-angled triangle \( ABC \). If \( BD = 6 \text{ in} \) and \( DC = 2 \text{ in} \), and if the area of \( \triangle ABC \) is 12 sq. in, find the length of \( AD \).

29. In fig. 475, \( CPQR \) is a square and \( AQB \) is a straight line; \( CA = 6 \text{ cm} \), \( CB = 10 \text{ cm} \). If \( CP = x \text{ cm} \), write down in terms of \( x \) the areas of \( \triangle CAQ \), \( \triangle CBQ \). What is the area of \( \triangle ABC \)? Find the value of \( x \).

30. \( ABCD \) is a square, side 4 in.; \( AB \) is produced to \( P \), and \( AD \) is produced to \( Q \) so that \( AP = 12 \text{ in} \), \( AQ = 6 \text{ in} \). Calculate the areas of \( \triangle APC \), \( \triangle ACQ \), \( \triangle APQ \) and then prove that \( PCQ \) is a straight line.

31. If in fig. 475, \( CPQR \) is a rectangle and \( AQB \) is a straight line, and if \( CP = x \text{ in} \), \( CR = y \text{ in} \), \( CA = a \text{ in} \), \( CB = b \text{ in} \), find an equation connecting \( x, y, a, b \).

32. In fig. 476, not drawn to scale, in which all the corners are right-angled, represents the plan of an estate of area 5 acres. The given measurements are in inches. On what scale, inches to the mile, is the plan drawn? A line perpendicular to \( AB \) cuts \( AB \) at \( Q \) and is drawn so that it bisects the area of the estate. Find the length of \( AQ \) on the plan.

33. Draw on squared paper a circle of radius 2 in. and find approximately its area by counting small squares. Find the ratio of the area of the circle to the area of the square on the radius.

34. Find the area of a pentagon whose vertices are the points \((0, 0); (4, 0); (9, 3); (9, 5); (3, 7)\).
New Geometry

Distance between Two Parallel Straight Lines

The following facts are easy to prove:

1. If \(AB\) and \(CD\) are parallel, and if from any point \(P\) on \(AB\) the perpendicular \(PX\) is drawn to \(CD\), then \(PX\) is also perpendicular to \(AB\).

**Proof:** \(\angle APX = \angle PXD\), alt. \(\angle s\), \(AB \parallel CD\).

But \(\angle PXD = 1\) rt. \(\angle\), given,

\(\therefore \angle APX\) is a right angle.

2. If \(AB\) and \(CD\) are parallel, and if \(PX\), \(QY\) are the perpendiculars from points \(P\), \(Q\) on \(AB\) to \(CD\), then \(PX = QY\).

**Proof:** Since \(\angle PXD = \angle QYD\) rt. \(\angle\), const.

\(PX\) is parallel to \(QY\).

Also \(PQ\) is parallel to \(XY\) given.

\(\therefore PQXY\) is a parallelogram,

\(\therefore PX = QY\) opp. sides \(\parallel\)gram.

These facts may be expressed as follows:

If two straight lines \(AB\), \(CD\) are parallel, the length of the perpendicular from any point \(P\) on EITHER line to the other line is the same for all positions of \(P\).

The length of this perpendicular is called the distance between the parallel lines.

The statement is often expressed by the words:

Parallel lines are everywhere the same distance apart.

But it must be realised that a proof is necessary.

Altitude or Height. If any side of a parallelogram is taken as its base, the distance between that side and the parallel side is called an altitude or height of the parallelogram.

Thus, in fig. 478, if \(AB\) is taken as the base of the parallelogram \(ABCD\), the altitude is the length of any one of the equal perpendiculars \(DH\), \(PQ\), \(P'Q'\). And in fig. 479, if \(BC\) is taken as the base of the same parallelogram \(ABCD\), the altitude is the length of any one of the equal perpendiculars \(AK\), \(RS\), \(R'S'\).

In general, a parallelogram has two distinct bases, i.e. bases of different lengths, and two distinct altitudes.

If any side of a triangle is taken as its base, the perpendicular to that side, produced if necessary, from the opposite vertex, is called an altitude or height of the triangle. Thus, in fig. 480, the perpendiculars \(AD\), \(BE\), \(CF\) to \(BC\), \(CA\), \(AB\) represent the altitudes of the triangle \(ABC\) corresponding to the bases \(BC\), \(CA\), \(AB\) respectively.
In general, a triangle has three distinct bases, i.e. bases of different lengths, and three distinct altitudes.

**Equal Altitudes**

If \( PX, QY \) are two parallel straight lines, and if

(i) parallelograms are drawn with bases along \( PX \) and their opposite sides along \( QY \),

(ii) triangles are drawn with bases along \( PX \) and opposite vertices on \( QY \),

then the parallelograms and triangles are all said to be between the same parallels, \( PX \) and \( QY \).

In fig. 481, if \( AB, EF \) are taken as the bases of the parallelograms \( ABCD, EFGH \), and if \( MN, ST \) are taken as the bases of the triangles \( LMN, RST \), the corresponding altitudes are all equal because each is equal to the distance between the parallel lines \( PX, QY \). Thus:

Parallelograms and triangles between the same parallels have equal altitudes.

Conversely, if we are given parallelograms and triangles of equal altitudes, and if we draw two parallel lines \( PX, QY \) whose distance apart is equal to that of the altitudes, we can construct parallelograms and triangles, between the parallels \( PX, QY \), congruent to the given parallelograms and triangles.

**Examples for Oral Discussion**

1. From a rectangular sheet of paper \( ABCD \) cut off a right-angled triangle \( BNC \), see fig. 482. Turn \( \triangle BNC \) round through 2 right angles and fit it on to the figure \( ANCD \), first with \( BN \) along \( B'C \) in the position \( NB'C \), and then with \( BN \) along \( DQ \) in the position \( ADQ \). What fact does this illustrate?

![Fig. 482](image)

2. Take a packet of sheets of paper. First hold it so that the front face is a rectangle. Then slide the sheets sideways so that the front face is a parallelogram. Is the height of the packet altered? What fact does this illustrate?

3. Fig. 483 represents two parallelograms \( ABCD, ABPQ \) on the same base \( AB \) and between the same parallels \( AB, DCQP \). Prove that area of \( ABCD = \) area of \( ABPQ \).

   (i) Use the SAA test to prove that \( \triangle DAQ \equiv \triangle CBP \).

   (ii) What remains if \( \triangle DAQ \) is taken away from the whole figure \( DABP \) and if \( \triangle CBP \) is taken away from \( DABP \)? Complete the proof.

   (iii) Draw your own figure, and draw in it a rectangle on the same base \( AB \) and between the same parallels as the parallelogram \( ABCD \). How does this help you to calculate the area of a given parallelogram?
4. In fig. 484, \( \triangle BXC = \triangle BYC \). What result is obtained by taking each triangle in turn from the whole figure?

![Fig. 484]

5. In fig. 485, \( \triangle ABC = \triangle BPQ \). What can you say about the area of the quadrilateral \( ACQP \)?

![Fig. 485]

6. Fig. 486 represents a triangle \( ABC \) and a rectangle \( PQBC \) on the same base \( BC \) and between the same parallels \( BC, QPA \). Prove that area of \( \triangle ABC = \frac{1}{2} \) area of rect. \( PQBC \).

![Fig. 486]

Complete the parallelogram \( AB \times X \).

(i) What do you know about the area of \( \triangle ABC \)?
   Give reasons.

(ii) Use Example 3 to complete the proof.

(iii) How does this help you to calculate the area of a given triangle?

(iv) If in fig. 486, \( BC = 3 \) in., \( BQ = 5 \) in., find the area of the parallelogram \( BC \times Xa \) and the area of \( \triangle ABC \).

Example 3 shows that the area of a parallelogram is equal to that of a rectangle on the same base and between the same parallels, and can be calculated by constructing this rectangle.

---

**Area of Parallelogram**

If \( ABCD \) is a parallelogram in which \( AB = x \) units and the distance between the parallel sides \( AB, CD \) is \( h \) units, the area of parallelogram \( ABCD = xh \) units of area.

This result may be expressed trigonometrically:

If \( AD = y \) units and if \( \angle DAB = \theta \),

\[
\begin{align*}
h &= \sin \theta \\
y &= \sin \theta \quad \text{or} \quad h = y \sin \theta \\
\therefore \text{area of parallelogram } ABCD &= xy \sin \theta \text{ units of area.}
\end{align*}
\]
NEW GEOMETRY

Area of Triangle

If \(ABC\) is a triangle in which \(BC = a\) units and the length of the perpendicular from \(A\) to \(BC\) is \(h\) units,

\[
\text{area of triangle } ABC = \frac{1}{2} ah \text{ units of area.}
\]

The result may be expressed trigonometrically:

If \(AC = b\) units and if \(\angle ACB\) is acute and is denoted by \(C\),

\[
\frac{h}{b} = \sin C \quad \text{or} \quad h = b \sin C,
\]

\[
\therefore \text{area of triangle } ABC = \frac{1}{2} ab \sin C \text{ units of area.}
\]

If \(\angle ACB\) is obtuse, see fig. 490, the same argument shows that the area is

\[
\frac{1}{2} ab \sin (180^\circ - C) \text{ units of area.}
\]

On pp. 75, 79, definitions were given only for the sines, cosines, tangents of acute angles. It is convenient to define the sine of an obtuse angle as equal to the sine of its supplement, e.g. by definition, \(\sin 150^\circ = \sin 30^\circ\), \(\sin 110^\circ = \sin 70^\circ\), etc.

Therefore by definition if \(\angle ACB\) is obtuse,

\[
\sin C = \sin (180^\circ - C).
\]

Hence, whether \(\angle ACB\) is acute or obtuse,

\[
\text{area of triangle } ABC = \frac{1}{2} ab \sin C \text{ units of area.}
\]

Area of Trapezium

\(ABCD\) is a trapezium in which \(AB\) is parallel to \(DC\). If \(AB = a\) units, \(DC = b\) units, and if the distance between the parallel lines \(AB, DC\) is \(h\) units,

\[
\text{area of trapezium } ABCD = \frac{1}{2} h(a + b) \text{ units of area.}
\]

Join \(BD\) and draw the altitudes \(DX, BY\) of \(\triangle DAB, \triangle BDC\).

\[
\begin{align*}
\text{area of } \triangle DAB &= \frac{1}{2} DX, \quad AB = \frac{1}{2} h a \text{ units of area,} \\
\text{area of } \triangle BDC &= \frac{1}{2} BY, \quad DC = \frac{1}{2} h b \text{ units of area,} \\
\therefore \text{area of trapezium } ABCD &= \frac{1}{2} ha + \frac{1}{2} hb \text{ units of area,} \\
&= \frac{1}{2} h(a + b) \text{ units of area.}
\end{align*}
\]

This result may be stated as follows:

The area of a trapezium is measured by the product of the measures of half the sum of the parallel sides and the distance between them,

\[
\text{or, more shortly, height } \times \text{average width.}
\]

Area of Quadrilateral

The area of any quadrilateral \(ABCD\) can be found by joining \(BD\), and finding the areas of each of the triangles \(ABD, CBD\). But it is better to construct a triangle whose area is equal to that of the given quadrilateral. The method is given on p. 263.
Example for Oral Discussion

If the lengths of the sides of a triangle are \(a\), \(b\), \(c\) units, and if \(s = \frac{1}{2}(a+b+c)\), i.e., if the semi-perimeter is \(s\) units, it can be proved that the area of the triangle is

\[
\sqrt{s(s-a)(s-b)(s-c)} \text{ units of area.}
\]

Find the area of a triangle whose sides are 15 in., 14 in., 13 in., and calculate the length of each altitude.

Draw your own figure and mark the data on it. Copy and complete the following:

\[
a = 15 \quad s - a = \ldots \quad \text{Why is the sum of} \\
b = 14 \quad s - b = \ldots \quad s - a, s - b, s - c \text{ equal} \\
c = 13 \quad s - c = \ldots \quad \text{to} \ s! \ \text{This provides} \\
\]

\[
a + b + c = \ldots \quad \text{Add} \ldots = \ldots \\
s = \ldots
\]

\[
\therefore \text{area of } \triangle ABC = \sqrt{(\ldots)} \text{ sq. in.} = \ldots \text{ sq. in.}
\]

If the perpendicular from \(A\) to \(BC\) is \(p\) in.,

area of \(\triangle ABC = \frac{1}{2} \times 15 \times p\) sq. in.

Hence find \(p\).

Find in a similar way the other two altitudes.

NUMERICAL EXAMPLES

EXERCISE 50

(The unit of length in each diagram is 1 in.)

1. Draw a parallelogram \(ABCD\) in which \(AB = 5\) cm., \(AD = 4\) cm., \(\angle BAD = 60^\circ\). Construct two rectangles equal in area to it, one with \(AB\) as base, the other with \(AD\) as base. Measure the heights of the two rectangles and then find the area of \(ABCD\) in two ways.

If you know the trigonometrical formula for the area of a parallelogram, find the area of \(ABCD\) in a third way.
8. In fig. 497, find
(i) the areas of $ABCD$, $\triangle PBQ$, $\triangle BCP$;
(ii) the lengths of the perpendiculars from $B$ to $AD$, from $Q$ to $BP$, from $C$ to $PB$;
(iii) the area of $AQPD$.

9. The distance between the parallel sides $AB$, $DC$ of a trapezium $ABCD$ is 4 in. If $AB = 8$ in., $DC = 5$ in., find the area of $ABCD$.


(i) Join $BG$, $CF$ and find the area of $BCFG$.
(ii) Join $CG$, $DH$ and find the area of $CGHD$.

11. $ABXY$ is a parallelogram of area 18 sq. cm.; $AB = 6$ cm., $AY = 4$ cm.; $C$ is a point on $YX$ produced such that $BC = 5$ cm. Find (i) area of $\triangle ABC$; (ii) the length of the perpendicular from $B$ to $AY$; (iii) the length of the perpendicular from $A$ to $CB$, produced if necessary.

12. $ABCD$ is a parallelogram of area 24 sq. cm.; $AB = 6$ cm., $AD = 5$ cm. Calculate the lengths of the corresponding altitudes. Draw the parallelogram and measure one of its acute angles. If you know the trigonometrical formula for the area of a parallelogram, find also this angle by calculation.

[13] $PQRS$ is a parallelogram of area 18 sq. cm.; $PQ = 5$ cm., $QR = 4$ cm. Calculate the lengths of the corresponding altitudes.

[14] $ABCD$ is a parallelogram in which $AB = 3$ in., $BC = 12$ in., and the perpendicular from $B$ to $AD$ is 2.5 in. Find the length of the perpendicular from $A$ to $CD$.

[15] $BE$ and $CF$ are two altitudes of $\triangle ABC$. If $AB = 6$ in., $AC = 5$ in., $CF = 4$ in., find the length of $BE$.

16. $P$ is a point on the base $BC$ of $\triangle ABC$ such that $BP = 3$ in., $PC = 2$ in. If the area of $\triangle APC$ is 5 sq. in., find the area of $\triangle APB$.

[17] A triangle and a rectangle have the same base and the height of the rectangle is 6 in. If the area of the triangle is $\frac{1}{2}$ of that of the rectangle, find the height of the triangle.

18. Find the area of a rhombus whose diagonals are 5 in., 6 in. long.

19. The area of a rhombus is 25 sq. cm. and one diagonal is twice as long as the other. Find the length of each diagonal.

20. $ABCD$ is a trapezium in which $AB$, $DC$ are parallel. If $AB = 7$ in., $DC = 3$ in., and if the area of $ABCD$ is 18 sq. in., find the area of $\triangle BCD$.

21. The lengths of the sides of a triangle are 13 in., 29 in., 21 in. Use the formula on p. 248 to calculate its area and find the length of the shortest altitude.

[22] Repeat No. 21 for a triangle whose sides are of lengths 10 in., 17 in., 21 in.

23. In $\triangle ABC$, $AB = 8$ cm., $AC = 9$ cm., and $D$ is the mid-point of $BC$. If the area of $\triangle ABC$ is 36 sq. cm., find the lengths of the perpendiculars from $D$ to $AB$ and $AC$.

24. The lengths of the two altitudes of a parallelogram are 2 in., 3 in., and the perimeter of the parallelogram is 29 in. Find the lengths of the sides of the parallelogram.

25. The lengths of the three altitudes of a triangle are 6$\sqrt{2}$, 4$\sqrt{3}$, 3 in., and the perimeter of the triangle is 18 in. Find the lengths of the sides of the triangle.

26. $D$, $E$ are points on the sides $AC$, $AB$ respectively of $\triangle ABC$ such that $AD = \frac{1}{4} AC$ and $AE = \frac{1}{2} AB$. If the area of $\triangle ABC$ is 18 sq. cm., find the areas of (i) $\triangle BCE$, (ii) $\triangle CDE$, (iii) $\triangle ADE$.

27. $P$, $Q$, $R$ are points on the sides $BC$, $CA$, $AB$ respectively of $\triangle ABC$ such that $BP = 6$ in., $PC = 8$ in., $CQ = 6$ in., $QA = 9$ in., $AR = 10$ in., $RB = 3$ in. Prove that the area of $\triangle ABC$ is 54 sq. in. and find the areas of (i) $\triangle APC$, (ii) $\triangle QPC$, (iii) $\triangle RPB$, (iv) $\triangle RQA$, (v) $\triangle PQR$.

28. The Sine Formula. If in any triangle $ABC$, $BC = a$ units, $CA = b$ units, $AB = c$ units, prove that $\frac{\sin A}{\sin B} = \frac{\sin B}{\sin C}$.

(See p. 246.)
THEOREM 35

The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels.

Given a parallelogram \(ABCD\) and a rectangle \(ABHK\) on the same base \(AB\) and between the same parallels \(AB, KHDC\).

To prove that \(\text{area of } ABCD = \text{area of } ABHK\).

**Proof.** With the notation in the figure in \(\triangle AKD, BHC\),

\[
\begin{align*}
m_1 &= m_2 \quad \text{corr. } \angle s, AK \parallel BH. \\
n_1 &= n_2 \quad \text{corr. } \angle s, AD \parallel BC. \\
AK &= BH \quad \text{opp. sides stdogram.}
\end{align*}
\]

\(\therefore\) \(\triangle AKD\) and \(\triangle BHC\) are congruent \(\text{AAS}\).

\(\therefore\) area of \(\triangle AKD\) = area of \(\triangle BHC\).

From the whole figure \(ABCK\), subtract each triangle in turn, then the remainders \(ABCD, ABHK\) are equal in area.

**Note.** In order to obtain a proof which holds for all figures, i.e., whether \(D\) lies between \(K\) and \(H\) or on \(KH\) produced, and which holds if \(ABHK\) is any parallelogram, it is necessary

(i) to use the \(\text{AAS}\) test for proving the triangles congruent,

(ii) to \text{subtract} the areas of these triangles in turn from the whole figure.
THEOREM 36

The area of a triangle is equal to one-half of the area of a rectangle on the same base and between the same parallels.

Fig. 499

Given a triangle \( \triangle ABC \) and a rectangle \( PQBC \) on the same base \( BC \) and between the same parallels \( BC, PA \).

To prove that \( \text{area } \triangle ABC = \frac{1}{2} \text{ area } PQBC \).

Construction. Through \( C \) draw \( CX \) parallel to \( BA \) to meet \( QA \) or \( QA \) produced at \( X \).

Proof. \( CX \) is parallel to \( BA \) \( \text{const.} \),
\( BC \) is parallel to \( AX \) \( \text{given} \),
\( \therefore \triangle ABCX \) is a parallelogram.

\( \therefore \triangle ABC, \triangle PQBC \) are parallelograms on the same base \( BC \) and between the same parallels \( BC, QPAX \),
\( \therefore \text{area } \triangle ABC = \text{area } \triangle PQBC \).

Since the diagonal \( AC \) bisects the area of the parallelogram \( \triangle ABC \),
\[ \text{area } \triangle ABC = \frac{1}{2} \text{ area } \triangle XABC, \]
\( \therefore \text{area } \triangle ABC = \frac{1}{2} \text{ area } \triangle PQBC \).
THEOREM 37

Triangles on the same base and between the same parallels are equal in area.

\[ \text{Fig. 500} \]

**Given** two triangles \( \triangle ABC, \triangle DBC \) on the same base \( BC \) and between the same parallels \( BC, AD \).

**To prove that** area of \( \triangle ABC = \text{area of } \triangle DBC \).

**Construction.** From \( A, D \) draw the perpendiculars \( AH, DK \) to \( BC \), produced if necessary.

**Proof.**
- Area of \( \triangle ABC = \frac{1}{2} BC \cdot AH \) \( \frac{1}{2} \) base \times altitude.
- Area of \( \triangle DBC = \frac{1}{2} BC \cdot DK \) \( \frac{1}{2} \) base \times altitude.

Since \( AH \) and \( DK \) are perpendicular to \( BC \),
- \( AH \) is parallel to \( DK \),
- also \( HK \) is parallel to \( AD \) given,
- \( \therefore \) \( AHKD \) is a parallelogram,
- \( \therefore AH = DK \) opp. sides \parallel.
- \( \therefore \) area of \( \triangle ABC = \text{area of } \triangle DBC \).

**Corollary.** Triangles on equal bases and between the same parallels are equal in area.

The proof is similar.

---

THEOREM 38

Triangles of equal area on the same base and on the same side of it are between the same parallels.

\[ \text{Fig. 501} \]

**Given** two triangles \( \triangle ABC, \triangle DBC \) on the same side of the same base \( BC \) and such that \( \triangle ABC = \triangle DBC \).

**To prove that** \( AD \) is parallel to \( BC \).

**Construction.** From \( A, D \) draw the perpendiculars \( AH, DK \) to \( BC \), produced if necessary.

**Proof.**
- Area of \( \triangle ABC = \frac{1}{2} BC \cdot AH \) \( \frac{1}{2} \) base \times altitude.
- Area of \( \triangle DBC = \frac{1}{2} BC \cdot DK \) \( \frac{1}{2} \) base \times altitude.

But area of \( \triangle ABC = \text{area of } \triangle DBC \) given,
- \( \therefore AH = DK \).

Since \( AH \) and \( DK \) are perpendicular to \( BC \),
- \( AH \) is parallel to \( DK \),
- \( \therefore AH \) and \( DK \) are equal and parallel.
- \( \therefore AHKD \) is a parallelogram.
- \( \therefore AD \) is parallel to \( HK \) or \( BC \).

**Corollary.** Triangles of equal area on equal bases in the same straight line and on the same side of it are between the same parallels.

The proof is similar.
THEOREM 39

If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to half that of the parallelogram.

Given a triangle ABC and a parallelogram PQBC on the same base BC and between the same parallels BC, AQ.

To prove that area of $ABC = \frac{1}{2}$ area of PQBC.

Construction. Join CQ.

Proof. The triangles ABC, QBC are on the same base BC and between the same parallels BC, AQ.

\[ \therefore \text{area of } ABC = \text{area of } QBC. \]

Since the diagonal CQ bisects the area of the parallelogram PQBC,

\[ \text{area of } QBC = \frac{1}{2} \text{area of } PQBC. \]

\[ \therefore \text{area of } ABC = \frac{1}{2} \text{area of } PQBC. \]

Note. If preferred, this theorem may be proved by using the construction and method of Theorem 37. Alternatively, the proof of Theorem 36 (with the construction) applies, word for word, to this theorem.

Area Theorems and Rider Work. Although the properties of areas of triangles have been deduced from those of parallelograms, it will be found that the theorems about the areas of triangles are more useful than those about the areas of parallelograms in rider work.

Most of the riders in the next exercise can be solved by using one or more of the following theorems (proved in the previous pages) with which pupils must be familiar:

1. If, in fig. 503, AD is parallel to BC, then $\triangle ABC = \triangle DBC$.
2. If, in fig. 503, $\triangle ABC = \triangle DBC$, then AD is parallel to BC.

3. If, in fig. 504, BCD is a straight line and BC = CD, then $\triangle ABC = \triangle ADC$.

4. If, in fig. 505, BCPQ is a parallelogram and if A lies on PQ, or PQ produced, then $\triangle ABC = \frac{1}{2} \text{llgram PQBC}$.

EXERCISE 51

[Arrows indicate that lines are given parallel.]

Nos. 1–5 refer to Fig. 506.

1. If AD is parallel to BC, name a triangle equal to $\triangle ABC$. Give reasons.
2. If AD is parallel to BC, prove that $\triangle AKB = \triangle DKC$.
3. If $\triangle KDA = \triangle KCB$, prove that BC is parallel to AB.
4. If DK = KB, explain why $\triangle DKC = \triangle BKC$ and prove that $\triangle DAC = \triangle BAC$.
5. If $\triangle DAK = \triangle DKC$, prove that $\triangle BAK = \triangle BCK$. 
6. If in fig. 507, \( YP = PQ \) and \( XQ = QP \), prove that \( \triangle QXY = \triangle QPZ \).

[7] If in fig. 507, \( YP = PQ \), prove that \( \triangle XYQ = \triangle XQZ \).

[8] \( K \) is a point on the side \( BC \) of \( \triangle ABC \). What can you say about \( BK \) if \( \angle ABK = \frac{1}{3} \angle \triangle ABC \)?

9. What can you say about the altitudes \( BE, CF \) of \( \triangle ABC \) if \( AB = 2AC \)?

10. If in fig. 508, \( PQ \) is parallel to \( BC \), prove that \( \triangle APC = \triangle AQB \).

11. If in fig. 508, \( AP = PB \) and \( AQ = QC \), what can you say about \( \triangle BPQ \) and \( \triangle CPQ \)? Prove that \( PQ \) is parallel to \( BC \).

[12] If in fig. 508, \( BP = 2PA \) and \( CQ = 2QA \), what can you say about \( \triangle BPQ \) and \( \triangle CPQ \)? Prove that \( PQ \) is parallel to \( BC \).

[13] If in fig. 509, \( AB = EC \), prove that \( \triangle DAB = \triangle BEC \).

14. In fig. 510, \( ABP \) is a straight line, prove that \( \text{quad. } ABCD = \triangle ADP \).

Use this result to construct a triangle equal in area to any given quadrilateral.

15. If in fig. 511, \( DC = FE \), prove that the trapeziums \( ABCD, ABED \) are equal, without assuming the formula for the area of a trapezium.

[16] \( ABCD \) is a parallelogram; \( P \) is any point on \( AB \) between \( A \) and \( B \). Prove that \( \triangle APD + \triangle BPC = \triangle DPC \).

[17] \( ABCD \) is a parallelogram; \( P \) is any point on \( AD \); \( Q \) is any point on \( AB \) produced. Prove that \( \triangle CPB = \triangle CQD \).

18. In fig. 512, \( \triangle ABC = \triangle XYZ \), prove that \( \text{quad. } ABXY = \text{quad. } BCZY = \text{quad. } ACZX \).

19. In fig. 513, prove that \( \triangle XBC = \triangle YAD \).

[20] \( P \) and \( Q \) are any points on the sides \( AB, DC \) of the trapezium \( ABCD \) in fig. 513. Prove that \( \text{quad. } XPQ = \triangle BXC \).

[21] \( ABCD \) is a parallelogram; \( P \) is any point on \( BC \); \( DQ \) is the perpendicular from \( D \) to \( AP \). Prove that the area of \( ABCD = \triangle DQ \), \( AP \).

[22] \( ABCD \) is a parallelogram; a line through \( A \) cuts \( CB \) produced at \( P \) and cuts \( CD \) produced at \( Q \). If \( \triangle BAP = \triangle DAQ \), prove that \( AP = AQ \). [Join \( AC \).]

[23] \( BE, CF \) are medians of \( \triangle ABC \) and cut at \( G \). Prove that \( \text{quad. } AEGF = \triangle BGC \).

24. In fig. 514, \( D \) is the mid-point of \( BC \). Prove that \( PQ \) bisects the area of \( \triangle ABC \). [Join \( AD \).]

25. \( P, Q \) are points on the sides \( CB, CD \) of the parallelogram \( ABCD \) such that \( PQ \) is parallel to \( BD \). Prove that \( \triangle ABP = \triangle ADQ \). [Join \( PD, BQ \).]
In fig. 515, prove that $\triangle ABP = \triangle AQC$. Join $CP, BQ$.

**Fig. 515**

27. In fig. 516, $ABP$, $CBQ$ are straight lines and $ABCD, PBQR$ are parallelograms. Prove that $\triangle ABCD = \triangle PBQR$. Join $AC, PQ$.

28. $ABCD$ is a parallelogram; any line through $A$ cuts $DC$ at $Y$ and cuts $BC$ produced at $Z$. Prove that $\triangle BCY = \triangle DYZ$. [Prove that $\triangle DYZ = \triangle AYC$.]

29. $AB$, $DC$ are the parallel sides of a trapezium $ABCD$; $AC$ cuts $BD$ at $K$. If the line through $K$ parallel to $BA$ cuts $AD$ at $P$, prove that $\triangle ABC = 2 \triangle KAD$.

30. $E$ is a point on the side $CD$ of the parallelogram $ABCD$, and $CD$ is produced to $F$ so that $DF = CE$. $BE$ produced cuts $AD$ produced at $G$ and cuts the line through $F$ parallel to $AD$ at $H$. Prove that $\triangle KFG = \triangle KAD$.

Nos. 31-33 refer to fig. 517 in which $ABCD$ is a parallelogram, and $PQ, RS$ are parallel to $AD, AB$.

31. Prove that $\triangle SPA = \triangle SPQ = \triangle SPD$.

32. Prove that $\triangle AFR = \triangle ASQ = \triangle ADQ$.

33. Prove that $\triangle AQP = \triangle ABQ = \triangle BKP$.

34. $ABCD$ is a parallelogram; $DC$ is produced to $P$; $AP$ cuts $BD$ at $Q$. Prove that $\triangle DQP = \triangle ABQ = \triangle BPD$.

35. The diagonals $AC, BD$ of the parallelogram $ABCD$ cut at $K$. $CA, DB$ are produced to $P, Q$ respectively so that $AP = CK$ and $BQ = DK$. Prove that quad. $ABCD = \triangle KPQ$.

36. $P$ is a variable point inside a given equilateral triangle $ABC$. $PX, PY, PZ$ are the perpendiculars from $P$ to $BC, CA, AB$. Prove that $PX + PY + PZ$ is constant.
EXERCISE 52

[Perform the constructions in this exercise without making calculations, whenever possible.]

1. Draw a triangle ABC in which BC = 6 cm., CA = 5 cm., AB = 7 cm. Construct an obtuse-angled triangle KBC equal in area to △ABC and such that KB = 8 cm. Measure ∠KBC.

[2] Copy △ABC in No. 1. Construct an equivalent triangle PBC such that PB = PC. Measure PB.

3. Copy △ABC in No. 1. Construct an equivalent triangle AQR in which AQ = 5 cm. and ∠AQR = 70°. Measure QR.

[First draw equivalent ∆ABQ, with AQ = 5 cm.]

[4] Draw a parallelogram ABCD in which AB = 6 cm., AD = 5 cm., ∠BAD = 62°. Construct an equivalent parallelogram with sides equal to 6 cm., 5 cm., and measure one of its acute angles.

5. Draw a parallelogram ABCD in which AB = 4 cm., AD = 6 cm., ∠BAD = 70°. Construct an equivalent parallelogram with sides equal to 6 cm., 5 cm., and measure an acute angle. [First draw equivalent trapezium ADKH with AH = 5 cm.]

6. Draw an equilateral triangle, side 3 in. Construct an equivalent rhombus in which each side is 2 5/8 in. long. Measure the acute angle of the rhombus.

[7] Draw a regular hexagon ABCDEF, such that AB = 4 cm. Construct an equivalent rectangle AEFP. Measure EP and find the area of the hexagon.

8. Draw a triangle with sides 5 cm., 6 cm., 8 cm. Construct an equivalent acute-angled triangle with two of its sides 5 cm., 6 cm. Measure the third side.

[9] Construct a parallelogram ABCD of area 4 sq. in. such that AC = 5 in., BD = 4 in. Measure AB and BC.

10. Construct a triangle ABC of area 4 5/8 sq. cm., given that the radius of the circle ABC is 5 cm. and that BC = 3 cm. Explain shortly your method.

11. Draw quadr. ABCD given AB = 3 cm., BC = 5 cm., CD = 6 cm., DA = 4 cm., BD = 5 cm. Construct a point K on AB produced such that △DAK = quad. ABCD. Find the area of ABCD.

[12] Draw a quadrilateral ABCD in which AB = 6 cm., BC = 5 cm., CD = 4 cm., AD = 7 cm., ∠ABC = 110°, ∠BCD = 95°. Construct an equivalent triangle ABP such that P lies on BC produced and find the area of ABCD.

CONSTRUCTIONS

13. Draw any triangle ABC and take a point D on BC produced. Construct a point Q on AB such that △BDQ is equal to △ABC. State your method and prove that it is correct.

[14] Draw any parallelogram ABCD and take a point E on AB produced. Construct a point P on AD such that the parallelogram PAEQ is equivalent to ABCD. State your method and prove that it is correct. [△EAP = △ABD]

[15] Draw any triangle ABC and take a point D inside it; join DB, DC. Construct a point P on AC such that △APB equals the area of the re-entrant quadrilateral ABCD. State your method and prove that it is correct.

16. Draw a quadrilateral ABCD in which △ABC is greater than △ADC. Construct a line through A which bisects the area of ABCD. [Find K on BC produced such that △ABK = quad. ABCD.]

[17] Draw a parallelogram ABCD. Construct points P, Q on BC, CD such that AP, AQ divide ABCD into three parts of equal area. State your method and prove that it is correct.

18. (i) In fig. 520, Q is any point on AC. Use the fact that the area of a parallelogram is bisected by a diagonal to prove that XQRD = PQYB.

(ii) Draw a parallelogram XQRD in which XQ = 6 cm., XD = 4 cm., ∠DX Q = 70°. Construct an equivalent parallelogram in which one side is 5 cm. and one angle is 70°. Measure the other side.

19. Draw an equilateral triangle, side 3 in. Construct an equivalent rectangle having one side 2 5/6 in. long. Measure the other side. [Use the fact stated in No. 18 (i)].

20. Draw a parallelogram with sides 2 1/2 in., 2 9/10 in., and one angle 72°. Construct a rectangle equal in area to the parallelogram and with one side 2 1/2 in. long. Measure the other side.

21. Construct a rhombus having each of its sides equal to 5 cm. and equal in area to a rectangle whose sides are 6 cm., 3 cm. long. Measure the acute angle of the rhombus.

22. Draw a triangle ABC in which BC = 6 cm., CA = 5 cm., AB = 7 cm. Take a point P on AC such that AP = 2 cm. Construct a point Q on BC such that PQ bisects the area of △ABC. Measure CQ. [See Ex. 51, No. 24].
PYTHAGORAS' THEOREM

Fig. 521 represents part of a floor tiled with equal tiles, each in the shape of an isosceles right-angled triangle.

Can you see any connection between the area of the square on the hypotenuse and the sum of the areas of the squares on the other two sides of one of these right-angled triangles?

If the right-angled triangle is not isosceles, the connection can be examined as follows:

Fold a rectangular sheet of paper and then fold it again so as to get four rectangular layers and then cut out four duplicate right-angled triangles.

Suppose the hypotenuse is c in., and that the other two sides are a in. and b in. Draw (or cut out) two equal squares of side (a + b) in. and arrange the triangles on the top of each square, firstly as in fig. 522 (i), secondly as in fig. 522 (ii). In fig. 522 (i), the uncovered shaded area is the square on the hypotenuse of the triangle labelled 1; in fig. 522 (ii), the uncovered shaded area is made up of the squares on the other two sides of the triangle labelled 1.

What does this experiment suggest?

If fig. 522 is used to obtain a proof of the fact suggested, it is necessary to show that the shaded figures are square; this is not difficult.

Perigal's Dissection

Fig. 523 shows a method of cutting up a square by means of which the squares on the two sides containing the right angle of a right-angled triangle can be made to cover exactly the square on the hypotenuse: a kind of jigsaw puzzle.

Draw a triangle ABC, right-angled at A, on thin cardboard or stiff paper, and construct the three squares on its sides. Take the centre P of the square on AB and draw through P lines parallel and perpendicular to BC. Cut out the squares on AB, AC and cut up the square on AB into the four parts indicated. Arrange the pieces to cover the square on BC.

It must be realised that a proof is essential. If one relies merely on appearances, it is easy to make mistakes. For example, draw on squared paper with $\frac{1}{4}$ inch as unit, a square, side 8 units long, and a rectangle 13 units long, 5 units high. Cut up the square into the four pieces shown and fit them on to the rectangle in the positions indicated. Do they cover up the rectangle? If so, this suggests that the area of the square is equal to that of the rectangle. Is this true?

Pythagoras lived in the sixth century B.C. Special cases of his theorem were known to the Egyptians much earlier,
Examples for Oral Discussion

1. In fig. 525, $ABHK$, $BCPQ$ are squares. Prove that
   
   (i) $\angle HBC = \angle ABQ$;  
   (ii) $\triangle HBC \equiv \triangle ABQ$.

   If the triangle $HBC$ is rotated clockwise about $B$ through a right angle, what is its new position? What can you say about the directions of $CH$ and $AQ$?

2. In fig. 526, $ABHK$, $ACMN$, $BCPQ$ are squares. Make a sketch of the figure. By joining two points in your sketch obtain a triangle congruent to $\triangle ACP$. Similarly, obtain a triangle congruent to $\triangle ABN$. Give reasons.

   In figs. 527, 528, $ABHK$, $BCPQ$ are squares. In fig. 527, $\angle BAC$ is acute; in fig. 528, $\angle BAC$ is a right angle. Make sketches of both figures and use them for Nos. 3-5.

3. Name 3 points which are collinear in fig. 528 but not collinear in fig. 527. Give reasons.

4. (i) By joining points in your sketches of figs. 527, 528, obtain three triangles with base $BQ$ which are equal to half the rectangle $BQYX$.

   (ii) Similarly, obtain triangles with base $HB$ which are equal to half the square $HBAK$. Are they the same for both figures?

5. (i) In your sketch of fig. 527, construct a rectangle $HBDE$ whose area is double that of $\triangle HBC$.

   (ii) Use the fact that $\triangle ABQ \equiv \triangle HBC$ to find a rectangle in your sketch of each figure which is equal in area to $BQYX$. Is it the same rectangle in the two figures?

   (iii) Construct in the same way rectangles which have the same area as $CQYX$ in fig. 527 and in fig. 528.

   (iv) What can you now say about the area of the square $BCPQ$ in fig. 527 and in fig. 528?
The results obtained in these examples may be summarised as follows:—

If in \( \triangle ABC \), \( \angle BAC \) is acute, see fig. 529, the area of the square on BC is less than the sum of the areas of the squares on AB and AC by the sum of the areas of the rectangles ADEK, AFGN.

Explain why rect. ADEK = rect. AFGN.

Prove that the area of each of the rectangles ADEK, AFGN is \( AB \cdot AC \cos BAC \).

Hence with the usual notation for \( \triangle ABC \), if \( \angle BAC \) is acute,

\[
a^2 = b^2 + c^2 - 2bc \cos A.
\]

If in \( \triangle ABC \), \( \angle BAC \) is a right angle, see fig. 530, the square on BC can be divided into two rectangles whose areas are equal to those of the squares on AB and AC respectively. This provides a proof of Pythagoras' theorem.

In a right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

*Similarly, if you draw a figure like fig. 529, but so that \( \angle BAC \) is obtuse, you will see that the square on BC can be divided into two rectangles, the area of one of which is greater than the area of the square on AB, and the area of the other is greater than the area of the square on AC.

Therefore in \( \triangle ABC \), if \( \angle BAC \) is obtuse, the area of the square on BC is greater than the sum of the areas of the squares on AB and AC by the sum of the areas of two rectangles, the area of each of which is \( AB \cdot AC \cos (180^\circ - \angle BAC) \).

Hence with the usual notation for \( \triangle ABC \), if \( \angle BAC \) is obtuse,

\[
a^2 = b^2 + c^2 + 2bc \cos (180^\circ - A).
\]

On p. 79, a definition was given for the cosine of an acute angle only. It is convenient to define the cosine of an obtuse angle to be minus the cosine of its supplement, e.g., by definition \( \cos 150^\circ = -\cos 30^\circ \), \( \cos 110^\circ = -\cos 70^\circ \), etc. Therefore by definition, if \( \angle BAC \) is obtuse,

\[
\cos A = -\cos (180^\circ - A).
\]

Hence in \( \triangle ABC \), if \( \angle BAC \) is obtuse,

\[
a^2 = b^2 + c^2 - 2bc \cos A.
\]

Thus the same formula holds for obtuse-angled triangles as for acute-angled triangles.

**Acute-angled, Right-angled, and Obtuse-angled Triangles.**

If the lengths of the sides of a triangle are given, it is easy to find out whether the triangle is acute-angled or not:

If \( \angle BAC \) is acute, then \( BC^2 < BA^2 + AC^2 \).

If \( \angle BAC \) is a right angle, then \( BC^2 = BA^2 + AC^2 \).

If \( \angle BAC \) is obtuse, then \( BC^2 > BA^2 + AC^2 \).

These tests cover every possibility, and therefore the converse statements are true.

**Oral Example.** Is a triangle obtuse-angled if the lengths of its sides are (i) 8 cm., 9 cm., 12 cm.? (ii) 7 cm., 12 cm., 14 cm.?

* This paragraph may be omitted at a first reading.
Areas associated with Pythagoras' Theorem. This theorem is fundamentally a property of areas, although its usefulness lies mainly in its application to the calculation of lengths and to Trigonometry. In order to make sure that the theorem is understood, it is desirable that in the first instance some applications should be discussed in which the reference is to actual areas; there is no risk of the arithmetical aspect of the theorem being overlooked. The following examples are intended to serve this purpose and to emphasize the importance of the theorem, which, with the notation of fig. 530, p. 276, can be expressed in the form \( BA^2 = BX \cdot BC \).

Examples for Oral Discussion. Make a sketch of fig. 530, without the shading, for each of Nos. 1-3.

1. If \( \angle BAC = 90^\circ \), \( AB = 4 \) in., \( BC = 5 \) in., show on your sketch the areas in succession of \( \text{sq. AH, rect. BY, rect. CY, sq. AM} \). Find the lengths of \( BX, CX, AC \).
2. If \( \angle BAC = 90^\circ \), \( AC = 6 \) in., \( BC = 10 \) in., show on your sketch the areas in succession of \( \text{sq. AM, rect. CY, rect. BY, sq. AH} \). Find the lengths of \( CX, XB, AB \).
3. If \( \angle BAC = 90^\circ \), \( AB = 12 \) in., \( AC = 5 \) in., show on your sketch the areas in succession of \( \text{sq. AH, sq. AM, rect. BY, rect. CY} \). What is the length of \( BC \)?
4. In fig. 530, if \( \angle BAC = 90^\circ \), \( AC = 2 \) in., \( BC = 3 \) in., find the area of the square \( ABHK \). Calculate the length of \( AB \), correct to \( 1/10 \) in.
5. Construct a square of area 10 sq. in. Then find by measurement the value of \( \sqrt{10} \) approximately.
6. Construct a square of area 7 sq. in. Then find by measurement the value of \( \sqrt{7} \) approximately.
7. In fig. 530, if \( \angle BAC = 90^\circ \), \( \angle AXC = 90^\circ \), \( AX = 4 \) in., \( XC = 3 \) in., find the length of \( XY \), and the lengths of \( CX, BX, BA \).
8. In fig. 530, if \( \angle BAC = 90^\circ \), \( \angle ABC = 60^\circ \), prove that \( \text{sq. ABHK is 3 times sq. ACMN} \). [Note that \( \triangle BAC \) is half an equilateral triangle, see p. 121.]

What are the values of \( \sin 60^\circ \) and \( \tan 60^\circ \)?

**NUMERICAL APPLICATIONS**

Example. \( \triangle ABC \) is a triangle such that

\[
AB = AC = 10 \text{ in., } BC = 12 \text{ in.}
\]

Calculate the area of \( \triangle ABC \).

If \( N \) is the mid-point of \( BC \), \( AN \) is perpendicular to \( BC \), and \( BN = 6 \) in.

Let \( AN = h \) in.

Since \( \angle ANB \) is a right angle,

\[
h^2 + 6^2 = 10^2;
\]

\[
\therefore \ h^2 = 100 - 36 = 64;
\]

\[
\therefore \ h = 8.
\]

\( \therefore \) the altitude \( AN \) of \( \triangle ABC \) is 8 in.

\( \therefore \) area of \( \triangle ABC = \frac{1}{2} \times 12 \times 8 \text{ sq. in.} \)

\( = 48 \text{ sq. in.} \)

Note.—For square-root tables see p. xiv.

**EXERCISE 53**

[Give answers which are approximate, correct to 3 figures. Unless otherwise stated, the unit of length in each diagram is 1 in.]

Nos. 1-9 refer to fig. 532.

1. \( b = 8 \), \( c = 15 \), find \( a \).
2. \( b = 5 \), \( c = 12 \), find \( a \).
3. \( b = 9 \), \( c = 3 \), find \( a \).
4. \( a = 25 \), \( b = 15 \), find \( c \).
5. \( a = 6 \), \( b = 8 \), find \( b \).
6. \( \angle C = 45^\circ \), \( a = 4 \), find \( b \).
7. \( \angle C = 60^\circ \), \( a = 6 \), find \( b \) and \( c \).
8. \( \angle C = 60^\circ \), \( b = 4 \), find \( a \) and \( c \).
9. \( AB = AC = 1 \text{ in., } BC = AB = 2 \text{ in., find } AB \).
10. A 20-foot ladder is leaning against a wall. The end on the ground is 12 ft. from the wall. How high up the wall does the ladder reach?

11. A ladder just reaches the top of a wall 18 ft. high when its end on the ground is 8 ft. from the wall. Find the length of the ladder.

12. A man travels 15 miles due north and then 5 miles due east. How far is he from his starting-point?

13. The sides of a rectangle are 5 in., 7 in.; find the length of a diagonal.

14. Find the side of a rhombus whose diagonals are 6, 10 cm.

15. An aeroplane heads north-west at 150 miles an hour and is carried north-east by a wind at 60 miles an hour. Find the distance from the starting-point after 1 hour.

16. In fig. 533, if $AD = 10$ cm., $AB = 8$ cm., $BC = 4$ cm., find the length of $CD$.

17. In fig. 533, if $AD = 8$ in., $BC = 3.5$ in., $CD = 7.5$ in., find the length of $AB$.

18. $PQRS$ is a quadrilateral in which $\angle PQR$ and $\angle PRS$ are right angles. If $PQ = 12$ cm., $QR = 9$ cm., $RS = 8$ cm., find the length of $PS$ and the area of $PQRS$.

19. $ABCD$ is a quadrilateral in which $\angle B = \angle D = 90^\circ$. If $AB = 6$ in., $BC = 8$ in., $CD = 5$ in., find the length of $AD$ and the area of $ABCD$.

20. $AD$ is an altitude of the acute-angled triangle $ABC$. If $AB = 20$ in., $BD = 16$ in., $DC = 5$ in., find the length of $AC$.

21. In $\triangle ABC$, $\angle B = 90^\circ$, $AB = 8$ cm., $AC = 17$ cm.; $D$ is a point on $BC$ such that $BD = 6$ cm. Find the area of $\triangle ACD$.

22. In $\triangle ABC$, $\angle C = 90^\circ$, $AC = 3$ in., $AB = 8$ in. Find the length of the median $AX$.

23. In $\triangle ABC$, $AB = AC = 13$ in., $BC = 10$ in. Find the area of $\triangle ABC$ and the length of the perpendicular from $C$ to $AB$.

24. Repeat No. 23 if $AB = AC = 10$ in., $BC = 12$ in.

25. $PQ$ is a chord of a circle, centre $O$, radius 6 cm.; $PQ = 8$ cm. Find the length of the perpendicular from $O$ to $PQ$.

26. A gun, whose effective range is 9000 yd., is 5000 yd. from a straight railway. What length of the railway is commanded by the gun?

27. Construct a square of area 13 sq. in. Measure the side of the square.

28. Construct a square of area 11 sq. in. Measure the side of the square.

29. In fig. 534, find the distance of $D$ from $A$.

30. Mark on squared paper the points whose co-ordinates are $(1, 2)$ and $(3, 5)$, and calculate the distance between them.

31. Prove that the points whose co-ordinates are $(5, 11), (6, 10), (7, 7)$ lie on a circle whose centre is $(2, 7)$, and find its radius.

32. Prove that the triangle whose sides are $37$ cm., $35$ cm., $12$ cm. is right-angled, and find its area.

33. $ABCD$ is a quadrilateral in which $AB = 9$ in., $BC = 12$ in., $AD = 25$ in., $CD = 20$ in., $\angle ABC = 90^\circ$. Find the area of $ABCD$.

34. Find out whether the triangle whose sides are $5$ cm., $6$ cm., $8$ cm. is acute-angled or obtuse-angled.

35. Repeat No. 34 for the triangle whose sides are $7$ cm., $10$ cm., $12$ cm.

36. Prove that the points whose co-ordinates are $(0, 0), (7, 17), (12, 6)$ are the vertices of an isosceles right-angled triangle.

37. In fig. 535, $O$ is the centre of the circular arc $AB$. Find the radius $r$ in. of the circle. [Note that $OC = (r - 8)$ in.]

38. In fig. 536, $AB = AC = 10$ in., $BC = 12$ in. Find the radius $r$ in. of the circle $ABC$. 

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Fig. 533

Fig. 534

Fig. 535

Fig. 536
[39] In $\triangle ABC$, $AB = AC = 10$ in., $BC = 16$ in. Find the radius of the circle which passes through $A$, $B$, $C$.

40. In $\triangle ABC$, $AB = 4$ in., $BC = 5$ in., $\angle B = 45^\circ$. Find $AC$.

41. In $\triangle ABC$, $AB = 8$ in., $BC = 3$ in., $\angle B = 60^\circ$. Find $AC$.

*42. $AD$ is an altitude of $\triangle ABC$. If $AB = 5$ cm, $BC = 9$ cm, $CA = 7$ cm, prove that $\angle BAC$ is obtuse and find the lengths of $BD$ and $AD$. [Let $BD = x$ cm, $AD = y$ cm, and use Pythagoras twice.]

*43. $AD$, $BC$ are two vertical poles, $D$ and $C$ being the ends on the ground which is level; $AD = 12$ ft., $AB = 10$ ft., $BC = 3$ ft.; find the length of $AD$.

*44. $AD$ is an altitude of $\triangle ABC$ in which $\angle B$ and $\angle C$ are acute. If $BD = x^2$ cm, $DC = y^2$ cm, $AD = xy$ cm, find expressions for $AB^2$ and $AC^2$ in terms of $x$ and $y$, and prove that $\angle BAC$ is a right angle.

45. Prove that the triangle whose sides are of lengths $2xy$, $x^2 - y^2$, $x^2 + y^2$ in. is right-angled. Also, if $a^2 + ab + b^2 = c^2$, prove that the three triangles obtained by putting (i) $x = a$, $y = a$, (ii) $x = b$, $y = b$, (iii) $x = a + b$, $y = c$, are all equal in area.

Prove also that if $a = 2mn + n^2$, $b = m^2 - n^2$, $c = m^2 + mn + n^2$, then $a^2 + ab + b^2 = c^2$. What result is obtained by putting $m = 2$, $n = 1$?

*46. Fig. 537 represents a horizontal section of the axles and wheels of a carriage with four wheels at full lock. The inner back wheel is travelling round a circle of radius 9 ft. Find the radius of the track traced out by the outer front wheel.

**47.** $AB$ is a fixed vertical wire and $C$ is a small fixed peg, 8 in. from $AB$, see fig. 538. $PCQ$ is a string 3 ft. long supporting a marble $Q$ and attached to a small ring $P$ which can slide on $AB$. Initially $Q$ is 19 in. below $C$. If $Q$ is pulled down 7 in., how much does $P$ rise?

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**SOLID GEOMETRY**

Examples for Oral Discussion

**CUBOID**

Standard Notation for Cuboid. The notation $ABCD.EFGH$ for a cuboid means that $ABCD$, $EFGH$ are parallel faces joined by $edges$ $AE$, $BF$, $CG$, $DH$.

1. A room is 17 ft. long, 14 ft. wide, 9 ft. high. Find the distance from a corner $A$ of the floor to the opposite corner $Q$ of the ceiling.

Let the diagonal $AC$ of the floor be $x$ ft. long, and the diagonal $AQ$ of the room be $y$ ft. long.

(i) Find $x^2$ from $\triangle ABC$.

(ii) Find $y^2$ from $\triangle ACG$.

Notice that it is unnecessary to find the value of $x$.

2. Find the length of a diagonal of a cuboid whose edges are 7 cm., 3 cm., 3 cm.
3. If, in fig. 541, ABCD is a rectangle whose diagonals intersect at N, and if V is any point on the perpendicular through N to the plane ABCD, prove that $\mathbf{VA} = \mathbf{VB} = \mathbf{VC} = \mathbf{VD}$. Prove that $\triangle VNA = \triangle VNB$.

Fig. 541 represents a right pyramid, vertex V, whose base is the rectangle ABCD; the length of VN is called the height of the pyramid.

4. In fig. 541, which represents a right pyramid on a rectangular base ABCD, if AB = 10 in., BC = 6 in., VA = 12 in., find the height VN. Make a sketch of fig. 541 and mark on it the mid-point P of AB.

(i) Find $\mathbf{VP}^2$ from the right-angled triangle VPA.
(ii) Find VN from the right-angled triangle VNP.

5. If, in fig. 542, ABC is an equilateral triangle whose medians intersect at N, and if V is any point on the perpendicular through N to the plane ABC, prove that $\mathbf{VA} = \mathbf{VB} = \mathbf{VC}$.

(i) Explain why $\mathbf{NA} = \mathbf{NB}$.
(ii) Explain why $\triangle VNA = \triangle VNB$.

Fig. 542 represents a right pyramid, vertex V, whose base is the equilateral triangle ABC; the length of VN is called the height of the pyramid.

Since N is the median-centre or centroid of $\triangle ABC$, $\mathbf{AN} = \frac{1}{2} \mathbf{AD}$, see p. 192. By using this fact, the length of AN can be found when the length of AB is given.

6. In fig. 542, which represents a right pyramid whose base is an equilateral triangle ABC, if BC = 6 cm., VA = 8 cm., find the height VN.

(i) Find $\mathbf{AD}^2$ from the right-angled triangle ADB.
(ii) Complete: $\mathbf{AN} = \frac{1}{2} \mathbf{AD}$, $\therefore \mathbf{AN}^2 = \ldots = \ldots$.
(iii) Find VN from the right-angled triangle VNA.

Alternatively, in place of (i) and (ii), find $\mathbf{AN}^2$ by using the fact that $\triangle ANF$ is "half an equilateral triangle," see p. 121.

7. Fig. 543 represents a sphere, centre O, and APB is a section of the surface made by any plane. NOS is the diameter of the sphere perpendicular to the plane APB and cuts this plane at K. Prove that the section APB is a circle, centre K.

8. In fig. 543, if the diameter of the sphere is 20 cm., and if the distance of the plane APB from the centre of the sphere is 6 cm., find the radius of the circular section APB. Use the right-angled triangle OKA.
NEW GEOMETRY

NUMERICAL EXAMPLES

EXERCISE 54

[Give answers which are approximate, correct to three figures.]

1. A room is 20 ft. long, 16 ft. wide, 8 ft. high. Find the length of the line joining a corner of the floor to the opposite corner of the ceiling.

2. Find the diagonal of a cube whose edge is 5 in.

3. What is the length of the longest straight rod, measuring a whole number of inches, that can be put into a rectangular box whose internal measurements are 6 ft. by 5 ft. by 4 ft.?

4. The edge of a cube is 6 in. Find the distance between the centres of two adjacent faces.

5. AB runs due east, BC runs due north, CD is vertical. If AB = 12 ft., BC = 6 ft., CD = 12 ft., find the distance of D from A.

6. Fig. 544 represents a closed box with a sloping lid; the base is a horizontal rectangle, 18 in. by 12 in., and the side faces are vertical. Find the area of the lid.

7. ABCD is the rectangular floor of a room of height 10 ft.; E is the corner of the ceiling above A. If AB = 24 ft., BC = 20 ft., find the area of ΔEBC.

8. Fig. 545 represents a circular cone, vertex V; VN is the axis of the cone, i.e. the line joining the vertex to the centre of the base. If the diameter of the base is 10 cm., and if the height VN of the cone is 12 cm., find the slant height VA of the cone.

9. If the slant height of a circular cone is 10 in. and the diameter of its base is 8 in., find the height of the cone.

10. If the slant height of a circular cone is 6 in. and if the height is 4 in., find the diameter of the base.

11. The base of a right pyramid of height 4 in. is a square whose diagonal is 6 in. Find the length of a sloping edge.

12. The base ABCD of a right pyramid, vertex V, is a square of side 6 cm. and the length of the perpendicular from V to AB is 5 cm. Find the height of the pyramid and the length of a sloping edge.

13. The base of a right pyramid of height 8 in. is a square whose side is 12 in. Find the area of a side face and the length of a sloping edge.

14. If, in fig. 541, p. 278, which represents a right pyramid on a rectangular base, AB = 8 cm., BC = 6 cm., VA = 13 cm., find the height VN of the pyramid.

15. If, in fig. 541, AB = 6 in., BC = 4 in., VN = 5 in., find the length of VA.

16. If, in fig. 542, p. 278, which represents a right pyramid whose base is an equilateral triangle ABC, AB = 9 cm., VA = 6 cm., find the height VN of the pyramid.

17. If, in fig. 542, AB = 12 in., VN = 4 in., find the length of VA.

18. A hemispherical bowl, internal diameter 20 in., contains some water. If the greatest depth of the water is 4 in., find the radius of the circle formed by the water-surface.

19. A circle of radius 2 in. is drawn on the surface of a sphere of diameter 6 in. Find the distance of the plane of the circle from the centre of the sphere.

20. ABCD is the base of a cuboid, and AP, BQ, CR, DS are the edges at right angles to the base; AB = 8 in., BC = 6 in., AP = 5 in.; H is the mid-point of BC, K is the mid-point of PQ. Find the distances (i) HP, (ii) HK, (iii) DK.

21. ABCD is a rectangle; AB = 6 in., BC = 8 in.; it is folded about BD so that the planes of the two parts are at right angles. Find the new distance of A from C.

22. Three points A, B, C on the surface of a sphere of diameter 8 in. are the vertices of an equilateral triangle of side 6 in. Find the distance of the plane of the triangle from the centre of the sphere.
THEOREM 40 (Pythagoras' Theorem)

In any right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the sides containing the right angle.

\[ \text{Fig. 546} \]

Given a triangle \( \triangle ABC \) in which \( \angle BAC = 90 \) right angle.

To prove that \( \text{sq. on BC} = \text{sq. on BA} + \text{sq. on AC} \).

Construction. Draw the squares \( BCPQ, ABHK, ACMN \), outside the triangle \( ABC \).

Through \( A \) draw a line \( AXY \) parallel to \( BQ \), and therefore also parallel to \( CP \), to meet \( BC \) at \( X \) and \( QP \) at \( Y \). Join \( AQ, CH \).

Proof. \( \angle BAC = 90 \) rt. \( \angle \) given,
\( \angle BAK = 90 \) rt. \( \angle \) of square

\[ \therefore \angle BAC + \angle BAK = 2 \times 90 \angle \delta. \]

\[ \therefore KA \text{ and } AC \text{ are in the same straight line.} \]
\[ \angle HBA = \angle CBQ \quad \text{rt. } \angle \delta \text{, } \angle \delta \text{ of squares.} \]

Add to each \( \angle ABC \),
\[ \therefore \angle HBC = \angle ABQ. \]

\[ \therefore \Delta HBC \text{ and } \Delta ABQ \text{ are congruent } \text{SAS}. \]

But \( \triangle HBC \) and sq. \( HBAK \) are on the same base \( HB \) and between the same parallels \( HB, KA, CH \),
\[ \therefore \text{area } HBC = \frac{1}{2} \text{ area } HBAK. \]

Also \( \triangle BQA \) and rect. \( BQYX \) are on the same base \( BQ \) and between the same parallels \( BQ, AX \),
\[ \therefore \text{area } BQA = \frac{1}{2} \text{ area } BQYX. \]
\[ \therefore \text{area } HBAK = \text{area } BQYX, \]

that is,
\[ \text{sq. on } BA = \text{rect. } BQYX. \]

Similarly, by joining \( AP, BM \) it can be proved that
\[ \text{sq. on } AC = \text{rect. } CQYX. \]
\[ \therefore \text{sq. on } BA + \text{sq. on } AC = \text{rect. } BQYX + \text{rect. } CQYX = \text{sq. on } BC. \]

Corollary. If the triangle \( ABC \) is right-angled at \( A \) and if \( AX \) is an altitude, then
\[ BA^2 = BX \cdot BC \text{ and } CA^2 = CX \cdot CB. \]

Area of square on \( AB = \text{area of rect. } BQYX = BX \cdot BQ. \)

But \( BQ = BC \) \( \text{side of square,} \)
\[ \therefore BA^2 = BX \cdot BC. \]

Similarly, area of square on \( AC = \text{area of rect. } CQYX. \)
\[ \therefore CA^2 = CX \cdot CP = CX \cdot CB. \]
Trigonometrical Proof of Pythagoras' Theorem

Let \( \triangle ABC \) be a triangle right-angled at \( A \), and \( AX \) an altitude.

With the notation of fig. 547:

from \( \triangle ABC \), \( \cos B = \frac{c}{a} \);

from \( \triangle AXB \), \( \cos B = \frac{p}{c} \);

\[ \therefore \frac{c}{a} = \frac{p}{c} \]

\[ \therefore \quad c^2 = ap \]

Similarly, \( b^2 = aq \). \( \therefore b^2 + c^2 = ap + aq = a(p + q) = a^2 \).

Note. The previous proof, pp. 282-3, depended on showing that \( BAX = BX \). \( BC \), that is \( c^2 = px^2 \); this was done by proving that the areas of the square and rectangle represented by these expressions are equal. The trigonometrical proof depends on the fact that equiangular triangles are the same shape.

Trigonometrical Equivalent of Pythagoras' Theorem

With the notation of fig. 547, \( b = a \sin B \) and \( c = a \cos B \).

\[ \therefore \quad a^2(\sin B)^2 + a^2(\cos B)^2 = a^2 \]

\[ \therefore \quad \sin B \cos B = 1 \]

This is written in the form \( \sin^2 B + \cos^2 B = 1 \).

By means of this formula, the cosine of an acute angle can be calculated when the sine is given, and vice versa.

For example, \( \cos 60^\circ = \frac{1}{2} \), see p. 121,

\[ \therefore \quad \sin^2 60^\circ = 1 - \cos^2 60^\circ = 1 - \frac{1}{2} = \frac{1}{2} \]

But \( \sin 60^\circ \) is positive, \( \therefore \quad \sin 60^\circ = \frac{\sqrt{3}}{2} \).

The Converse of Pythagoras' Theorem. The difficulty in remembering the construction used in Theorem 41 is diminished if the reason for it is understood. We cannot apply Pythagoras' theorem to the given \( \triangle ABC \) which we have to prove is right-angled. We therefore construct a right-angled \( \triangle XYZ \) as much like \( \triangle ABC \) as possible, and start the proof by applying Pythagoras' theorem to \( \triangle XYZ \).
EXERCISE 55

1. ABCD is a square. Prove that $AC^2 = 2AB^2$.

2. AD is an altitude of the equilateral triangle ABC. Prove that $4AD^2 = 3BC^2$. [Let BC = 2x units.]

3. ABCD is a quadrilateral in which $AB = BC = 2CD$ and $\angle ABD = \angle BCD = 1$ rt. $\angle$. Prove that $AD = 3CD$.

4. $APQ$ is a right line such that $AP = AB$ and $BQ = BA$. Prove that the triangle whose sides are equal to $AP$, $PQ$, $QB$ respectively is right-angled.

5. AB is an altitude of $\triangle ABC$. If $\angle C = 45^\circ$, prove that $AB^2 = BD^2 + DC^2$.

6. AD is an altitude of $\triangle ABC$. Prove that $AB^2 + CD^2 = AC^2 + BD^2$.

7. ABCD is a quadrilateral in which $\angle B = \angle D = 1$ rt. $\angle$. Prove that $AB^2 - AD^2 = CD^2 - CB^2$.

8. In fig. 549, AKD is perpendicular to BDC. Prove that $KB^2 - KC^2 = AB^2 - AC^2$.

9. In fig. 550, APB is perpendicular to AQC. Prove that $BQ^2 + CP^2 = BC^2 + PQ^2$.

10. If, in fig. 550, where $\angle A = 1$ rt. $\angle$, P and Q are the mid-points of AB and AC, prove that $BQ^2 + CP^2 = PQ^2$.

11. The diagonals of the quadrilateral ABCD cut at right angles. Prove that $AB^2 + CD^2 = AD^2 + BC^2$.

12. ABCD is a rhombus. Prove that $AC^2 + BD^2 = 4AB^2$.

13. Given two squares, show how to construct a square whose area is equal to (i) the sum of the areas of the given squares, (ii) the difference of the areas of the given squares.

14. $P$ is a point inside a rectangle ABCD. Prove that $PA^2 + PC^2 = PB^2 + PD^2$. [Drop perpendiculars from P to the sides of ABCD.] Is the result true if P lies outside ABCD?

15. $P$ is a point on the base BC of the equilateral triangle ABC such that $BP = \frac{1}{2} BC$. Prove that $PA^2 = \frac{7}{4} AB^2$. [Draw the altitude AD; let BC = 2x units.]

16. In $\triangle ABC$, PQR, $\angle B = \angle Q = 1$ rt. $\angle$. If $PQ = AB + BC$ and $QR = AB - BC$, prove that $PR$ is equal to the diagonal of the square on AC.

17. In $\triangle ABC$, $\angle B$ is a right angle, X is the mid-point of BC and $\angle X$ is the perpendicular from X to AC, prove that $AX^2 - NC^2 = AB^2$. [Join AX.]

18. ABCD is a quadrilateral in which $\angle B = \angle D = 1$ rt. $\angle$; $AH$, $CK$ are the perpendiculars from A, C to BD. Prove that $AH^2 + BK^2 = DH^2 + DK^2$.

19. ABCD is a quadrilateral in which $\angle ACB = \angle ADB = 1$ rt. $\angle$; AP, BQ are the perpendiculars from A, B to CD. Prove that $CP^2 + CQ^2 = DP^2 + DQ^2$.

*20. In fig. 546, p. 238, $\angle BAC = 1$ rt. $\angle$ and $AB = 2AC$, prove that $BX = 4XC$. [What do you know about the areas of rect. BY, rect. CY?]

*21. In fig. 546, p. 238, $\angle BAC = 1$ rt. $\angle$ and $BX = 2XC$, prove that $AB = 1\frac{1}{2} AC$.

*22. In $\triangle ABC$, $AB = AC = 2BC$. If $BF$ is an altitude, prove that $CF = \frac{1}{2} CB$. [In fig. 529, p. 270, what do you know about rect. FM?]

*23. In fig. 547, p. 238, $\angle BAC = 1$ rt. $\angle$ and AX is perpendicular to BC. Prove that $AX^2 = BX \cdot XC$. [Let $AX = h$ units; use the small letters in the diagram and apply Pythagoras to each of the three right-angled triangles.]

Can you also prove this result trigonometrically?

24. The base BC of $\triangle ABC$ is produced to any point P. If $AB = AC$, prove that $AP^2 = AB^2 + PB \cdot PC$. [Draw the altitude AD; let $BC = 2x$ units, $CP = y$ units.]

*25. ABC is a straight line; ABXY, BCPF are squares on the same side of AC. Prove that $PX^2 + CY^2 = 3(AB^2 + BC^2)$.

*26. O is any point inside $\triangle ABC$; OP, OQ, OR are the perpendiculars from $O$ to $BC$, $CA$, $AB$ respectively. Prove that $BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2$. 

PYTHAGORAS’ THEOREM

287
27. If, in fig. 529, p. 270, which represents the squares on the sides of any triangle ABC, T is any point inside ∆ABC, prove that \( TQ^2 + TM^2 + TK^2 = TP^2 + TN^2 + TH^2 \).

28. Given a line AB, construct, when possible, a point P in AB such that the sum of the squares on AP and PB is equal to the area of a given square. When is it impossible? [Use the fact stated in No. 5.]

29. Given a line AB, construct a point P in AB such that \( AP^2 = 2PB^2 \). [Draw ∆AOB so that \( ∠QAB = 45°, ∠QBA = 22\frac{1}{2}° \), and draw the perpendicular bisector of BQ.]

30. Prove that, if A is any point on the circle whose diameter is BC, then \( ∠BAC \) is a right angle. Use this fact and fig. 530, p. 270, to construct a square equal in area to a given rectangle.

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**REVISION PAPERS 35-42 (Theorems 1-39)**

Including areas of parallelograms and triangles.

35

1. In ∆ABC, \( AB = AC \) and D is a point on AC such that \( BD = BC \). If \( ∠CBD = x^\circ \), find \( ∠DBA \) in terms of x.

2. ABC is an equilateral triangle; P, Q are points on BC, CA respectively, such that \( BP = CQ \); AP cuts BQ at R. Prove that (i) \( AP = BQ \); (ii) \( ∠BRA = 120° \).

3. Construct a parallelogram ABCD such that \( ∠B = 60° \), \( BC = 3 \text{ in.} \), and area of ABCD is 6 sq. in. Explain your method and measure \( AB \).

4. In fig. 551, ABCD is a parallelogram, and PR, QS are parallel to AB, AD. Prove that area \( ABCD = \text{twice area} \ PQRS \).

36

1. The angles of a triangle are \( x^\circ, x^\circ + y^\circ, x^\circ - 2y^\circ \), and one of the exterior angles is \( 3x^\circ - y^\circ \). Find the angles of the triangle. [Three sets of answers.]

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**REVISION PAPERS 37**

37

1. The base BC of ∆ABC is produced to D; DA is joined and produced to E. If \( AB = AC = CD \), prove that \( ∠BAE = 3∠BDA \).

2. H is the mid-point of the side CD of the parallelogram ABCD; AH, BH produced meet BC, AD produced at X, Y respectively. Prove that (i) \( AH = AX \); (ii) \( XY \) is parallel to \( AB \).

3. Draw a parallelogram ABCD in which \( AB = 3 \text{ in.}, BC = 2 \text{ in.}, ∠B = 42° \). Construct ∆PAB equal in area to ABCD and such that \( ∠PBA = 45° \). Measure PB.

4. The side BC of the parallelogram ABCD is produced to any point K. Prove that \( ∠ABK = \text{quad.} ASDK \).

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42

1. In fig. 552, \( ∠PQC = ∠RQA \), \( ∠QRC = ∠SRB \), and \( \text{CQA}, \text{CRB, PRS} \) are straight lines. Prove that \( ∠QPR = 2∠ACB \).

2. K is any point inside an equilateral triangle ABC; BKD, CKE are equilateral triangles on the same sides of the bases BK, CK as A is. Prove that (i) \( ∠ADB = ∠ACB \); (ii) \( AD = KE \); (iii) \( ADKE \) is a parallelogram.

3. Draw ∆ABC such that \( AB = 5 \text{ cm.}, BC = 3 \text{ cm.}, ∠B = 30° \). Construct one position of a point P such that \( PA = PB \) and the area of ∆PBC equals 4.5 sq. cm. Explain your method shortly.

4. The diagonals of the quadrilateral ABCD cut at K. If \( ∠AKB = ∠BKC = ∠CKD \), prove that ABCD is a parallelogram.
39
1. P is a point on the side AB of \( \triangle ABC \) such that \( AP = PC = CB \). If \( CP \) bisects \( \angle ABC \), calculate \( \angle BAC \).

2. BDEC is a square on the side BC of \( \triangle ABC \) outside the triangle. Lines through B, C parallel to AD, AE meet at P. Prove that (i) \( PA = BC \); (ii) \( PA \) is perpendicular to BC.

3. M is the mid-point of the side BC of \( \triangle ABC \); MP, MQ are the perpendiculars from M to AB, AC respectively. If \( AB = 8 \text{ in.} \), \( AC = 12 \text{ in.} \), MP = 6 in., calculate the length of MQ.

4. ABCD is a parallelogram; Q is any point on AB produced. Prove that \( \triangle AQD - \triangle BQC = \triangle DQC \). [Join BD.]

40
1. In \( \triangle ABC \), AB = AC and \( \angle B = 51^\circ \). K is a point on AC produced such that \( \angle AKB = 25^\circ \). Which is the longer, (i) BC or KC, (ii) BK or AK?

2. In \( \triangle ABC \), \( \angle BAC = 90^\circ \); BDEC is a square outside \( \triangle ABC \); DX is the perpendicular from D to AC. Prove that DX = AB + AC. [Draw the perpendicular BY from B to DX.]

3. Draw a convex quadrilateral ABCD in which AB = 2 in., BC = 3 in., CD = 4 in., \( \angle B = 70^\circ \). Construct a point X on BC produced such that \( \angle ABX = \text{quad. ABCD} \). Find the area of ABCD.

4. P, Q, R are points on the sides BC, CA, AB respectively of \( \triangle ABC \) such that PQ is parallel to AB, and QR is parallel to BC. Prove that \( \angle ABP = \angle ACQ \).

41*
1. The sides AB, AC of \( \triangle ABC \) are produced to D, E; AH, AK are lines parallel to the bisectors of \( \angle BGE, \angle CBD \) meeting BC in H, K. Prove that (i) \( AC = CH \); (ii) \( AB + AC = BC + AH \).

2. In Fig. 553, KE, KF are the bisectors of \( \angle AEB, \angle AFD \). If \( \angle AEB, \angle AFD \) are supplementary, prove that (i) \( \angle KEF + \angle KEF = \angle EAF \); (ii) \( \angle EKF \) is a right angle.

REVISION PAPERS 43–50 (Theorems 1–41)

(Including Pythagoras' Theorem.)

43
1. ABC is an acute-angled triangle; BAHK, CAXY are squares outside \( \triangle ABC \). If BH and CX when produced meet at P, prove that (i) \( \angle BPA = \angle CPA \); (ii) \( \triangle BPC + \triangle BAC = 90^\circ \).

2. (i) \( \triangle ABC \) is a rectangle in which AB = 18 in., AC = 10 in.; C is a point on XY such that \( \angle X = 90^\circ \). Find (i) the area of \( \triangle ABC \), (ii) the length of AC, (iii) the length of the perpendicular from C to XY.

(ii) In \( \triangle ABC \), \( \angle A = 90^\circ \), \( \angle B = 30^\circ \), BC = 6 in.; calculate the area of \( \triangle ABC \).
NEW GEOMETRY

3. \( \triangle ABP, \triangle ABQ \) are triangles of equal area on opposite sides of \( AB \). Prove that \( AB \) (produced if necessary) bisects \( PQ \).

4. \( AD \) is an altitude of the acute-angled triangle \( ABC \). If \( BD = 2DC \), prove that \( AB^2 = AC^2 + 3CD^2 \).

44

1. In \( \triangle ABC \), \( \angle A = 120^\circ \), \( P, Q \) are points on \( BC \) such that \( BP = BA \) and \( CQ = CA \). Prove that \( \angle PAQ = 30^\circ \).

2. \( APQR \) is a rectangle in which \( AP = 4 \text{ in.} \), \( AR = 11 \text{ in.} \). \( B, C \) are points on \( QR, QP \) respectively such that \( RB = 3 \text{ in.} \), \( QC = 8 \text{ in.} \). Calculate (i) the area of \( \triangle ABC \); (ii) the length of the line joining the mid-points of \( BA \) and \( BC \).

3. \( ABCD \) is a parallelogram; \( P \) is any point on \( BD \). Prove that \( \triangle PAB = \triangle PBC \). [Let \( AC \) cut \( BD \) at \( K \).]

4. \( ABC \) is an equilateral triangle; \( BC \) is bisected at \( D \) and produced to \( E \) so that \( CE = DC \). Prove that \( AE^2 = 7EC^2 \). [Let \( BC = 2x \text{ units.} \]

45

1. (i) In \( \triangle ABC \), \( AB = 5 \text{ in.} \), \( BC = 6 \text{ in.} \), \( \angle ABC = 45^\circ \). Calculate the area of \( \triangle ABC \) and the length of \( AC \).

(ii) \( ABCD \) is a quadrilateral in which \( \angle A = \angle B = 90^\circ \), \( AB = 6 \text{ in.} \), \( BC = 10 \text{ in.} \), \( AD = 18 \text{ in.} \). Prove that \( BD \) bisects \( \angle ADC \).

2. In \( \triangle ABC \), \( AB > AC \); \( CT \) is the perpendicular from \( C \) to the line bisecting \( \angle BAC \); \( D \) is the mid-point of \( BC \). Prove that (i) \( DT \) is parallel to \( BA \); (ii) \( DT = \frac{1}{2}(AB - AC) \). [Produce \( CT \) to meet \( AB \) at \( K \).]

3. \( K \) is the mid-point of the diagonal \( BD \) of the quadrilateral \( ABCD \). Prove that the difference between the areas of \( \triangle ABC \) and \( \triangle ADC \) is equal to twice the area of \( \triangle AKC \) [Let \( AC \) cut \( BD \) at \( N \).]

4. In \( \triangle ABC \), \( \angle ACB \) is a right angle. If \( D \) is the mid-point of \( BC \), prove that \( AB^2 = AD^2 + 3BD^2 \).

46

1. In \( \triangle ABC \), \( AB = AC \); \( P \) is any point on \( BC \). If the perpendicular bisectors of \( PB, PC \) meet \( AB, AC \) respectively at \( X, Y \), prove that \( AXPY \) is a parallelogram.

2. \( ABCD \) is a trapezium of area 36 sq. cm, in which \( AB \) is parallel to \( DC \), and \( AD = BC \). If \( AB = 12 \text{ cm.} \), and \( CD = 6 \text{ cm.} \), find the length of \( BC \).
NEW GEOMETRY

4. AD is an altitude of ΔABC; P, Q are points on AD produced such that DP = AB and DQ = AC. Prove that BQ = CP.

49*

1. (i) Find the area of a triangle whose sides are \( (p^2 - q^2), 2pq, (p^2 + q^2) \) in. Give reasons.
   (ii) ABCD is a parallelogram in which AB = 4 in., BC = 5 in., \( \angle ABC = 60^\circ \). Calculate the area of ABCD, the distance of A from CD, and the length of BD.

2. In fig. 557, \( \angle ABC = 90^\circ \), AK = BC, and E, F are the mid-points of AC, KB. Prove that \( \angle AFE = 90^\circ \). [Draw CQ parallel to EF to cut AB produced at Q.]

3. ABCD is a quadrilateral in which \( \angle ABC = \angle ADC = 90^\circ \).
   Find a relation between the lengths of the sides of ABCD.
   If AP, AQ are drawn parallel to CD, CB cutting CB, CD at P, Q respectively, prove that QA, AB = PA, AD.
   [Use area-formulae.]

4. ABCD is the base of a cuboid, and AP, BQ, CR, DS are the edges perpendicular to the base; AB = 4 in., AP = 3 in. If K is the mid-point of PQ and if AK = AD, find the length of AB.

50*

1. ABCD is a rhombus in which \( \angle BAD = 120^\circ \) and AB = 10 cm. Calculate the lengths of AC and BD.

2. ABCD is a square, side 3 cm., in a horizontal plane; AP is a vertical line 4 cm. long. Find the area of ΔAPC and the length of PC.

3. ABCD is a parallelogram; any line parallel to RA cuts BC. AC, AD at X, Y, Z respectively. Prove that ΔAXY = ΔDYZ.
   [Join CZ.]

4. AKB, CKD are two fixed lines such that \( \angle BKD = 50^\circ \); P is a variable point such that its distance from AKB exceeds its distance from CKD by 1 cm. Draw a figure and show on it the precise locus of P. [Find two straight lines from which P is equidistant.]

PART II

(Section 2)

THE CIRCLE

Symmetrical Properties of the Circle

The chief definitions relating to the circle were given on p. 10, and many properties of the circle have already been discussed or given as examples, particularly in connection with loci.

It has been proved, see pp. 200, 201, that the locus of a point equidistant from two fixed points is the perpendicular bisector of the line joining the two fixed points.

This property involves two distinct theorems:

1. If PA = PB, then P lies on the perpendicular bisector of AB.

2. If Q lies on the perpendicular bisector of AB, then QA = QB.

If A and B are any two points on the circumference of a circle, centre O, then OA = OB radii,

\( \therefore \) O lies on the perpendicular bisector of AB.

In words,

The centre of a circle lies on the perpendicular bisector of any chord of the circle.

This statement is substantially equivalent to the facts expressed in Theorems 42, 43; but independent proofs of these theorems are given on pp. 300, 301 to meet examination requirements.

By using both facts established in the locus theorem, it can be proved, see p. 304, that one and only one circle can be drawn through three given points, not in the same straight line. The size and position of a circle is therefore fixed completely if three points A, B, C on its circumference are given; and so we may speak of the circle ABC without ambiguity because there is only one circle which passes through A, B, C.
Examples for Oral Discussion

1. If in fig. 558, \(AB = 8\) cm. and the radius of the circle is 5 cm., find the distance \(OH\) of the centre \(O\) from \(AB\).
   (i) What is the length of \(AH\)? Give reasons.
   (ii) Use Pythagoras to find the length of \(OH\).

2. If in fig. 558, the distance \(OK\) of the centre \(O\) from \(CD\) is 6 in. and the radius is 7.5 in., find the length of \(CD\).

3. If two chords of a circle are unequal, prove that the greater is nearer the centre.

Given in fig. 558 that \(AB > CD\) and that \(OH, OK\) are the perpendiculars from the centre \(O\) to \(AB, CD\), prove that \(OH < OK\).
   (i) Explain why \(AH > CK\).
   (ii) Explain why \(AH^2 + KO^2 = CK^2 + KO^2\).

4. Calculate the circumradius, \(r\) cm., of \(\triangle ABC\), given that \(AB = AC = 15\) cm., \(BC = 18\) cm.
   (i) Explain why the circumcentre \(O\) lies on the line joining \(A\) to the mid-point \(N\) of \(BC\).
   (ii) Find the length of \(AN\); then express the length of \(ON\) in terms of \(r\).
   (iii) Use \(\triangle ONB\) to find \(r\).

NUMERICAL EXAMPLES

EXERCISE 56

[Give answers which are approximate, correct to 3 figures.]

1. A chord of length 10 cm. is at a distance of 12 cm. from the centre of the circle. Find the radius.

2. A chord of a circle of radius 6 cm. is 8 cm. long. Find the distance of the chord from the centre.

3. A chord of a circle of radius 7 cm. is at a distance of 4 cm. from the centre. Find the length of the chord.

4. \(PQ\) is a variable chord of a given circle of radius 7.5 cm. If \(PQ = 9\) cm., find the locus of the mid-point of \(PQ\).

5. \(N\) is a point on the diameter \(AB\) of a circle \(APBQ\); \(PNQ\) is the chord through \(N\) perpendicular to \(AB\). If \(AN = 8\) cm., \(NB = 2\) cm., find the length of \(PQ\).

6. In a circle of radius 5 cm., there are two parallel chords of lengths 6 cm., 4 cm. Find the distance between the chords. [Two answers.]

7. Two concentric circles are of radii 7 in., 4 in.; a line \(PQRS\) cuts one circle at \(P, QR\) and the other at \(Q, R\). If \(QR = 6\) in., find the length of \(PS\).

8. A chord of a circle is 10 cm. long and is 4 cm. from the centre. Find the length of a chord which is 3 cm. from the centre.

9. \(AB, CD\) are two chords of a circle, centre \(O\), radius 7 in., which meet at \(P\) at right angles. If \(AB = 6\) in., \(CD = 10\) in., find the length of \(OP\).

10. The length of the common chord of two equal intersecting circles is 10 cm., and the distance between the two centres is 6 cm. Find the radius of each circle.

11. \(AB, CD\) are two chords of a circle at distances \(d\) in., 7d in. from the centre. If \(AB = 2CD\), find the length of \(CD\) in terms of \(d\).

12. A hemispherical bowl, internal diameter 12 in., is partly full of water. If water-surface is 4 in. below centre of bowl, find the diameter of the circular water-surface.

E.G.
13. If fig. 560 represents a section of a circular cone, vertex V, by a plane through its axis VN, and the section of the sphere of radius 5 in. in which the cone is inscribed, and if the base-radius of the cone is 3 in., find the height VN of the cone.

14. In fig. 560, if VA = VB = 10 cm. and if the diameter of the circle VAB is 12 cm., find the lengths of VN and AB. [Let ON = h units, AN = d units, where O is the centre.]

[15] In fig. 560, if VA = VB = 13 in. and AB = 10 in., find the radius of the circle VAB. [Use the right-angled triangle OAN where O is the centre; let the radius = r in.]

[16] If, in fig. 560, VAB is an equilateral triangle of side 6 cm., find the radius of the circle VAB. [Note that if O is the centre, ∆AON is "half an equilateral triangle."]

17. The perpendicular bisector ND of a chord AB of a circle, centre O, cuts AB at N and the circle at D, see fig. 561. If AB = 6 in., ND = 1 in., find the radius of the circle.

18. If, in fig. 561, AN = NB = r in. and ND = h in., prove that the diameter of the circle ADB is \( \sqrt{r^2 + h^2} \) in.

19. In fig. 562, AB, CD are parallel chords of a circle, 3 in. apart. If AB = 4 in., CD = 10 in., find the radius of the circle. [Take the centre O; join OA, OC; let ON = x in., OC = r in.]

20. A crescent is formed of two circular arcs ACB, ADB, of equal radius, centres E, F, see fig. 563; the perpendicular bisector of AB cuts the crescent at C, D; CD = 5 cm., AB = 12 cm. Prove that EF = CD and find the radius of the arcs.

21. Two spheres, radii 6 in., 8 in., have their centres 10 in. apart. Find the radius of the circle in which the spheres cut one another, and the distances of the plane of this circle from the centres of the spheres.

22. A thin rectangular plate, 3 in. by 4 in., rests horizontally in a hemispherical bowl, 13 in. in diameter. Find the height of the centre of the plate above the lowest point of the bowl.

23. O is the centre of a horizontal circle, diameter 3 in., and OA is a horizontal line of length 2½ in.; P is a point 2 in. vertically above A. If C is the centre of the sphere whose surface passes through the circle and the point P, find the length of CO and the radius of the sphere.
THEOREM 42

The straight line which joins the centre of a circle to the middle point of a chord, which is not a diameter, is perpendicular to the chord.

Given a circle, centre O, and a chord AB whose mid-point is M, where M and O are different points.

To prove that \( \angle OMA \) is a right angle.

Construction. Join OA, OB.

Proof. In \( \triangle OMA \), \( OMB \),
\[
\begin{align*}
OA &= OB \quad \text{radii}, \\
AM &= BM \quad \text{given}, \\
OM &= OM
\end{align*}
\]
\[\therefore \triangle OMA \cong \triangle OMB.\]
\[\therefore \angle OMA = \angle OMB.\]

But these are adjacent angles on a straight line,
\[\therefore \angle OMA \text{ is a right angle.}\]

Corollary. The perpendicular bisector of a chord of a circle passes through the centre of the circle.

If the chord is not a diameter, the line joining the centre of the circle to the mid-point of the chord is the perpendicular bisector of the chord.

If the chord is a diameter, its mid-point is the centre of the circle.

CHORDS OF A CIRCLE

THEOREM 43

The straight line drawn from the centre of a circle perpendicular to a chord bisects the chord.

Given a circle, centre O, and the perpendicular ON from O to a chord AB.

To prove that \( AN = NB \).

Construction. Join OA, OB.

Proof. In \( \triangle OAN \), \( ONB \),
\[
\begin{align*}
OA &= OB \quad \text{radii}, \\
ON &= ON \\
\angle ONA &= \angle ONB \quad \text{rt. \( \angle s \), given},
\end{align*}
\]
\[\therefore \triangle ONA \cong \triangle ONB \quad \text{RHS}.
\]
\[\therefore AN = BN.\]

Corollary. The locus of the mid-points of parallel chords of a circle is a diameter of the circle.

Since the chords are parallel, the diameter perpendicular to one of the chords is perpendicular to each of the others and therefore bisects it.

Conversely, any point on this diameter is the mid-point of the chord through this point perpendicular to the diameter.
THEOREM 44

If two chords of a circle are equal, they are equidistant from the centre.

**Given** a circle O and two equal chords AB, CD.

**To prove that** the perpendiculars OH, OK from O to AB, CD are equal.

**Construction.** Join OA, OC.

**Proof.** Since the perpendicular from the centre of a circle to a chord bisects the chord,

\[ AH = \frac{1}{2} AB \text{ and } CK = \frac{1}{2} CD; \]

but \( AB = CD \) given,

\[ \therefore \ AH = CK. \]

In \( \triangle OHA, OKC \),

\[ OA = OC \text{ radii,} \]
\[ AH = CK \text{ proved,} \]
\[ \angle OHA = \angle OKC \text{ rt. \( \angle s \) given,} \]

\[ \therefore \ \triangle OHA \text{ and } OKC \text{ are congruent \( RHS \).} \]

\[ \therefore \ OH = OK. \]

**Corollary.** In equal circles, equal chords are equidistant from the centres.

The proof is similar.

---

THEOREM 45

If two chords of a circle are equidistant from the centre, their lengths are equal.

**Given** a circle centre O and two chords AB, CD such that the perpendiculars OH, OK from O to AB, CD are equal.

**To prove that** \( AB = CD \).

**Construction.** Join OA, OC.

**Proof.** In \( \triangle OHA, OKC \),

\[ OH = OK \text{ given,} \]
\[ OA = OC \text{ radii,} \]
\[ \angle OHA = \angle OKC \text{ rt. \( \angle s \) given,} \]
\[ \therefore \ \triangle OHA \text{ and } OKC \text{ are congruent \( RHS \).} \]

\[ \therefore \ AH = CK. \]

Since the perpendicular from the centre of a circle to a chord bisects the chord,

\[ AH = \frac{1}{2} AB \text{ and } CK = \frac{1}{2} CD, \]

\[ \therefore \ AB = CD. \]

**Corollary.** In equal circles, chords which are equidistant from the centres are equal.

The proof is similar.
THEOREM 46

There is one circle and only one circle which passes through three given points not in the same straight line.

Given three points A, B, C not in the same straight line.

To prove that one, and only one, circle can be drawn to pass through A, B, C.

Construction. Join AB, BC.

Draw the perpendicular bisectors XH, YK of AB, BC.

Proof. Since AB and BC are not in the same straight line, the perpendiculars XH, YK are not parallel and therefore intersect at a point O.

Since XH is the perpendicular bisector of AB, it is the locus of points equidistant from A and B.

Similarly, YK is the locus of points equidistant from B and C.

∴ the point of intersection O of XH, YK is equidistant from A, B, C.

But any point which is equidistant from A, B, C must lie on XH and on YK,

∴ O is the only point which is equidistant from A, B, C.

∴ the circle, centre O, radius OA, passes through A, B, C, and there is no other circle which passes through A, B, C.

Corollary. Two distinct circles cannot intersect in more than two points.

If two distinct circles cut at 3 points A, B, C, there would be more than one circle passing through A, B, C.

The Circumcircle of a Triangle. Given a triangle ABC, the circle ABC is called the circumcircle of \( \triangle ABC \), its centre is called the circumcentre and its radius is called the circumradius, see p. 202. The method for constructing the circumcentre and circumcircle is given in Theorem 46.

To construct the centre of a circle an arc of which is given, it is sufficient to take any three points A, B, C on the arc and construct the circumcentre of \( \triangle ABC \), because only one circle can be drawn through three given points.

Symmetry about an Axis. If one part of a figure can be made to coincide with the rest of the figure by folding it about a straight line AB, the figure is said to be symmetrical about AB, and the straight line AB is called an axis of symmetry of the figure. It then follows that the length of any line or the size of any angle in one-half of the figure is equal to the length of the corresponding line or the size of the corresponding angle in the other half of the figure.

If, in fig. 569, P coincides with Q when the figure is folded about AB, and if PQ cuts AB at N, \( \angle PNA \) coincides with \( \angle QNA \), and therefore each is a right angle; also PN = QN.

Therefore, if P and Q are corresponding points for an axis of symmetry AB, the perpendicular bisector of PQ is AB. Conversely, if AB is the perpendicular bisector of PQ, then P and Q are corresponding points for the axis of symmetry AB; and we say that Q is the image of P in AB, and that P is the image of Q in AB.

If, in fig. 570, O is the centre and AB is a diameter of the circle AHBE, and if the figure is folded about AB, all points on the semicircle AHB in their new positions are still at a distance from O equal to the radius, and therefore coincide with points on the semicircle AHK. Therefore

A circle is symmetrical about any diameter.
Many properties of the circle which are proved by congruent triangles can also be proved by using this fact:

*eg. Theorems 42, 43, and the property that the common chord of two intersecting circles is bisected at right angles by the line joining the centres.*

**EXERCISE 57**

1. A straight line $PQRS$ cuts two concentric circles at $P$, $S$ and $Q$, $R$. Prove that $PQ = RS$. [Draw the perpendicular from the centre of the circles to $P$; no other construction.]

2. If $AB$ and $CD$ are equal chords of a circle, centre $O$, prove that $\angle AOB = \angle COD$.

3. $M$ is the mid-point of a chord $AB$ of a circle, centre $O$; $MO$ is produced to any point $T$. Prove that $TM$ bisects $\angle ATB$.

4. $O$ is the centre of the circle $PQRS$. If $PQ = RS$, prove that $PR = QS$. [Join $O$ to $P$, $Q$, $R$, $S$.]

5. Given a point $K$ inside a given circle, show how to construct the chord $AKB$ such that $AK = KB$.

6. Two circles, centres $A$, $B$, intersect at $X$, $Y$. Prove that $AB$ bisects $XY$ at right angles.

7. Two circles intersect at $X$, $Y$; a line $PQRS$ parallel to $XY$ cuts one circle at $P$, $S$ and the other circle at $Q$, $R$. Prove that $PQ = RS$. [Use No. 6.]

8. Two circles, centres $A$, $B$, intersect at $C$, $D$; $PCQ$ is a line parallel to $AB$ cutting the circles at $P$, $Q$. Prove that $PQ = 2AB$.

9. $APB$, $CPD$ are intersecting chords of a circle, centre $O$. If $OP$ bisects $\angle APC$, prove that $AB = CD$. [Draw $OH$, $OK$ perpendicular to $AB$, $CD$.]

10. $AB$ and $CD$ are two equal chords of a circle; $M$, $N$ are their mid-points. Prove that $MN$ makes equal angles with $AB$ and $CD$.

11. $PQ$ is a variable chord of given length of a given circle. Find the locus of the mid-point of $PQ$.

12. A chord $AB$ of a circle, centre $O$, is produced to $P$ so that $BP = 2AB$. Prove that $OP^2 = OA^2 + 6AB^2$. [Draw $ON$ perpendicular to $AB$; let $ON = d$ units, $AN = t$ units.]

13. $A$, $B$, $C$ are three points on a circle such that $\angle ABC$ is a right angle. Prove that the mid-point of $BC$ is the centre of the circle. [Draw the perpendicular bisectors of $AB$, $AC$.]

14. Two chords of a circle bisect each other at $K$. Prove that $K$ is the centre of the circle. [If possible, join $K$ to the centre.]

15. $ABCD$ is a given quadrilateral. Show how to construct two concentric circles, one of which passes through $A$, $B$ and the other through $C$, $D$. What can you say about $ABCD$ (i) if there is more than one solution, (ii) if two distinct circles cannot be drawn, (iii) if there is no solution?

16. The diagonals of the quadrilateral $ABCD$ meet at $O$; circles are drawn through $A$, $O$, $B$, $D$, $O$, $A$. Prove that their four centres are the vertices of a parallelogram.

17. In fig. 571, if $PXQ$ is parallel to $RYS$, prove that $PQ = RS$. [Draw perpendiculars from the centres to $PQ$, $RS$.]

18. If $A$, $B$ are the centres of the circles in fig. 571, prove that $PQ$ is not greater than $2AB$.

19. A variable circle cuts off equal intercepts from two given lines $AB$, $AC$, produced if necessary. Find the locus of its centre.

20. $ABC$ is a given triangle. Show how to construct a circle to pass through $B$ and $C$ and to cut off equal intercepts from $AB$, $AC$, produced if necessary.

21. In fig. 572, $AX$, $BY$ are the perpendiculars to a chord $PQ$ from the ends $A$, $B$, $D$, $S$. Prove that $XP = XQ$.

Is the same result true if $P$, $Q$ lie on opposite sides of $AB$? [Use the intercept theorem.]

22. Two circles, centres $A$, $B$, cut at $X$, $Y$; $M$ is the mid-point of $AB$. If the line through $X$ perpendicular to $MX$ cuts the circles again at $P$ and $Q$, prove that $XP = XQ$. [Draw the perpendiculars from $A$, $B$ to $PQ$ and use the intercept theorem.]
23. In fig. 573, A, C, B are the centres of three unequal circles. If $AC=CB$, prove that $PQ=RS$.

[PS is not parallel to AB.]

*24. Two variable chords $PAQ$, $RAS$ of a fixed circle intersect at right angles at a fixed point $A$. Prove that $PQ^2 + RS^2$ is constant.

*25. $X$, $Y$ are mid-points of the chords $AB$, $CD$ of a circle, centre $O$. $XN$, $YM$ are the perpendiculars from $X$, $Y$ to $CD$, $AB$ respectively. If $XN$ cuts $YM$ at $P$, prove that $OP$ and $XY$ bisect each other.

*26. $P$ is any point on a diameter $AB$ of a circle; $QPR$ is a chord such that $\angle APQ = 45^\circ$. Prove that $AB^2 = 2PQ^2 + 2PR^2$.

**Angle Properties of a Circle**

Definitions. (1) Any part of the circumference of a circle is called an arc of the circle: it is called a minor arc if it is less than half the circumference and a major arc if it is greater than half the circumference.

(2) The plane figure bounded by a chord of a circle and either of the arcs it cuts off is called a segment of the circle: it is called a minor segment if it is less than a semicircle and a major segment if it is greater than a semicircle. In fig. 574, the figure bounded by the chord $AB$ and the minor arc $APB$ is a minor segment; the figure bounded by the chord $AB$ and the major arc $AOB$ is a major segment.

(3) The figure bounded by two radii of a circle and either of the arcs they cut off is called a sector of a circle.

In fig. 575, the figure bounded by the radii $OC$, $OD$ and the arc $CPD$ is a sector; so also is the figure bounded by the radii $OC$, $OD$ and the arc $CQD$.

(4) Any number of points are said to be concyclic if a circle can be drawn to pass through all of them. If the four vertices of a quadrilateral are concyclic, the quadrilateral is called a cyclic quadrilateral.

**Examples for Oral Discussion**

1. In fig. 576, $O$ is the centre of the circle and $PON$ is a straight line.

(i) Name two equal angles in the figure.

(ii) If $\angle APN = 24^\circ$, find $\angle AON$.

(iii) If $\angle AON = 52^\circ$, find $\angle APN$.

(iv) If $\angle APN = x^\circ$, find $\angle AON$.

**Give reasons for each answer.**

2. In fig. 577, $O$ is the centre of the circle and $QON$ is a straight line.

(i) Name two pairs of equal angles in the figure.

(ii) If $\angle AON = 20^\circ$ and $\angle BQN = 50^\circ$, find $\angle AON$ and $\angle BON$. Find also $\angle AQB$ and $\angle AOB$.

(iii) If $\angle AON = x^\circ$ and $\angle BQN = y^\circ$, find $\angle AQB$ and $\angle AOB$ in terms of $x$ and $y$.

(iv) What can you say about $\angle AQB$ if $AOB$ is a straight line?
3. In fig. 578, O is the centre of the circle and RON is a straight line.

(i) Name two pairs of equal angles in the figure.
(ii) If $\angle ARN = 30^\circ$ and $\angle BRN = 70^\circ$, find $\angle AON$ and $\angle BON$. Find also $\angle ARB$ and $\angle AOB$.
(iii) If $\angle ARN = x^\circ$ and $\angle BRN = y^\circ$, find $\angle ARB$ and $\angle AOB$ in terms of $x$ and $y$.

![Fig. 578](image)

![Fig. 579](image)

4. In fig. 579, O is the centre of the circle AQRB. Use the facts proved in 2 (iii) and 3 (iii) to state the connection between the sizes of $\angle AOB$, $\angle AQB$, $\angle ARB$. Also express the fact in the form of a general statement.

**Angles at the Centre and Circumference of a Circle**

![Fig. 580](image)

In fig. 580 (i), (ii), $\angle p$ is an angle whose vertex is at the centre O of the circle and is called the angle at the centre standing on the arc XKY; $\angle a$ is an angle whose vertex A is a point on the circumference of the circle and is called an angle at the circumference standing on the arc XKY.

Thus in fig. 580, $\angle p$ and $\angle a$ are angles standing on the same arc, one at the centre and the other at the circumference.

It is necessary to use three letters to name the arc XKY on which the angles stand in order to distinguish it from the arc XHY. The angle at the centre standing on the arc XHY is $\angle q$, see fig. 581 (i), (ii), and an angle at the circumference standing on the arc XHY is $\angle b$.

![Fig. 581](image)

We also say that the arc XHY subtends $\angle q$ at the centre and subtends $\angle b$ at the point B on the circumference.

It is important to learn how to see at a glance the arc on which an angle stands:

Look at the two points in which the arms of the angle cut the circumference.

The arms of $\angle q$ in fig. 581 cut the circumference in X and Y, and the arc which lies inside the angle $q$ is XHY, not XKY, and we therefore say that $\angle q$ stands on the arc XHY.

Similarly, the arms of $\angle b$ in fig. 581 cut the circumference in X and Y, and the arc which lies inside the angle $b$ is XHY, and we therefore say that $\angle b$ stands on the arc XHY.
Angles in a Segment

Fig. 582 (i), (ii) show a circle XHYK; XHY is a segment of the circle bounded by the chord XY and the arc XHY; A, B, C are any points on the arc XHY.

We say that \( \angle XAY, \angle XBY, \angle XCY \) are angles in the segment XHY.

The vertex of any angle in a segment is a point on the arc which bounds the segment, and the arms of the angle pass through the ends of the chord which bound the segment.

Be careful to notice that any angle in the segment XHY stands on the arc XKY, not on the arc XHY which bounds the segment.

Example for Oral Work

Draw on the blackboard a large circle and mark (say) eight points A, B, C, D, E, F, G, H at irregular intervals on its circumference. Join each pair of points by a straight line, and discuss such questions as:

(i) On what arc do the angles BCE, HAB, etc., stand?
(ii) What angles stand on the arc BCD, EGA, etc.?
(iii) What angles stand on the same arc as \( \angle EAC, \angle HBD \), etc.?
(iv) What angles stand on the chord EC, HD, etc.?
(v) Name two angles which stand on the chord AE but do not stand on the same arc.
(vi) What angles are in the segment ABD, HEB, etc.?

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Given a minor arc AHB of a circle, centre O, and a point P on the remaining part of the circumference.

To prove that \( \angle AOB = 2 \angle APB \).

Construction. Join PO and produce it to any point N.

Proof.

\[
\begin{align*}
\angle OAP &= \angle OPA \quad \text{(base } \angle s, \text{ isos. } \triangle) \\
\angle NOA &= \angle OAP + \angle OPA \\
\angle NOA &= 2 \angle OPA \\
\angle NOB &= 2 \angle OPA.
\end{align*}
\]

But \( \angle NOA \) is an exterior angle of \( \triangle AOP \),

\[
\angle NOA = \angle OAP + \angle OPA \\
\angle NOA = 2 \angle OPA.
\]

Similarly, \( \angle NOB = 2 \angle OPA \).

\[
\therefore \text{ adding in fig. 583 (i) and subtracting in fig. 583 (ii), } \angle AOB = 2 \angle APB.
\]

In fig. 583 (iii), where PO produced passes through B.

\[
\angle BOA = \angle NOA = 2 \angle OPA \quad \text{proved}.
\]

In fig. 584, the arc AHB is a major arc and the angle AOB at the centre standing on the arc AHB is reflex. The proof in this case, and in the case where the arc AHB is a semicircle, is the same as for fig. 583 (i).
Examples for Oral Discussion

1. Fig. 585 represents a circle, centre O, and the small letters denote angles.
   (i) If \( p = 110^\circ \), find \( a \) and \( b \).
   (ii) If \( b = 62^\circ \), find \( p \) and \( a \).
   (iii) If \( a = 52^\circ \), find \( b \). Give reasons.
   (iv) If \( c = 125^\circ \), find \( q \) and \( d \).
   (v) If \( q = 200^\circ \), find \( c \) and \( d \).
   (vi) If \( a = 50^\circ \), find \( p \); then find \( q, c \). How much is \( a + c \)?
   (vii) If \( p = 108^\circ \), find \( b \) and \( d \). Give reasons. How much is \( b + d \)?
   (viii) If \( a = 48^\circ \), find \( d \). (ix) If \( c = 138^\circ \), find \( b \).
   (x) If \( b \) were \( 90^\circ \), what would \( p \) be? How must the figure be drawn to give this result?
   (xi) What points must be joined in the figure to form an angle double \( \angle AXB \)?
   (xii) Name angles in the figure which equal half \( \angle DOC \).

2. In fig. 586, the angles \( p, q, r \) are angles in the major segment \( AKB \) of a circle, centre \( O \). Prove that \( p = q = r \).
   (i) What do you know about \( p \)? Give reasons and complete the proof.
   (ii) Draw a figure showing three angles \( p, q, r \) in a minor segment \( AKB \) of a circle and explain why they are equal.
   (iii) Express the fact that has been proved in the form of a general statement.
   (iv) If \( p = 40^\circ \), find the angles of \( \triangle AOB \). Hence on a given line \( AB \) 5 cm. long, construct a segment of a circle to contain an angle of \( 40^\circ \).

3. In fig. 587 (i), \( AKB \) is a major segment of a circle; in fig. 587 (ii), \( AKB \) is a semicircle; in fig. 587 (iii), \( AKB \) is a minor segment of a circle. Prove that
   (i) the angle \( p \) is acute;
   (ii) the angle \( q \) is a right angle;
   (iii) the angle \( r \) is obtuse.
Express these facts in the form of general statements.

4. (i) In fig. 588 (i), the angles \( b, d \) are the opposite angles of a cyclic quadrilateral \( ABCD \). Prove that \( b + d = 2 \text{ rt. \( \angle \)} \).
   (ii) In fig. 588 (ii), \( p \) is the exterior angle of the cyclic quadrilateral formed by producing \( AD \). Prove that \( p = b \).
   (i) In fig. 588 (i), what do you know about \( b \), about \( d \)?
   (ii) In fig. 588 (ii), what do you know about \( p \) and \( d \)?
   (iii) Express the two facts that have been proved in the form of general statements.
5. In fig. 589, $PABQ, RDCS$ are straight lines cutting the circle $ABCD$. Give short reasons for each answer to the following:

(i) Which angle equals
   $\angle DAC, \angle BDC$?

(ii) Which angle equals
   $\angle QBC, \angle BAD$?

(iii) What follows if $BD$ is a diameter?

(iv) If $\angle ABD = \angle CBD$, name another pair of equal angles.

(v) If $KC = KD$, name four equal angles.

6. Draw a large circle and mark four points $A, B, C, D$ in order on its circumference. Use your figure for the following:

(i) Mark a point $E$ on the circumference such that $\angle AEC = \angle ABC$.

(ii) Mark a point $F$ on the circumference such that $\angle AFB, \angle ACB$ are supplementary.

(iii) Draw an angle $p$ in the minor segment cut off by $BC$.

(iv) Draw an angle $q$ in the major segment cut off by $CD$.

(v) Mark by the letters $r, s$ the two exterior angles of quadrilateral $ABCD$ which are equal to $\angle BCD$.

7. Make a sketch of fig. 589 for each of the following examples and show the answers on it.

(i) Find all the angles you can if $\angle PAD = 100^\circ, \angle DAC = 30^\circ$.

(ii) Find all the angles you can if $\angle ADB = 60^\circ, \angle DBC = 25^\circ$.

(iii) Find all the angles you can if $\angle RDB = 140^\circ, \angle QBD = 130^\circ$.

(iv) Find the remaining angles if $\angle RDA = 72^\circ, \angle ABD = 30^\circ, \angle ACB = 40^\circ$.

(v) Find the remaining angles if $\angle PAD = 115^\circ, \angle ADB = 50^\circ, \angle BKC = 85^\circ$.

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**NUMERICAL EXAMPLES**

**EXERCISE 58**

1. Two chords $AB, CD$ of a circle intersect at right angles at a point inside the circle. If $\angle BAC = 30^\circ$, find $\angle ABD$.

2. $AB$ is a diameter of the circle $ABCD$. If $\angle ADC = 127^\circ$, find $\angle BAC$.

3. $AC$ is a diameter of the circle $ABCD$. If $\angle BDC = 25^\circ$, find $\angle ACD$.

4. $ABCD$ is a cyclic quadrilateral. If $\angle ADC = 70^\circ$ and $\angle ACD = 50^\circ$, find $\angle BCD$.

5. Two chords $AB, CD$ of a circle meet, when produced, at $K$. If $\angle KAD = 31^\circ$ and $\angle AKC = 42^\circ$, find $\angle KBC$.

6. The diagonals of the cyclic quadrilateral $ABCD$ cut at $N$. If $\angle BAC = 42^\circ, \angle BNC = 114^\circ$ and $\angle ADB = 33^\circ$, find $\angle BCD$.

7. $ABCD$ is a cyclic quadrilateral and $EABF$ is a straight line. If $\angle EAD = 82^\circ, \angle FBC = 74^\circ$ and $\angle BDC = 50^\circ$, find the acute angle between $AC$ and $BD$.

8. $ABC$ is a triangle inscribed in a circle, centre $O$. If $\angle AOC = 130^\circ, \angle BOC = 150^\circ$ and if $O$ lies inside $\triangle ABC$, find $\angle ACB$.

9. In fig. 590, $O$ is the centre of the circle; $ABN$ is a straight line. Find $\angle AOC$.

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10. In fig. 591, $AB$ is a diameter of the circle $APBR$; $APQ$ and $RBP$ are straight lines. Find $\angle BPR$. 
11. P, Q, R are points on a circle, centre O; $\angle PQO=54^\circ$, $\angle QOR=30^\circ$, and P, R are on opposite sides of OQ. Find $\angle QPR$ and $\angle PQR$.

12. P is a point on the minor arc AB of a circle, centre O. If $\angle APB=x^\circ$ and $\angle AOB=y^\circ$, find $x$ in terms of $y$.

13. ABCD is a convex quadrilateral in which $AB=AC=AD$. If $\angle BAD=140^\circ$, find $\angle BCD$.

14. In fig. 592, O is the centre of the circle, OABC is a parallelogram, and BCP is a straight line. Find $\angle OAP$.

15. In fig. 593, AB is a diameter and O is the centre of the circle. Find $\angle OPQ$.

16. O is the centre of the circle ABC and $\angle ABC=30^\circ$. If BC is parallel to OA, prove that AB is perpendicular to OC.

17. Two chords AB, DC of a circle, centre O, are produced to meet at E. If $\angle AOB=100^\circ$, $\angle EBC=72^\circ$, and $\angle ECB=84^\circ$, find $\angle CDE$.

18. D is a point on the base BC of $\triangle ABC$; H, K are the centres of the circles ADB, ADC. If $\angle AHD=70^\circ$ and $\angle AKD=90^\circ$, find $\angle BAC$. [Two answers.]

19. Two circles APRB, AQSB intersect at A, B; PAQ, RBS are straight lines. If $\angle QPR=80^\circ$ and $\angle PRS=70^\circ$, find $\angle PQS$ and $\angle QSR$. [Join AB.]

20. In fig. 594, AB is a diameter; ABRS and PQR are straight lines. Prove that PQ = QB.

21. In fig. 595, if $e=32^\circ$ and $f=40^\circ$, find $p$.

22. In fig. 595, prove that $e+f=180^\circ-2p$.

23. A chord QR and a diameter AB of a circle AQRB, when produced, meet at P. If $\angle QPA=28^\circ$ and $\angle QAR=42^\circ$, find $\angle QRA$.

24. In fig. 595, if AC cuts BD at K, and if $e=40^\circ$, $f=20^\circ$, $\angle BKC=100^\circ$, prove that $\angle BAC=2\angle BCA$.

25. ABCDE is a pentagon inscribed in a circle. If $\angle BDC=20^\circ$, $\angle CAD=28^\circ$, $\angle ABD=70^\circ$, and $\angle CD=DE$, find the angles of the pentagon.

26. AB is a diameter of a circle APB, radius 5 cm. If AP = 6 cm, find the length of PB.

27. Draw a circle of radius 4 cm. and mark a point A on the circumference. Inscribe a rectangle ABCD in the circle such that AB = 6 cm. Measure and calculate the length of BC.

28. Draw a triangle ABC such that AB = 10 cm., AC = 4 cm., $\angle BAC=90^\circ$. Construct a triangle ABP equal in area to $\triangle ABC$ and such that $\angle APB=90^\circ$. Measure PA and PB.

29. Construct a cyclic quadrilateral ABCD such that AB = 24 in., BC = 18 in., $\angle ABC=90^\circ$ and AD = DC. Measure AD.

30. Draw a rectangle 3 in. by 2 in. and construct without any arithmetical calculations a square equal in area to the rectangle. Measure the side of the square. [Use fig. 530, p. 270.]
THEOREM 48

Angles in the same segment of a circle are equal.

Given a segment $AKB$ of a circle $AKBH$ and any two angles $APB$, $AQB$ in the segment.

To prove that $\angle APB = \angle AQB$.

Construction. Let $O$ be the centre of the circle. Join $OA$, $OB$.

Proof. With the notation in the figures, since $\angle$ at centre = twice $\angle$ at $O$, standing on same arc,

$\angle x = 2 \angle p$ arc $AHB$, and $\angle x = 2 \angle q$ arc $AHB$,

$\therefore \angle p = \angle q$.

Corollary. The angle in a major segment of a circle is acute, and the angle in a minor segment of a circle is obtuse.

$\angle x$ is less than 2 rt. $\angle s$ if $AHB$ is a minor arc, and is greater than 2 rt. $\angle s$ if $AHB$ is a major arc.

For reference. Same arc $AHB$ or same segment $AKB$.

THEOREM 49

The angle in a semicircle is a right angle.

Given a semicircle $AKB$ of a circle $AKBH$ and any angle $APB$ in the semicircle.

To prove that $\angle APB = 1$ right angle.

Proof. Since $AKB$ is a semicircle, the centre $O$ of the circle lies on $AB$.

With the notation in the figure, since $\angle$ at centre = twice $\angle$ at $O$, standing on same arc,

$\angle x = 2 \angle p$ arc $AHB$.

But since $AOB$ is a straight line,

$\angle x = 2$ rt. $\angle s$.

$\therefore \angle p = 1$ rt. $\angle s$.

For reference. $\angle$ in semicircle.

This fact was probably discovered by Thales (640-546 B.C.), one of the "Seven Wise Men." He introduced the study of geometry into Greece, and when in Egypt he showed King Amasis how to find the heights of the Pyramids by measuring their shadows and comparing the lengths with that of a shadow cast by a vertical pole.
THEOREM 50

(1) The opposite angles of a cyclic quadrilateral are supplementary.

(2) If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle.

\[ \angle p + \angle d = 2 \text{ rt. } \angle s \] 
adj. \( \angle s \) on st. line,

but \[ \angle b + \angle d = 2 \text{ rt. } \angle s \] 
proved,

\[ \therefore \angle p + \angle d = \angle b + \angle d, \]

\[ \therefore \angle p = \angle b. \]

For reference: (1) opp. \( \angle s \), cyclic quad.

(2) ext. \( \angle \), cyclic quad.

Note. If Theorem 50 (2) is set by itself in an examination, the proof of Theorem 50 (1) must be included to secure full credit.

Important Hints. (i) In rider work, Theorem 50 (2) is more often of use than Theorem 50 (1).

(ii) In problems on intersecting circles, it is usually advisable to draw the common chord.

EXERCISE 59

Nos. 1–4 refer to fig. 599, in which \( AB \) cuts \( BC \) at \( N \).

1. If \( AB \) is parallel to \( XY \), prove that \( \angle ANX = 2 \angle ABX \).
2. If \( AB \) is parallel to \( XY \), prove that \( NX = NY \).
3. If \( NY = YB \), prove that \( XN = XA \).
4. If \( XA \) and \( YB \) are produced to meet at \( K \), and if \( AB \) is parallel to \( XY \), prove that \( KK = KY \).

Nos. 5–7 refer to fig. 600 in which \( PQM \), \( RSM \) are straight lines.

5. Prove that \( \angle PSM = \angle RQM \).

6. If \( MQ = MS \), prove that \( MP = MR \).

7. If \( PS = SM \), prove that \( \angle PQR = 2 \angle QRS \).
11. In fig. 602, the centre $O$ of circle $BPC$ lies on circle $BQC$. Find a relation between $\angle BPC$ and $\angle BQC$. [Join $OB$, $OC$.]

12. In fig. 603, $CAL$, $CBM$ are straight lines. If $CA$ is a diameter of the circle $ABC$, prove that $\angle ALM$ is a right angle.

13. Two circles intersect at $A$, $B$; $AP$, $AQ$ are diameters of the circles. Prove $\angle PBQ$ is a straight line.

14. $AP$ is a chord of a circle, centre $O$. If the circle on $AO$ as diameter cuts $AP$ at $N$, prove that $AN = NP$.

15. $AB$ is a diameter of the circle $ADCB$; the chord $DC$ is produced to $E$. Prove that $\angle ABD + \angle BCE = 90^\circ$.

16. $AR$ is a diameter of the circle $APQR$. Prove that $\angle APQ + \angle QRB = 270^\circ$. [Join $AR$.]

17. $ABCD$ is a hexagon inscribed in a circle. Prove that $\angle FAB + \angle BCD + \angle DEF$ is equal to 4 right angles.

18. $ABC$ is a triangle inscribed in a circle, centre $O$; $N$ is the mid-point of $BC$. Prove that $\angle BON$, $\angle BAC$ are equal or supplementary.

19. In fig. 604, $PAQ$, $RBS$ are straight lines. Prove that $\angle PAQ = \angle S$.

20. If, in fig. 604, $PS$ cuts the circle $PAB$ at $K$ and cuts the circle $QAB$ at $H$, prove that $\angle PAK = \angle KBS$.

21. Two straight lines $ABCD$, $PQRS$ are drawn to cut two circles $ABQP$, $CDSR$. If $AP$ is parallel to $CR$, prove that $BQ$ is parallel to $DS$.

22. In fig. 605, $O$ is the centre of the circle $ABC$. If $\angle AOB = \angle OAB$, prove that $\angle AOB$ is a right angle.

23. $OP$, $OQ$, $OR$ are three equal lines. If $\angle POQ = 90^\circ$, prove that $\angle PRO$ is either $30^\circ$ or $150^\circ$.

24. The side $BA$ of the cyclic quadrilateral $ABCD$ is produced to $E$. If $AD$ bisects $\angle CAD$, prove that $\angle DB = DC$.

25. $ABCD$ is a parallelogram. The circle through $A$, $B$, $C$ cuts $CD$, produced if necessary, at $E$. Prove that $AE = AD$.

26. Two chords $AB$, $CD$ of a circle, centre $O$, intersect at right angles at a point inside the circle. Prove that $\angle AOD + \angle BOC$ equals two right angles.

27. Two chords $AB$, $CD$ of a circle, centre $O$, intersect at a point $N$ inside the circle. If $\angle ANC$ is acute, prove that $\angle AOC + \angle BOC = 2\angle ANC$.

28. If $O$ is the centre of the circle in fig. 600, p. 323, prove that $\angle POR = \angle QOS = 2\angle PMR$.

29. $ABCP$ is a cyclic quadrilateral. Prove that a triangle whose sides are parallel to $PA$, $PB$, $PC$ is equiangular to $\triangle ABC$.

[Draw $HK \parallel PC$ to cut $PA$, $PB$ at $H$, $K$.]
30. ABCD is a rectangle; any circle through A cuts AB, AC, AD at X, Y, Z. Prove that \( \triangle XYZ \) is equiangular to \( \triangle CBA \).

31. In fig. 606, PQRS is a straight line. If BA bisects \( \angle PAS \), prove that \( BQ = BR \).

32. In fig. 607, PAQ and BHK are straight lines. Prove that PH is parallel to KQ. [Join AB.]

33. D is any point on the side AB of \( \triangle ABC \); points E, F are taken on AC, BC respectively so that \( \angle EDA = 60^\circ = \angle FDB \); a circle is drawn through D, E, F and cuts AB again at Q. Prove that \( \triangle DEF \) is equilateral.

34. P is any point on the minor arc AC of the circle ABC; BP cuts AC at Q. If \( \angle PABC = \angle AQB \), prove that \( \angle PAB = \angle QAQ \).

35. P is any point on the minor arc AC of the circle ABC; AP produced meets BC produced at Q. If \( \angle ABP = \angle AQP \), prove that \( \angle AQB = \angle ABP \).

36. The bisectors of \( \angle ABC, \angle ACB \) of \( \triangle ABC \) meet AC, AB at X, Y respectively and intersect at Z. If \( \angle AXZ \) is a cyclic quadrilateral, prove that \( \angle BAC = 60^\circ \).

37. K is a point inside \( \triangle ABC \); BK, CK produced meet AC, AB at E, F respectively. If AEK and BFEC are cyclic quadrilaterals, prove that BE and CF are altitudes of \( \triangle ABC \).

38. If, in fig. 600, p. 323, PR and QS are produced to meet at N and if the circles MQS, NRS cut again at X, prove that MXN is a straight line.

39. The bisectors of \( \angle ABC, \angle ACB \) meet at I; CI produced cuts the circle ABC at P. Prove that

(i) \( \angle PBC = \frac{1}{2}(\angle ABC + \angle ACB) \);

(ii) \( PB = PI \).

40. In fig. 608, the centre O of the circle APB lies on the circle ABQ, and OPQ is a straight line. Prove that BP bisects \( \angle ABQ \).

41. ACB, ADB are two arcs on the same side of AB and such that the centre of the circle ADB lies on the arc ACB. If a straight line ACD cuts the arcs at C, D, prove that \( \angle C = \angle D \).

42. In \( \triangle ABC \), \( \angle B = \angle C \). If the circle, centre E, radius EC, cuts AB, AC at D, E, prove that DE is parallel to the line bisecting \( \angle ABC \).

43. O is the centre of the circle ABC; NO, produced if necessary, cuts AC at P. Prove that \( \angle PON = \angle OBC \). [Produce CO to cut the circle at Q; join QB.]

44. If O is the centre of the circle in fig. 600, p. 323, and if QS cuts QR at N, prove that \( \angle QNP + \angle PMR = \angle PQR \).

45. AB, CD are perpendicular chords of a circle, centre O. Prove that \( \angle DAC = \angle CDA \).

46. The bisectors of the angles ABC, ACB of \( \triangle ABC \) meet at I; the circle BIC cuts AB, AC again at P, Q respectively. Prove that \( \angle BAP = \angle APC \).

47. The bisectors of the angles ABC, DBC of \( \triangle ABC \) intersect at I and cut AC, AB at Y, Z respectively; the circles BIZ, CIY meet again at X. Prove that \( \angle YZX + \angle BIC \) equals two right angles.

48. In fig. 609, A, B are the centres of the circles DCH, DECK; AEB, DEH, HCK are straight lines. Prove that the triangles HDK, AEB are equiangular.
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49. AKB, CKD are perpendicular chords of a circle ACBD. Prove that the perpendicular from K to AD bisects, when produced, BC.

50. Two given circles ABP, ABQ intersect at A, B; a variable line through A meets the circles at P, Q, as in fig. 610. Prove that \( \angle PBQ \) is of constant size.

51. In fig. 610, the circles ABP, ABQ are given; PAQ is a variable line through A; PSB, QBT are straight lines; QS meets PT at R. Prove that \( \angle PRQ \) is of constant size.

52. Two circles PAB, QASB cut at A, B; PAQ, PSB are straight lines, as in fig. 610. If O is the centre of the circle PAB, prove that OP is perpendicular to QS, produced if necessary. [Join AB, AO.]

53. If in fig. 600, p. 328, QS and PR are produced to meet at N, prove that the bisectors of the angles PMR, QNP cut one another at right angles.

54. CD is a rod hinged at a fixed point C and loosely jointed at D to the rod ADB, see fig. 611. If AD = DB = DC and if A moves along the straight line CN, and if D moves in a given plane through AC, prove that B will also move along a straight line.

CONCYCLIC POINTS

Tests for Conyclic Points

Any three points lie either on a straight line or a circle. In general, if any four points are taken in a plane, it is impossible to draw either a straight line or a circle to pass through all of them. We can obtain tests for determining whether four points are conyclic by proving that the converses of Theorems 48, 50 are true.

1) If, in fig. 612, \( n_1 = n_2 \), then the points A, P, Q, B are conyclic.

If, in fig. 612, a circle is drawn through A, P, B, and if it does not pass through Q it will cut BQ at a point X which lies either on BQ produced or between B and Q or on QB produced, or it will not meet BQ at any other point except B, see fig. 613 (i), (ii), (iii), (iv). It can be proved that if \( n_1 = n_2 \), each of these four alternatives is impossible and hence the circle through A, P, B must also pass through Q. But if we proceed in this way, each of these four cases must be considered.

The object of the argument used in the proof of Theorem 52 is to show that a method can be used in which only two alternative cases need be considered.
(2) If in fig. 614, \( \angle A + \angle C = 2 \text{ rt. } \angle s \), then the points A, B, C, D are concyclic.

If, in fig. 614, a circle is drawn through B, C, D, and if it does not pass through A it will cut DA at a point X which lies either on DA produced or between D and A or on AD produced, or it will not meet DA at any other point except D.

Draw four figures to show these alternatives. It can be proved that, if \( \angle A + \angle C = 2 \text{ rt. } \angle s \), each of these four alternatives is impossible, and hence the circle through B, C, D must also pass through A. But if we proceed in this way, each of these four cases must be considered. The object of the argument used in the proof of Theorem 53 is to show that a method can be used in which only two alternative cases need be considered.

In connection with this discussion of the various possible forms a construction may take, the reader may examine the following fallacy:

Given any triangle ABC, prove that AB = AC.

In fig. 615, the bisector of \( \angle BAC \) cuts the perpendicular bisector of BC at K; \( \perp \) KB, KY are perpendicular to AB, AC; KB, KC are joined.

(i) From \( \triangle AKX, AKY \), prove that \( AX = AY \)
and \( KX = KY \).

(ii) From \( \triangle BNX, CNK \), prove that \( KB = KC \).

(iii) From \( \triangle KXB, KYC \), prove that \( XB = YC \).

\[ \therefore AB = AX + XB = AY + YC = AC. \]

What is wrong with this proof?

THEOREM 51

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

Given a triangle ABC in which \( \angle BAC = 1 \text{ rt. } \angle \).

To prove that the circle on BC as diameter passes through A.

Construction. Bisect BC at O and bisect BA at X.

Join OA,OX.

Proof. Since BO = OC and BX = XA,

\( \perp \) BX is parallel to CA \( \text{ mid-point theorem} \).

But \( \angle CAB = 1 \text{ rt. } \angle \) given,
\( \perp \) OXB = 1 rt. \( \angle \) cor., \( \perp \) BC, OX \( \parallel \) CA,
\( \therefore \) OX is the perpendicular bisector of BA,
\( \therefore OB = OA \).

But \( OB = OC \) \( \text{ constr.} \),
\( \therefore OA = OB = OC \).

\( \therefore \) the circle on BC as diameter passes through A.

Corollary. The line joining the mid-point of the hypotenuse of a right-angled triangle to the opposite vertex is equal to half the hypotenuse.

Since \( OA = OB = OC \), \( OA = \frac{1}{2} BC \).

Note. This theorem is a special case of Theorem 32, making use of Theorem 49.
THEOREM 52

If the straight line joining two points subtends equal angles at two other points on the same side of it, then the four points lie on a circle.

Given two points $P, Q$ on the same side of a straight line $AB$ such that $\angle APB = \angle AQB$.

To prove that $P, A, P, Q, B$ lie on a circle.

Construction and Proof. Since $\triangle PAB, QAB$ are on the same side of $AB$, either the angles $PAB, QAB$ are unequal or the angles $PBA, QBA$ are unequal, because otherwise $P$ would coincide with $Q$.

Suppose $\angle PAB > \angle QAB$,

then $AQ$ lies in the angle $PAB$.

Draw the circle through $P, A, B$ and suppose, if possible, it does not pass through $Q$.

Since $AQ$ lies in the angle $PAB$, the circle must cut $AQ$ or $AQ$ produced, at $X$, say.

Join $BX$.

Then $\angle APB = \angle AXB \angle s in same segment$,

but $\angle APB = \angle AQB \angle given$.

$\therefore \angle AXB = \angle AQB$.

But one of these is the exterior angle and the other is the interior opposite angle of $\triangle BQX$, therefore they cannot be equal.

Therefore the original supposition is false.

$\therefore$ the circle through $P, A, B$ must pass through $Q$.

Corollary. Given the base $AB$ of a triangle $ABP$ in magnitude and position, and given the size of $\angle APB$, then the locus of $P$ is two equal arcs of equal circles on opposite sides of $AB$.

Hence it follows that, if the length of one side of a triangle and the size of the angle opposite to that side are given, the circumradius of the triangle is fixed and can be found either by measurement or by using trigonometry.

Let $O$ be the circumcentre of $\triangle ABC$ and $R$ its circumradius. Draw the diameter $BOK$ of the circumcircle; join $AK$. In fig. 619 (i), explain why

$c = BK \sin C$; deduce that $R = \frac{c}{2 \sin C}$.

In fig. 619 (ii), explain why $c = BK \sin (180^\circ - C)$ and (see p. 246) deduce that $R = \frac{c}{2 \sin C}$. Write down two similar expressions for $R$, compare p. 251, No. 28.
Theorem 53

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

\[ \angle ABC + \angle ADC = 2 \text{ rt. } \angle s. \]

Given a quadrilateral \(ABCD\) in which

To prove that \(A, B, C, D\) lie on a circle.

Construction and Proof. Draw the circle through \(B, C, D\) and suppose, if possible, that it does not pass through \(A\). Since \(CA\) lies in the angle \(BCD\), the circle \(BCD\) must cut either \(CA\) or \(CA\) produced, at \(X\), say.

Join \(XB, XD\).

\[ \angle XBC + \angle XDC = 2 \text{ rt. } \angle s \quad \text{opp. } \angle s, \text{ cyclic quad. } \]

\[ \angle ABC + \angle ADC = 2 \text{ rt. } \angle s \quad \text{given.} \]

\[ \therefore \angle XBC + \angle XDC = \angle ABC + \angle ADC. \]

But one side of this equation is part of the other side, therefore the two sides cannot be equal. Therefore the original supposition is false.

\[ \therefore \text{the circle through } B, C, D \text{ must pass through } A. \]

Numerical Examples

Exercise 60

1. In fig. 621, find whether the points \(A, B, C, D\) are concyclic if (i) \(m = 130^\circ\), (ii) \(m = 140^\circ\).

2. In fig. 622, find whether the points \(E, F, G, H\) are concyclic if (i) \(n = 55^\circ\), (ii) \(n = 45^\circ\).

3. Draw a quadrilateral \(PQRS\) and its diagonals \(PR, QS\). If \(\angle PQR = 70^\circ, \angle PRQ = 35^\circ, \angle QSR = 75^\circ\), prove that \(P, Q, R, S\) are concyclic and find \(\angle PSQ\).

4. The diagonals \(AC, BD\) of the quadrilateral \(ABCD\) intersect at \(K\). If \(\angle BAC = 50^\circ, \angle CAD = 45^\circ, \angle ACD = 55^\circ, \angle BKC = 105^\circ\), prove that \(A, B, C, D\) are concyclic and find \(\angle CBD\).

5. \(ABCD\) is a quadrilateral in which \(\angle ABD = 30^\circ, \angle ADB = 40^\circ, \angle BCD = 70^\circ\). Find \(\angle ACB\).

6. Draw a quadrilateral \(ABCD\) and its diagonals \(AC, BD\). If \(\angle DAC = 65^\circ, \angle CAR = 50^\circ, \angle CDB = 65^\circ\), state the sizes of any other angles in the figure that can be calculated.

7. \(ABCD\) is a quadrilateral in which \(AB = AD\) and \(DB = DC\). If \(\angle DBC = 25^\circ\) and \(\angle BDC = 25^\circ\), prove that \(A, B, C, D\) are concyclic.

8. \(BE, CF\) are altitudes of \(\triangle ABC\). If \(\angle AEF = 65^\circ\), find \(\angle BCF\).
9. P, Q, R are points on the sides BC, CA, AB of \( \triangle ABC \) such that \( \angle RPQ = 30^\circ \), \( \angle QPC = 20^\circ \), \( \angle PRQ = 10^\circ \). If \( \angle ABC = 85^\circ \) and \( \angle ACB = 70^\circ \), prove that \( PQAB \) and \( BRQC \) are cyclic quadrilaterals.

10. AD, BE are altitudes of \( \triangle ABC \). If \( \angle ADE = 30^\circ \) and \( \angle BED = 20^\circ \), find the angles of \( \triangle ABC \).

[11] In \( \triangle ABC \), \( AB = AC \) and \( \angle BAC = 56^\circ \). If the line bisecting \( \angle ABC \) meets the line through C parallel to BA at D, prove that A, B, C, D are concyclic.

12. A, B are fixed points such that \( AB = 4 \) cm.; \( P \) is a variable point such that \( \angle APB = 70^\circ \). Construct the complete locus of \( P \). [See p. 314, No. 2 (iv).]

[13] In \( \triangle ABC \), \( BC = 2 \) in. and \( \angle BAC = 30^\circ \). Find the radius of the circle \( \triangle ABC \).

14. Construct \( \triangle ABC \) given that \( \angle ACB = 50^\circ \), \( AB = 5 \) cm. and that the distance of \( C \) from \( AB \) is 4 cm.

**EXERCISE 61**

Nos. 1-5 refer to fig. 623 in which BE, CF are altitudes of \( \triangle ABC \) and intersect at H.

1. Prove that (i) B, F, E, C are concyclic; (ii) \( \angle AEF = \angle ABC \).

[2] Prove that (i) A, E, H, F are concyclic; (ii) \( \angle AHE = \angle ACB \).

3. Prove that \( \angle HAE = \angle EBC \) and deduce that AH produced cuts BC at right angles.

4. If \( X \) is the mid-point of BC, prove that \( EX = XF \).

[5] If \( X \) is the mid-point of BC, prove that \( \angle FXE = 180^\circ - 2\angle BAC \).

6. ABCD is a parallelogram; any circle through A and D cuts AB, DC at P, Q. Prove that B, C, Q, P are concyclic.

[7] ABCD is a parallelogram in which \( \angle ABC = 60^\circ \). Prove that the centre of the circle \( \triangle ABD \) lies on the circle \( \triangle CBD \).

CONCYCLIC POINTS

8. In fig. 624, P, Q, R are points on BC, CA, AB. Prove that \( AQKR \) is a cyclic quadrilateral. [Join KP.]

[Fig. 624]

9. In fig. 625, \( AB = AC \) and \( \angle ABE = \angle ACE \). Prove that \( \angle AEC = 60^\circ + \frac{1}{2} \angle BAC \).

[10] If in fig. 625, p. 338, AC is a diameter of the circle and if AD = DQ, prove that (i) \( CA = CQ \); (ii) \( DB = DQ \).

[11] In \( \triangle ABC \), \( AB = AC \) and \( BC > AB \); \( P \) is a point on CA produced and \( Q \) is a point on BC such that \( \angle BQP = 2 \angle QPC \). Prove that A, P, B, Q are concyclic.

12. In fig. 626, \( O \) is the centre of the circle and \( OLM \) is perpendicular to \( \triangle AOB \). Prove that (i) \( A, O, M, P \) are concyclic; (ii) \( \angle OPA = \angle OMB \).

[Fig. 626]

[Fig. 627]

13. In fig. 627, \( CAD, CPQ, CBE \) are straight lines. If \( DE \) cuts \( CQ \) at \( K \), prove that \( A, P, K, D \) are concyclic. [Join \( AP, AB \).]
In $\triangle ABC$, $AB > AC$ and $P$ is a point on $AB$ such that $AP = AC$. The bisector of $\angle BAC$ cuts $BC$ at $Q$ and cuts the circle $ABC$ at $R$. Prove that $B, Q, R$ are concyclic.

[15] A, B, C are any three points on a circle. The internal and external bisector of $\angle BAC$ cut the circle again at $H, K$. Prove that $HK$ is a diameter of the circle.

[16] Prove that the quadrilateral formed by the external bisectors of the angles of any quadrilateral is cyclic.

17. In fig. 628, $PBA, PCD, QDA, QCB$ are straight lines, If $\angle APD = \angle BQA$, prove that $AC$ is a diameter.

In fig. 629, $AB$ is a diameter of the circle. If $\angle APK = \angle ASK$, prove that (i) $\angle PKB = 90^\circ$; (ii) $STKB$ is a cyclic quadrilateral. [Join $PB$]

19. The base $AB$ and the angle $ACB$ of a triangle $ABC$ are given. If $\angle ABC$ is acute, prove that $AC$ is greatest when $\angle ABC = 90^\circ$.

20. The circle $BCGF$ lies inside the circle $ADHE$. $OABCD$ and $OEFH$ are two straight lines cutting the circles. If $A, B, F, E$ are concyclic, prove that $C, D, H, G$ are concyclic.

21. $PQ, PR$ are any two chords of a circle, centre $O$. If the diameter perpendicular to $PQ$ cuts $PR$ at $K$, prove that $Q, O, K, R$ are concyclic.

22. $ABCD$ is a cyclic quadrilateral; $AP, DQ$ are the perpendiculars from $A, D$ to $CD, AB$ respectively. Prove that $PQ$ is parallel to $CB$.

23. In fig. 630, the side $BA$ of the equilateral triangle $ABC$ is produced to $Y$, and $AX$ is parallel to $BC$. If $\angle CYX = 60^\circ$, prove that $\triangle CXY$ is equilateral.

24. In fig. 631, $AB = AC$, $QB = QP$ and $\angle BAC = \angle BQP$. Prove that $QA$ is parallel to $BC$.

25. Two circles $APRB, ASQB$ intersect at $A, B$; $PAQ$ and $RAS$ are straight lines. $RP$ and $QS$ are produced to meet at $O$. Prove that $O, P, B, Q$ are concyclic.

26. $AOB, COD$ are two perpendicular diameters of a circle. Two chords $CP, CQ$ cut $AB$ at $H, K$. Prove that $H, K, Q, P$ are concyclic. [Join $CA, CB, AQ, PQ$]

27. If any five circular arcs are drawn intersecting as in fig. 632, prove that a sixth circle can be drawn to pass through $P, Q, R, S$.

28. In fig. 633, $\triangle PXK \equiv \triangle PHK$ and the lines $XH, KY$ meet when produced at $A$. Prove that the circle $XPY$ passes through $A$. 

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Fig. 628
Fig. 629
Fig. 630
Fig. 631
Fig. 632
Fig. 633
**EQUAL ARCS of the same or Equal Circles**

Fig. 634 represents two minor arcs \( AB, PQ \) of a circle \( ABPQ \), centre \( O \), such that \( \angle AOB = \angle POQ \).

If we draw the diameter \( XOY \) which bisects \( \angle BOP \), it also bisects \( \angle AOP \). But the circle is symmetrical about the diameter \( XOY \), see p. 305; therefore if we fold the figure about \( XOY \), \( B \) can be made to coincide with \( P \), and \( A \) with \( Q \), and the minor arc \( AB \) will coincide with the minor arc \( PQ \). Thus

If \( O \) is the centre of the circle \( ABPQ \) and if \( \angle AOB = \angle POQ \), then

\[
\text{minor arc } AB = \text{minor arc } PQ.
\]

Suppose an arc \( AB \) of a circle, centre \( O \), is given. What measurements must be taken in order to copy the arc \( AB \)?

If the radius of the circle is measured, a copy of the circumference of the circle can be made.

Since arcs of a circle which subtend equal angles at the centre are of equal length, the length of the arc \( AB \) is fixed if the size of \( \angle AOB \) is known. Therefore, if \( \angle AOB \) is measured, a copy of the arc can be made.

Consequently if two circular arcs are drawn which agree with each other as regards

(i) the length of the radius,
(ii) the angle subtended at the centre of the circle,
then they will agree completely in size and shape, in other words they will be congruent.

Thus, see fig. 635,

If \( H, K \) are the centres of two equal circles \( ABP, CDQ \), and if \( \angle AHB = \angle CKD \), then minor arc \( AB = \text{minor arc } CD \); and conversely.

If \( H, K \) are the centres of two equal circles \( ABP, CDQ \), and if arc \( AB = \text{arc } CD \), then \( \angle AHB = \angle CKD \).

---

**Examples for Oral Discussion**

1. Fig. 635 represents two equal circles, centres \( H, K \).
   (i) If arc \( AB = \text{arc } CD \), prove that \( \angle APB = \angle CQD \).
   (ii) If \( \angle APB = \angle CQD \), prove that minor arc \( AB = \text{minor arc } CD \).

2. Fig. 635 represents two equal circles, centres \( H, K \).
   (i) If arc \( AB = \text{arc } CD \), prove that chord \( AB = \text{chord } CD \).
   (ii) If chord \( AB = \text{chord } CD \), prove that minor arc \( AB = \text{minor arc } CD \).

3. If two arcs of two circles are of equal length, must the arcs be congruent?
Calculation of Length of Arc

If the circumference of a circle is divided into 360 equal parts, each arc subtends an angle of $1^\circ$ at the centre. Hence the length of the arc which subtends an angle of $x^\circ$ at the centre is $\frac{x}{360}$ of the circumference.

If the radius of a circle is $r$ in., the length of the circumference is $2\pi r$ in. where $\pi \approx 3.1416$.

Hence the length of an arc of a circle of radius $r$ in., which subtends an angle $x^\circ$ at the centre is $\frac{x}{360} \times 2\pi r$ in.

Calculation of Area of Sector

If the area of a circle is divided into 360 sectors of equal area, the angle of each sector, i.e., the angle between the radii bounding it, is $1^\circ$. Hence the area of a sector of angle $x^\circ$ is $\frac{x}{360}$ of the area of the circle.

If the radius of a circle is $r$ in., the area of the circle is $\pi r^2$ sq. in. where $\pi \approx 3.1416$.

Hence the area of a sector, angle $x^\circ$, of a circle of radius $r$ in.

is $\frac{x}{360} \times \pi r^2$ sq. in.

\[ \therefore \text{area of sector} = \frac{1}{2} \times \text{radius} \times \text{length of arc}. \]

Area of the Curved Surface of a Circular Cone

Let the slant height of the cone be $l$ in., and the radius of the base of the cone be $r$ in.

If we cut down a slant edge of the cone and fold out flat the curved surface, we obtain a sector of a circle, radius $l$ in., length of arc $2\pi r$ in.

But the area of this sector is $\frac{1}{2} l \times 2\pi r$ sq. in.,

\[ \therefore \text{area of curved surface of cone} = \pi rl \text{ sq. in.} \]

The results which have been discussed are expressed by the following theorems which are proved in the Appendix, pp. 548-9.

THEOREM 54

(i) In equal circles (or in the same circle), equal angles at the centres (or centre) stand on equal arcs.

(ii) In equal circles (or in the same circle), equal angles at the circumferences (or circumference) stand on equal arcs.

THEOREM 55

(i) In equal circles (or in the same circle), equal arcs subtend equal angles at the centres (or centre).

(ii) In equal circles (or in the same circle), equal arcs subtend equal angles at the circumferences (or circumference).

NUMERICAL EXAMPLES

EXERCISE 62 (Oral)

Nos. 1-8 refer to fig. 636, in which arc $AB = \frac{1}{6}$ circumference, arc $AC = \frac{1}{6}$ circumference, arc $APD = \frac{1}{6}$ circumference. Find the following angles:

1. $\angle AQB$.
2. $\angle BQC$.
3. $\angle APD$.
4. $\angle CPD$.

Express as fractions of the circumference:

5. Minor arc $BC$.
6. Major arc $CQD$.

7. Find the ratio of minor arc $AB$ to minor arc $BC$.
8. If minor arc $AQ$ = twice arc $QD$, find $\angle QAD$. 

\[ \text{Fig. 636} \]
NUMERICAL EXAMPLES

EXERCISE 63

[In this exercise, unless otherwise stated, take \( \pi = 3.142 \) and give those answers which depend on the value of \( \pi \) to three figures.]

1. The value of \( \pi \) is found experimentally by wrapping a piece of thread 5 times round a cylinder of diameter 5 inches. When unwrapped the thread measures 78.55 in. What value does this give for \( \pi \)?

Find the lengths of the circumferences of the circles Nos. 2-4.

2. Radius, 4 in. 3. Diameter, 7 cm. 4. Radius, 100 yd.

Find the radii of the circles, Nos. 5-8, to 2 figures. Take \( \pi = \frac{22}{7} \).

5. Circumference, 11 in. 6. Circumference, 8.8 cm.

7. Circumference, 440 yd. 8. Circumference, 6 ft.

9. The diameter of a semi-circular protractor is 3-5 in. Find its perimeter, correct to \( \frac{1}{10} \) in.

10. Find the length of an arc of a circle of radius 4 cm., which subtends 50° at the centre.

11. \( \triangle ABC \) is an equilateral triangle inscribed in a circle of radius 4 cm. Find the length of the major arc \( AB \).

12. If the length of an arc of a circle of radius 5 cm. is 4 cm., find to the nearest degree the angle which the arc subtends at the centre. [Take \( \pi = \frac{31}{10} \).]

13. If a circular arc, 6 cm. long, subtends 80° at the centre, find the radius of the circle correct to 2 figures. [Take \( \pi = \frac{31}{10} \).]

14. A circle of radius 2 in. is drawn on squared paper. By counting squares, a boy estimates the area of the circle to be 12.57 sq. in. What value does this give for \( \pi \)?

Find the areas of the circles, Nos. 15-17.

15. Radius, 10 in. 16. Diameter, 8 cm. 17. Radius, 7 ft.

Find the radii of the circles, Nos. 18, 19, correct to 2 figures. Take \( \pi = \frac{22}{7} \).

18. Area, 616 sq. in. 19. Area, 38.5 sq. cm.

20. Find the area of the ring between two concentric circles of radii 3 in. and 4 in.

21. The angle of a sector of a circle, radius 2-5 cm., is 108°. Find the area of the sector.

22. A square \( ABCD \) is inscribed in a circle of radius 4 in. Find the area of the minor segment cut off by \( AB \).

23. \( AB \) is a chord of a circle \( AKB \), radius 4 in. If \( \angle AKB = 30^\circ \), find the area of the segment \( AKB \).

24. \( ABCD \) is a square and \( AEF \) is an equilateral triangle inscribed in the circle \( ABCFD \). Find the angles of \( \triangle ECD \).

25. \( ABCDE \) is a regular pentagon inscribed in a circle. Find the angles of \( \triangle ABD \).

26. \( AB \) is a side of a regular hexagon and \( AC \) of a regular octagon, inscribed in the same circle. Find the angles of \( \triangle ABC \). [Two sets of answers.]

27. Find the angles of the triangle formed by joining the points \( I, II, VI, IX \) on the face of a clock.

28. \( A, B \) are points on the circle \( ABCD \) such that the minor arc \( AB \) is half the major arc \( AB \); \( \angle DAB = 74^\circ \); arc \( BC \) = arc \( CD \). Find \( \angle ABD \) and \( \angle BDC \).
NEW GEOMETRY

[29] A, B, C are 3 points on a circle such that \( \angle ABC = 38^\circ \), \( \angle ACB = 88^\circ \); P, Q are the midpoints of the minor arcs AC, AB respectively. Find \( \angle BCP \) and \( \angle CPQ \).

30. ABCD is a square and APQ is an equilateral triangle inscribed in the circle APQCD. Prove that \( BD = \frac{1}{2} \) AR. 

31. ABC is a triangle inscribed in a circle; T is a point on BC produced. If \( \angle BAT = 120^\circ \), \( \angle CAT = 15^\circ \), \( \angle ATB = 30^\circ \), find the ratio of the arc AB to the arc AC.

32. On a clock-face, prove that the line joining the points IV, VII is perpendicular to the line joining the points V, XII.

33. If, in fig. 638, arc BD is four times the arc AC, find \( \angle ADC \). 

34. With the data of fig. 638, find what fraction are \( AD + \text{arc } BC \) is of the circumference. [Join AC.]

*35. Draw a circle of radius 5 cm. and place in it a chord AB of length 4 cm. Find the area of the major segment cut off by AB, making any necessary measurements.

*36. A piece of wire is in the form of an arc of a circle of radius 6 cm., subtending 100° at the centre. It is bent into a complete circle. Find the radius of the circle.

*37. A piece of wire 1 yd. long is bent into the form of a semi-circular arc and its diameter. Find the radius. [Take \( \pi \approx \frac{22}{7} \).]

*38. Find the radius of a circle whose area is equal to the sum of the areas of two circles of radii, 3 in. and 4 in.

*39. ABCD is a quadrilateral inscribed in a circle. If \( \angle ADB = 26^\circ \) and \( \angle DCB = 85^\circ \), prove that

\[
\text{arc } AB + \text{arc } CD = \text{arc } BC + \text{arc } AD.
\]

*40. ABC is an equilateral triangle and PQRS is a square inscribed in the circle APQBCD. If AB is parallel to PQ, prove that A, P, Q, B, R, C, S are some of the vertices of a regular 24-sided polygon.

*41. ABCD is a quadrilateral inscribed in a circle; AC cuts BD at K; DA, CB when produced meet at E; AB, DC when produced meet at F. If \( \angle AEB = 55^\circ \), \( \angle BFC = 35^\circ \), \( \angle DKA = 88^\circ \), prove that arc BC is twice arc AB.

CHORDS AND ARCS

THEOREM 56

In equal circles (or in the same circle) equal chords cut off equal arcs.

![Fig. 639](image)

Given two equal circles, centres H, K, and two equal chords AB, CD of these circles.

To prove that

- minor arc \( AB = \text{minor arc } CD \),
- major arc \( AB = \text{major arc } CD \).

Construction. Join HA, HB, KC, KD.

Proof. In \( \triangle HAB, KCD \):

\[
\begin{align*}
HA &= KC & \text{radii of equal circles, given,} \\
HB &= KD & \text{radii of equal circles, given,} \\
AB &= CD & \text{given,}
\end{align*}
\]

\[\therefore \triangle HAB \equiv \triangle KCD.\]

\[\therefore \angle AHB = \angle CKD.\]

But, in equal circles, equal angles at the centres stand on equal arcs.

\[\therefore \text{minor arc } AB = \text{minor arc } CD.
\]

But the circumference of the two circles are equal,

\[\therefore \text{major arc } AB = \text{major arc } CD.\]
THEOREM 57

In equal circles (or in the same circle) the chords of equal arcs are equal.

![Diagram](image)

**Fig. 640**

**Given** two equal circles, centres H, K, and two equal arcs AB, CD of these circles.

**To prove that** chord $AB = $ chord $CD$.

**Construction.** Join $HA, HB, KC, KD$.

**Proof.** In equal circles the angles at the centres which stand on equal arcs are equal,

$$\therefore \angle ABH = \angle CKD,$$

$$\therefore \text{in } \triangle HAB, KCD,$$

$$HA = KC \quad \text{radius of equal circles, given},$$

$$HB = KD \quad \text{radius of equal circles, given},$$

$$\angle AHB = \angle CKD \quad \text{proved},$$

$$\therefore \triangle HAB \cong \triangle KCD \quad \text{SAS},$$

$$\therefore AB = CD.$$
EXERCISE 64

1. AB, CD are equal chords of the circle ABCD. Prove that (i) AC = BD; (ii) BC is parallel to AD.

2. ABCD is a cyclic quadrilateral. If AB = CD, prove that \( \angle ABC = \angle BCD \). [On what angles do these angles stand?]

3. ABCDEF is a hexagon inscribed in a circle.
   If \( \angle ABC = \angle DEF \), prove that AF is parallel to CD.
   [Join AD. What are the equal angles?]

4. A, B, C, D, E are five consecutive vertices of a regular polygon, with more than 5 sides, inscribed in a circle CDEFAB, centre K. Prove that \( \angle CPE = \angle AKB \).

5. The chords AB, DC of a circle ABCD meet when produced at E. If AB = CD, prove that EA = ED.

6. ABCD is a rectangle inscribed in a circle; DP is a chord equal to DC. Prove that (i) \( DP = AC \); (ii) PB = AD.

7. In fig. 644, X, Y are the mid-points of the arcs AB, AC. Prove that AP = AQ.
   [Join AX, AY.]

8. A circle AOBP passes through the centre O of a circle ABQ. Prove that OP bisects \( \angle APB \).

9. PQ, RS are parallel chords of the circle PRSQ, centre O. If PS cuts QR at K, prove that \( \angle PKR = \angle POR \).

10. In fig. 645, the chords BX, CY are perpendicular to AC, AB respectively. Prove that AC = AB.

11. If, in fig. 646, \( \angle PMA = \angle QNB \), prove that PQ = AB.
   [PQ is not parallel to BC.]

CHORDS AND ARCS

12. APQB is an arc of a circle. If the bisectors of \( \angle PAQ, \angle PBQ \) meet at R, prove that R lies on the circle.

13. ABCD is a quadrilateral inscribed in a circle; CD is produced to F; the bisector of \( \angle ABC \) cuts the circle at E. Prove that DE bisects \( \angle ADF \).

14. A hexagon is inscribed in a circle. If two pairs of opposite sides are parallel, prove that the third pair are also parallel. [Let the lengths of the minor arcs cut off by the sides be \( a, b, c, d, e, f \). Use Example 2, p. 340.]

15. The sides AD of the cyclic quadrilateral ABCD is produced to E so that DE = AB. If AC bisects \( \angle BAD \), prove that CE = CA.

16. The bisectors of \( \angle ABC, \angle ACB \) meet at I, and the circle BIC cuts AB, AC, produced if necessary, at P, Q. Prove that PI = IC and QI = IB.

17. In fig. 647, the circles are equal and BXY is a straight line. Prove that AX = AY.

18. P is any point on a chord BC of a circle, centre O. Prove that the circles OPB, OPQ are equal.

19. In \( \triangle ABC, AB = AC; D \) is any point on BC produced. Prove that the circles ADB, ADC are equal.

20. ABCD is a cyclic quadrilateral. If \( \angle ABC = 2 \angle BAC \) and if AB is parallel to DC, prove that AD = DC = CB.

21. A variable triangle PQR is inscribed in a given circle. If the angle QPR is of given size, what else can you say about \( \triangle PQR \)?

22. CD is a quadrant of the circle ACDB; AB is a diameter. If AD cuts BC at P, prove that AC = CP.
23. ABCD is a cyclic quadrilateral; BC and AD meet when produced at E. If the circle ACE cuts AB, CD, produced if necessary, at P, Q, prove that EP = EQ.

24. AB, BC are two chords of a circle, AB > BC. The minor arc AB is folded over about the chord AB and cuts AC at D. Prove that BD = BC.

25. ABC is an equilateral triangle inscribed in a circle; H, K are the mid-points of the minor arcs AB, AC. Prove that HK is bisected by AB, AC. [Join AH, AK, CH, BK.]

26. ABCD is a quadrilateral inscribed in a circle; X, Y, Z, W are the mid-points of the minor arcs AB, BC, CD, DA respectively. Prove that XZ is perpendicular to YW. [Join XY, YZ.]

27. ABC is a triangle inscribed in the circle BPACO, centre O; PQ is the diameter perpendicular to BC. Prove that \( \angle ACB = \angle ABC = \angle AOP. \)

28. ABC is an equilateral triangle inscribed in a circle; D, E are points on the minor arcs AB, BC such that AD = BE. Prove that AD + BE = AE. [Draw DK parallel to BE to meet AE at K.]

29. In fig. 648, three circles are equal, AD = BC, and XAB, DCY are straight lines. Prove that XBYD is a parallelogram.

30. Two fixed circles cut at A, B; P is a variable point on one circle; PA, PB when produced cut the other circle at Q, R. Prove that QR is of constant length.

Secants and Tangents

Definitions. If a straight line cuts a circle at two distinct points, it is called a secant.

If a straight line has one point, and only one point, in common with a circle, however far either way it is produced, the straight line is called a tangent to the circle, and the common point is called the point of contact.

The words "touching" and "meeting" must not be confused. If a line meets a circle, it may when produced meet it at a second distinct point, and if it does so, the line is a secant. It may, however, have only one point in common with the circle, however far either way it is produced, and if this is so, the line is a tangent.

A discussion of the properties of tangents by the use of limits is given in the Appendix, pp. 550–552.

Examples for Oral Discussion

1. A is any point on a circle, centre O, and BAC is drawn at right angles to OA through A. Prove that BAC is a tangent to the circle.

If P is any point, other than A, on the line BAC, prove that OP > OA, and complete the proof.

If we assume that there is only one tangent to a circle at any given point on it, it follows that the converse of Example 1 is true. It is, however, easy to draw curves for which this assumption is untrue, see fig. 651. It is therefore desirable to prove that this assumption is true for a circle.
2. If a straight line \( BAC \) touches at \( A \) the circular arc \( AEF \), centre \( O \), prove that \( OA \) is perpendicular to \( BAC \).

![Diagram](image)

*Fig. 652*

*If possible, suppose \( OA \) is not perpendicular to \( BAC \) and let \( ON \) be the perpendicular from \( O \) to \( BC \). Produce \( AN \) to \( Q \) so that \( AN = NQ \). Join \( OQ \) and prove that \( OQ = OA \). Explain why this is contrary to the data.*

3. \( T \) is any point outside a given circle, centre \( O \). Construct the tangents from \( T \) to the circle.

![Diagram](image)

*Fig. 653*

Let the circle on \( TO \) as diameter cut the given circle at \( P, Q \). Then \( TP, TQ \) are the required tangents.

(i) Explain why \( \angle OPT \) is a right angle.

(ii) Prove that \( TP \) and \( TQ \) are tangents.

4. If \( TP, TQ \) are the tangents from \( T \) to a circle, centre \( O \), prove that (i) \( TP = TQ \); (ii) \( OT \) bisects \( \angle PTQ \) and \( \angle POQ \).

Explain why \( \triangle OPT = \triangle OQT \).

---

**Exercise 65**

Nos. 1–4 refer to fig. 654 in which \( TAB \) is the tangent at \( A \) to the circle, centre \( O \).

1. If \( \angle PAB = 25^\circ \), find \( \angle OAP \) and \( \angle AOP \).

2. If \( \angle AOP = 70^\circ \), find \( \angle PAB \).

3. If \( OA = 6 \text{ cm.}, OT = 10 \text{ cm.} \), find \( TA \).

4. If \( TA = 12 \text{ cm.}, OA = 5 \text{ cm.} \), find \( OT \).

Nos. 5–7 refer to fig. 653, p. 354, in which \( TP, TQ \) are the tangents from \( T \) to the circle, centre \( O \).

5. If \( \angle PTQ = 70^\circ \), find \( \angle POQ \).

6. If \( \angle TOP = 56^\circ \), find \( \angle PTQ \).

7. If \( \angle PTQ = 36^\circ \), find \( \angle TPQ \).
8. **ABC** is a minor arc of a circle; the tangents at **A**, **C** meet at **T**. If $\angle ATC = 54^\circ$, find $\angle ABC$.

Nos. 9-11 refer to fig. 655 in which **AC** is a diameter of the circle and **TAB** is a tangent.

9. If $\angle QAB = 38^\circ$, find $\angle QAC$ and $\angle QCA$.

10. If $\angle APQ = 42^\circ$, find $\angle QAB$.

11. If $\angle QAT = 155^\circ$, find the angle in the minor segment **AQ**.

12. The tangent to a circle of radius 4.5 in. from an external point **T** is 8 in. long. Find the distance of **T** from the nearest point of the circumference.

13. The radii of two concentric circles are 5 cm., 3 cm. Find the length of a chord of the larger circle which touches the smaller circle.

Nos. 14-17 refer to fig. 656, in which a circle, centre **I**, touches the sides of $\triangle ABC$ at **X**, **Y**, **Z**.

14. If $\angle B = 50^\circ$, $\angle C = 70^\circ$, find the angles of $\triangle XYZ$.

15. If $\angle XYZ = 64^\circ$, $\angle XZY = 48^\circ$, find $\angle XIZ$ and the angles of $\triangle ABC$.

16. If $AB = 8$ cm., $BC = 7$ cm.,

   $CA = 5$ cm., find $BX$.

   [Let $BX = BZ = x$ cm., then $CZ = CX = (7 - 2x)$ cm.;
   also $AY = AZ$.]

17. Repeat No. 16 if $AB = 4$ in., $BC = 4.5$ in., $CA = 3.5$ in.

18. In $\triangle ABC$, $\angle B = 60^\circ$, $\angle C = 70^\circ$; a circle touches $BC$, $AC$ produced, $AB$ produced at $P$, $Q$, $R$ respectively. Find $\angle QPR$.

19. Three of the angles of a quadrilateral circumscribing a circle are $70^\circ$, $84^\circ$, $96^\circ$ in order. Find the angles of the quadrilateral whose vertices are the points of contact.

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**TANGENTS**

20. In $\triangle ABC$, $BC = 3$ cm., $CA = 6$ cm., $AB = 7$ cm.; a circle is drawn to touch $AB$ produced at $R$, $AC$ produced at $Q$, and $BC$ at $P$. Find the lengths of $AR$ and $CP$.

21. Repeat No. 20 if $BC = 3$ in., $CA = 5$ in., $AB = 4$ in.

22. In fig. 657, not drawn to scale, $QR$ touches each of the circles, centres **A**, **B**, radii 8 cm., 3 cm. respectively. If $AB = 13$ cm., find $QR$.

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*23. In fig. 658, $AN$ is a tangent to the circle and $ANP$ is a right angle. If $AN = 13$ cm. and $PN = 9$ cm., find the radius of the circle. [Draw $PK$ perpendicular to the radius $OA$; let $OA = r$ cm.]

*24. In $\triangle ABC$, $AB = 2$ in., $BC = 3$ in., $\angle ABC = 90^\circ$. Find the radius of the circle which touches $AB$ at $A$ and passes through $C$. [Draw $CK$ perpendicular to the radius $AO$ produced; join $OC$.]

*25. $AD$ is an altitude of $\triangle ABC$. If $AB = 3$ in., $AC = 4$ in., $AD = 2.4$ in., and if $D$ lies between $B$ and $C$, prove that $AC$ touches the circle, centre $B$, radius $BA$.

*26. A hemispherical bowl of diameter 25 in. and negligible thickness, rests on a horizontal table. Water is poured into the bowl till the surface of the water is $2\frac{1}{4}$ in. below the rim. If the bowl is tilted slowly, find the height above the table of the highest point of the rim when the water is about to overflow.
THEOREM 58

The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

Given a circle centre O, a radius OA, and the straight line BAC perpendicular to OA.

To prove that BAC is a tangent.

Construction. Take any point P on BC. Join OP.

Proof.
\[ \angle OAP = 1 \text{ rt. } \angle \text{ given}, \]
\[ \therefore \text{ each other angle of } \triangle OAP \text{ is less than } 1 \text{ rt. } \angle. \]
\[ \therefore \angle OPA < \angle OAP, \]
\[ \therefore OA < OP \text{ greater side opposite greater angle.} \]

But OA is a radius,
\[ \therefore OP \text{ is greater than a radius,} \]
\[ \therefore P \text{ lies outside the circle.} \]

Similarly, every point on BC except A lies outside the circle.
\[ \therefore \text{BC touches the circle at A.} \]

THEOREM 59

A tangent to a circle is perpendicular to the radius drawn through the point of contact.

Given a tangent BAC at a point A on a circle, centre O.

To prove that \( \angle OAC \) is a right angle.

Construction and Proof. If possible, suppose that OA is not perpendicular to BC and draw the perpendicular ON from O to BC.

Produce AN to Q so that AN = NQ. Join OQ.

By construction, ON is the perpendicular bisector of AQ,
\[ \therefore OA = OQ \text{ locus theorem}, \]
\[ \therefore Q \text{ lies on the circle,} \]
\[ \because BAC \text{ cuts the circle at two points, A and Q.} \]

But this is impossible because BAC is a tangent. Therefore the original supposition is false.
\[ \therefore OA \text{ must be perpendicular to BC.} \]

Corollary 1. At every point of a circle, one and only one tangent can be drawn to the circle.

Corollary 2. The perpendicular to a tangent at its point of contact passes through the centre of the circle.
THEOREM 60

If two tangents are drawn to a circle from an external point, then:
1. the tangents are equal;
2. the tangents subtend equal angles at the centre;
3. the line joining the centre to the external point bisects the angle between the tangents.

![Fig. 661]

Given a circle centre $O$ and the tangents $TP$, $TQ$ from an external point $T$ to the circle.

To prove that:
1. $TP = TQ$,
2. $\angle TOP = \angle TOQ$,
3. $\angle OTP = \angle OQT$.

Proof. In $\triangle OPT$, $OQT$:
- $OP = OQ$ (radii),
- $OT = OT$ (common side),
- $\angle OPT = \angle OQT$ (rt. $\angle$s, tangent perp. to radius).

$\therefore \triangle OPT \cong \triangle OQT$ (RHS).

$\therefore TP = TQ$,
and $\angle TOP = \angle TOQ$,
and $\angle OTP = \angle OQT$.

Examples for Oral Discussion

1. $PQ$ is a variable chord of a given circle, centre $O$. If $PQ$ is of constant length, prove that it touches a fixed concentric circle.

If $ON$ is the perpendicular from $O$ to $PQ$, explain why the length of $ON$ is constant.

2. In fig. 663, $ABCD$ is a quadrilateral circumscribing a circle. Prove that $AB + CD = AD + BC$.

Let the points of contact be $P$, $Q$, $R$, $S$.

What do you know about $AP$? about $BP$?

3. If, in fig. 663, $O$ is the centre of the circle, prove that $\angle AOB + \angle COD = 2$ right angles.

What do you know about $\angle OAB$? about $\angle OBA$?
What do you know about $\angle A + \angle B + \angle C + \angle D$?

4. Fig. 664 represents three circles, centres $A$, $B$, $C$, touching a straight line $XY$ at the point $P$. Prove that $P$, $A$, $B$, $C$ are collinear.

What do you know about $\angle CPX$?

EXERCISE 66

1. $AB$ is a diameter of the circle $APB$. Prove that the circle, centre $A$, radius $AP$, touches $PB$.

[2] $AB$ is a diameter of a circle. Prove that the tangents at $A$ and $B$ are parallel.
3. AP is a chord of a circle, centre O. PN is the perpendicular from P to the tangent at A. Prove that AP bisects \( \angle OPN \).

Nos. 4–7 refer to fig. 666 in which PQ, PR are chords of a circle, centre O, which touch a concentric circle at A, B.

4. Prove that PA = AQ.

[5] Prove that PQ = PR.

[6] Prove that QR = 2AB.

7. Prove that OP is the perpendicular bisector of AB.

8. AB is a diameter of the circle APB; AX, BY are the perpendiculars from A, B to the tangent at P. Prove that PX = PY. [Join P to the centre.]

Nos. 9–12 refer to fig. 666, p. 356, in which a circle touches the sides of \( \triangle ABC \).

9. Prove that \( \angle YXZ = 90^\circ - \frac{1}{2} \angle BAC \). [Join Y, Z to the centre.]

[10] If Y, Z are the mid-points of AC, AB, prove that BX = CX.

11. Prove that AY = \( \frac{1}{2}(AB + AC - BC) \).

[12] Prove that the inscribed circles of \( \triangle AXY, \triangle AXC \) touch AX at the same point.

13. ABCDEF is a hexagon circumscribing a circle. Prove that AB + CD + EF = BC + DE + FA.

14. P and Q are points on two circles which have the same centre O; the tangents at P and Q meet at T. Prove that (i) O, P, Q, T are concyclic; (ii) \( \angle OQP \) and \( \angle OTP \) are either equal or supplementary.

[15] AP, AQ are two chords of a circle and AB is a diameter. The tangent at B meets AP produced, AQ produced at X, Y. Prove that P, Q, X, Y are concyclic.

[16] If a parallelogram circumscribes a circle, prove that it is a rhombus and that the diagonals intersect at the centre of the circle.

17. In fig. 666, O is the centre of the circle, \( \angle AOT \) is a right angle, and TP is a tangent. Prove that TP = TQ. [Join OP.]

[18] O is the centre of the circle ABC. If \( \angle ABC = 90^\circ \) and if the tangent at A cuts OC produced at T, prove that OC = CT.

[19] AB is a diameter of a circle APB, centre O; the tangents at A and B meet at T. Prove that TG is parallel to PB. [Let AP cut QT at N; join OP.]

[20] O and A are fixed points; AP is the tangent from A to a variable circle, centre O. Prove that the locus of P is a circle.

[21] The tangents at points P, Q on a circle, centre O, meet at T; the line through T perpendicular to QT meets OP, produced if necessary, at K. Prove that KT = KO. [Join OQ, OT.]

22. Two parallel tangents to a circle, centre O, are cut by a third tangent at P, Q. Prove that \( \angle POQ \) is a right angle.

23. In \( \triangle ABC \), \( \angle ABC = 90^\circ \). A circle, centre X, is drawn to touch AB produced, AC produced, and BC. Prove that \( \angle AXC = 45^\circ \).

24. PQ is a chord of a circle, centre O; the tangents at P, Q meet at T; TR is drawn perpendicular to TP and so that \( \angle QTR \) is acute. If \( \angle QTR > \angle OTP \), prove that \( \angle POQ > 120^\circ \).

25. In fig. 667, APB is a semicircle, centre O, touching three sides of the rectangle ABCD; ANPC is a straight line and AN = NP. Prove that (i) O, B, C, N are concyclic; (ii) \( \angle ANB = 135^\circ \).

26. In fig. 668, A, B are the centres of two circles touching the parallel lines EPF, GQH; PQ touches each circle. Prove that (i) APBQ is a rectangle; (ii) PQ = AB.
27. If in fig. 663, p. 361, BC, AD produced meet at H, and AB, DC produced meet at K, prove that HB + BK = HD + DK.

28. PQRS is a minor arc of a circle; the tangents at P, S meet at B; the tangents at Q, S meet at D; BD produced cuts BS at C. Prove that AB - CD = BC - AD.

29. A diameter AB of a circle APB is produced to C so that AB = 2BC. CT is the perpendicular from C to the tangent at P. Prove that (i) BP = BT; (ii) \( \angle PBT = 2\angle ABP \).

Alternate Segment

Examples for Oral Discussion

Nos. 1, 2 refer to fig. 669, in which BAC is a tangent and AD is any chord.

1. If \( \angle c = 52^\circ \), find \( \angle APD \).

Draw the diameter AE. Write down, with reasons, the sizes of \( \angle e, \angle p_1, \angle p \).

2. Prove that \( \angle c = \angle p \).

Draw the diameter AE. Explain why \( \angle c + \angle e \) and \( \angle p_1 + \angle e \) each equal 1 rt. \( \angle \). Complete the proof.

Fig. 669

Nos. 3, 4 refer to fig. 670, in which BAC is a tangent and AD is any chord.

3. If \( \angle c = 52^\circ \), find \( \angle b \) and \( \angle q \).

Use the fact, proved in No. 2, that \( \angle c = \angle p \).

4. Prove that \( \angle b = \angle q \).

Explain why \( \angle b + \angle c \) and \( \angle p + \angle q \) each equal 2 rt. \( \angle \).

Fig. 670

Fig. 671 shows the facts proved in Nos. 2, 4.

If a straight line BAC touches a circle at A, and if AD is any chord through the point of contact A, \( \angle DAC \) and \( \angle DAB \) are the two angles formed by the chord AD with the tangent at A.

Since BAC is a straight line, these angles are supplementary. In fig. 671 (i), \( \angle DAC \) is on the right of the chord AD, and it is equal to the angle in the segment cut off by AD on the left of AD.

In fig. 671 (ii), \( \angle DAB \) is on the left of the chord AD, and it is equal to the angle in the segment cut off by AD on the right of AD.

Fig. 671

The segment APD of the circle on the opposite side of AD to \( \angle DAC \) is called the alternate segment corresponding to \( \angle DAC \). Similarly, the segment AQD of the circle on the opposite side of AD to \( \angle BAD \) is called the alternate segment corresponding to \( \angle BAD \).
The results given in Nos. 2, 4 may therefore be stated as follows:—

The angles which a tangent to a circle makes with any chord through the point of contact are equal to the angles in the alternate segments of the circle.

5. Given a circle and a point A on it, inscribe a triangle ABC in the circle, such that \( \angle B = 55^\circ, \angle C = 75^\circ \).

Draw the tangent SAT. Explain how AB and AC must be drawn.

6. Given a straight line AB, 2 in. long, construct a segment of a circle APB such that \( \angle APB = 40^\circ \). Measure the radius.

Draw AT so that \( \angle BAT = 40^\circ \). Explain how to find the centre of a circle which (i) touches AT at A and (ii) passes through A and B.

7. If, in fig. 674, \( \angle TAB = \angle APB \), prove that AT touches the circle APB at A.

If possible, suppose AT is not a tangent and draw the tangent AK so that AK and AT are on the same side of AB. Explain why this is impossible.

Example 7 is the converse of Nos. 2, 4 and is often useful in rider work when it is necessary to prove that a line touches a circle.
THEOREM 61

If a straight line touches a circle and from the point of contact a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments of the circle.

![Diagram](image)

**Fig. 680**

Given a straight line BAC touching a circle at A and a chord AD forming the two segments APD, AQD.

To prove (1) \( \angle DAC = \angle APD \) in alternate segment APD,  
(2) \( \angle DAB = \angle AQD \) in alternate segment AQD.

(1) **Construction.** Draw the diameter AE. Join ED.

**Proof.** With the notation in fig. 680 (i), since AE is a diameter and AC is a tangent,
\[
\angle C + \angle e = 1 \text{ rt. } \angle; \quad \angle \text{ in semicircle,}
\]
also \( \angle ADE = 1 \text{ rt. } \angle \)
\[
\therefore \angle p + \angle e = 1 \text{ rt. } \angle \quad \angle \text{ sum of } \Delta,
\]
\[
\therefore \angle c = \angle p + \angle e = \angle p \quad \therefore \angle c = \angle p.
\]
But \( \angle p_1 = \angle p \quad \angle s \text{ in same segment,}
\]
\[
\therefore \angle c = \angle p.
\]

(2) **Proof.** With the notation in fig. 680 (ii),
\[
\angle b + \angle c = 2 \text{ rt. } \angle s \quad \text{adj. } \angle s \text{ on st. line,}
\]
and \( \angle q + \angle p = 2 \text{ rt. } \angle s \quad \text{opp. } \angle s \text{ cyclic quad.,}
\]
\[
\therefore \angle b + \angle c = \angle q + \angle p.
\]
But \( \angle c = \angle p \quad \therefore \angle b = \angle q.
\]

--

**EXERCISE 67 (continued)**

Note. 13, 14 refer to fig. 678 in which QAPK, RBP are straight lines and PT is a tangent.

13. If \( \angle PQR = 75^\circ \), what other angles in the figure can be found? Find them.

14. If \( \angle TPK = 72^\circ \), \( \angle TPR = 65^\circ \), find \( \angle PQR \) and \( \angle PRQ \).

![Diagram](image1)

**Fig. 678**

In fig. 679, AS, CT are tangents and SBKT is a straight line. If \( \angle ASY = 82^\circ \), \( \angle OKT = 55^\circ \), \( \angle DCO = 165^\circ \), find \( \angle DAC \) and \( \angle DTS \).

16. In \( \triangle ABC \), \( AB = AC \) and \( \angle B = 70^\circ \); BC is produced to D so that \( \angle CAD = 30^\circ \). Prove that AB touches the circle ADC.

17. **ABCD** is a quadrilateral in which \( \angle BCD = 95^\circ \), \( \angle CDA = 60^\circ \). If \( \angle CAB = 55^\circ \) and \( \angle ACD = 45^\circ \), prove that AD touches the circle ABC.
THEOREM 62

If a straight line be drawn from an extremity of a chord of a circle making with the chord an angle equal to the angle in the alternate segment, the straight line touches the circle.

![Diagram showing a circle with a chord AD and a tangent AT]

**Fig. 681**

Given a chord $AD$ of a circle and a straight line $AC$ such that $\angle DAC = \angle APD$ in alternate segment $APD$.

**To prove** that $AC$ touches the circle at $A$.

**Construction and Proof.** If possible, suppose that $AC$ does not touch the circle at $A$ and draw the tangent $AT$ on the same side of $AD$ as $AC$.

\[ \angle TAD = \angle APD \quad \angle \text{in alt. segment,} \]

but $\angle CAD = \angle APD \quad \text{given,}$

\[ \therefore \angle TAD = \angle CAD. \]

But this is impossible because $AC$ and $AT$ are on the same side of $AD$.

Therefore the original supposition is false.

\[ \therefore AC \text{ must touch the circle at } A. \]
9. In fig. 686, AP, AQ are tangents and PHKQ is a straight line. Prove that AH = AK.

[10] If in fig. 686, in which AP, AQ are tangents, QB is produced to meet the circle ABP in R, prove that PR is parallel to AQ. [Join AB.]

[11] In \( \triangle ABC \), \( \angle BAC \) is a right angle; D is any point on BC. If DP, DQ are the tangents at D to the circles ABD, ACD, prove that \( \angle PDQ \) is a right angle.

12. The tangent at B to a circle ABC meets a circle ABD at D, and the tangent at A to the circle ABD meets the circle ABC at C. Prove that AD is parallel to BC. [Join AB.]

[13] The tangent at B to the circle ABQ meets a circle ABP at T. If \( \angle PAQ \) is a straight line, prove that \( \angle BQ \) is parallel to TP.

14. In fig. 687, BT is a tangent and ACT is a straight line. If the bisector of \( \angle ABC \) cuts AC at P, prove that \( TP = TB. \)

15. In fig. 687, BT is a tangent and ACT is a straight line. If \( \angle ABC = \angle T \), prove that \( AB = CT. \)

[16] AB, AC are equal chords of a circle and such that CA produced meets at D the tangent BDE to the circle. Prove that \( \angle EDA = 2\angle EBA. \)

17. In \( \triangle ABC \), \( AB = AC \); a circle is drawn to touch BC at B and to pass through A; if it cuts AC again at D, prove that \( BC = BD. \)

18. A line AD is trisected at B, C; BPC is an equilateral triangle. Prove that AP touches the circle PBD.

[19] ABCDE is a regular pentagon; BD cuts CE at K. Prove that BC touches the circle BKE.

20. In fig. 688, AP, AQ are tangents and \( \angle PAQ \) is acute. Prove that \( \angle PBQ = \frac{1}{2} \angle PAQ. \) What happens if \( \angle PAQ \) is obtuse?

[21] In fig. 688, AP, AQ are tangents. If P, B, Q are collinear, prove that AP, AQ are diameters of the circles ABP, ABQ.

22. ABC is a triangle inscribed in a circle. Any line parallel to AC cuts BC at P and the tangent AQ at Q. Prove that A, B, P, Q are concyclic.

23. The straight line PQ touches the circles ABP, ABQ at P and Q. PA is produced to cut BQ at H; QA is produced to cut PB at K. Prove that A, H, B, K are concyclic.

[24] AB is a diameter of a circle ABC; TC is the tangent from a point T on AB produced; TD is drawn perpendicular to TA and meets AC produced at D. Prove that \( \angle ADC = \angle TD. \)

[25] ABCD is a cyclic quadrilateral; the line DE parallel to CB cuts AB, produced if necessary, at E. Prove that BE touches the circle DAE.

[26] ABCD is a cyclic quadrilateral such that the tangent at A to the circle is parallel to BD; AC cuts BD at E. Prove that (i) \( \angle AC \) bisects \( \angle BCD \), (ii) \( AB \) touches the circle CBE.

27. ABC is a minor arc of the circle ABCD. If the tangents at A and C meet at T, prove that \( \angle ATC = \angle ABC = \angle ADC. \)

[28] ABC, ABD are two equal circles. If \( AB = BC \), prove that AC touches the circle ABD.

29. In fig. 689, BPE bisects \( \angle ABC \); APQ is a straight line such that \( AP = AE \). Prove that \( AB \) touches the circle AQG.

[30] AB is a diameter of the circle AQRB, centre O. If AQ, BR meet when produced at P, prove that \( OQ, OR \) are tangents to the circle PQR. [Produce OQ to K.]

31. P is any point on the base BC of \( \triangle ABC \). A circle is drawn to touch AB at B and to pass through P. If it cuts the circle ABC again at Q, prove that AC touches the circle PQR. [Produce AC to K; join QB, QC, QP.]

[32] The bisector of \( \angle BAC \) cuts BC at D; a circle is drawn through D and to touch AC at A. Prove that its centre lies on the perpendicular from D to AB. [Let the circle cut AB at Q; join DQ.]
33. \( AB \) is a diameter of the circle \( ACB \). If the tangents at \( A \) and \( C \) meet at \( T \) and if \( TC \), \( AB \) are produced to meet at \( N \), prove that \( \angle BCN = \angle ATN \).

34. In fig. 690, \( AS \), \( AT \) are tangents to the circles \( APB \), \( AQB \), and \( APQ \) is a straight line. Prove that \( \angle SAT = \angle PBQ \).

*35. Two chords \( AOB \), \( COD \) of a circle cut at \( O \); the tangents at \( A \) and \( C \) meet at \( X \); the tangents at \( B \) and \( D \) meet at \( Y \). Prove that \( \angle AXC + \angle BYD = 2 \angle AOD \).

*36. The diameter \( AB \) of a circle \( APB \), centre \( O \), is produced to \( T \) so that \( OB = BT \); \( TP \) is a tangent to the circle. Prove that \( TP = PA \). [Draw \( BN \) parallel to \( OP \) to cut \( TP \) at \( N \).]

*37. \( BAC \), \( BAD \) are two circles such that the tangents at \( C \) and \( D \) meet at \( T \) on \( AB \) produced. If \( CBD \) is a straight line, prove (i) \( TCAD \) is a cyclic quadrilateral; (ii) \( \angle TAC = \angle TAD \); (iii) \( TC = TD \).

*38. \( ABCD \) is a minor arc of a circle such that \( AB = BC \); \( AB \) and \( DC \) when produced meet at \( P \), and \( DB \) is produced to meet the tangent \( AT \) at \( T \). Prove that \( TP = TA \). [Prove that \( TADP \) is a cyclic quadrilateral.]

*39. Assuming the result of Exercise 58, No. 19, p. 325, what special cases can be obtained by taking (i) \( P \) very close to \( R \), (ii) \( P \) very close to \( A \), (iii) \( A \) very close to \( B \).

*40. \( OA \) is a chord of a circle, centre \( C \); \( T \) is a point on the tangent at \( O \) such that \( OA = OT \) and \( \angle AOT \) is acute; \( TA \) is produced to cut \( OC \) at \( B \). Prove that \( \angle OBA = \angle OCA \). Find the position of \( B \) when \( A \) is very close to \( O \).

*41. \( PQRS \) is a cyclic quadrilateral such that the sides \( P Q \), \( QR \), \( RS \), \( SP \) touch a circle at \( A \), \( B \), \( C \), \( D \) respectively. Prove that (i) \( AC \) is perpendicular to \( BD \); (ii) the mid-points of \( AB \), \( BC \), \( CD \), \( DA \) lie on a circle.

*42. \( PQ \), \( CD \) are parallel chords of a circle; the tangent at \( D \) cuts \( PQ \) at \( T \); \( B \) is the point of contact of the other tangent from \( T \). Prove that \( BC \) bisects \( PQ \). [If \( BC \) cuts \( PQ \) at \( K \) and \( O \) is the centre, prove that (i) \( T \), \( B \), \( D \), \( K \) are concyclic; (ii) \( T \), \( B \), \( D \), \( O \) are concyclic.]

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**Contact of Circles.** If two circles touch the same straight line at the same point, they are said to touch each other at that point. If the circles lie on opposite sides of the line, they are said to touch **externally**; if they lie on the same side of the line, they are said to touch **internally**.

1. In fig. 691, \( A \) and \( B \) are the centres of two circles which touch at \( P \). Prove that \( A \), \( B \), \( P \) are collinear.

What do you know about \( \angle APX \) and about \( \angle BPX \)? Give reasons and complete the proof.

2. If two circles touch externally, the distance between the centres is equal to the sum of the radii.

3. If two circles touch internally, the distance between the centres is equal to the difference of the radii.

4. Fig. 692 represents three circles, centres \( A \), \( B \), \( C \), radii 4 cm., 3 cm., 2.5 cm. respectively, touching one another. Find the lengths of the sides of \( \triangle ABC \). Using instruments, draw \( \triangle ABC \) and then draw the circles.
5. Fig. 693 represents three circles, centres F, G, H, radii 2 in., 1·2 in., 0·6 in. respectively, touching one another. Which of the contacts are internal? Find the lengths of GH, HF, FG. Using instruments, draw $\triangle FGH$ and then draw the circles.

6. If, in fig. 692, $AB = 3·5$ in., $BC = 2·7$ in., $CA = 3·2$ in., calculate the radii of the three circles and then draw the figure.

Let the radius of the circle, centre A, be $x$ in.; express the other radii in terms of $x$.

7. What is the complete locus of the centre of a variable circle of radius 3 cm. which touches a fixed circle of radius 5 cm?

8. What is the complete locus of the centre of a variable circle of radius 7 cm. which touches a fixed circle of radius 4 cm?

9. In fig. 694, C is a point on AB such that $AC = 7$ cm., $CB = 3$ cm. Find the radius of the circle, centre Q, which touches AB at C and also touches the circle, centre O, diameter AB.

If the radius is $r$ cm., explain why $OQ = (5 - r)$ cm.; and use the right-angled triangle $OCQ$.

10. In fig. 695, not drawn to scale, $ABCD$ is a rectangle, $AB = 4$ cm., $BC = 3$ cm.; and A is the centre of a circle, radius 2 cm. Find the radius of the two circles which touch this circle and also touch BC at C.

Let P, Q be the centres of the required circles, radii $x$ cm., $y$ cm. respectively. Find in terms of $x$ the sides of $\triangle ADP$; then use Pythagoras to find $x$. Similarly find $y$.

NUMERICAL EXAMPLES

EXERCISE 69

1. A circle, radius 5 cm., touches two concentric circles and encloses the smaller. The radius of the larger circle is 7 cm., find the radius of the smaller.

2. The distance between the centres of two circles of radii 4 cm., 7 cm. is 13 cm. Find the radius of the least circle that can be drawn to touch them and enclose the smaller circle.

3. In fig. 696, AB is a quadrant touching AD at A and the quadrant BC at B. If $\angle ADC = 90^\circ$, $AD = 12$ cm., $DC = 3$ cm., find the radii of the circles.

4. Three circles, centres A, B, C, touch each other externally. If $AB = 4$ in., $BC = 6$ in., $CA = 7$ in., find the radii of the circles.

5. Three circles, centres P, Q, R, touch each other externally. If $QR = 6$ cm., $RP = 7$ cm., $PQ = 8$ cm., find the radii of the circles.

6. Two circles, centres B, C, touch each other externally; a circle, centre A, touches the others internally. If $AB = 4$ in., $BC = 7$ in., $CA = 6$ in., find the radii of the circles.

7. Fig. 697 is formed of three circular arcs of radii 6 cm., 2·2 cm., 3·1 cm., touching one another; X, Y, Z are the centres of the circles. Find the lengths of the sides of $\triangle XYZ$.

8. State, without proof, the complete locus of the centre of a variable circle of radius 3·5 cm. which touches a fixed circle, centre A, of radius 2·3 cm.

9. Draw a line $AB$, 4 cm. long; draw a circle, centre A, radius 6 cm. and a circle, centre B, radius 2 cm. Construct a circle of radius 3 cm. to touch the larger circle internally and the smaller circle externally.

10. Draw two circles, each of radius 2·5 cm., with their centres 6 cm. apart. Construct a circle of radius 7 cm. to touch one circle internally and the other externally.

x.e.
11. Draw a circle, radius $\frac{1}{4}$ in., and draw one of its diameters. Construct a circle of radius $\frac{1}{2}$ in. to touch the first circle and this diameter, and construct a circle of radius $\frac{1}{4}$ in. to touch the first circle and the diameter produced.

12. C is a point on AB such that $AC = 5$ in., $CB = 3$ in. Calculate the radius of the circle which touches AB at C and also touches the circle on AB as diameter.

[13] C is the midpoint of AB. Three semicircles are drawn on the same side of AB having AC, CB, AB as diameters. If $AB = 12$ cm., calculate the radius of the circle which touches the three semicircles.

14. A, B are the centres of two circles of radii 5 cm., 3 cm.; $AB = 12$ cm.; BC is a radius perpendicular to BA. Calculate the radius of a circle which touches the larger circle and also touches the smaller circle at C. [Two answers.]

15. In fig. 698, AB, BC are two equal quadrants touching at B; $AC = 12$ cm.; find the radius of the circle which touches are AB, arc BC, AC.

[16] ABCD is a square of side 7 in.; C is the centre of a circle of radius 3 in.; find the radius of each circle which touches this circle and touches AB at A.

*17. Six circles each of radius 5 cm. are drawn with their centres at the vertices of a regular hexagon of side 10 cm. Find the radius of the circle which touches and encloses all of them.

*18. In one corner of a square frame, side 5 ft., is placed a disc of radius 1 ft., touching both sides. Find the radius of the largest disc which will fit into the opposite corner.

*19. $OA = a$ in., $OB = b$ in., $\angle AOB = 90^\circ$. Two variable circles are drawn touching each other externally, one of them touches OA at A, and the other touches OB at B. If their radii are $x$ in., $y$ in., prove that $(x + a)(y + b)$ is constant. If $a = b$, $b = 5$, $x = 4$, calculate $y$.

Important Hint. In solving problems about circles which touch each other, either internally or externally, it will often be found useful to draw the common tangent at their point of contact, as is done for proving Theorem 63. It is also useful, owing to Theorem 63, to join the centres to the point of contact.

**Theorem 63**

If two circles touch one another, the line joining their centres (produced if necessary) passes through the point of contact.

Given two circles, centres A, B, touching each other at P. To prove that A, B, P lie on a straight line.

Construction and Proof. Since the circles touch each other at P, they have a common tangent $XYP$ at P. Since the tangent to a circle is perpendicular to the radius through the point of contact, $\angle APX$ and $\angle BPX$ are right angles. \therefore A and B lie on the line through $P$ perpendicular to $XYP$; \therefore A, B, P lie on a straight line.

Corollary 1. If two circles touch each other EXTERNALLY, the distance between the centres is equal to the SUM of the radii. In fig. 699 (i), $AB = AP + BP$.

Corollary 2. If two circles touch each other INTERNALLY, the distance between the centres is equal to the DIFFERENCE of the radii. In fig. 699 (ii), $AB = AP - BP$. 
EXERCISE 70

Nos. 1-6 refer to fig. 700 in which the circles touch at A; the line HK touches the circles at H, K and meets at T the tangent at A.

1. Prove that TH = TK.

[2] If P is any point on AT, prove that the tangents from P to the circles are equal.

3. Prove that \( \angle HAK \) is a right angle.

[4] If AB is a diameter of the circle AHB, prove that AB touches the circle HAK.

5. If HA is produced to cut the circle AKC at M, prove that KM is a diameter of the circle AKC.

6. If any line LAS through A cuts the circles at R, S, prove that the tangents at R and S are parallel.

7. ABCD is a parallelogram. If the circles on AB and CD as diameters touch each other, prove that AB = BC. [Join the centres of the circles.]

8. ABCD is a parallelogram; AC cuts BD at K. Prove that the circles AKB, CKD touch each other. [Draw the tangent KT to the circle AKB.]

[9] ABCD is a straight line and P is a point such that \( \angle APB = \angle CPD \). Prove that the circles PAD, PBC touch each other.

10. Two circles touch internally at A; a chord PQ of one touches the other at R. Prove that \( \angle PAR = \angle QAR \). [Produce the tangent at A and the line PQ to meet at K.]

[11] Two circles touch internally at A; any line PQRS cuts one circle at P, S and the other at Q, R. Prove that \( \angle PAQ = \angle QAS \).

12. Two circles touch internally at A. The tangent at any point P on the inner circle cuts the outer at Q, R; AQ, AR cut the inner at H, K. Prove that the triangles PPH, AKR are equiangular to one another.

13. Two circles touch externally at A; a tangent to one of them at P cuts the other at Q, R. Prove \( \angle PAQ + \angle PAR = 180^\circ \). [Let the tangent at A cut PQ at T.]

14. T is any point on the tangent at A to the two circles AHP, AKQ which touch externally at A; TP, TQ are the other tangents from T to the circles and PHKQ is a straight line. Prove that (i) the tangents at P and K are parallel, (ii) \( \angle PAH = \angle QAK \).

15. Two circles touch externally at A; AB is a diameter of one circle; BR is the tangent from B to the other circle. Prove that \( \angle ARB = 45^\circ - \frac{1}{2} \angle ABR \).

16. Two equal circles touch externally at A; AB is a diameter of one circle; BR is the tangent from B to the other circle and cuts the first circle at Q. Prove that \( SQ = \frac{1}{2} BR \). [Join AQ; join R to centre of second circle; use the intercept theorem.]

17. O is the centre of a fixed circle. Two variable circles, centres P, Q, touch the fixed circle internally and each other externally. Prove that the perimeter of \( \triangle OPQ \) is constant.

18. Four circular coins of unequal sizes lie on a table so that each touches two, and only two, of the others. Prove that the four points of contact are concyclic.

*19. Two circles, centres A, B, touch externally at P; a third circle, centre C, encloses both, touching the first at Q and the second at R. Prove that \( \angle BAC = 2 \angle PQR \). [Draw tangent at R.]

*20. Two circles, centres B and C, touch externally at A; PQ is a line touching the circles at P and Q. Prove that the circle on BC as diameter touches PQ.

*21. C is the mid-point of AB; semicircles are drawn with AC, CB, AB as diameters and on the same side of AB. Prove that the radius of the circle which touches these three semicircles is equal to \( \frac{1}{2} AC \).

*22. OA, OB are two radii of a circle such that \( \angle AOB = 60^\circ \); a circle touches OA, OB and the arc AB internally; prove that its radius is equal to \( \frac{1}{2} OA \).

*23. In \( \triangle ABC \), \( AB = p \) in., \( AC = q \) in., \( \angle BAC = 90^\circ \) and \( \angle ABC > \angle ACB \). O is the mid-point of BC. Circles are drawn with AB and AC as diameters. Prove that two circles can be drawn with O as centre to touch each of these circles, and find their radii in terms of p, q.
CONSTRUCTIONS

The methods for performing simple tangent constructions have been given in previous exercises; for formal statements and proofs, see pp. 384-7.

Definition. If a straight line touches each of two circles, it is called a common tangent to the two circles; it is called a direct or exterior common tangent if the circles lie on the same side of it (fig. 705), and is called a transverse or interior common tangent if they lie on opposite sides of it (fig. 706).

EXERCISE 71

1. Draw a circle, centre A, radius 3 cm., and take a point T 5 cm. from A. Construct the tangents from T to the circle, and calculate their lengths.

2. Draw a line AB, 4 cm. long. Construct a line AK such that its distance from B is 3 cm.

3. Draw a line AB, 5 cm. long. Construct a circle, centre A, such that lengths of tangents from B to the circle are 3-5 cm.

4. Draw a circle, centre A, radius 4 cm., and a straight line BC at a distance 5 cm. from A. Construct a point T on BC such that the angle between the tangents from T to the circle is 90°.
CONSTRUCTION 10

Construct the tangent to a given circle at a given point on the circumference.

Given a circle, centre $O$, and a point $A$ on its circumference.

To construct the tangent at $A$ to the circle.

Construction. Join $AO$.

Through $A$, construct the line $AT$ perpendicular to $AO$.

Then $AT$ is the required tangent.

Proof. The tangent is perpendicular to the radius through the point of contact.

But $AO$ is a radius and $\angle OAT$ is a right angle,

$\therefore AT$ is the tangent at $A$.

CONSTRUCTION 11

Construct the tangents to a given circle from a given point outside the circle.

Given a circle, centre $O$, and a point $T$ outside the circle.

To construct the tangents from $T$ to the circle.

Construction. Join $OT$ and bisect it at $F$.

With centre $F$ and radius $FT$, describe a circle and let it cut the given circle at $P$, $Q$.

Join $TP$, $TQ$.

Then $TP$, $TQ$ are the required tangents.


Since $TF = FO$, the circle, centre $F$, radius $FT$, passes through $O$, and $TO$ is a diameter.

$\therefore \angle TPQ = 90^\circ \quad \angle$ in semicircle,

$\therefore PT$ is perpendicular to the radius through $P$,

$\therefore PT$ is the tangent at $P$ to the given circle.

Similarly, it may be proved that $QT$ is also a tangent to the given circle.
CONSTRUCTION 12

Construct the exterior common tangents to two circles.

Given two circles, centres $A$, $B$, radii $a$, $b$, and suppose $a > b$.

To construct the exterior common tangents to the two circles.

Construction. With centre $A$, radius $a - b$, describe a circle, and construct the tangents $BP$, $BP'$ from $B$ to this circle.

Join $AP$, $AP'$ and produce them to cut the circle, radius $a$, at $Q$, $Q'$.

Through $B$, draw $BR$, $BR'$ parallel to $AQ$, $AQ'$ to meet the circle, centre $B$ at $R$, $R'$.

Join $QR$, $Q'R'$.

Then $QR$, $Q'R'$ are the required common tangents.

Proof. By construction, $AP = a - b$ and $AQ = a$,

$\therefore PQ = AQ - AP = b = BR$,

$\therefore PQ$ and $BR$ are equal and parallel,

$\therefore PQRB$ is a parallelogram.

But by construction $PB$ is the tangent at $P$,

$\therefore \angle APB$ is a right angle,

$\therefore PQRB$ is a rectangle,

$\therefore \angle AQR$ and $\angle BRQ$ are right angles,

$\therefore QR$ is perpendicular to the radii through $Q$ and $R$.

$\therefore QR$ is a tangent to each circle.

Similarly, it may be proved that $Q'R'$ is a common tangent.

CONSTRUCTION 13

Construct the interior common tangents to two non-intersecting circles.

Given two circles, centres $A$, $B$, radii $a$, $b$.

To construct the interior common tangents to the two circles.

Construction. With centre $A$, radius $a + b$, describe a circle, and construct the tangents $BP$, $BP'$ from $B$ to this circle.

Join $AP$, $AP'$, cutting the circle, radius $a$, at $Q$, $Q'$.

Through $B$, draw $BR$, $BR'$ parallel to $QA$, $Q'A$ to meet the circle, centre $B$, at $R$, $R'$.

Join $QR$, $Q'R'$.

Then $QR$, $Q'R'$ are the required common tangents.

Proof. By construction, $AP = a + b$ and $AQ = a$,

$\therefore PQ = AP - AQ = b = BR$,

$\therefore PQ$ and $RB$ are equal and parallel,

$\therefore PQRB$ is a parallelogram.

But by construction $PB$ is the tangent at $P$,

$\therefore \angle APB$ is a right angle,

$\therefore PQRB$ is a rectangle,

$\therefore \angle AQR$ and $\angle BRQ$ are right angles,

$\therefore QR$ is perpendicular to the radii through $Q$ and $R$.

$\therefore QR$ is a tangent to each circle.

Similarly, it may be proved that $Q'R'$ is a common tangent.
CONSTRUCTION 14

Construct the inscribed circle of a given triangle.

\[ \text{Fig. 707} \]

Given a triangle $ABC$.

To construct the inscribed circle of the triangle $ABC$.

**Construction.** Construct the lines $B_1C_1$ bisecting the angles $ABC, ACB$ and intersecting at $I$.

Draw $IX$ perpendicular to $BC$.

With centre $I$, radius $IX$, describe a circle.

This is the required circle.

**Proof.** Draw $IY$, $IZ$ perpendicular to $AC$, $AB$.

Since $I$ lies on the line bisecting $\triangle ABC$,

$I$ is equidistant from $BA$, $BC$,

$\therefore \ IZ = IX$.

Similarly, $IY = IX$.

$\therefore$ the circle, centre $I$, radius $IX$, passes through $X$, $Y$, $Z$.

Also $\angle IXC$, $\angle IYA$, $\angle IZA$ are right angles,

$\therefore$ $BC$, $CA$, $AB$ are tangents to this circle.

**Note.** The definition of the inscribed circle of a triangle was given on p. 210, and the proof of Construction 14 is mainly a repetition of that of Theorem 33, p. 213. For the definition of the escribed circles of a triangle, see p. 211.

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CONSTRUCTION 15

Construct an escribed circle of a given triangle.

\[ \text{Fig. 708} \]

Given a triangle $ABC$.

To construct the circle which touches $AB$ produced, $AC$ produced, and $BC$.

**Construction.** Produce $AB$, $AC$ to $H$, $K$.

Construct the lines $B_1C_1$ bisecting the angles $HBC$, $KCB$ and intersecting at $I_1$.

Draw $I_1X_1$ perpendicular to $BC$.

With centre $I_1$, radius $I_1X_1$, describe a circle.

This is the required circle.

**Proof.** Draw $I_1Y_1$, $I_1Z_1$ perpendicular to $ACK$, $ABH$.

Since $I_1$ lies on the bisector of $\angle HBC$,

$I_1$ is equidistant from $BH$, $BC$,

$\therefore \ I_1Z_1 = I_1X_1$.

Similarly, $I_1Y_1 = I_1X_1$.

$\therefore$ the circle, centre $I_1$, radius $I_1X_1$, passes through $X_1$, $Y_1$, $Z_1$.

Also $\angle I_1X_1C$, $\angle I_1Y_1C$, $\angle I_1Z_1B$ are right angles,

$\therefore$ $BC$, $ACK$, $ABH$ are tangents to this circle.
CONSTRUCTION 16

On a given straight line, construct a segment of a circle containing an angle equal to a given angle.

**Fig. 709**

Given a straight line $AB$ and an angle $X$.

To construct a segment on $AB$ containing an angle equal to $X$.

**Construction.** At $A$ make an angle $BAC$ equal to $X$.
- Draw $AD$ perpendicular to $AC$.
- Draw the perpendicular bisector of $AB$ and let it cut $AD$ at $O$.
- With $O$ as centre and $OA$ as radius describe a circle.
- Then the segment of this circle on the side of $AB$ opposite to $C$ is the required segment.

**Proof.** Join $OB$.

Since $O$ lies on the perpendicular bisector of $AB$,

$$OA = OB.$$  

$\therefore$ the circle passes through $B$.

Since $AC$ is perpendicular to the radius through $A$,$ AC$ is a tangent,

$\therefore \angle X = \angle CAB$  

$= \text{angle in alternate segment } APB,$

$\therefore$ the segment $APB$ is the required segment.

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REGULAR POLYGONS

A shorter method can be used for Construction 16 if the use of a protractor is allowed.

Suppose, for example, with the data of fig. 709, $\angle X = 38^\circ$.

Then $\angle OAB = 90^\circ - 38^\circ = 52^\circ$.

Also $OA = OB$,

$\therefore \angle OBA = \angle OAB = 52^\circ$.

Therefore the centre $O$ of the required segment is found by drawing an isosceles triangle $OAB$, with base $AB$ and each base angle equal to $52^\circ$.

I nscribed and Circumscribed Regular Polygons

If a regular polygon of $n$ sides is inscribed in a circle or circumscribed about a circle, each side subtends an angle of $\frac{360}{n}$ degrees at the centre of the circle.

For the values $n = 3, 4, 6, 8$, these angles are respectively $120^\circ, 90^\circ, 60^\circ, 45^\circ$, and it has already been shown that angles of these magnitudes can be constructed without using a protractor.

Fig. 710 represents a regular octagon inscribed in a circle and one circumscribed about a circle.

The reader should construct inscribed and circumscribed regular figures of 3, 4, 6 sides, and, using a protractor, regular polygons of 5 sides and 7 sides.

The simplest way of inscribing a regular hexagon in a circle is to make use of the fact that the length of each side is equal to the radius.

To circumscribe a regular hexagon about a circle, draw tangents at the corners of the inscribed regular hexagon.

Alternate vertices of a regular hexagon are the vertices of an equilateral triangle.
Examples for Oral Discussion

1. In a given circle, inscribe a triangle equiangular to a given triangle PQR.

![Figure 711]

Draw the tangent SAT at any point A on the given circle. Draw AC, AB so that \( q_1 = q \) and \( r_1 = r \), and prove that \( \triangle ABC \) is equiangular to \( \triangle PQR \).

2. About a given circle, circumscribe a triangle equiangular to a given triangle PQR.

Let I be the centre of the given circle; draw radii IX, IY, IZ so that \( m_1 = m \) and \( m_2 = m \).

Draw the tangents at X, Y, Z to form \( \triangle ABC \), and prove that \( \triangle ABC \) is equiangular to \( \triangle PQR \).

3. Construct a circle to pass through a given point A and to touch a given circle, centre O, at a given point B.

Draw the perpendicular bisector LM of AB, and let it cut OB, produced if necessary, at P.

With centre P, radius PB, describe a circle, and prove that this is the required circle.

![Figure 712]

4. Construct a triangle ABC, such that \( AB = 6 \) cm., \( \angle ACB = 50^\circ \), and the median CX = 5.5 cm.

Draw AB and bisect it at X. Draw BC a segment APB containing an angle of 50°. Draw a circle, centre X, radius 5.5 cm.; this cuts the arc APB at C (two positions).

5. Construct a circle to touch a given line AB and a given circle, centre C, at a given point D. (Two answers.)

![Figure 714]

Draw CN perpendicular to AB and let it cut the given circle at H, H'.

(i) HD produced cuts AB at K.

Draw KP parallel to NCH and let it meet CD produced at P.

Then the circle, centre P, radius PD, is the required circle. Explain why PK = PD and complete the proof.

(ii) By using H' instead of H in (i), obtain a second circle touching AB and the given circle at D (internally).

6. Construct a circle to touch a given line AB at a given point K and a given circle, centre C. (Two answers.)

Use the construction indicated in fig. 714. State the method and prove that it is correct.
7. Construct a circle to touch a given circle, centre A, at a given point B and a second given circle, centre C. (Two answers.)

Draw the diameter H'CH parallel to AB.

(i) Join BH and let it cut the circle, centre C, again at K. Produce AB and CK to meet at P. Then the circle, centre P, radius PB, is the required circle.

(ii) By using H' instead of H in (i), obtain a second circle touching the circle, centre A, at B and touching the circle, centre C.

EXERCISE 72

1. Draw $\angle BAC = 35^\circ$ and make $AB = 4$ cm. Construct a circle to touch AC at A and pass through B. Measure its radius.

2. Draw a line AB and take a point C 3 cm. from AB. Construct two circles of radius 4 cm. to pass through C and touch AB.

3. Draw a circle of radius 4 cm. and inscribe an equilateral triangle in the circle. Measure its side.

4. Draw a circle of radius 2 cm. and circumscribe an equilateral triangle about the circle. Measure its side.

5. Draw $\angle BAC = 65^\circ$. Construct a circle of radius 3 cm. to touch AB and AC.

6. Take two points A, B, 4 cm. apart. Construct a circle to pass through A and B and such that the tangents at A and B include an angle of 100°. Measure its radius.

CONSTRUCTION OF CIRCLES

7. Draw a circle, radius 3 cm., and take a point A 4 cm. from its centre. Construct a circle of radius 2 cm. to touch this circle and to pass through A.

8. Take a point C 1 cm. from a line AB and draw a circle, centre C, radius 3 cm. Construct a circle of radius 2 cm. to touch this circle and AB.

9. The centre C of a given circle, radius 6 cm., is 8 cm. from a given line AB. Under what conditions is it impossible to draw a circle of radius 6 cm. to touch this circle and AB?

10. Take two points A, B, 6 cm. apart. With A, B as centres and radii 3 cm., 2 cm. respectively, describe two circles. Construct a circle of radius 5 cm. to touch each of these circles internally or externally. Give all possible answers.

11. Draw $\triangle ABC$ such that $AB = AC = 7$ cm., $BC = 5$ cm. Draw a circle, centre A, radius 3 cm. Construct a circle to touch this circle and pass through B and C.

12. Draw a circle, radius 4-5 cm., and a diameter AB. Construct a circle of radius 1-5 cm. to touch AB (or AB produced) and to touch the given circle (i) internally, (ii) externally.

13. Draw a circle, radius 5 cm. Construct two circles, radii 1-5 cm., 2-5 cm. touching each other externally and touching the given circle internally.

14. Draw $\triangle ABC$ such that $BC = 4$ cm., $CA = 3$ cm., $AB = 2$ cm. Construct the four circles which touch the sides of $\triangle ABC$ and measure their radii.

15. Draw $\triangle ABC$ such that $AB = 4$ cm., $BC = 6$ cm., $\angle B = 90^\circ$. Construct the circle escribed to BC and measure its radius.

16. Draw a quadrilateral such that its sides in order are 4, 5, 7, 6 cm. Inscribe a circle in it to touch three of the sides. Does it touch the fourth side?

17. Draw $\angle AOB = 40^\circ$ and make $OA = 4$ cm. Construct a circle touching OA at A and touching OB. Measure its radius.

18. Draw $\triangle ABC$ such that $BC = 7$ cm., $CA = 6$ cm., $AB = 5$ cm. Construct a circle to touch AB and AC and to have its centre on BC. Measure its radius.

19. Draw two parallel lines AB, CD, 6 cm. apart and take a point E between them, 2 cm. from AB. Construct a circle to touch AB and CD and to pass through E.

20. Draw two parallel lines AB, CD and any circle cutting AB. Construct a circle to touch AB, CD and the given circle.
[21] Given two points A, B and a point D on a line CDE, show how to construct two concentric circles one of which passes through A and B, and the other touches C at D. When is this impossible?

Construct the figures in Nos. 22-29. Arrows which meet are tangential where they meet unless otherwise indicated.

Do not rub out any of the construction lines.

22. Fig. 716 shows three arcs each of radius 3 cm. and each 1/4 of a complete circumference. The arcs are not tangential at B or C.

![Fig. 716](image)

![Fig. 717](image)

[23] Fig. 717 shows four equal quadrants AB, BC, CD, DE; AE = 6 cm.

[24] Fig. 718 shows three arcs, each of radius 3 cm.

![Fig. 718](image)

Fig. 719

25. Fig. 719 shows a rectangle 6 cm. by 8 cm. and the four outer circles are equal.

26. In fig. 720, the radii of the arcs AB, BC, CA are 3-5 cm., 2-5 cm., 7 cm.

![Fig. 720](image)

![Fig. 721](image)

[27] In fig. 721, the radii of the circles are 1 cm., 2 cm., 2 cm., 3 cm., and the centre of the smallest circle lies on the largest.

28. In fig. 722, AP, AQ are arcs of radii 4 cm., not tangential at A; PQ is an arc of radius 8 cm.; AB is perpendicular to CD and equals 3 cm.

![Fig. 722](image)

![Fig. 723](image)

[29] In fig. 723, ABCD is a square of side 2 cm.; BE, EF are arcs with C, A as centres respectively.

[30] On a line of length 5 cm., construct a segment of a circle containing an angle of 70°. Measure its radius.

21. On a line of length 2 in., construct a segment of a circle containing an angle of 140°. Measure its radius.

32. In a circle of radius 3 cm., inscribe a triangle whose angles are 40°, 60°, 75°. Measure its longest side.

33. Circumscribe about a circle of radius 2 cm. a triangle whose angles are 50°, 55°, 75°. Measure its longest side.

[34] Construct \( \triangle ABC \) such that \( BC = 6 \text{ cm.} \), \( \angle BAC = 90° \), altitude \( AD = 2 \text{ cm.} \). Measure \( AB, AC \).

35. Construct \( \triangle ABC \) such that \( BC = 5 \text{ cm.} \), \( \angle BAC = 55° \), altitude \( AD = 4 \text{ cm.} \). Measure \( AB, AC \).

36. Construct \( \triangle ABC \) such that \( BC = 6 \text{ cm.} \), \( \angle BAC = 52° \), median \( AX = 5 \text{ cm.} \).

[37] Draw \( \triangle ABC \) such that \( AB = 1-8 \text{ in.} \), \( BC = 2-6 \text{ in.} \), \( \angle ABC = 130° \). Find a point \( P \) within \( \triangle ABC \) such that \( \angle BPC = 50° \) and area of \( \triangle PAB = 2-7 \text{ sq. in.} \).

38. Construct the quadrilateral \( ABCD \) such that \( AD = 5 \text{ cm.} \), \( BC = 4-6 \text{ cm.} \), \( \angle ABD = \angle ACD = 55° \), \( \angle CBD = 43° \). Measure CD.

[39] The vertices of the pentagon \( ABCDE \) are concyclic: \( AB = 2 \text{ in.} \), \( BC = 3 \text{ in.} \), \( \angle ADB = 36° \), area of \( \triangle ABC = \text{area of} \triangle ABE \), and BD bisects \( \angle CBE \). Construct the pentagon.

40. A is a point on a circle of radius 5 cm.; \( P \) is a point on the tangent at \( A \) such that \( AP = 8 \text{ cm.} \). Construct a circle to touch the given circle and touch \( AP \) at \( P \). Measure its radius.
41. Draw $\angle ABC = 55^\circ$ and make $BC = 7$ cm; draw a circle, centre $C$, radius $3$ cm, and let it cut $CB$ at $X$. Construct a circle to touch the given circle externally at $X$ and to touch $AB$. Measure its radius.

Construct the figures in Nos. 42-45. Areas which meet are tangential where they meet.

Do not rub out any of the construction lines.

42. In fig. 724, $AB$, $AD$ are arcs of radii $6$ cm.; the distance of $C$ from $A$ is $6$ cm. and the line $AC$ is an axis of symmetry.

[Image: Fig. 724 and Fig. 725]

43. In fig. 725, $AB$ is an axis of symmetry; $PAQ$ is a semicircle of radius $2$ cm.; $RBS$ is an arc of radius $1$ cm.; $AB = 7$ cm.

[Image: Fig. 726 and Fig. 727]

44. In fig. 726, $AB$ is a quadrant of radius $2.5$ cm. with its centre on $AC$; $AC = 7$ cm.

45. In fig. 727, $AB$ is a semicircle, radius $3$ cm., centre $O$; $OP$, $OQ$ are arcs each of radius $1$ cm.

[Image: Fig. 728]

46. $ABC$ is an equilateral triangle; $AB = 4$ cm.; $A$, $B$ are the centres of two circles, each of radius $2.5$ cm.; $CA$ is produced to meet the first circle at $D$. Construct a circle touching the first circle internally at $D$ and touching the second circle externally.

47. Draw a circle centre $O$ and two radii $OA$, $OB$. Show how to inscribe a circle in the sector $AOB$, i.e. to touch $OA$, $OB$, and arc $AB$.

*48. Draw any triangle $ABC$. Without making any measurements, construct three circles, centres $A$, $B$, $C$, so that each touches the other two. Give the four possible answers.

Examples for Oral Discussion

The In-circle and Ex-circles of a Triangle. Nos. 1-4 refer to fig. 728, in which $I$ is the in-centre and $I_1$ is an ex-centre of $\triangle ABC$ (see pp. 388, 389).

$BC = a$, $CA = b$, $AB = c$, $s = \frac{1}{2}(a + b + c)$, area of triangle $ABC = \Delta$.

1. Prove that $AY = s - a$. Write down corresponding expressions for $BX$, $CX$.

2. Prove that $AR = s$, and find $BP$, $PC$ in terms of $a$, $b$, $c$, $s$.

3. Prove that the radius $r$ of the in-circle equals $\frac{\Delta}{s}$.

$[\triangle IBC + \triangle ICA + \triangle IAB = \triangle ABC; \triangle IBC = \frac{1}{2}r(a + b + c)]$

4. Prove that the radius $r_1$ of the ex-circle, centre $I_1$, equals $\frac{\Delta}{s - a}$.

$[\triangle I_1 AB + \triangle I_1 AC - \triangle I_1 BC = \triangle ABC]$.

The Orthocentre of a Triangle.

Nos. 5-10 refer to fig. 729, in which $H$ is the orthocentre of $\triangle ABC$ (see p. 203).

5. Where is the orthocentre of $\triangle HBC$?

6. What can you say about the circles whose diameters are (i) $AH$, (ii) $BC$?

7. Prove that $\angle BHC = 180^\circ - \angle BAC$.

8. Prove that $\triangle AEF$ is equiangular to $\triangle ABC$.

9. Prove that $\angle HDE = \angle HDF = 90^\circ - \angle BAC$. 

[Image: Fig. 729]
10. (Nine-point Circle) If \( AD, BE, CF \) are the altitudes of \( \triangle ABC \), if \( H \) is the orthocentre, and if \( X, Y, Z, P, Q, R \) are the mid-points of \( BC, CA, AB, HA, HB, HC \), respectively, prove that

(i) \( PZ \) is parallel to \( BE \), and \( ZX \) is parallel to \( AC \);
(ii) \( \angle PZX = 90^\circ \) and \( \angle PYX = 90^\circ \);
(iii) \( P, Z, X, D, Y \) lie on a circle;
(iv) the circle through \( X, Y, Z \) passes through \( P, Q, R, D, E, F \) (this circle is called the nine-point circle of \( \triangle ABC \));
(v) the radius of the nine-point circle of \( \triangle ABC \) is half the circumradius of \( \triangle ABC \). [\( \frac{1}{2}BC \), etc.]

Loci

11. If the base \( BC \) of \( \triangle ABC \) is given in magnitude and position and if \( \angle BAC \) is given in magnitude, find the locus of the orthocentre \( H \) of \( \triangle ABC \).

![Fig. 730](image_url)

(i) Prove that \( \angle BHC = 180^\circ - \angle BAC \) in fig. 730 (i) where \( \angle ABC \) and \( \angle ACB \) are acute and that \( \angle BHC = \angle BAC \) in fig. 730 (ii) where \( \angle ABC \) is obtuse.

(ii) What is the position of \( H \) when \( \angle ABC \) is a right angle?

(iii) What can you say about the path traced out by \( H \) when \( \angle ABC \), \( \angle ACB \) are acute? What can you say if one of them is obtuse? Are these paths part of the same circle?

(iv) What is the position of \( H \) when \( A \) is very close to \( B \)?

(v) What is the complete locus of \( H \)?

(vi) Draw the locus of \( H \) if \( BC = 3 \) cm. and \( \angle BAC = 30^\circ \).

12. If the base \( BC \) of \( \triangle ABC \) is given in magnitude and position and if \( \angle BAC \) is given in magnitude, find the locus of the in-centre \( I \) of \( \triangle ABC \).

(i) What is the complete locus of \( A \)?

(ii) Prove that \( \angle BIC = 90^\circ + \frac{1}{3} \angle BAC \).

(iii) What is the position of \( I \) when \( A \) is very close to \( B \)?

(iv) What is the complete locus of \( I \)?

**EXERCISE 73**

Nos. 1-9 refer to fig. 728, p. 399, representing the in-circle and one ex-circle of \( \triangle ABC \).

1. Prove that (i) \( YQ = ZR \); (ii) \( BP = XC \).

2. If \( AB > AC \), prove that \( PX = AB - AC \).

3. Prove that \( B, I, C, I_i \) are concyclic.

4. Prove that \( \angle AIC = 90^\circ + \frac{1}{3} \angle ABC \).

5. If \( \angle BIC = 100^\circ \), find \( \angle BAC \).

6. Prove that \( AZ + BX + CY = \frac{1}{2}(BC + CA + AB) \).

7. Prove that \( AB - AC = BX - XC \).

8. If \( AD \) is an altitude of \( \triangle ABC \) and if \( AB > AC \), prove that \( \angle IAD = \frac{1}{2}(\angle ACB - \angle ABC) \).

9. If \( AD \) is an altitude of \( \triangle ABC \) and if \( O \) is the circumcentre, prove that \( A \) bisects \( \angle OAD \).

10. If \( AB \) is a chord of a circle; the tangents at \( A, B \) meet at \( T \). Prove that the in-centre of \( \triangle TAB \) lies on the circle.

Nos. 11-14 refer to fig. 729, p. 399, representing a triangle \( ABC \) and its orthocentre \( H \).

11. Prove that \( \triangle BDF, EDC \) are equiangular.

12. If \( O \) is the circumcentre of \( \triangle ABC \), prove \( \angle HBA = \angle OBC \).

13. Prove that \( H \) is the in-centre of \( \triangle DEF \). What points are the ex-centres of \( \triangle DEF \)?

14. Prove that the circumcircles of \( \triangle AHB, \triangle AHC \) are equal.
[15] If $I$ is the in-centre and $I_1$, $I_2$, $I_3$, $I_4$ are the ex-centres of $\triangle ABC$, prove that $I_1$ is the orthocentre of $\triangle I_1I_2I_3$.  

[16] If $I_1$, $I_2$, $I_3$, $I_4$ are the ex-centres of $\triangle ABC$, what is the nine-point circle of $\triangle I_1I_2I_3$ and what follows from this fact?  

[17] In $\triangle ABC$, $\angle BAC = 90^\circ$. Prove that the diameter of the in-circle of $\triangle ABC$ equals $AB + AC - BC$.  

[18] If the in-circle of $\triangle ABC$ touches $BC$ at $X$, prove that the in-circles of $\triangle ABX$, $\triangle ACX$ touch each other.  

[19] ABCD is a quadrilateral circumscribing a circle. Prove that the in-circles of $\triangle ABC$, $\triangle ADC$ touch each other.  

[20] If in fig. 728, p. 396, $I_1$ cuts the circumcircle of $\triangle ABC$ at $K$, prove that $I_1$, $I_2$, $B$, $C$ lie on a circle, centre $K$.  

[21] If in fig. 729, p. 399, $AD$ produced cuts the circumcircle of $\triangle ABC$ at $P$, prove that $HD = DP$.  

[22] If in fig. 729, p. 399, $BH$ produced cuts the circumcircle of $\triangle ABC$ at $K$, prove that $AH = AK$.  

[23] If in fig. 728, p. 399, the circumcircle of $\triangle BIC$ cuts $AB$ at $M$, prove that $AM = AC$.  

[24] If in fig. 729, p. 399, $O$ is the circumcentre of $\triangle ABC$, prove that $OA$ is perpendicular to $EF$.  

[25] If in fig. 728, p. 399, $AM$, $AN$ are the perpendiculars from $A$ to $B_1$, $C_1$, prove that $MN$ is parallel to $BC$.  

[26] $H$ is the orthocentre and $O$ is the circumcentre of $\triangle ABC$; $AK$ is a diameter of the circumcircle. Prove that  

(i) $BHCK$ is a parallelogram,  

(ii) $CH$ equals twice the distance of $O$ from $AB$.  

[27] A is a fixed point on a fixed circle, centre C; $AP$ is a variable chord of the circle. Prove that the locus of the mid-point of $AP$ is the circle on $OA$ as diameter.  

[28] A variable chord $PQ$ of a given circle pass through a fixed point. Find the locus of the mid-point of $PQ$.  

[29] A and B are fixed points; $AP$ is the tangent from $A$ to a variable circle, centre $B$. Find the locus of $P$.  

[30] A and $B$ are fixed points; $ABPQ$ is a variable parallelogram; the bisectors of $\angle QAB$, $\angle BPA$ meet at $R$. Find the locus of $R$.  

[31] The extremities of a line $PQ$ of given length move along two given perpendicular lines $OA$, $OB$. Prove that the locus of the mid-point of $PQ$ is a circle, centre $O$.  

[32] A and $B$ are fixed points; $P$ is a variable point on the given circle $ABP$; $AP$ is produced to $Q$ so that $PQ = PB$. Find the complete locus of $Q$.  

[33] If the base $BC$ of $\triangle ABC$ is given in magnitude and position and if $\angle BAC$ is given in magnitude, find the complete locus of $I_1$, the centre of the ex-circle escribed to $BC$.  

[34] With the data of No. 33 find the complete locus of $I_1$, the centre of the ex-circle escribed to $AC$.  

[35] Two given circles cut at $A$, $B$; $P$ is a variable point on one of the circles; $PA$, $PB$, produced if necessary, cut the other circle again at $Q$, $R$. If $AR$ and $BQ$, produced if necessary, meet at $S$, find the locus of $S$.  

[36] A and $B$ are fixed points on a given circle; $PQ$ is a variable chord of the circle of given length; $AP$, $BQ$, produced if necessary, meet at $R$. Find the complete locus of $R$.  

[37] In fig. 732, $ABC$ is a given triangle. If $\angle APB = \angle BAC$ and $\angle BQA = \angle BAC$, find the locus of $P$. [Join $PB$.]  

[38] $P$, $Q$ are variable points on the fixed lines $AB$, $AC$ (not produced in the sense $BA$, $CA$); the perpendiculars at $P$, $Q$ to $AB$, $AC$ respectively meet at $R$. If $PQ$ is of given length, prove that the locus of $R$ is part of a circle, centre $A$, and state what part of this circle belongs to the locus.  

[39] $ABC$ is a given triangle such that $AB = AC$: $P$ is a variable point such that $\angle APB = \angle APC$. Prove that the locus of $P$ consists of one complete straight line, part of a second straight line, and part of the circle $ABC$. State the precise locus.  

[40] $PQ$ is a variable chord of given length of a given circle; $A$ is a fixed point on the circle. Prove that the locus of the orthocentre of $\triangle APQ$ is a circle, centre $A$.  

**Fig. 732**
MISCELLANEOUS CONSTRUCTIONS (Revision)

EXERCISE 74

1. Use a coin to draw a circle and construct its centre.

2. Draw a line $AB$, 3 cm. long. Construct a circle, radius 5 cm., to pass through $A$ and $B$.

3. Construct two circles of radii 4 cm., 5 cm., such that their common chord is of length 6 cm. Measure the distance between their centres.

4. Draw $\triangle ABC$ so that $AB = AC = 5$ cm., $\angle BAC = 90^\circ$. Construct a point $D$ on $AC$ such that $BD = BC$. Construct the circle $ABD$ and measure its radius. Does it touch $BC$?

5. Given two points $A$, $B$ and a line $CD$, construct a circle to pass through $A$ and $B$ and have its centre on $CD$.

6. Draw two lines $AOB$, $COD$ so that $\angle AOC = 80^\circ$, $AO = 3$ cm., $OR = 4$ cm., $CO = 5$ cm., $OD = 2.4$ cm. Construct a circle to pass through $A$, $B$, $C$. Does it pass through $D$?

7. Given a circle and two points $A$, $B$ inside it, construct a circle through $A$ and $B$, with its centre on the given circle.

8. Draw a circle of radius 4 cm. and take a point 6 cm. from the centre. Construct the tangents from this point to the circle and measure their lengths.

9. Draw a circle of radius 3 cm. and construct two tangents which include an angle of 75°.

10. Draw a circle of radius 3 cm. Construct a parallelogram circumscribing the circle and having one angle equal to 110°.

11. Draw a circle of radius 1-5 in. and construct a chord of the circle of length 2-5 in. Take a point $A$ 1 in. from the centre; construct a chord of length 2-5 in. passing through $A$.

12. $A$, $B$, $C$ are given points on a given circle. Construct a chord of the circle equal to $AB$ and parallel to the tangent at $C$.

13. Draw a circle of radius 3 cm. and take a point 5 cm. from the centre. Construct a chord of length 4 cm. which, when produced, passes through this point.

14. Draw a line $AB$ of length 5 cm. and describe a circle with $AB$ as diameter. Construct a point on $AB$ produced such that the tangent from it to the circle is of length 3 cm.
AREAS OF RECTANGLES

Many of the theorems relating to areas of rectangles are proved most easily by using algebraic (or trigonometric) methods. Conversely some algebraic identities may be illustrated geometrically by reference to areas of rectangles.

Definitions. If $AB$ and $CD$ are two given straight lines, any rectangle having two adjacent sides equal to $AB$, $CD$ respectively, is called a rectangle contained by $AB$ and $CD$: all such rectangles are congruent.

A rectangle contained by $AB$ and $CD$ is described as the rect. $AB$, $CD$ or as $AB \cdot CD$, because the area of the rectangle is measured by the product of the measures of two adjacent sides.

The rectangle contained by $AB$ and $CD$ is said to be equal to the rectangle contained by $PQ$ and $RS$ if their areas are equal,—that is, if $AB \cdot CD = PQ \cdot RS$.

Geometrical Illustrations of Algebraic Identities

Examples for Oral Discussion.

1. Illustrate the identity

\[ k(a + b + c + d) = ka + kb + kc + kd. \]

Draw the rectangles shown in fig. 734.
What is the area of rectangle $P$?
What are the areas of the rectangles $A$, $B$, $C$, $D$?

The corresponding geometrical theorem may be stated as follows:

If two straight lines are given, one of which is divided into any number of parts, then the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and each part of the divided line.

\[ \frac{AB^2}{PQRST} \]

Fig. 735

In fig. 736, if $AB$ and $PQ$ are straight lines,

\[ AB \cdot PT = AB \cdot PQ + AB \cdot QR + AB \cdot RS + AB \cdot ST. \]

2. Illustrate the identity

\[ (a + b)^2 = a^2 + 2ab + b^2. \]

Draw the squares in fig. 736 and divide one of them into rectangular compartments as shown.

What is the area of $P$?
What are the areas of $A$, $B$, $C$, $D$?

The corresponding geometrical theorem may be stated as follows:

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by these parts.

In fig. 737, if $AQB$ is a straight line,

\[ AB^2 = AQ^2 + QB^2 + 2AQ \cdot QB. \]

Fig. 737
3. Illustrate the identity
\[(a - b)^2 = a^2 - 2ab + b^2\].

Draw the squares A and B shown in fig. 738 (i).

Explain why the areas of fig. 738 (i), and (ii) are equal.

What are the areas of C, P, Q?

Hence
\[a^2 + b^2 = (a - b)^2 + 2ab\].

The corresponding geometrical theorem may be stated as follows:

If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole line and that part together with the square on the other part.

In fig. 737, if \(AB\) is a straight line,
\[AB^2 + QB^2 = 2AB \cdot QB + AQ^2\].

4. Illustrate the identity
\[a^2 - b^2 = (a + b)(a - b)\].

Draw the square, side \(a\) units, shown in fig. 739 (i) and divide it into a square side \(b\) units and the two rectangles \(P\) and \(Q\).
**Example 2.** If $A$, $B$, $C$, $D$ are four points in order on a straight line, prove that

$$\overrightarrow{AC} \cdot \overrightarrow{BD} = \overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{AD} \cdot \overrightarrow{BC}.$$

**EXERCISE 75**

Illustrate by a figure the following identities:

1. $(a + b)(c + d) = ac + ad + bc + bd$.
2. $k(a - b) = ka - kb$.
3. $(2a)^2 = 4a^2$.
4. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
5. A straight line $AB$ is bisected at $O$; $P$ is any point on $AC$. Prove that $OP = \frac{1}{2}(PB - AP)$.
6. A straight line $AB$ is bisected at $O$ and produced to $P$. Prove that $OP = \frac{1}{2}(AP + BP)$.
7. $ABCD$ is a straight line; $X$, $Y$ are the mid-points of $AB$, $CD$. Prove that $AD + BC = 2XY$.
8. $AD$ is bisected at $B$, $C$. Prove that $AD^2 = AB^2 + 2BD^2$.
9. $AB$ is bisected at $O$ and produced to $P$. Prove that $AO \cdot AP = OB \cdot BP - 2AO^2$.
10. $AB$ is bisected at $C$ and produced to $Q$. Prove that $AQ^2 = 4AC \cdot CQ + BC^2$.
11. $ABCD$ is a straight line. If $AB = CD$, prove that $AD^2 + BC^2 = 2AB^2 + 2BD^2$.
12. $X$ is a point on $AB$ such that $AB \cdot BX = AX^2$. Prove that $AB^2 + BX^2 = 3AX^2$.

**PROJECTIONS**

13. $C$ is a point on $AB$ such that $AB \cdot BC = AC^2$. Prove that $AC \cdot BC = AC^2 - BC^2$.
14. $X$ is a point on $AB$ such that $AB \cdot SX = AX^2$; $O$ is the mid-point of $AX$. Prove that $OB^2 = 3OA^2$.
15. $AB$ is bisected at $O$ and produced to $P$ so that $OB \cdot OP = BP^2$. Prove that $PA^2 = 4PB^2$.
16. $AB$ is produced to $P$ so that $PA^2 = 4PB^2 + AB^2$. Prove that $2PA^2 = 5PB$.

**Projections**

**Definition.** If $AB$ and $CD$ are any two straight lines, and if $AH$, $BK$ are the perpendiculars from $A$, $B$ to $CD$, produced if necessary, then $HK$ is called the projection of $AB$ on $CD$.

**Examples for Oral Discussion**

1. In fig. 530, p. 270, which represents the squares on the sides of the right-angled triangle $ABC$, what lines represent the projection of (i) $BA$ on $QP$, (ii) $AC$ on $BC$, (iii) the line joining $A$ to $Q$ on $BC$?
2. In fig. 529, p. 270, which represents the squares on the sides of an acute-angled triangle $ABC$, what lines represent the projections of (i) $BC$ on $AB$, (ii) $AB$ on $NM$, (iii) $AC$ on $KH$?
3. In fig. 529, p. 270, what lines represent the projections of (i) $AC$ on $EC$, (ii) the line joining $B$ to $N$ on $AC$, (iii) the line joining $C$ to $K$ on $CD$?
Co-ordinates

An important example of the use of projections is the method of fixing the position of a point in a plane by co-ordinates.

If Ox, Oy are directed axes of reference, Oy making an angle +90° with Ox, and if P is any point in the plane of Ox and Oy,

the x co-ordinate of P is the projection OM of OP on Ox, and

the y co-ordinate of P is the projection ON of OP on Oy.

Suppose that OP is of unit length and makes an acute angle $\theta$ with Ox.

Then by the definition of the sine and cosine of an acute angle on p. 79,

$$OM = \cos \theta \quad \text{and} \quad ON = \sin \theta.$$

$\therefore$ P is the point $(\cos \theta, \sin \theta)$.

Next suppose that OP is of unit length and makes with Ox an angle $\theta$ of any magnitude, then we make the following definitions:

the x co-ordinate of P is $\cos \theta$, and

the y co-ordinate of P is $\sin \theta$.

In fig. 744, $\theta$ is obtuse, and it is then evident that in this case the x co-ordinate of P is negative and the y co-ordinate of P is positive.

Hence, since $\triangle MOP = 180^\circ - \theta$,

$$\cos \theta = -\cos (180^\circ - \theta) \quad \text{and} \quad \sin \theta = \sin (180^\circ - \theta).$$

These results agree with the statements on pp. 246, 271. The sines and cosines of reflex angles can be discussed in a similar way but are not needed here.

Examples for Oral Discussion

1. With the notation of fig. 745, where $\angle BAC$ is ACUTE, prove that

$$a^2 = b^2 + c^2 - 2bc \cos \theta.$$

In fig. 745 (i), BN = $c - x$; in fig. 745 (ii), BN = $x - c$.

In each case, apply Pythagoras to $\triangle BNC$ and $\triangle ANC$.

2. Use the fact that AN is the projection of AC on AB in fig. 745 to express the result of No. 1 in words.

3. Prove that the result of No. 1 may be written

$$a^2 = b^2 + c^2 + 2bc \cos \theta.$$

4. With the notation of fig. 746, where $\angle BAC$ is OBTUSE, prove that

$$a^2 = b^2 + c^2 + 2bc \cos \theta.$$

Apply Pythagoras to $\triangle BNC$ and $\triangle ANC$.

5. Express the result of No. 4 in words.

6. Prove that the result of No. 4 may be written

$$a^2 = b^2 + c^2 + 2bc \cos \theta.$$

Since $\angle CAN = 180^\circ - A$, $y = b \cos (180^\circ - A) = -b \cos A$.

The Cosine Formula. The results established in Examples 3, 6 may be combined into the following single statement:

In ANY triangle $ABC$, if $BC = a$ units, $CA = b$ units, $AB = c$ units, $\angle ABC = A$, $\angle BCA = B$, and $\angle CAB = C$.

Similarly,

$$b^2 = c^2 + a^2 - 2bc \cos B \quad \text{and} \quad c^2 = a^2 + b^2 - 2ab \cos C.$$
This formula may be used to calculate
(i) the length of the third side of a triangle if the lengths of
two sides and the size of the included angle are given;
(ii) any angle of a triangle if the lengths of the three
sides are given.

Use the cosine formula for Nos. 7-10:
7. (i) $b = 2$, $c = 3$, $A = 50^\circ$, find $a$;
(ii) $b = 5$, $c = 3$, $A = 135^\circ$, find $a$.
8. (i) $a = 5$, $b = 7$, $C = 35^\circ$, find $c$;
(ii) $a = 5$, $b = 7$, $C = 145^\circ$, find $c$.
9. $a = 6$, $b = 5$, $c = 4$, find $A$ and $C$.
10. $a = 7$, $b = 5$, $c = 4$, find $A$ and $C$.

11. If $D$ is the mid-point of the side $BC$ of $\triangle ABC$, prove that

$$AB^2 + AC^2 = 2AD^2 + BD^2.$$ 

Draw $AN$ perpendicular to BC.

Complete the following:

From $\triangle ADB$ where $\angle ADB$ is obtuse,

$$AB^2 = \ldots$$

From $\triangle ADC$ where $\angle ADC$ is acute,

$$AC^2 = \ldots$$

$$\therefore \ AB^2 + AC^2 = \ldots$$

12. Prove the statement in No. 11 by applying the cosine formula to $\triangle ADB$ and $\triangle ADC$.

Let $\angle ADC = \theta$; $\therefore \angle ADB = 180^\circ - \theta$.

Illustrative Example. If in $\triangle ABC$, $BC = 11$ in., $CA = 9$ in.,
$AB = 6$ in., find whether $\angle BAC$ is acute or obtuse.

$$BC^2 = 11^2 \text{ sq. in.} = 121 \text{ sq. in.},$$
$$AB^2 + AC^2 = (36 + 81) \text{ sq. in.} = 117 \text{ sq. in.};$$
$$\therefore \ BC^2 > AB^2 + AC^2,$$
$$\therefore \angle BAC \text{ is obtuse.}$$

14. Fig. 748 represents a crane supporting a load $L$; $BC$ is vertical. Find the height of $L$ above the horizontal plane through $C$. 

[Give answers, which are approximate, correct to three figures.]

Find whether the triangles, the lengths of whose sides are
given in Nos. 1-4, are obtuse angled or acute angled:
1. 4 in., 5 in., 7 in.
2. 8 cm., 9 cm., 12 cm.
3. 7 in., 8 in., 11 in.
4. 10 in., 16 cm., 22 cm.

5. $ABC$ is an acute-angled triangle in which $AB = 12$ in.,
$AC = 15$ in., and the length of $BC$ is a whole number of inches.
Find the greatest possible length and the least possible length of $BC$.

8. In $\triangle ABC$, $AB = 9$ in., $AC = 11$ in., $\angle BAC > 90^\circ$. Prove
that $BC > 14$ in.

In Nos. 7-10, $CN$ is an altitude of $\triangle ABC$. Find the lengths
of $AN$ and $CN$ and the area of $\triangle ABC$.

7. $BC = 8$ in., $CA = 9$ in., $AB = 10$ in.
8. $BC = 6$ cm., $CA = 3$ cm., $AB = 4$ cm.
9. $BC = 7$ in., $CA = 13$ in., $AB = 10$ in.
10. $BC = 11$ cm., $CA = 9$ cm., $AB = 10$ cm.
11. If $AB = 6$ cm., $BC = 5$ cm., $CA = 7$ cm., find the length
of the projection of $AB$ on $BC$.

12. If $AB = 15$ in., $BC = 24$ in., $CA = 19$ in., prove that the foot
of the perpendicular from $A$ to $BC$ is a point of trisection of $BC$.

13. $ABCD$ is a parallelogram; $AB = 5$ in., $AD = 3$ in., and
the projection of $AC$ on $AB$ is $6$ in. Find $AC$. 

Fig. 748
15. The sides of a triangle are 9 cm., 7 cm., 14 cm.; find the length of the shortest median.

16. The sides of a triangle are 8 in., 9 in., 11 in.; find the length of the two shorter medians.

[17] Find the lengths of the three medians of a triangle whose sides are 6 cm., 8 cm., 9 cm.

18. AD is a median of \( \triangle ABC \). If \( AB = 6 \) in., \( AC = 8 \) in., \( AD = 5 \) in., find the length of \( BC \).

[19] In \( \triangle ABC \), \( AB = 4 \) cm., \( BC = 5 \) cm., \( CA = 8 \) cm.; \( BC \) is produced to \( D \) so that \( BC = CD \), find the length of \( AD \).

20. The sides of a parallelogram are 5 cm., 7 cm., and one diagonal is 8 cm.; find the length of the other diagonal.

21. In \( \triangle ABC \), \( BC = 24 \) cm., \( CA = 13 \) cm., \( AB = 17 \) cm.; \( BC \) is trisected at \( Y, Z \). Find the lengths of \( AY, AZ \).

22. In \( \triangle ABC \), \( AC = 8 \) cm., \( BC = 6 \) cm., \( \angle ACB = 120^\circ \). Find the length of \( AB \).

23. In \( \triangle ABC \), \( AB = 8 \) cm., \( AC = 7 \) cm., \( BC = 3 \) cm. Prove that \( \angle ABC = 60^\circ \).

[24] In \( \triangle ABC \), \( AB = 14 \) in., \( BC = 10 \) in., \( CA = 6 \) in. Prove that \( \angle ACB = 120^\circ \).

*25. In \( \triangle ABC \), \( BC = (2a + 4b) \) in., \( CA = (4a + b) \) in., \( AB = (2a + 3b) \) in., find in terms of \( a, b \), the length of the median \( AD \).

*26. \( AD \) is a median of \( \triangle ABC \), and \( DN \) is the perpendicular from \( D \) to \( AB \). If \( AB = 12 \) in., \( AC = 8 \) in., \( AD = 6 \) in., find the length of \( AN \).

*27. In \( \triangle ABC \), \( AB = AC = 3 \) cm., \( BC = 2 \) cm.; \( D \) is taken on \( BC \) produced so that \( AD = 6\sqrt{2} \) cm. Prove that \( \angle BAD \) is a right angle.

*28. In \( \triangle ABC \), \( AB^2 - AC^2 = 66 \) sq. in. and \( BC = 6 \) in. Find the length of the projection of \( AC \) on \( BC \).

Historical Note. Theorem 66, by means of which it is possible to calculate the lengths of the medians of a triangle, whose sides are given, is associated with the name of Apollonius of Perga (247–205 B.C.), known among the ancients as the “Great Geometer.” His writings, which dealt mainly with the properties of the ellipse, parabola, and hyperbola, see p. 216, and those of Euclid, dominated geometry up to the beginning of the nineteenth century.

**THEOREM 64**

In an obtuse-angled triangle, the square on the side opposite the OBTUSE angle is equal to the sum of the squares on the sides containing it PLUS twice the rectangle contained by one of those sides and the projection on it of the other.

![Fig. 749](image)

Given a triangle \( ABC \) in which \( \angle BAC \) is obtuse, and the perpendicular \( CN \) from \( C \) to \( BA \) produced.

To prove that \( BC^2 = BA^2 + CA^2 + 2BA \cdot AN \).

**Proof.** (Put in a small letter for each length that comes in the answer and also for the altitude.)

Let \( BC = a \) units, \( CA = b \) units, \( AB = c \) units, \( AN = y \) units, \( CN = h \) units.

Since \( \triangle BNC \) is a right angle,

\[ a^2 = (c + y)^2 + h^2 \]  
*Pythagoras,*

\[ a^2 = c^2 + 2cy + y^2 + h^2. \]

Since \( \triangle ANC \) is a right angle,

\[ b^2 = y^2 + h^2 \]  
*Pythagoras,*

\[ b^2 = c^2 + 2cy + b^2, \]

that is, \( BC^2 = BA^2 + 2BA \cdot AN + CA^2. \)

Note. Some pupils find it difficult to remember this result; it is a useful mnemonic to observe that the vertex \( A \), opposite the side \( BC \), is one extremity of each length that occurs on the right side.
THEOREM 65

In any triangle, the square on the side opposite an ACUTE angle is equal to the sum of the squares on the sides containing it MINUS twice the rectangle contained by one of those sides and the projection on it of the other.

![Diagram](image)

**Fig. 750**

**Given** a triangle ABC in which \( \angle BAC \) is acute, and the perpendicular CN from C to AB or AB produced.

**To prove** that \( BC^2 = BA^2 + CA^2 - 2BA \cdot AN \).

**Proof.** (Put in a small letter for each length that comes in the answer and also for the altitude.)

Let \( BC = a \) units, \( CA = b \) units, \( AB = c \) units, \( AN = z \) units, \( CN = h \) units.

In fig. 750 (i), \( BN = c - a \) units;

in fig. 750 (ii), \( BN = z - c \) units.

Since \( \angle C NB \) is a right angle,

\[
a^2 = (c - x)^2 + h^2 \quad \text{in fig. 750 (i)}
\]

\[
a^2 = (x - c)^2 + h^2 \quad \text{in fig. 750 (ii)}
\]

\[
\therefore \text{in each case, } a^2 = c^2 - 2cx + x^2 + h^2.
\]

Since \( \angle CNA \) is a right angle,

\[
b^2 = x^2 + h^2 \quad \text{Pythagoras},
\]

\[
\therefore a^2 = c^2 - 2cx + b^2,
\]

that is, \( BC^2 = BA^2 - 2BA \cdot AN + CA^2 \).

Note. As for Theorem 64, it is a useful mnemonic to observe that the vertex A, opposite to the side BC, is one extremity of each length that occurs on the right side.

---

THEOREM 66 (Apollonius’ Theorem)

In any triangle, the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.

![Diagram](image)

**Fig. 751**

Given a triangle ABC in which D is the mid-point of BC.

To prove that \( AB^2 + AC^2 = 2AD^2 + 2BD^2 \).

**Proof.** Either (i) \( \angle ADB \) and \( \angle ADC \) are both right angles or (ii) one is obtuse and the other acute.

(i) If \( \angle ADB \) and \( \angle ADC \) are right angles,

\[
AB^2 = AD^2 + BD^2 \quad \text{and} \quad AC^2 = AD^2 + DC^2 \quad \text{Pythagoras.}
\]

But \( BD = DC \) given,

\[
\therefore BD^2 = DC^2.
\]

\[
\therefore AB^2 + AC^2 = 2AD^2 + 2BD^2.
\]

(ii) Suppose \( \angle ADB \) is obtuse, then \( \angle ADC \) is acute.

Let AN be the perpendicular from A to BC.

From \( \triangle ADB \), since \( \angle ADB \) is obtuse,

\[
AB^2 = AD^2 + BD^2 + 2BD \cdot DN.
\]

From \( \triangle ADB \), since \( \angle ADC \) is acute,

\[
AC^2 = AD^2 + DC^2 - 2DC \cdot DN.
\]

But \( BD = DC \) given,

\[
\therefore BD^2 = DC^2 \quad \text{and} \quad BD \cdot DN = DC \cdot DN.
\]

\[
\therefore AB^2 + AC^2 = 2AD^2 + 2BD^2.
\]

If \( \angle ADB \) is acute, then \( \angle ADC \) is obtuse, and the proof is the same as before, except that B and C are interchanged.
EXERCISE 77

1. BE, CF are altitudes of the acute-angled triangle ABC. Write down two expressions for BC² not involving BE or CF. Hence prove that \( AF : AB = AE : AC \).

[2] If the altitudes BE, CF of the acute-angled triangle ABC cut at H, prove that BH : HE = CH : HF. [Use \( \triangle BHC \) to write down two expressions for BC².]

3. \( \triangle ABC \) is an equilateral triangle; BC is produced to D so that \( BC = CD \). Prove that \( AD^2 = 3AB^2 \). [Use Apollonius.]

[4] In \( \triangle ABC \), \( AB = AC \); AB is produced to D so that \( AB = BD \). Prove that \( CD^2 = AB^2 + 2BC^2 \).

5. ABCD is a parallelogram. Prove that AC² + BD² = 2AB² + 2BC².

[Let AC cut BD at K.]

[6] ABCD is a rectangle; P is any point in the same plane or in any other plane. Prove that \( PA^2 + PC^2 = PB^2 + PD^2 \).

[Let AC cut BD at K; join PK.]

7. A, B are fixed points; P is a variable point such that \( PA^2 + PB^2 \) is constant. Find the locus of P.

[Join P to the mid-point of AB.]

8. AX, BY are medians of \( \triangle ABC \). Prove that \( AX^2 = BY^2 = \frac{1}{2}(AC^2 - BC^2) \).

9. In \( \triangle ABC \), \( AB = AC \); CD is an altitude. Prove that \( BC^2 = 2AB \cdot BD \).

[10] In \( \triangle ABC \), \( \angle A = 90^\circ \); AC is produced to D so that \( CD = BC \). Prove that \( BD^2 = 2BC \cdot AD \).

[11] A, B are fixed points; P is a variable point such that \( PA^2 + PB^2 \) is constant. Prove that the area of \( \triangle PAB \) is greatest when \( PA = PB \).

12. (i) If AD is a median of \( \triangle ABC \) and if the lengths of the sides BC, CA, AB are a, b, c respectively, find AD² in terms of a, b, c.

(ii) If AD, BE, CF are the medians of \( \triangle ABC \), prove that \( 4(AD^2 + BE^2 + CF^2) = 3(BC^2 + CA^2 + AB^2) \).

13. The base BC of \( \triangle ABC \) is trisected at X and Y. Prove that \( AX^2 + XY^2 + YB^2 = AB^2 + AC^2 \).

14. In \( \triangle ABC \), \( \angle C = 90^\circ \); AB is trisected at P and Q. Prove that \( PC^2 + CQ^2 + PQ^2 = \frac{3}{2}AB^2 \).

SEGMENTS OF A STRAIGHT LINE

15. BC is a diameter of a circle, centre O; A is any point inside the circle on the radius OE. Prove that \( 2AE^2 = AB^2 + AC^2 - 2AO \cdot BC \).

[16] In \( \triangle ABC \), \( AB = AC \). From any point D on AB a line DE is drawn parallel to BC to cut AC at E. Prove that \( BE^2 = CE^2 + BC \cdot DE \). [Draw DH, EL perpendicular to BC.]

[17] BE, CF are altitudes of the acute-angled triangle ABC. Prove that \( BA : BF = CA : CE = BC^2 \).

[18] ABC is a triangle; ABPQ, ACXY are squares outside \( \triangle ABC \). Prove that \( BC^2 + QY^2 = AP^2 + AX^2 \).

[19] ABCD is a quadrilateral; \( X, Y \) are the mid-points of AC, BD. Prove that \( AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4XY^2 \).

[20] ABCD is a tetrahedron; \( \angle BAC = \angle CAD = \angle DAB = 90^\circ \). Prove that BCD is an acute-angled triangle.

[21] D is a point on the side BC of \( \triangle ABC \) such that \( BD = 2DC \). Prove that \( AB^2 + 2AC^2 = 6CD^2 + 3AD^2 \).

Segments of a Straight Line

If \( \triangle ABC \) is any straight line and if \( P \) is any point on \( AB \) between the points \( A \) and \( B \), \( AB \) is said to be divided internally at \( P \), see fig. 752 (i).

![Fig. 752](image)

If \( P \) is any point beyond \( B \) on \( AB \) produced, fig. 752 (ii), or beyond \( A \) on \( BA \) produced, fig. 752 (iii), \( AB \) is said to be divided externally at \( P \).

In each case, that is whether \( P \) lies on \( AB \) or on \( AB \) produced or on \( BA \) produced, \( PA \) and \( PB \) are called segments of the straight line \( AB \); and a rectangle, whose adjacent sides are equal to \( PA \) and \( PB \), is said to be contained by the segments \( PA \), \( PB \) of the straight line divided at \( P \).

Note. Each segment of the line \( AB \), divided at \( P \), is measured from \( P \).
Intersecting Chords of a Circle

Examples for Oral Discussion.

1. Through any point $X$ inside a circle, centre $O$, radius $r$, a line is drawn cutting the circle at $A$ and $B$. Prove that

$$X'A \cdot X'B = r^2 - OX^2$$

With the notation in fig. 753, express the lengths of $X'A, X'B$ in terms of $a, x$. Use Pythagoras for $\triangle OX'N, \triangle OXN$.

![Fig. 753](image)

2. Two chords $AB, CD$ of a circle intersect at a point $X$ inside the circle. Prove that $X'A \cdot X'B = X'C \cdot X'D$.

Express this fact in words.

Note that the segments of the chord $AB$, divided at $X$, are the lines $X'A, X'B$ measured from $X$.

3. From any point $X$ outside a circle, centre $O$, radius $r$, a line is drawn cutting the circle at $A$ and $B$. Prove that

$$X'A \cdot X'B = OX^2 - r^2$$

![Fig. 755](image)

Use the method given above for (1).

4. Two chords $AB, CD$ of a circle intersect, when produced, at a point $X$ outside the circle; $XT$ is the tangent from $X$ to the circle. Prove that

$$X'A \cdot X'B = X'C \cdot X'D = XT^2$$

Express these results in words.

![Fig. 756](image)

Note that the segments of the chord $AB$, divided at $X$, are the lines $X'A, X'B$ measured from $X$.

5. State the converse of No. 2 and prove that it is true.

6. State the converse of No. 4 and prove that it is true.

NUMERICAL EXAMPLES

EXERCISE 78

Nos. 1–5 refer to fig. 754 in which $AX'B, CX'D$ are two chords of a circle, centre $O$.

1. If $AX = 6$ in., $XB = 2$ in., $CX = 3$ in., find $XD$.

2. If $AB = 11$ cm., $BX = 3$ cm., $CX = 4$ cm., find $CD$.

3. If $AX = 12$ cm., $AB = 15$ cm., $CX = XD$, find $CD$.

4. If $OA = 6$ in., $AX = 5$ in., $OX = 4$ in., find $BX$.

5. If $OA = 7$ in., $OX = 3$ in., find $AX \cdot XB$. 


Nos. 8-9 refer to fig. 756 in which the chords \( AB, CD \) of a circle, centre \( O \), meet at a point \( X \) outside the circle and \( XT \) is a tangent.

8. If \( CD = 2 \text{ in.}, DX = 6 \text{ in.}, BX = 3 \text{ in.} \), find \( AB \).
9. If \( AB = 9 \text{ cm.}, BX = 3 \text{ cm.} \), find \( TX \).
10. If \( TX = 5 \text{ in.}, DX = 2 \frac{1}{2} \text{ in.} \), find \( CD \).
11. If \( OA = 3 \text{ cm.}, OX = 9 \text{ cm.} \), find \( KA \cdot KB \).

10. From a point \( P \) on a circle \( PN \) is drawn perpendicular to a diameter \( AB \); \( AN = 4 \text{ in.}, NB = 16 \text{ in.} \), find \( PN \).

11. In \( \triangle ABC, \angle BAC = 90^\circ, AB = 4 \text{ cm.}, AC = 3 \text{ cm.} \); \( AD \) is an altitude; find \( BD \).
12. In \( \triangle ABC, AB = 9 \text{ cm.}, AC = 12 \text{ cm.} \), \( F \) is the mid-point of \( AC \). If the circle through \( B, F \), \( C \) cuts \( AB \) at \( E \), find \( SE \).
13. \( AOB, OCD \) are two straight lines such that \( AB = 20 \text{ cm.}, CD = 19 \text{ cm.}, AC = 6 \text{ cm.}, CD = 7 \text{ cm.} \). Prove that \( ACBD \) is cyclic.

14. \( OA, OB \), \( OC \) are two straight lines such that \( OA = 3 \text{ cm.}, AB = 12 \text{ cm.}, OC = 5 \text{ cm.}, CD = 4 \text{ cm.} \). Prove that \( ABDC \) is cyclic.

15. In \( \triangle ABC, AB = 4 \text{ cm.}, BC = 5 \text{ cm.} \); \( D \) is a point on \( BC \) such that \( DC = 6 \text{ cm.} \). Prove that \( AD \) touches the circle \( ADC \).

In Nos. 16-19, figs. 757-760, \( PT \) represents the tangent at \( T \). The unit of length is 1 cm. Find the unknown lengths.

20. The diagonals \( AC, BD \) of the parallelogram \( ABCD \) are of lengths 8 cm., 10 cm. The circle \( BCD \) cuts \( CA \) at \( F \). Find \( AF \).

21. The roadway \( AB \) of a bridge is a circular arc resting on supports at \( A, B \) at the same level; the highest point \( C \) of the roadway is 4 ft. above \( AB \), and \( AB \) is 8 yd. Find the radius of the arc.

22. \( ABC \) is a triangle inscribed in a circle; \( AB = AC = 10 \text{ cm.}, BC = 16 \text{ cm.} \); \( AD \) is drawn perpendicular to \( BC \) and is produced to meet the circle at \( E \). Find \( DE \) and the radius of the circle.

23. Given that the straight line which joins two points on the surface of level water is two miles long and is 3 in. below the surface of the water at its middle point, find the radius of the Earth in miles.

24. \( AXB, CXD \) are two perpendicular chords of a circle, centre \( O \); \( AX = 3 \text{ in.}, CX = 5 \text{ in.}, XD = 6 \text{ in.} \). Find \( OX \) and the radius of the circle.

25. A small heavy body is suspended from a fixed point by a string 6 ft. long; it is pulled aside, the string remaining taut, so that it rises 6 in. Through what horizontal distance does it move?

26. In \( \triangle ABC, AB = 9 \text{ in.}, AC = 15 \text{ in.}, \angle A = 90^\circ \). Find the diameter of the circle which touches \( AC \) at \( C \) and passes through \( E \).

27. In \( \triangle ABC, AB = 6 \text{ cm.}, AC = 4 \text{ cm.}, \angle A = 90^\circ \). Find the radius of the circle which touches \( AB \) at \( B \) and passes through \( C \).

28. (i) If the mean radius of the Earth is \( r \) miles and if a man stands on a hill of height \( h \) miles above mean level, show that the distance he can see is about \( V(2rh) \) miles.

(ii) Taking the radius of the Earth as 3960 miles, show that at a height of \( x \) feet above sea-level, the distance that can be seen across level ground is about \( V(\frac{x}{2}) \) miles.

(iii) Find the approximate distance of the horizon for a height of (i) 6 ft., (ii) 600 ft.
THEOREM 67

If two chords of a circle intersect at a point inside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Given two chords $AB$, $CD$ of a circle, intersecting at a point $X$ inside the circle.

To prove that $XA \cdot XB = XC \cdot XD$.

Construction. Draw the perpendicular $ON$ from the centre $O$ to $AB$. Join $OA$, $OX$.

Proof. $AN = NB$ perp. from centre bisects chord,

$\therefore XA \cdot XB = (AN + NX)(NB - NX)$

$= (AN + NX)(AN - NX)$

$= AN^2 - NX^2$.

But $AN^2 = OA^2 - ON^2$ and $NX^2 = OX^2 - ON^2$.

$\therefore AN^2 - NX^2 = OA^2 - OX^2$.

$\therefore XA \cdot XB = OA^2 - OX^2$.

Similarly, it may be proved that $XC \cdot XD = OC^2 - OX^2$.

But $OA = OC$ radii,

$\therefore XA \cdot XB = XC \cdot XD$.

Corollary. If $X$ is any point inside a circle, centre $O$, radius $r$, the rectangle contained by the segments of any chord drawn through $X$ equals $r^2 - OX^2$.

Fig. 761

THEOREM 68

If two chords of a circle, when produced, intersect at a point outside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other and each rectangle is equal to the square on the tangent from the point of intersection to the circle.

Given two chords $AB$, $CD$ of a circle, intersecting at a point $X$ outside the circle, and the tangent $XT$.

To prove that $XA \cdot XB = XC \cdot XD = X^2T^2$.

Construction. Draw the perpendicular $ON$ from the centre $O$ to $AB$. Join $OA$, $OX$, $OT$.

Proof. $AN = NB$ perp. from centre bisects chord,

$\therefore XA \cdot XB = (XN + NA)(XN - NB)$

$= (XN + NA)(XN - NA)$

$= XN^2 - NA^2$.

But $XN^2 = XO^2 - ON^2$ and $NA^2 = AO^2 - OX^2$.

$\therefore XN^2 - NA^2 = XO^2 - AO^2$.

$\therefore XA \cdot XB = XO^2 - AO^2$.

But the tangent $XT$ is perpendicular to the radius $OT$,

$\therefore XT^2 = XO^2 - TO^2$ Pythagoras,

$= XO^2 - AO^2$ $OT = OA$, radii,

$= XA \cdot XB$.

Similarly, it may be proved that $XT^2 = XC \cdot XD$.

Corollary. If $X$ is any point outside a circle, centre $O$, radius $r$, the rectangle contained by the segments of any chord drawn through $X$ equals $OX^2 - r^2$.

Fig. 762
THEOREM 69

If two straight lines $AB$ and $CD$ are divided both internally or both externally at the same point $X$ such that $XA \cdot XB = XC \cdot XD$, the four points $A$, $B$, $C$, $D$ are concyclic.

Fig. 763

Construction and Proof. Draw the circle $ABC$ and let it cut the line $CD$ produced if necessary, at $K$.

If $X$ divides $AB$ internally, fig. 763 (i), $X$ lies inside the circle $ABC$. $\therefore X$ divides the chord $CK$ internally.

If $X$ divides $AB$ externally, fig. 763 (ii), $X$ lies outside the circle $ABC$. $\therefore X$ divides the chord $CK$ externally.

In each case, $K$ and $D$ are on the same side of $X$.

Also $XA \cdot XB = XC \cdot XK$ and $XA \cdot XD = XC \cdot XD$.

$\therefore XC \cdot XK = XC \cdot XD$, $\therefore XK = XD$.

But $K$ and $D$ are on the same side of $X$.

$\therefore K$ and $D$ are the same point.

$\therefore$ the circle $ABC$ passes through $D$.

Corollary. If the straight line $AB$ is divided externally at $X$, and if $C$ is a point, not on $AB$, such that $XA \cdot XB = XC^2$, the circle $ABC$ touches $XC$ at $C$.

The proof is the same as for fig. 763 (ii).

INTERSECTING CHORDS

There are two other important results which it is convenient to refer to at this stage: they are given later as corollaries to Theorem 78, p. 504.

If $AB$ is a diameter of a circle and if $PN$ is the perpendicular to $AB$ from any point $P$ on the circumference, then

1. $PN^2 = AN \cdot NB$;
2. $AP^2 = AN \cdot AB$ and $BP^2 = BN \cdot BA$.

Fig. 764

(1) Produce $PN$ to meet the circle again at $Q$.

Since $PQ$ is perpendicular to the diameter $AB$, $PN = NQ$;

$\therefore AN \cdot NB = PN \cdot NQ = PN \cdot PN = PN^2$.

(2) Since $\angle PNB$ is a right angle,

the circle on $PB$ as diameter passes through $N$.

Since $\angle APB$ is a right angle,

$AP$ is the tangent at $P$ to the circle on $PB$ as diameter.

$\therefore AP^2 = AN \cdot AB$.

Similarly, it may be proved that $BP^2 = BN \cdot BA$.

Note. Result (2) also follows at once from the figure and proof of Pythagoras' theorem, see p. 283, and it is a useful exercise for the reader to prove result (1) by applying Pythagoras to the three triangles $APB$, $ANP$, $BNP$. [Let $AN = x$, $NB = y$, $PN = h$, $PA = a$, $PB = a$.]

Both results can be proved easily by using trigonometry.
Exercise 79

1. Two circles cut at A and B; P is any point on AB produced. Prove that the tangents from P to the two circles are equal.

2. Prove that the common chord of two intersecting circles, when produced, bisects the common tangents.

3. The tangent at P to the circle APB meets AB produced at T. If TA = TP, prove that AB = BT.

4. P and Q are points on the chords AB and AC respectively of the circle ABC, centre O, such that AP = AQ = QC. Prove that OP = OQ. [Use the Corollary of Theorem 67.]

5. If the altitudes BE, CF of ∆ABC intersect at H, prove that (i) BH : HE = CH : HF; (ii) AF : AB = AE : AC.

6. AB is a diameter of the circle APB; a line perpendicular to AB cuts AB at H, K respectively. Prove that AH : AB = AK : AP.

7. K is a point inside ∆ABC; BK, CK, produced, cut AC, AB at Q, R. If BK = CK, KQ = CK, KR, prove AR : AB = AQ : AC.

8. BE, CF are altitudes of ∆ABC; M, N are the mid-points of AC, AB respectively. Prove that AM : AE = AN : AF. What follows from this fact?

9. AB, AC are two chords of a circle; any line parallel to the tangent at A cuts AB, AC at D, E respectively. Prove that AB : AD = AC : AE.

10. AB is a diameter of the circle APB; the tangent at B meets AP, AQ produced at X, Y respectively. Prove that AP : AX = AQ : AY = AB².

11. AB, AC are two equal chords of a circle; AP is another chord of the circle which cuts BC at Q. Prove AP : AQ = AB².

12. Two lines XB, XCD cut a circle at A, B, C, D; through X a line is drawn parallel to BC to meet DA produced at Y. Prove that XA² = YA · YD.

13. In ∆ABC, AB = AC and ∠A = 90°. If the bisector of ∠ABC meets AC at P, prove that AC : CP = BC² : AP².

14. In ∆ABC, ∠BAC = 90° and AB = 2AC. If AD is an altitude, prove that BD = DC.

15. Any two circles being given, a third circle is drawn cutting one of the circles at A, B and the other at C, D; AB and CD, when produced, meet at X. Prove that the tangents from X to the three circles are equal.

16. ABXY and ABZ are two circles; AB and XY, when produced, meet at T; TZ is the tangent from T to the circle ABZ. Prove that the circle XYZ touches TZ.

17. PQ is a chord of a circle, centre O; the tangents at P, Q meet at X; OX cuts PQ at N. Prove that ON : OX = OP².

18. Two circles intersect at A, B; X is a point such that the tangents from X to the circles are equal. Prove that X must lie on AB produced or BA produced.

19. In fig. 765, CE touches the circle BAED, and CF touches the circle CAB. If CAD is a straight line, prove that CE² + DF² = CD².

20. Three circles are drawn so that each intersects the other two. Prove that the three common chords are concurrent. (If AB, CD, EF are the common chords and if AB cuts CD at X, suppose if possible EX when produced cuts the circles at distinct points P, Q.)

21. TP, TQ are tangents to a circle HPKQ, centre O; TO cuts PQ at N. If HKQ is a straight line, prove that the circle TKQ passes through O. [Prove that O, P, T, Q are concyclic.]

22. Two circles cut at A, R; X is any point on AR produced; a circle, centre X, cuts one circle at P, Q and the other at L, M; XP, XM, produced if necessary, cut the circles PQA, LMA at S, T. Prove that PS = TM.

23. AH, AK are diameters of the circles AQH, APK. If PAH, QAK are straight lines, prove that PA : AH = QA : AK.

[This exercise is continued on p. 433.]
CONSTRUCTION 17

Construct a square equal in area to a given rectangle.

Given a rectangle ABCD.

To construct a square equal in area to ABCD.

Construction. Produce AB to E making BE = BC.
On AE as diameter describe a semicircle.
Produce CB to meet the semicircle at P.
On BP describe a square.
This is the required square.

Proof. Since AE is a diameter of the circle APE and since
PB is the perpendicular from P to AE,
\[ BP^2 = AB \cdot BE. \]
But \[ BE = BC \quad \text{constr.} \]
\[ \therefore BP^2 = AB \cdot BC \]
\[ \text{area of ABCD.} \]

Note. The proof of Construction 17 depends on the
property proved on p. 429.
The method is the same as that used for constructing the
mean proportional between two given lines, see p. 514,
first method.

CONSTRUCTIONS

Construction 17 may be used to construct a square equal in
area to any polygon.

Construction. By the method of Construction 9, p. 263,
reduce the polygon to an equivalent triangle XYZ.
Draw the altitude XK and bisect YZ at Q.
Use Construction 17 to construct a square equal in
area to the rectangle contained by YQ and XK.
This is the required square.

Proof. Area of polygon = area of \( \triangle XYZ \)
\[ = \frac{1}{2} YZ \cdot XK \]
\[ = YQ \cdot XK \]
\[ = \text{square.} \]

EXERCISE 79 (continued)

*24. The triangle ABC is such that AC equals the diagonal of
the square described on AB; D is the mid-point of AC. Prove
\( \angle ABD = \angle ACB. \)

*25. In \( \triangle ABC, \angle BAC = 90^\circ; \) E is a point on BC such that
AE = AB. Prove that BE \cdot BC = 2AB^2. [Draw AN perpendicular
to BC.]

*26. P, Q, R are points on the sides BC, CA, AB of \( \triangle ABC \)
such that BP \cdot PC = CQ \cdot QA = AR \cdot RB. Prove that the circles
ABC, PQR are concentric.

*27. AB is a diameter of the circle APQB; AP = \( \frac{1}{2} \) AB; N is
the mid-point of PB and ANQ is a straight line. Prove that
NQ = \( \frac{1}{2} \) AN. [Use Apollonius for \( \triangle APB \).]
CONSTRUCTION 18

Construct a circle to pass through two given points and to touch a given line, not parallel to the line joining the given points. [Two solutions.]

Given two points A, B and a line CD not parallel to AB.

To construct a circle to pass through A, B and touch CD.

Construction. Join BA and produce it to meet CD at O. Describe a semicircle with OB as diameter. Through A draw AG perpendicular to OB to meet the semicircle at G. Join OG. With centre O, radius OG, describe a circle to cut CD at P, Q. Construct the circles through A, B, P and A, B, Q. These are the required circles.

Proof. Since OB is a diameter of the circle OGB and since GA is the perpendicular from G to OB,

\[ OA \cdot OB = OG^2 \]

\[ = OP^2 \quad \text{constr.} \]

\[ \therefore OP \text{ is the tangent at } P \text{ to the circle } ABP. \]

Similarly, it may be proved that OQ is the tangent at Q to the circle ABQ.

Note. This method fails if AB is parallel to CD. This special case forms an easy exercise: only one circle can then be drawn through A and B to touch CD.

EXERCISE 80

1. Draw a rectangle, 5 cm. by 8 cm., and construct a square of equal area. Measure its side.

2. Construct a line of length \( \sqrt{31} \) cm. and measure it.

3. Construct a square equal in area to an equilateral triangle of side 3 in. Measure its side.

4. Draw a regular hexagon of side 5 cm. and construct a square of equal area. Measure its side.

5. Draw a quadrilateral ABCD in which \( AB = 3 \) cm., \( BC = 4 \) cm., \( CD = DA = 6.5 \) cm., \( \angle ABC = 90^\circ \). Construct a square equal in area to ABCD and measure its side.

6. Use construction 17 to solve the simultaneous equations,

\[ x + y = 11, \quad xy = 24. \]

7. Use construction 17 to solve the equation, \( x(9 - x) = 12. \)

8. A, B are given points and CD is a given line parallel to AB. Show how to construct a circle to pass through A, B and to touch CD.

9. Draw a line AB. Construct a point P on AB such that \( AP^2 = \frac{1}{3} AB^3. \)

10. Given a rectangle ABCD, construct when possible a point P on AB such that \( AP \cdot PB = BC^2. \) When is this impossible?

11. Construct a circle to pass through two given points A, B and to touch a given circle. [Draw any circle through A, B and let it cut the given circle at P, Q; let AB and PQ meet at O. From O draw OH, OK to touch the given circle, and prove that \( OH^2 = OA \cdot OB. \)]

12. Given four points A, B, C, D on a straight line, construct when possible a point P on the line such that \( PA \cdot PD = PB \cdot PC. \)

13. Given a rectangle ABCD, construct a point P on AB produced such that \( PA \cdot PB = BC^2. \) [Draw a tangent QT to the circle AQD, diameter AB, making QT = BC; find P on AB so that the tangent PK equals QT. Or bisect AB at N and use the fact that \( PA \cdot PB = PN^2 - AN^2. \)]

14. Show how to construct a circle to pass through two given points and to cut a given circle so that the common chord of given length.

15. Construct a circle to pass through a given point A and to touch two given lines BC, BD. [Draw the bisector BE of \( \angle CBD \) and take the image A' of A in BE.]
**REVISION OF GEOMETRICAL FACTS FOR ORAL DISCUSSION**

**EXERCISE 81**

Nos. 1–10 refer to fig. 769 which represents a quadrilateral and its diagonals.

1. (i) What is the definition of a parallelogram?
   (ii) If $ABCD$ is a parallelogram, what facts do you know about lengths and about areas?
   (iii) State as many tests as you can for $ABCD$ to be a parallelogram.

2. What facts are true (i) for a rectangle, (ii) for a rhombus, which are not true for all parallelograms?

3. What triangles are equal in area if $AD$ is parallel to $BC$?

4. What triangles are equal in area if $BK = KD$?

5. What angles are equal if a circle can be drawn through $A, B, C, D$?

6. What can you say about the sizes of angles if $A$ lies outside the circle on $BD$ as diameter and if $C$ lies on it?

7. What angles are equal if $DA$ touches the circle $DKC$?

8. If $AK = KC$, what do you know about $DA + DC$?

9. What follows if $AB$ is greater than $BC$?

10. What follows if $BA = BC$ and $DA = DC$?

Nos. 11–14 refer to fig. 770.

11. State all the facts you know relating to lengths and areas if $AP = PB$ and $AQ = QC$.

12. If $AP = PB$ and $AQ = QC$ and if $AG$ produced cuts $BC$ at $X$, what facts about lengths do you know?

13. What can you say about the figure if the areas of $\triangle APC$, $\triangle AQB$ are equal?

14. State all the facts you know relating to lengths and areas if $AP = 2PB$ and if $PQ$ is parallel to $BC$.

**REVISION OF RIDER WORK FOR ORAL DISCUSSION**

**EXERCISE 82**

Nos. 1–12 refer to fig. 771 which represents a triangle and three concurrent lines through its vertices.

1. If $y = z$, prove that $m = n$.

2. What angles are equal if $y = z = 2$ right angles?

3. If $\angle PDC = \angle DBC$, find an angle equal to $\angle DPC$.

4. Express $\angle BDC - \angle BAC$ as the sum of two angles.

5. If $DB = DC$ and $DR = DQ$, prove that $AB = AC$.
6. Is it possible to draw the figure so that BD = DQ and also CD = DR?

7. Is it possible to draw the figure so that BP = PC and also BD = DQ?

8. If RD = RB and BD = DC, compare the sizes of ∠RBC and ∠RCB.

9. If BD = DC = CP, find the relation between ∠BDP and ∠CDP.

10. How must the figure be drawn if

(i) △BPA, △BQA are equal in area?

(ii) △PDC, △RDA are equal in area?

(iii) △BOD, △BDA are equal in area?

11. If BD and AD are the bisectors of ∠CBA, ∠CAB, find the relation between ∠BDA and ∠BCA.

12. If AR = AD, express ∠BCR in terms of ∠ABP and ∠APB.

Nos. 13-17 refer to fig. 772 in which ABCD is a parallelogram and Y is the mid-point of CD.

13. Prove that BY = 2AD.

14. Prove that DZ is parallel to AC.

15. Prove that

△ABZ = quadr. ABCD.

16. Prove that

△ABD = ∠ quadr. ACZD.

17. Join KY and prove that

KY = ½ AD.

18. If in fig. 772, ABCD is a parallelogram and if Y is any point on CD, prove that △DYZ = △YBC.

19. If in fig. 772, ABCD is a parallelogram and if Y is a point on CD such that ABZD is a cyclic quadrilateral, prove that DZ = AB.

20. With the data of No. 18, prove that DB touches the circle DYZ.

21. In fig. 773, AY = YC, AZ = ZB, BY cuts C at G, GH = HB, GK = KC. Find out all the facts you can about this figure. Give reasons.

Nos. 22-31 refer to fig. 774 in which PAQ, RBS, AXY are straight lines.

22. Prove that PR is parallel to QS.

23. What results can be deduced from No. 22 by taking (i) S very close to Q, (ii) S very close to B, (iii) A very close to B?

24. What points in the figure must be joined to give a line parallel to RX? Give reasons.

25. If AR is a diameter of circle ABP, prove that AS is a diameter of circle ABQ.

26. If PQR is a cyclic quadrilateral, prove that PQ = RS.

27. If AP = BX, prove that XP = AB.

28. Prove that ∠XBY = ∠PBQ.

29. If the circles are equal, prove that BX = BY.

30. Prove that ∠XBY is equal to one of the angles between the tangents at A to the two circles.

31. If AR, AS are tangents to the circles ABS, ABR, prove that they are diameters of the circles ABR, ABS.

Nos. 32-35 refer to fig. 775 which represents two circles, centres A, B, radii a, b units, touching externally at C. DRTQ, D' Q'T'Q', TCT' are common tangents.
Xex. 36-50 refer to fig. 776 in which \(ABCD\) is a cyclic quadrilateral whose opposite sides meet at \(E\) and \(F\) and whose diagonals meet at \(G\).

36. If \(BD\) bisects the angles at \(B\) and \(D\), prove that \(\angle BAD = \angle DCB\).  
\[\text{Fig. 776}\]

37. What is the relation between the angles \(EAF\), \(ECF\)?

38. If \(\angle BAC = 24^\circ\), \(\angle AFB = 42^\circ\), \(\angle BGC = 72^\circ\), find \(\angle EAD\) and \(\angle ACD\) and \(\angle DBF\).

39. What change must be made in the drawing of the figure if \(AD\), \(DC\), \(CB\) are all equal? In this case, if \(\angle ACB = \theta\), find \(\angle CAF\) and \(\angle ABC\) and \(\angle AFB\) in terms of \(\theta\).

40. Prove that \(BG > GC\). Which is the greater, \(AG\) or \(DG\)?

41. What change must be made in the drawing of the figure if \(G\) is the centre of the circle?

42. Prove that the triangle whose sides are parallel to \(DA\), \(DB\), \(DC\) is equiangular to the triangle \(ABC\).

43. Prove that \(\angle E + \angle F = 180^\circ - 2 \angle ABC\).

44. Prove that the line joining \(E\) to the circumcentre of the triangle \(EAC\) is perpendicular to \(BD\).

45. Prove that the circumcircles of \(\triangle EAD\) and \(\triangle FCD\) cut again at a point on \(EF\).

46. Prove that the bisectors of \(\angle BAC\), \(\angle AFB\) are at right angles.

47. Prove that the bisectors of \(\angle BAC\), \(\angle AFB\) are right angles.

48. If \(AC\) is perpendicular to \(BD\), prove that \(BC\) is twice the perpendicular from the centre of the circle to \(AD\).

49. If \(AC\) is perpendicular to \(BD\), prove that the perpendicular from \(G\) to \(AD\) when produced bisects \(BC\).

50. Prove that the sum of the squares of the tangents from \(E\) and \(F\) to the circle is equal to \(EF^2\). [Use No. 45.]

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**Revision Papers 51-58 (Theorems 1-50)**

**(Including angles in a segment)**

51. \(ABCD\) is a square; the bisector of \(\angle ACD\) cuts \(BD\) at \(Q\).

- Prove that \(BQ = CD\).

2. \(AD\), \(BC\) are the parallel sides of the trapezium \(ABCD\);
   - \(AB = 6\text{ in.}\), \(BC = 9\text{ in.}\), \(CD = 5\text{ in.}\), \(AD = 14\text{ in.}\)
   - Find the area of \(ABCD\).

3. A chord of a circle at a distance 11 cm. from the centre is 6 cm. long. Calculate the length of a chord of the same circle which is at a distance 9 cm. from the centre.

4. If \(O\) is the centre of a circle; \(CB\) is a chord parallel to the radius \(OA\); \(OB\) cuts \(AC\) at a point \(K\) inside the circle.

- Prove that \(\angle AKB = 3\angle ACB\).

52. **In \(\triangle ABC\), \(\angle ABC = 54^\circ\), \(\angle BAC = 78^\circ\). If the bisector of \(\angle BCA\) cuts \(AB\) at \(X\), prove that \(CA = CX\)**

2. Two straight lines \(ABC\), \(PQR\) are cut by three parallel lines \(AP\), \(BQ\), \(CR\). Prove that \(\angle AQC\) is equal in area to \(\angle FBR\).

3. \(ABC\) is a straight line such that \(AB = 1\text{ in.}\), \(BC = 4\text{ in.}\);
   - \(PBQ\) is the chord of the circle on \(AC\) as diameter perpendicular to \(AC\).
   - Find the length of \(PQ\).

4. \(AB\) is a quadrant of a circle; \(AC\) is any chord. If \(BN\) is the perpendicular from \(B\) to \(AC\), prove that \(BN = NC\).

53. **In \(\triangle ABC\), \(AB = 10\text{ cm.}\), \(CD = 5\text{ cm.}\), calculate the radius of each arc.**

- [Fig. 777]
3. **ABCD** is a quadrilateral inscribed in a circle, centre 
\[\angle ABD = 30^\circ, \angle BDC = 43^\circ, \angle CDO = 49^\circ;\] calculate \[\angle ADB.\]

4. Two circles \[\text{ABPH, ABKQ}\] intersect at \[A\] and \[B\]. If \[PBQ\] is a straight line and if \[PH\] is parallel to \[QK\], prove that \[HAK\] is a straight line.

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54

1. **ABCD** is a parallelogram; \[\text{ABHK, ADPQ}\] are squares outside \[ABCD\]. Prove that \[QK\] is equal to one of the diagonals of \[ABCD\].

2. Construct a parallelogram of area 21 sq. cm such that one side is 6 cm and one angle is 50°. Measure the other side.

3. \[AB\] is the diameter of a semicircle \[APQB\]; parallel lines \[PC, CQ\] cut \[AB\] at points \[C, D\] between \[A\] and \[B\]. If \[AC = DB\], prove that \[\angle CPQ\] is a right angle.

What special case is obtained by making \[Q\] coincide with \[B\]?

4. \[AB\] is a chord of a circle, centre \[O\], such that \[\Delta OAB\] is equilateral. The line which bisects \[\angle OAB\] cuts the circle again at \[Q\]. Prove that \[AB = BQ\].

---

55

1. **ABCD** is a square; any line is drawn through \[A\] outside the square; \[BH, DK\] are the perpendiculars from \[B, D\] to this line. Prove that (i) \[\Delta ABH \cong \Delta DAK\]; (ii) \[BH + DK = HK\].

2. In \[\Delta ABC\], \[AB = 6\ cm, AC = 8\ cm, \angle BAC = 90^\circ\]; \[D\] is the mid-point of \[BC\]. Find (i) the area of \[\Delta ABD\], (ii) the length of the perpendicular from \[B\] to \[AD\].

3. **ABCD** is a quadrilateral inscribed in a circle; \[AC\] is a diameter. If \[\angle BAC = 43^\circ\], find \[\angle ADB\].

4. \[AB\] is a diameter of a circle; \[PQ\] is any chord. The perpendicular from \[A\] to \[PQ\] is produced to any point \[H\]. Prove that the perpendicular bisector of \[PQ\] bisects \[BH\].

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56

1. **APRT, AQSC** are straight lines such that \[\triangle PRT = \triangle QRT\]. Prove that \[\angle PST = 5\angle QST\].

2. A segment of a circle is cut off by a chord of length 6 cm; the height of the segment is 2 cm. Calculate the radius of the circle.

3. In fig. 778, \[A, B, C\] are the centres of the circles \[CDQ, CPD; CPQ\] is a straight line and \[BP\] cuts \[AQ\] at \[K\]. Prove that \[\angle AKB = \angle AGB\].

4. \[AB, AC\] are equal chords of the circle \[ABC\] and \[AP, BQ\] are parallel chords. Prove that \[\angle PBQ, \angle ABC\] are either equal or supplementary.

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57*

1. **ABC** is an equilateral triangle; \[P, Q, R\] are points on \[BC, CA, AB\] such that \[PQR\] is also an equilateral triangle. Prove that (i) \[\Delta BPR \cong \Delta CQP\], (ii) \[AQ + AR = BC\].

2. Draw a quadrilateral \[ABCD\] such that \[AB = 6\ cm, BC = 5\ cm, CD = 4\ cm, \angle ABC = 110^\circ, \angle BCD = 90^\circ\]. Reduce it to an equivalent triangle with \[AB\] as base and vertex on \[BC\]. Find its area.

3. \[AB\] is the diameter of the semicircle \[AQRB\]; \[RQ\] and \[BA\] when produced meet at \[P\]. If \[\angle QBR = 90^\circ\] and \[\angle APQ = 20^\circ\], calculate \[\angle RQB\].

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58*

4. In fig. 779, \[EFQG\] is a straight line cutting the circles \[CEP, CQH; PCQ\] is a straight line and \[PF, QH\] meet at \[A\]. If \[AP = AQ\], prove that \[CE = CG\].
58*

1. In the quadrilateral $ABCD$, $\angle DAB = \angle ABC = 60^\circ$ and $\angle ADC = 90^\circ$. Prove that $\overline{AB} + \overline{BC} = 2\overline{AD}$.

2. $E, F$ are the mid-points of the sides $AC, AB$ of $\triangle ABC$. If $BE$ cuts $CF$ at $G$, prove that the triangles $\triangle BGC, \triangle CAG, \triangle ABG$ are equal in area.

3. Two parallel chords $AB, CD$ of a circle are 1 in. apart; $AB = 4$ in., $CD = 6$ in. Find the radius of the circle correct to $\frac{1}{10}$ in.

**Fig. 780**

4. In fig. 780, $PQ$, $RS$ are parallel chords of the circle; $RS$ is produced to $T$ so that $QR = QT$. Prove that $ST = PQ$.

**REVISION PAPERS 59-66 (Theorems 1-57)**

[Including tests for concyclic points, equal area]

59

1. $ABCD$ is a parallelogram, $BCH, DCK$ are equilateral triangles outside it. Prove that (i) $\triangle ADK \equiv \triangle HBA$; (ii) if $AD$ is produced to $E$, $\angle EDK = \angle DAB - \angle KAH$; (iii) $\angle KAH = 60^\circ$; (iv) $KA = KH$.

2. $ABCD$ is a rectangle, 3 in. by 4 in.; a halfpenny (diameter 1 in.) is in the plane of the rectangle and is made to roll once completely round the outside of it. Find the distance travelled by its centre.

3. $O$ is the centre of the circle circumscribing $\triangle ABC$; $AD$ is an altitude of $\triangle ABC$. If $AO$ bisects $\angle BAD$, prove that $\angle ACB = \angle ABC = \frac{1}{2} \angle BAC$.

4. $ABCD$ is a quadrilateral, right-angled at $B, C$; a line perpendicular to $AD$ cuts $AD, BC$ at $P, Q$. Prove $\angle BPC = \angle AQD$.

60

1. The side $BC$ of an equilateral triangle $ABC$ is produced to $D$ so that $CD = 3BC$. Prove that $AD^2 = 13AB^2$.

2. $ABCD$ is a quadrilateral. If $\angle ABC + \angle ADC = 180^\circ$, prove that the perpendicular bisectors of $AC, BD, AB$ are concurrent.

3. In fig. 781, $A$ is the centre of the circle $BCP$; $PBQ$ is a straight line. Prove that $QP = QC$.

**Fig. 781**

What can you say about the position of the centre of the circle $BCQ$?

4. $ABCD$ is a rectangle; the line through $C$ perpendicular to $AC$ cuts $AB, AD$ produced at $P, Q$. Prove that the points $P, B, D, Q$ are concyclic.

61

1. $H, K, L$ are the mid-points of the sides $AB, AC, BC$ of $\triangle ABC$; $P, Q$ are points on $BC$ such that $BP = \frac{1}{4}BC = \frac{1}{4}BQ$. Prove that (i) $PH$ is parallel to $AL$; (ii) $PH = QK$.

2. $ABCD$ is a quadrilateral such that $\angle BAD = 127^\circ$, $\angle BCD = 53^\circ$, $\angle ABD = 31^\circ$. Calculate $\angle ACB$.

3. $AB, AC$ are equal chords of the circle $ABDC$, and $BD = BC$. If $BA, DC$ when produced meet at $K$, and if $AD$ cuts $BC$ at $X$, prove that $\triangle CAX, \triangle XCD$ are equiangular.

4. $A, B, C, P, Q, R$ are points on the circle $ABPCQ$ such that $\angle ABC$ and $\angle PQR$ are right angles. Prove that $\triangle AP$ is equal and parallel to $\triangle CR$.

62

1. $ABCD$ is a square; $P$ is a point on $AB$ such that $AP = \frac{1}{4}AB$; $Q$ is a point on $PC$ such that $PQ = \frac{1}{4}PC$. Prove $PQCD = \frac{1}{4}ABCD$.

2. The side $AB$ of a cyclic quadrilateral $ABCD$ is produced to $E$; $\angle ABE = 140^\circ$, $\angle ADC = 100^\circ$, $\angle ACB = 40^\circ$. Find $\angle BAC$, $\angle CAD$. 


3. AB, AC are equal chords of a circle; BC is produced to D so that CD = CA; DA cuts the circle again at E. Prove that BE bisects \( \angle ABC \).

4. OY is the bisector of \( \angle XOZ \); P is any point; PX, PY, PZ are the perpendiculars from P to OX, OY, OZ. Prove that \( XY = YZ \). [Draw the circle on OP as diameter.]

63

1. In fig. 782, PN is perpendicular to AC, and PR is parallel to AC; also QR = 2AP. Prove that \( \angle CAR = \frac{1}{2} \angle CAB \). [If K is the mid-point of QR, KP = KQ = KR.]

2. In the quadrilateral ABCD,
   \( AB = 5 \text{ in.} \), \( BC = 12 \text{ in.} \), \( CD = 7 \text{ in.} \),
   \( \angle ABC = \angle BCD = 90^\circ \). If P is a point on BC such that \( \angle APD = 90^\circ \), calculate the length of BP. [Two answers.]

3. PAB, PBC, PCA are three unequal circles. From any point D on the circle PBC, lines DB, DC are drawn and produced to meet the circles PBA, PCA again at X, Y. Prove that XAY is a straight line.

4. AB, BC, CD are equal chords of the circle ABCD; AD is produced to E so that DE = DC. Prove that (i) CE = CA; (ii) CE is parallel to BD.

64

1. In \( \triangle ABC \), \( \angle ACB = 90^\circ \) and AC = 2CB; CD is an altitude. Prove by using the figure of Pythagoras' theorem, p. 282, that AD = 4DB.

2. Two chords AB, DC of a circle, centre O, are produced to meet at E; \( \angle CBE = 75^\circ \), \( \angle CEB = 25^\circ \), \( \angle AOD = 144^\circ \). Prove that \( \angle AOB = \angle BAC \).

3. In fig. 783, prove that QR is parallel to ST.

4. Five points A, B, C, D, E are taken in order on a circle so that the chords AB, AE are equal. If AC, AD meet BE at X, Y, prove that C, X, Y, D are concyclic. [Join CE, CD.]

REVISION PAPERS 67-74 (Theorems 1-63)

[Including tangent properties.]

67

1. In fig. 785, AB is a diameter; \( \angle HPQ = \angle KQP = 90^\circ \). Prove that AH = BK. [Draw the perpendicular from the centre to PQ.]

2. AOB is a chord of a circle ABC; T is a point on the tangent at A; the tangent at B meets TO produced at P; \( \angle ATO = 35^\circ \), \( \angle BTO = 115^\circ \). Find \( \angle BPT \).
3. APC is a minor arc of a circle, centre O; AQOC is another circular arc. Prove that $\angle APC = \angle PAQ + \angle PCQ$.

4. AB is a chord of a circle; AT is the tangent at A; AC is a chord bisecting $\angle BAT$. Prove that $AC = CB$.

68

1. ABCD is a parallelogram; P is any point on CD; PB, CB, AD cut any line parallel to AB at Y, Z, W. Prove that quad. DCZW = 2\triangle APY.

2. AB, CD are two intersecting chords of a circle; AP, CQ are the perpendiculars from A, C to CD, AB respectively. Prove that PQ is parallel to BD. [Join AC.]

3. The radii of two circles are 2 cm, 3 cm, and the distance between their centres is 9 cm. Calculate the lengths of the interior and exterior common tangents.

4. AB is a diameter of a circle; AC is any chord; P is the mid-point of the arc BC. Prove that AC is perpendicular to the tangent at P.

69

1. In $\triangle ABC$, $\angle ABC = 90^\circ$, $\angle BAC = 45^\circ$. The bisector of $\angle ACB$ meets AB at P. Prove that $AP^2 = 2PB^2$. [Draw PN perpendicular to AC.]

2. A circle is drawn touching the sides BC, CA, AB of $\triangle ABC$ at X, Y, Z. If $\angle B = 36^\circ$ and $\angle C = 66^\circ$, calculate $\angle XYZ$.

3. The circle ABCD passes through the centre O of the circle ABQ; APQ is a straight line. Prove that (i) PB = PQ; (ii) CP produces bisects BQ at right angles. [Join AC, BC.]

4. AOB is a diameter of a circle, centre O. The tangent at B meets any chord AP produced at T. Prove $\angle ATB = \angle OPB$.

70

1. ABCD is a parallelogram; AB, CB are produced to X, Y; P is any point within the angle XYB. Prove that $\triangle PCD = \triangle PAB = \triangle ABC$.

2. In fig. 786, AP, AQ are tangents to the circles ABQ, ABR; QBR is a straight line. Prove that RP is parallel to AQ.

71

1. ABC is a triangle; APQB, AXYC are square outside $\triangle ABC$. Prove that PC is perpendicular to SX.

2. PR is a chord of a circle, centre O. T is a point on the tangent at P and OT cuts PR at Q. If TP = TQ, prove that $\angle ROT = 90^\circ$.

3. ABCD is a quadrilateral inscribed in a circle, centre O. If AC bisects $\angle BAD$, prove that OC is perpendicular to BD.

4. In fig. 787, TP is a tangent to the circle, centre O, and TQ bisects $\angle OTP$. Prove that $\angle TQP = 45^\circ$.

72

1. ABCD is a parallelogram; P is the mid-point of AD; AB is produced to Q so that AB = BQ. Prove that $\triangle ABCD = \triangle PQQD$.

2. AB is a chord of a circle, centre O, such that $\angle AOB$ is obtuse; BC is a chord parallel to QA. If $\angle OAB = x^\circ$, find in terms of x the acute angle which BC makes with the tangent at B.

3. In $\triangle ABC$, $\angle BAC = 90^\circ$ and $\angle BAC < AC$. D is the mid-point of BC. A circle touches BC at D, passes through A and cuts AC again at E. Prove that AC = 2 AD and DE.

4. In fig. 788, TP is a tangent to the circle, centre O; PQ and PT are equally inclined to TO. Prove that $\angle QOT = \frac{1}{2} \angle POT$. 
73

1. ABCD is a quadrilateral; AB is parallel to CD; BP, CP are drawn parallel to AC, AD respectively to meet at P. Prove that \( \Delta PDC = \Delta ABD \).

2. In \( \triangle ABC \), \( AB = AC \); D is the mid-point of BC. Prove that the tangent at D to the circle ADC is perpendicular to AB. [Join D to A.]

3. Two circular cylinders of radii 2 in., 6 in. are bound tightly together with their axes parallel by an elastic band. Find the stretched length of the band.

4. In fig. 789, AB is the diameter of the semicircle APQR; ANQC is a straight line such that \( AB = AC \) and \( NQ = QC \). Prove that \( \angle BAR = \angle RAQ = \angle QAP \). [Join BQ.]

74

1. ABP, ABQ are equivalent triangles on opposite sides of AB; PR is drawn parallel to BQ to meet AB at R. Prove that QR is parallel to PB.

2. ABC is a triangle inscribed in a circle; BE, CF are altitudes of \( \triangle ABC \). Prove that EF is parallel to the tangent at A.

3. OBC is a straight line such that \( OB = 9 \text{ in.}, BC = 8 \text{ in.} \); OA is drawn perpendicular to OB. Calculate the radius of the circle which touches the circle, centre C, radius CB, and touches OB at a point between O and B and also touches OA.

4. In fig. 790, O is the centre of the circle; \( PQ = AO \), \( \angle AOQ = 90^\circ \). Prove that are BR = 3 arc AP.

REVISION PAPERS 75–80 (Theorems 1–69)

[Including extensions of Pythagoras and segments of chords.]

75

1. If each diagonal of a quadrilateral bisects the area of the quadrilateral, prove that the quadrilateral is a parallelogram.
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2. The tangents at points P, Q on a circle meet at T; points H, K are taken on TQ and TQ produced respectively so that PQ bisects ∠HPK. Prove that ∠HPK = ∠PKT.

3. In △ABC, AB = 6 cm., BC = 12 cm.; BC is trisected at Y and Z. Calculate the length of AY.

4. ABC is a triangle inscribed in a circle; the tangent at C meets AB produced at K. If BK = \frac{3}{4}CK, prove that BK = \frac{3}{4}AB.

PART III

SIMILAR FIGURES

Ratio

If the lengths of two straight lines are 4 cm. and 6 cm., the length of the first is \frac{2}{3} or \frac{2}{3} of that of the second, and we say that the ratio of the length of the first line to that of the second is 2 to 3, written 2 : 3, and this ratio is represented by the fraction \frac{2}{3}.

Ratios should be expressed as simply as possible; just as the fraction \frac{4}{8} is equivalent to \frac{1}{2}, so the ratio 20 : 25 is equivalent to 4 : 5. A ratio is unaltered if the two numbers or quantities in the ratio are both multiplied, or both divided, by the same number.

A ratio is a comparison of the magnitudes of two quantities which must be of the same kind; it is meaningless to compare 5 ounces with 10 shillings or to compare 6 inches with 4 sq. inches.

If two quantities have a common measure, their ratio can be expressed as the ratio of two integers, e.g. if the lengths of two straight lines are given to be 2-56 in., 1-12 in., since 2-56 256 16

1-12 " 112 = 7", the ratio of their lengths is 16 : 7. Here the common measure may be taken as \frac{1}{16} in. But we frequently meet with pairs of lines whose lengths have no common measure: if the side of a square is 1 in., the diagonal is \sqrt{2} in. (Pythagoras), and these two lengths have no common measure and are called incommensurable. The ratio of two such lengths cannot be expressed as the ratio of two integers, although two integers can be found whose ratio differs from this ratio by an amount as small as we please. Formal proofs of theorems involving the ratio of incommensurable quantities are very difficult, and we shall assume that if a theorem has been proved for all commensurable ratios, it is also true if the ratios are incommensurable.

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**Ratio of Segments of a Line**

If $P$ is any point on a straight line $AB$ or on $AB$ produced or on $BA$ produced, $PA$ and $PB$ are called segments of the line $AB$, see p. 421, and the line $AB$ is said to be divided at $P$ in the ratio $AP : PB$.

If $P$ lies between $A$ and $B$, see fig. 791 (i), the line $AB$ is said to be divided internally at $P$ in the ratio $AP : PB$.

If $P$ lies on $AB$ produced or on $BA$ produced, see fig. 791 (ii), (iii), the line $AB$ is said to be divided externally at $P$ in the ratio $AP : PB$.

It is important to notice that in all cases the whole line $AB$ does not appear in the ratio $AP : PB$ of the segments of $AB$. This definition may also be emphasised, if considered advisable, by a discussion of directed lengths and the interpretation of positive and negative ratios.

**Proportion**

If four quantities $a$, $b$, $c$, $d$ are such that

$$a : b = c : d$$

then $a$, $b$, $c$, $d$ are said to be in proportion.

Thus if $a$, $b$, $c$, $d$ are in proportion, we have

$$\frac{a}{b} = \frac{c}{d}$$

d and is called the fourth proportional to $a$, $b$, $c$.

If three quantities $a$, $b$, $c$ are such that

$$\frac{a}{c} = \frac{b}{c}$$

then $a$, $b$, $c$ are said to be in continued proportion. Further, $c$ is then called the third proportional to $a$, $b$; and $b$ is called a mean proportional between $a$, $c$.

Thus, if $b$ is a mean proportional between $a$, $c$,

$$b^2 = ac$$

Therefore, if a square, side $b$ units, is equal in area to a rectangle whose adjacent sides are $a$ units, $c$ units, then $b$ is a mean proportional between $a$ and $c$.

**Examples for Oral Discussion**

1. If $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{b} = \frac{2a - 3c}{2b - 3d}$

   If $\frac{a}{b} = \frac{c}{d} = k$, then $a = bk$, $c = dk$.

   Substitute for $a$ and $c$.

2. If $\frac{a}{b} = \frac{c}{d}$ express $\frac{a + b}{a - b}$ in terms of $c$ and $d$.

3. The line $AB$ is divided internally at $P$ in the ratio $x : y$; express in terms of $x$ and $y$, (i) $PB : AB$; (ii) $AB : AP$.

4. If, in fig. 792, $\frac{AP}{AQ}$ prove that $\frac{AP}{AQ}$

   (i) $\frac{AP}{AQ}$

   (ii) find a ratio equal to $\frac{AB}{PB}$.

5. $AB$ is divided externally at $X$ in the ratio $5 : 6$ and is divided externally at $Y$ in the ratio $5 : 3$. Is $X$ nearer to $A$ or $B$? Is $Y$ nearer to $A$ or $B$?

6. Are the following in proportion:

   (i) $3\frac{1}{2}$, 5, 8, 12; (ii) 8 in., 6 deg., 12 deg., 9 in.?

**EXERCISE 83**

Express the following ratios as simply as possible:


Find the values of $x$ in Nos. 4–7.


Express in the form of equal ratios the relations, Nos. 8-10.

8. \( pq = xy \). [9] \( XA : XB = XT : T \). 10. rect. \( ABCD = \text{rect. PQRS} \).

11. A line \( AB \), 8 in. long, is divided internally at \( P \) in the ratio 2 : 3. Find \( AP \).

12. A line \( AB \), 8 in. long, is divided externally at \( Q \) in the ratio 7 : 3. Find \( AQ \).

13. A line \( AB \), 6 in. long, is divided externally at \( R \) in the ratio 2 : 7. Find \( AR \).

14. A line \( AB \), 12 cm. long, is divided internally at \( P \) in the ratio 3 : 5, externally at \( Q \) in the ratio 4 : 9, and externally at \( R \) in the ratio 8 : 3. Find the lengths of \( PQ \) and \( PR \).

15. A line \( AB \), 8 in. long, is divided internally at \( C \) and externally at \( D \) in the ratio 9 : 5; \( O \) is the mid-point of \( AB \). Prove that \( OC : OD = OB \).

16. A line \( AB \), 6 in. long, is divided internally at \( C \) and externally at \( D \) in the ratio 4 : 1; \( O \) is the mid-point of \( CD \). Find the ratio \( AO : BO \).

17. A line \( AB \), 6 in. long, is divided internally at \( P \) in the ratio 2 : 1 and externally at \( Q \) in the ratio 5 : 2. Find the ratios in which \( PQ \) is divided by \( A \) and by \( B \).

18. A line \( AB \) is divided internally at \( P \) and externally at \( Q \) in the ratio \( c : d \). If \( AB = 2a \) in., find the lengths of \( AP \) and \( AQ \).

19. A line \( AB \) is bisected at \( O \) and divided internally at \( P \) in the ratio \( x : y \). Find the ratio \( OP : AB \) in terms of \( x \) and \( y \).

(Let \( AB = 2a \) in.)

20. If \( \frac{a}{b} : \frac{c}{d} \), state ratios equal to (i) \( b : a \); (ii) \( a : c \); (iii) \( b : d \).

21. If \( \frac{a}{b} = \frac{c}{d} \), state ratios equal to (i) \( \frac{a+b}{b} : \frac{a+b}{b+d} \); (ii) \( \frac{a}{a+b} \).

If \( \frac{a}{b} = \frac{c}{d} \) prove the relations in Nos. 22-24.

22. \( \frac{b}{a} = \frac{c}{c-d} \). 23. \( \frac{a-c}{b-d} = \frac{a-c}{b-d} \).

24. \( \frac{a}{b} = \frac{a^2-c^2}{b^2-d^2} \). 25. \( \frac{a}{b} = \frac{\ldots}{\ldots} \) fill up the blank spaces in Nos. 25-27.

26. \( \frac{a-c}{b+d} = \frac{\ldots}{\ldots} \). 27. \( \frac{b}{d} = \frac{\ldots}{\ldots} \). 28. \( \frac{a}{b} = \frac{\ldots}{\ldots} \).

28. \( ABCDE \) is a straight line such that \( AB : BC : CD : DE = 1 : 3 : 2 : 5 \).

Find the ratios (i) \( AB : AE \); (ii) \( AC : CE \); (iii) \( EB : AD \).

Find also the ratios in which \( BE \) is divided internally by \( D \) and externally by \( A \). If \( BE = 4 \) in., find \( AC \).

29. \( ABCDEF \) is a straight line such that \( AB : BC : CD : DE : EF = \ldots \).

Find the ratios (i) \( AB : AF \); (ii) \( BE : CF \); (iii) \( BE : CF \).

Find also the ratios in which \( CF \) is divided externally by \( A \) and internally by \( E \). If \( BD = x \) in., find \( AE \).

30. \( ABC \) is a straight line. If \( AC : AB = n : 1 \), find \( AB : BC \).

31. \( AB \) is divided internally at \( C \) and externally at \( D \) in the ratio \( x : y \). Find (i) the ratio \( CD : AB \), (ii) the ratio in which \( D \) divides \( CD \).

32. \( ABCD, AZY \) are two straight lines such that \( AB : BC : CD = AX : XY : YZ \). Fill up the blank spaces in the following: (i) \( \frac{AB}{AC} \); (ii) \( \frac{BC}{YD} \); (iii) \( \frac{YZ}{AY} \) = \( \frac{\ldots}{\ldots} \).

**EQUAL RATIOS**

**Examples for Oral Discussion**

1. In fig. 793, \( AB \) is divided internally at \( X \) in the ratio \( 3 : 5 \), and \( XY \) is drawn parallel to \( BC \) to meet \( AC \) at \( Y \). Prove that \( AC \) is divided at \( Y \) in the same ratio \( 3 : 5 \). Find the values of the ratios \( AX : AB \) and \( AY : AC \).

If \( AB \) is divided into 8 equal parts, \( AX \) contains 3 of these parts and \( XB \) contains 5 of them.

Use the intercept theorem.
2. The side \( AB \) of \( \triangle ABC \) is divided internally at \( P \) in the ratio \( 4 : 7 \), and \( PQ \) is drawn parallel to \( BC \) to meet \( AC \) at \( Q \). What construction can you use to prove that \( AC \) is divided at \( Q \) in the same ratio, \( 4 : 7 \)?

What are the values of the ratios \( PB : AB \) and \( QC : AC \)?

3. Mark on squared paper the points \( A (0, 0) \), \( B (3, 0) \), \( C (2, 3) \), the unit being 1 in., and draw \( \triangle ABC \). Draw the line joining \((0, 1-6)\) to \((3, 1-6)\) and let it cut \( CA, CB \) at \( X, Y \).

Into how many equal parts do the printed lines parallel to \( AB \) divide \( CX \) and \( XA \)?

What are the values of the ratios \( \frac{CX}{CY} \); \( \frac{XA}{YB} \); \( \frac{CX}{CY} \); \( \frac{CA}{CB} \)?

4. If in fig. 794 (i) \( XY \) is parallel to \( BC \), prove that \( \frac{AX}{AY} = \frac{XB}{YC} \).

Suppose that \( \frac{AX}{XB} \) can be expressed in the form \( \frac{p}{q} \), where \( p \) and \( q \) are integers.

5. If with the data of No. 4, \( \frac{AX}{AB} = \frac{AY}{AC} \), express the ratios in terms of \( p \) and \( q \).

6. What ratio of lengths in fig. 793 is equal to \( YC : AC \)?

7. If in fig. 794 (ii), \( XY \) is parallel to \( BC \), prove that \( AX : XB = AY : YC \).

The results established in Nos. 4, 7 may be expressed as follows:

A straight line drawn parallel to one side of a triangle divides the other sides proportionally.

8. If in fig. 795, \( AX : XB = AY : YC \), prove that \( XY \) is parallel to \( BC \).

Let the line through \( X \) parallel to \( BC \) cut \( AC \) at \( P \).

Explain why \( AP : AC = AY : AC \). This proves that \( P \) is the same point as \( Y \).

9. If in fig. 794 (ii), \( AX : XB = AY : YC \), prove that \( XY \) is parallel to \( BC \).

Use the construction and method of No. 8.

The results established in Nos. 8, 9 may be expressed as follows:

If two sides of a triangle are divided in the same ratio, both internally or both externally, the straight line joining the points of section is parallel to the third side.

These results may also be obtained by using the theorem that the area of a triangle is measured by half the product of the base and altitude.
12. In fig. 798, express as the ratio of two lengths,
\[ \frac{\triangle AXY}{\triangle BXY}; \; \frac{\triangle XAY}{\triangle XCY} \]
What can you say about these ratios if \( XY \) is parallel to \( BC \)?

13. Repeat No. 12 for fig. 799 (i).

14. Repeat No. 12 for fig. 799 (ii).

**NUMERICAL EXAMPLES**

**EXERCISE 84**

[Arrows indicate that lines are given parallel.]

Nos. 1–9 refer to fig. 800.
Name two ratios equal to each of the ratios in Nos. 1–6:

1. \( \frac{OA}{AB} \)
2. \( \frac{OE}{OF} \)
3. \( \frac{CD}{OD} \)
4. \( \frac{OA}{OC} \)
5. \( \frac{OD}{OF} \)
6. \( \frac{AB}{EF} \)

7. If \( OA = 10 \text{ cm}, \; AB = 4.5 \text{ cm}, \; OD = 7 \text{ cm} \), find \( CD \).
8. If \( OB = 10 \text{ cm}, \; OA = 12 \text{ cm} \), find \( OC \).
9. If \( OA = 12 \text{ cm}, \; AB = 9 \text{ cm}, \; OC = 8 \text{ cm}, \; EF = 4.5 \text{ cm}, \) find \( CD \) and \( OF \).

[This exercise is continued at the foot of p. 463.]
THEOREM 70

If two triangles are of equal altitude, the ratio of their areas is equal to the ratio of their bases.

*Fig. 896*

Given two triangles $ABC$, $XYZ$ in which the altitudes $AH$, $XK$ are equal.

To prove that

$$\frac{\triangle ABC}{\triangle XYZ} = \frac{BC}{YZ}$$

Proof. The area of a triangle is measured by half the product of the measures of its base and altitude.

$$.: \triangle ABC = \frac{1}{2} BC \cdot AH$$

and

$$\triangle XYZ = \frac{1}{2} YZ \cdot XK;$$

$$\triangle ABC = \frac{1}{2} BC \cdot AH$$

$$\triangle XYZ = \frac{1}{2} YZ \cdot XK$$

But

$$AH = XK \text{ given,}$$

$$\triangle ABC \sim \triangle XYZ$$

Note. This proof is only valid if the measures of the altitude and base are commensurable.

---

Corollary. If $BCD$ is a straight line and if $A$ is any point not on this line,

$$\frac{\triangle ABC}{\triangle ACD} = \frac{BC}{CD}$$

and

$$\frac{\triangle ABC}{\triangle ABD} = \frac{BC}{BD}.$$
THEOREM 71
If a straight line is drawn parallel to one side of a triangle, it divides the other sides, produced if necessary, proportionally.

Given a triangle ABC and a line parallel to BC cutting AB, AC, produced if necessary, at X, Y.

To prove that \( \frac{AX}{XB} = \frac{AY}{YC} \).

Construction. Join BY, CX.

Proof. The perpendicular from Y to AB is an altitude of each of the triangles AXY, BYX,

\[ \frac{AX}{\triangle AXY} \]
\[ \frac{XB}{\triangle BXY} \]

Similarly, \( \frac{AY}{\triangle AYX} \)
\[ \frac{YC}{\triangle CYX} \]

But \( \triangle BXY = \triangle CYX \).

\[ \frac{AX}{\triangle AXY} \]
\[ \frac{AY}{\triangle AYX} \]

\[ \frac{XB}{\triangle BXY} \]
\[ \frac{YC}{\triangle CYX} \]

Corollary. If a line XY parallel to BC cuts AB, AC at X, Y, then

\[ \frac{AX}{AY} \]
\[ \frac{XB}{YC} \]

\( \frac{AB}{AC} \) and \( \frac{AB}{AC} \)

EQUAL RATIOS
THEOREM 72
If two sides of a triangle are divided in the same ratio, both internally or both externally, the straight line joining the points of section is parallel to the third side.

Given a triangle ABC and two points X, Y dividing AB, AC, both internally or both externally, such that

\[ \frac{AX}{AY} \]
\[ \frac{XB}{YC} \]

To prove that XY is parallel to BC.

Construction. Join BY, CX.

Proof. The perpendicular from Y to AB is an altitude of each of the triangles AXY, BYX,

\[ \frac{AX}{\triangle AXY} \]
\[ \frac{XB}{\triangle BXY} \]

Similarly, \( \frac{AY}{\triangle AYX} \)
\[ \frac{YC}{\triangle CYX} \]

But \( \frac{AX}{AY} \) \( \frac{XB}{YC} \) given, \( \frac{\triangle AXY}{\triangle AYX} \)

\[ \frac{\triangle BXY}{\triangle CYX} \]

But these triangles are on the same base XY and on the same side of it,

\( \frac{AX}{AY} \)
\( \frac{XB}{YC} \)

\( \frac{AB}{AC} \)

\( \frac{XY}{BC} \) is parallel to BC.
EXERCISE 85

[Arrows indicate that lines are given parallel.]

1. With the data of fig. 800, p. 460, prove that and complete the relation ... What can you say about the lines \( AE \) and \( BF \)? Give reasons.

2. With the data of fig. 801, p. 461, prove that and complete the relation ...

[3] Three parallel lines \( AX, BY, CZ \) cut two lines \( ABC, XYZ \). Prove that \( AB : BC = XY : YZ \).

4. With the data of fig. 810, prove that \( QP \) is parallel to \( BC \). [What ratios must you try to prove equal?]

[5] \( P \) is any point on the side \( AB \) of the quadrilateral \( ABCD \); \( PX, PY \) are drawn parallel to \( AC, AD \), to cut \( BC, BD \) respectively at \( X, Y \). Prove that \( XY \) is parallel to \( CD \). [What ratios must you try to prove equal?]

6. If in fig. 811, \( BD = EC \), prove that \( PQ \) is parallel to \( BC \).

[7] \( AB, DC \) are the parallel sides of a trapezium \( ABCD \); \( H, K \) are points on \( AD, BC \) such that \( AH : HD = BK : KC \). Prove that \( HK \) is parallel to \( AB \).

8. \( ABC \) is a triangle; \( P, Q \) are points on \( AB, AC \) such that \( AP = \frac{1}{2} AB \) and \( CQ = \frac{1}{2} CA \). Prove that the line through \( C \) parallel to \( PQ \) bisects \( AB \).

9. In fig. 812, \( PBQ \) is a straight line. Prove that

\[
PB : BQ = QR : RC.
\]

[10] In fig. 812, \( PBQ \) is a straight line. Prove that

\[
PQ : QC = QB : CR.
\]

11. In fig. 813, \( XABC \) and \( XPQ \) are straight lines. Prove that \( \frac{XA}{XB} = \frac{XB}{XC} \).

12. In fig. 814, \( AD \) produced cuts \( BC \) at \( P \). Prove that

\[
\frac{\triangle ABD}{BP} = \frac{\triangle ACD}{PC}.
\]

13. In fig. 814, \( AD \) produced cuts \( BC \) at \( P \). Prove that

\[
\frac{\triangle ABC}{AP} = \frac{\triangle ABC}{DP}.
\]

14. A variable straight line \( RQP \) passes through a fixed point \( A \) and meets a fixed line \( BC \) at \( P \). If \( AQ = \frac{1}{2} AP \) and if \( RA = QP \), find (i) the locus of \( Q \), (ii) the locus of \( R \).

15. \( ABC \) is a triangle; three parallel lines \( AP, BQ, CR \) meet \( BC, CA, AB \) produced if necessary, at \( P, Q, R \) respectively. Prove that

\[
\frac{PB \times CQ \times AR}{QA \times RB} = 1.
\]

16. \( P \) is any point on the median \( AD \) of \( \triangle ABC \); \( AD \) is produced to \( Q \) so that \( PD = DQ \); \( BP, CQ \) produced meet \( AC, AB \) at \( E, F \) respectively. Prove that

(i) \( AP = AQ = AF : AB \);

(ii) \( EF \) is parallel to \( BC \).

17. In fig. 815, the lines \( AB, AC \) and the point \( D \) are given. Construct the line \( PDQ \) so that \( PD = DQ \).

*18. \( ABC \) is a triangle; a straight line cuts \( BC \) produced, \( CA, AB \) at \( P, Q, R \) respectively; \( CX \) is drawn parallel to \( PQ \) meeting \( AB \) at \( X \). Prove that

\[
\frac{BP}{BR} = \frac{BC \times CQ \times AR}{QA \times RB} = 1.
\]

This is called Menelaus' Theorem.
19. In fig. 816, if $AH = HB$ and $AK = 2KC$, prove that (i) $\triangle BOH = 3\triangle COK$; (ii) $CO = OH$.

20. In fig. 816, if $HB = \frac{1}{4} AB$ and $KC = \frac{1}{4} AC$, prove that (i) $BO = OK$; (ii) $CO = 2OH$.

21. If in fig. 816, $AO$ produced cuts $BC$ at $N$, prove that

$$\frac{BN \times CK \times AH}{NC \times KA \times HB} = 1.$$  
[Use the result in No. 12.]

This is known as Ceva's Theorem.

**Bisectors of an Angle of a Triangle**

**Examples for Oral Discussion**

1. In fig. 817, if the internal bisector of $\angle BAC$ cuts $BC$ at $D$, prove that

$$\frac{BD}{BA} = \frac{DC}{AC}.$$  

[Fig. 817]

Draw the perpendiculars $DH$, $DK$ from $D$ to $AB$, $AC$.

(i) Explain why $DH = DK$.

(ii) Express the ratio $\frac{\triangle ABD}{\triangle ACD}$ in two different ways.

2. In fig. 817, if $BC$ is divided internally at $D$ in the ratio $AB : AC$, prove that $AD$ is the internal bisector of $\angle BAC$.

Use the construction in No. 1 and express $\frac{\triangle ABD}{\triangle ACD}$ in two ways.

3. In fig. 818, if the external bisector of $\angle BAC$ cuts $BC$ produced at $D$, prove that

$$\frac{BD}{BA} = \frac{DC}{AC}.$$  

Draw the perpendiculars $DH$, $DK$ from $D$ to $AB$, $AC$, produced if necessary.

Express the ratio $\frac{\triangle ABD}{\triangle ACD}$ in two different ways.

4. In fig. 818, if $BC$ is divided externally at $D$ in the ratio $AB : AC$, prove that $AD$ is the external bisector of $\angle BAC$.

Use the construction in No. 3 and express $\frac{\triangle ABD}{\triangle ACD}$ in two ways.

5. Construct a line $BC$ of length $\sqrt{3}$ in. Perform the following construction for dividing $BC$ internally and externally in the ratio $5 : 3$ and prove that it is correct.

Draw $\triangle ABC$ so that $AB = 2.5$ in., $AC = 1.5$ in. Draw the internal and external bisectors of $\angle BAC$ and let them cut $BC$ at $P$, $Q$. Then $P$ and $Q$ are the required points of section.
NUMERICAL EXAMPLES

EXERCISE 86

1. In \(\triangle ABC\), \(BC = 6\) in., \(CA = 3\) in., \(AB = 5\) in. If the internal and external bisectors of \(\angle BAC\) meet \(BC\) and \(BC\) produced at \(X\) and \(Y\), find the lengths of \(XC\) and \(XY\).

2. \[ \text{In } \triangle ABC, \text{BC} = 5 \text{ cm., CA = 4 cm., AB = 6 cm. If the} \]
   \[\text{internal and external bisectors of } \angle BAC \text{ meet } BC \text{ and } BC \text{ produced at } P \text{ and } Q, \text{find the lengths of BP and BQ, and prove} \]
   \[\frac{1}{BP} + \frac{1}{BQ} = \frac{2}{BC}. \]

3. The perimeter of \(\triangle ABC\) is 45 in.; the internal bisector of \(\angle BAC\) cuts \(BC\) at \(P\) and the internal bisector of \(\angle ACB\) cuts \(AB\) at \(Q\). If \(BP = 9\) in., \(CP = 6\) in., find \(AQ\).

4. \[ \text{In } \triangle ABC, \text{AB} = 4 \text{ in., BC = 3 in., } \angle ABC = 90^\circ. \text{ If the} \]
   \[\text{bisector of } \angle ACB \text{ cuts } AB \text{ at } R, \text{find } CR. \]

5. \[ \text{In } \triangle ABC, \text{AB} = 12 \text{ cm., BC = 15 cm., CA = 8 cm. If } P \text{ is a point on} \]
   \[\text{BC such that } BP = 9 \text{ cm. Prove that AP bisects} \]
   \[\angle BAC. \text{ If the external bisector of } \angle ACB \text{ cuts } BC \text{ produced at } Q, \text{and if } D \text{ is the mid-point of } BC \text{, prove that} \]
   \[DP \cdot DQ = DC^2. \]

6. If the bisector of \(\angle BAC\) cuts \(BC\) at \(P\), and if the lengths of \(BC, CA, AB\) are \(a, b, c\) units, find the length of \(BP\) in terms of \(a, b, c\).

7. The internal and external bisectors of \(\angle BAC\) meet \(BC\) and \(BC\) produced at \(P\) and \(Q\). If \(BP = 5\) in., \(PC = 3\) in., find \(CQ\).

8. \[ \text{In } \triangle ABC, \text{AB} = 6 \text{ cm., AC = 10 cm., and the bisector of} \]
   \[\angle BAC \text{ cut BC at } P. \text{ If the area of } \triangle ABC \text{ is } 24 \text{ sq. cm., find the area of } \triangle ABP. \]

9. \[\text{ABCD is a straight line such that AB = 14 cm., BC = 6 cm., CD = 13 cm.; } K \text{ is a point such that } KA = 21 \text{ cm., } KC = 9 \text{ cm.} \]
   \[\text{Prove that } \angle BKD \text{ is a right angle.} \]

10. \(A, B\) are fixed points such that \(AB = 2.1\) in.; \(P\) is a variable point such that \(PA : PB = 5 : 2. \text{ If } AB \) is divided internally and externally at \(C\) and \(D\) in the ratio \(3 : 2\), prove that the locus of \(P\) is the circle on \(CD\) as diameter. \text{ Find the radius of this circle.} \]

11. \(ABCD\) is a rectangular sheet of paper; \(AB = 4\) in., \(BC = 3\) in. The edge \(BC\) is folded along \(BD\) and the corner is then cut off along the crease. \text{ Find the area of the remainder.} \]

12. \[ \text{In } \triangle ABC, \text{AB} = 6 \text{ in., AC = 4 in. The bisector of } \angle BAC \]
   \[\text{meets the median } BE \text{ at } O. \text{ If the area of } \triangle ABC \text{ is } 8 \text{ sq. in.,} \]
   \[\text{find the area of } \triangle AOB. \]

13. \[ \text{If } I \text{ is the in-centre of } \triangle ABC \text{ and if AI cuts BC at } P, \text{ and if the lengths of } BC, CA, AB \]
   \[\text{are } a, b, c \text{ units, find the ratio } A1 : IP \text{ in terms of } a, b, c. \]

14. \[ \text{AD is a median of } \triangle ABC \text{ and the bisector of } \angle ABC \text{ cuts } \]
   \[\text{AD at } P. \text{ If } BC = 16 \text{ cm., CA = 11 cm., AB = 13 cm., find DP.} \]

15. \[ \text{APB, CPD are intersecting chords of the circle ACBD} \]
   \[\text{and } C \text{ is the mid-point of the arc } AB. \text{ If } AP = 24 \text{ cm.,} \]
   \[PB = 16 \text{ cm., find the ratio } DA : DB. \]
   \[\text{Show how to construct another point E on the circle such that } \]
   \[EA : EB = DA : DB. \]

16. \[ \text{In a circle of radius } 1.6 \text{ in. inscribe a triangle } ABC \text{ such that } BC = 2.5 \text{ in. and} \]
   \[\text{AB : AC = 4 : 1. [Two solutions.]} \]

17. \[ \text{Construct } \triangle ABC, \text{given that } AB = 4 \text{ in., } AC = 2BC \text{ and} \]
   \[\angle ACB = 120^\circ. \]

18. \[ \text{ABCD is a quadrilateral such that } AB = 6 \text{ cm., } BC = 8 \text{ cm.,} \]
   \[CD = 12 \text{ cm., DA = 9 cm. What can you say about the point of intersection of (i) the bisectors of } \angle ABC, \angle ADC, \text{ (ii) the} \]
   \[\text{bisectors of } \angle BAD, \angle BCD? \]
THEOREM 73 (First Proof)

If the vertical angle of a triangle is bisected internally or externally by a straight line which cuts the base or the base produced, it divides the base internally or externally in the ratio of the other sides of the triangle.

Fig. 819

Given a triangle $ABC$ and the line $AD$ which bisects $\angle BAC$ internally, fig. 819 (i), or externally, fig. 819 (ii), and cuts $BC$ or $BC$ produced at $D$.

To prove that

$$\frac{BD}{DC} = \frac{AB}{AC}.$$ 

Construction. From $D$ draw the perpendiculars $DH$, $DK$ to $AB$, $AC$, produced if necessary.

Proof. Since $D$ is a point on the bisector of one of the angles formed by $AB$, $AC$, $D$ is equidistant from $AB$ and $AC$.

\[\therefore DH = DK \quad \text{locus theorem}.\]

But $DH$, $DK$ are altitudes of $\triangle DAB$, $\triangle DAC$,

\[\therefore \frac{AB}{AC} = \frac{\triangle DAB}{\triangle DAC}.
\]

Also the perpendicular from $A$ to $BC$ is an altitude of each of the triangles $\triangle ABD$, $\triangle ADC$,

\[\therefore HD = DK.\]

\[\therefore AD \text{ bisects } \angle BAC \text{ internally or externally}.\]

THEOREM 74 (First Proof)

If a straight line through the vertex of a triangle divides the base internally or externally in the ratio of the other sides, it bisects the vertical angle internally or externally.

Fig. 820

Given a triangle $ABC$ and a point $D$ on $BC$, fig. 820 (i), or on $BC$ produced, fig. 820 (ii), such that

$$\frac{BD}{DC} = \frac{AB}{AC}.$$ 

To prove that $AD$ bisects $\angle BAC$ internally or externally.

Construction. From $D$ draw the perpendiculars $DH$, $DK$ to $AB$, $AC$, produced if necessary.

Proof. The perpendicular from $A$ to $BC$ is an altitude of each of the triangles $\triangle ABD$, $\triangle ADC$,

\[\therefore \frac{BD}{DC} = \frac{\triangle ABD}{\triangle ADC}.
\]

But $DH$, $DK$ are altitudes of $\triangle ABD$, $\triangle ADC$,

\[\therefore \frac{BD}{DC} = \frac{AB}{AC} \quad \text{given},\]

\[\therefore DH = DK.\]

\[\therefore AD \text{ bisects } \angle BAC \text{ internally or externally}.\]
THEOREM 73 (Second Proof)

If the vertical angle of a triangle is bisected internally or externally by a straight line which cuts the base or the base produced, it divides the base internally or externally in the ratio of the other sides of the triangle.

Given a triangle ABC and the line AD which bisects \( \angle BAC \) internally, fig. 821 (i), or externally, fig. 821 (ii), and cuts BC or BC produced at D.

To prove that \[ \frac{BD}{DC} = \frac{AB}{AC}. \]

Construction. Through C draw a line parallel to DA to cut BA, produced if necessary, at X.

Proof. With the notation in the figures,

- \( m_1 = p \) corr. \( \angle s, DA \parallel CX \),
- \( m_2 = q \) alt. \( \angle s, DA \parallel CX \),
- \( m_1 = m_2 \) given,

\[ \therefore p = q. \]

\( \therefore \triangle AXC \) is isosceles and \( AX = AC \).

Since \( DA \) is parallel to \( CX \),

- \( BD \parallel BA \)
- \( DC \parallel AX \)
- \( BD \parallel BA \)
- \( DC \parallel AC \)

\[ \therefore \triangle AXC \]

Note. If \( AB = AC \), the external bisector of \( \angle BAC \) is parallel to BC.

THEOREM 74 (Second Proof)

If a straight line through the vertex of a triangle divides the base internally or externally in the ratio of the other sides, it bisects the vertical angle internally or externally.

Given a triangle ABC and a point D on BC, fig. 822 (i), or on BC produced, fig. 822 (ii), such that

\[ \frac{BD}{DC} = \frac{AB}{AC}. \]

To prove that \( AD \) bisects \( \angle BAC \) internally or externally.

Construction. Through C draw a line parallel to DA to cut BA, produced if necessary, at X.

Proof. With the notation in the figures, since \( DA \) is parallel to \( CX \),

- \( BD \parallel BA \)
- \( DC \parallel AX \)
- \( BD \parallel AB \)
- \( DC \parallel AC \)

\[ \therefore AX = AC, \]

\[ \therefore p = q, \quad \text{base } \angle s, \text{isos. } \triangle. \]

But \( m = p \) corr. \( \angle s, DA \parallel CX \),
and \( n = q \) alt. \( \angle s, DA \parallel CX \),

\[ \therefore m = n. \]

\[ \therefore AD \text{ bisects } \angle BAC, \text{ either internally or externally.} \]
EXERCISE 87

Nos. 1–5 refer to fig. 823 in which AD, AE are the bisectors of ∠BAC and cut BC at D, E.

1. If the line through D parallel to BA cuts CA at N, prove that BA : AN = AC : CN.

2. Prove that BD : DC = BE : CE.

3. If the line through C parallel to AB cuts AD produced at K, prove that DK = 2AC.

4. If the line through E parallel to AB cuts AC produced at P, prove that PA : PC = AB : AC.

5. If circles with B, C as centres are drawn through D and cut BA, CA at G, K, prove that GK is parallel to BC.

6. AX is a median of △ABC; the bisectors of ∠AXB, ∠AXC meet AB, AC at H, K. Prove that HK is parallel to BC.

7. A straight line cuts four lines OP, OQ, OR, OS at P, Q, R, S. If OP : PQ = QR : RS, prove that OR bisects ∠QOS.

8. H is any point inside △ABC; the bisectors of ∠BHC, ∠CHA, ∠AHB cut BC, CA, AB respectively at X, Y, Z. Prove that BX : CY : AZ = 1.

9. ABCD is a parallelogram. If the bisector of ∠BAD meets BD at X and CD at Y, prove that AX : XY = DC : DA.

10. ABCD is a quadrilateral in which AB = AD. The bisectors of ∠CAB, ∠CAD meet CB, CD respectively at H, K. Prove that HK is parallel to BD.

11. Two circles, centres A, B, touch at O. Any line parallel to AB cuts the circles at P, Q respectively. If AP and BQ are produced to meet at K, prove that KO is one of the bisectors of ∠AKB.

12. ABCD is a quadrilateral such that ∠B = ∠C, and AC bisects ∠BAD. If BA and CD, when produced, meet at E, prove that AD : DC = AE : BE.

13. D is the mid-point of the base BC of △ABC; AD is the bisector of ∠ABC. If the internal bisectors of ∠BDE, ∠CDE meet AB produced, AC produced at H, K respectively, prove that HK is parallel to BC.

14. The diagonals AC, BD of the cyclic quadrilateral ABCD cut at E. If AB = BC, prove that DA : DC = AE : EC.

15. AB is a chord of a circle perpendicular to a diameter CD; E is any point on AB; CE, DE when produced meet the circle again at P, Q respectively. Prove that PA : PB = QA : QB.

16. AB, AC are equal chords of a circle and ∠BAC > 60°. If the tangent at C meets BA produced at T, prove that CB : CA = TC : TA.

17. Prove the result in No. 16 if ∠BAC < 60°. [In this case, the tangent at C meets AB produced at T.]

18. Apollonius' Circle. A, B are fixed points and P is a variable point such that PA : PB is constant. Prove that

(i) the internal and external bisectors of ∠APB cut AB at fixed points,

(ii) the locus of P is a circle.

If PA : PB = 3 : 2 and AB = 6 cm., calculate the diameter of this circle.

19. The tangent at a point A of a circle, centre O, meets a radius OB at T; AD is the perpendicular from A to OB. Prove that DB : BT = AD : AT.

20. ABCD is a quadrilateral. If the bisectors of ∠DAB, ∠DCB meet DB, prove that the bisectors of ∠ABC, ∠ADC meet on AC.

21. Two circles touch internally at O; a chord OP of the larger circle touches the smaller at R. Prove that OP : OQ = PR : PQ.

22. ABCD is a parallelogram: the bisector of ∠BAD meets BD at K; the bisector of ∠ABC meets AC at L. Prove that KL is parallel to AB.

23. In △ABC, ∠BAC = 90° and AD is an altitude. If the bisector of ∠ABC meets AD, AC at L, K, prove that AL : LD = CK : KA.
CONSTRUCTION 19

Divide a given finite straight line in a given ratio, (i) internally, (ii) externally.

Given two lines of lengths \( p, q \) units and a finite line \( AB \).

To construct (i) a point \( X \) in \( AB \) such that \( \frac{AX}{XB} = \frac{p}{q} \);
(ii) a point \( X' \) in \( AB \) produced such that \( \frac{AX'}{BX'} = \frac{p}{q} \).

(i) Construction. Draw any line \( AC \) and cut off from it \( AH = p \) units, \( HK = q \) units. Join \( KB \). Through \( H \) draw \( HX \) parallel to \( KB \) to cut \( AB \) at \( X \).

Then \( AB \) is divided internally at \( X \) in the ratio \( p : q \).

Proof. Since \( XH \) is parallel to \( BK \),
\[
\frac{AX}{XB} = \frac{AH}{HK} = \frac{p}{q}.
\]

(ii) Construction. Draw any line \( AC \) and cut off from it \( AH = p \) units. From \( HA \) cut off \( HK' = q \) units. Join \( KB' \). Through \( H \) draw \( HX' \) parallel to \( KB' \) to meet \( AB \) produced at \( X' \).

Then \( AB \) is divided externally at \( X' \) in the ratio \( p : q \).

Proof. Since \( X'H \) is parallel to \( BK' \),
\[
\frac{AX'}{BX'} = \frac{AH}{HK'} = \frac{p}{q}.
\]

CONSTRUCTION 20

Construct a fourth proportional to three given lines.

Given three lines of lengths \( a, b, c \) units.

To construct a line of length \( d \) units such that \( \frac{a}{b} = \frac{c}{d} \).

Construction. Draw any two lines \( OX, OY \).

From \( OX \) cut off \( OP, OQ \) such that \( OP = a \) units, \( OQ = b \) units.

From \( OY \) cut off \( OR, OS \) such that \( OR = c \) units. Join \( PR \). Through \( Q \) draw \( QS \) parallel to \( PR \) to meet \( OY \) at \( S \).

Then \( OS \) is a fourth proportional to \( a, b, c \).

Proof. Since \( PR \) is parallel to \( QS \),
\[
\frac{OP}{OS} = \frac{OR}{OS'} = \frac{a}{c}.
\]

Note: Constructing a third proportional to two given lines of lengths \( a, b \) units is the same as constructing a fourth proportional to three lines of lengths \( a, b, c \) units. Therefore the method of Construction 20 gives the required result.
EXERCISE 88

1. Draw a line $AB$ and divide it internally in the ratio $2:3$.
2. Draw a line $AB$ and divide it externally (i) in the ratio $5:3$, (ii) in the ratio $3:5$.
3. Draw a line $AB$ and divide it internally and externally in the ratio $4:7$.
4. Given a line $AB$ and divide it internally and externally in the ratio $1.4$ cm.
5. Draw a line $AB$ and construct points $P$, $Q$ on it such that $AP: PQ: QB = 2: 6 : 3$.
6. Draw a line $AB$ and construct a point $X$ on $AB$ and a point $Y$ on $AB$ produced such that $AX: XB: BY = 4: 5 : 2$.
7. Construct and measure a fourth proportional to lines of lengths $4$, $5$, $6$ cm.
8. Construct and measure a third proportional to lines of lengths $5$, $6$ cm.
9. Use a construction to solve $\frac{x}{3} = \frac{7}{5}$.
10. Use a construction to find $x$ and $y$ such that $\frac{x}{3} = \frac{7}{5}$ and $x + y = 11$.

Find graphically the values of the following: $-\frac{3}{2}$, $\frac{5}{9}$, $\frac{12}{4}$, $\frac{10}{7}$, $\frac{7}{12}$, $\frac{5}{13}$, $\frac{8}{2.7}$.

11. $23 \times 5 = 9$
12. $23 \div 5 = 9$
13. $23 \times 2 = 7$.

14. Draw any triangle $ABC$ and a line $PQ$. Construct a triangle such that its perimeter equals $PQ$ and its sides are in the ratio $BC : CA : AB$.
15. Draw an angle $BAC$ and take a point $O$ on the bisector of $\angle BAC$, construct a line $POQ$ cutting $AB$, $AC$ at $P$, $Q$ so that $PO: OQ = 3: 5$.
16. Draw an equilateral triangle $ABC$, side $6$ cm., and take a point $N$ on $BC$ so that $BN = 1.5$ cm. Construct a rectangle equal in area to $\triangle ABC$ and such that one side is equal to $AN$. Measure an adjacent side.
17. Draw a line $AB$ 3 inches long and construct a point $P$ on $AB$ so that $AP = 2PB$.
18. Draw a line $AB$ 3 cm. long and construct a point $Q$ on $AB$ produced so that $AQ = 3QB$.

SIMILAR TRIANGLES

Similarity

The meaning of similarity has already been discussed, see pp. 70-72, where it was shown that to each test for the congruence of two triangles there corresponds a test for similarity. The tests for similarity were obtained by considering what groups of measurements of the sides and angles of a given triangle must be taken in order to draw elsewhere a triangle of the same shape but of any convenient size, i.e. in order to make a scale-drawing of the given triangle. Proofs that these tests are correct are given in Theorems 75-77.

Definitions. (1) Two polygons are said to be equiangular to one another if the angles of the first polygon, taken in order, are respectively equal to the angles of the second polygon taken in order.

(2) Two polygons are said to be similar to one another if (i) the polygons are equiangular to one another and if also (ii) the ratio of any side of the first polygon to the corresponding side of the second is the same, or can be proved to be the same, for every pair of corresponding sides, i.e. if corresponding sides are proportional.

It is obvious that polygons which are equiangular to one another need not be similar; consider, for example, a square and any rectangle. It is also obvious that polygons in which corresponding sides are proportional need not be similar; consider, for example, a square and any rhombus.

But the test for similarity of two triangles show that (i) triangles equiangular to one another must be similar; and (ii) triangles in which corresponding sides are proportional must be similar.

Notation. Similar triangles and polygons should always be named so that the order of the letters indicates the correspondence between the two figures.

Thus, the statement that $\triangle ABC \sim XYZ$ are similar means that $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$ and that $\frac{BC}{CA} = \frac{AB}{XY}$.
Examples for Oral Discussion

1. In fig. 827, if $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$, prove that
   $$\frac{AB}{AC} \times \frac{XY}{XZ} = \frac{AB}{BC} \times \frac{XY}{YZ}$$
   and that
   $$\frac{AB}{BC} = \frac{XY}{YZ}$$
   [Diagram]
   Fig. 827

From $AB$, $AC$ cut off $AH = XY$, $AK = XZ$; join $HK$.

(i) Explain why $\angle AHK = \angle XYZ$.
(ii) Explain why $HK \parallel BC$. What follows?
(iii) What construction is needed to prove $\frac{AB}{BC} = \frac{XY}{YZ}$?

2. In fig. 828, if $\frac{AB}{BC} = \frac{CA}{AX}$, prove that
   $$\angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z.$$  
   With the notation in fig. 828, take a point $K$ such that $b_2 = b_1$ and $c_2 = c_1$.
   [Diagram]
   Fig. 828

(i) Using the result proved in No. 1, write down two ratios equal to $\frac{BC}{YZ}$. What follows from the data?
(ii) Prove that $\triangle ABC$ are congruent.

3. In fig. 829, if $\angle A = \angle X$ and if $\frac{AB}{AX} = \frac{AC}{AY}$ prove that $\triangle ABC$ are similar.
   From $AB$, $AC$ cut off $AH = XY$, $AK = XZ$; join $HK$.
   (i) Explain why $HK \parallel BC$.
   (ii) Prove that $\triangle ABC$ are congruent.

4. In fig. 830, the triangles are not drawn accurately. The data are shown in the figure.
   [Diagram]
   Fig. 830
   (i) Explain why the triangles are similar and state the fact in the form $\triangle ABC \ldots$ are similar.
   (ii) Find the values of $a$, $b$, $c$, $d$.
   (iii) Sketch a triangle $DEF$ such that $EF = 6$ cm., $FD = 4.8$ cm., $DE = 4$ cm. Is this triangle similar to $\triangle ABC$? If so, give the reason and state the fact in the proper form.
5. What is the simplest way of proving that the triangle whose sides are 5.1 in., 6.8 in., 8.5 in. is right-angled?

![Diagram of triangle with sides 5.1, 6.8, and 8.5 inches]

Fig. 831

6. In fig. 831, $\triangle AKB$ and $\triangle CKD$ are straight lines.
   (i) Explain why the triangles are similar and state the fact in the proper form.
   (ii) Find the value of $e$.

7. If $\triangle ADK$ are similar, name one ratio equal to $\frac{AD}{QX}$.
   (i) If $\frac{BQ}{AK}$, name two ratios equal to $\frac{QB}{AK}$.

   Draw any triangle $\triangle ABC$ and mark points $D$, $E$ on $AB$, $AC$ respectively so that $\angle ADE = \angle C$. Complete the sentence, $\angle ADE = \angle C$. Complete the sentence, $\triangle ABC$ are similar. Name one ratio equal to $\triangle AD$ and two ratios equal to $DE : CB$.

8. In fig. 832, the chords $AD$, $GD$ cut at $K$ and the chords $AD$, $CB$ are produced to cut at $P$. Name in the proper form, with reasons, two pairs of similar triangles.
   Name one ratio equal to $\frac{KA}{KD}$ and two ratios equal to $\frac{PA}{PC}$.

![Diagram of chords cutting at K and P]

Fig. 832

9. A pole 10 ft. high casts a shadow $3\frac{1}{2}$ ft. long and at the same time a tower casts a shadow 42 ft. long, on level ground. Find the height of the tower.

Fig. 833

Exercises 89

NOS. 1-4 REFER TO FIG. 833, DIMENSIONS IN INCHES.

1. If $BH = 3''$, find $BK$, $HK$.
2. If $BK = 4''$, find $BH$, $HK$.
3. If $HK = 4''$, find $BH$, $BK$.
4. If $KC = 4''$, find $AH$, $HK$.
5. Find the value of $p : q$ in fig. 834.

![Diagram of triangle with ratios]

Fig. 834

6. Find the value of $y : x$ in fig. 835.

Find the marked lengths in Nos. 7-10, unit 1 cm.:

7. [Diagram of marked lengths]

8. [Diagram of marked lengths]

9. [Diagram of marked lengths]

10. [Diagram of marked lengths]
[12] In a photograph of a chest of drawers, the height measures 6 in. and the breadth 3 2/3 in. If the height is 7 1/2 ft, find the breadth.

13. A light is 9 ft. above the floor; a ruler 8 in. long is held horizontally 4 ft. above the floor. Find the length of its shadow.

[14] The slope of a railway is marked as 1 in 60. What height in feet does it climb in 1/4 mile?

15. In the quadrilateral ABCD, AB is parallel to DC; AB = 8 cm., AD = 3 cm., DC = 5 cm. If AD, BC produced meet at P, find PD.

[16] Show that the line joining the points (1, 1), (4, 2) is parallel to and half the line joining the points (0, 0), (6, 2).

[17] The bases of two equiangular triangles are 4 in., 6 in., and are corresponding sides. If the height of the first triangle is 5 in., find the area of the second.

18. A line parallel to BC meets AB, AC at X, Y; BC = 8 in., XY = 5 in.; the lines BC, XY are 2 in. apart. Find the area of \( \triangle AXY \).

19. The diameter of the base of a cone is 9 in. and its height is 15 in. Find the diameter of a section parallel to the base and 3 in. from it.

20. The diameter of the base of a cone is 8 in.; the diameter of a parallel section, 3 in. from the base, is 6 in. Find the height of the cone.

Find the marked lengths in Nos. 21–24, unit 1 cm.

[21]

[22]

[23]

[24]

26. Explain why there are two similar triangles in fig. 844 (i) and in fig. 844 (ii). Name them in the proper way. What angle is equal to \( \angle KCA \)? What can you say about the four points A, B, C, D? What is the length of BD, unit 1 in.?
35. In fig. 847, if \( AB = 6 \) cm., \( AQ = 20 \) cm., \( AE = 5 \) cm., 
\( CD = 9 \) cm., find \( QD \) and \( AP \).

36. \( ABCD \) is a quadrilateral in which \( \angle ABC = 90^\circ = \angle ACD \), 
\( AC = 5 \) in., \( BC = 3 \) in., \( CD = 10 \) in. 
Find the distances of \( D \) from \( BC \) and \( BA \).

37. How far in front of a pinhole camera must a man 6 ft. high stand in order that a full-length photograph may be taken on a film 2\( \frac{1}{2} \) in. high, 2\( \frac{1}{2} \) in. from the pinhole?

[38] A halfpenny (diameter 1 in.) at the distance of 3 yards appears nearly the same size as the sun or moon at its mean distance. Taking the distance of the sun as 93 million miles, find its diameter. Taking the diameter of the moon as 2160 miles, find its mean distance.

39. A rectangular table, 5 ft. wide, 8 ft. long, 3 ft. high, stands on a level floor under a hanging lamp. The shadow on the floor of the shorter side is 8 ft. long. Find the length of the shadow of the longer side and the height of the lamp above the table.

40. (i) A sphere, of radius 5 in., is placed inside a conical funnel whose slant height is 15 in. and whose greatest diameter is 18 in. Find the distance of the vertex of the funnel from the centre of the sphere.

(ii) Find the radius of a sphere which, when put in this funnel, touches the plane of the rim of the funnel.

[41] The length of each arm of a pair of nutcrackers is 6 in. Find the distance between the ends of the arms when a nut, diameter 1 in., is put with its nearer end 1 in. from the apex.

42. Draw a circle of radius 5 cm. and inscribe a triangle \( ABC \) in the circle such that \( BC : CA : AB = 5 : 6 : 7 \).

*43. In \( \triangle ABC \), \( \angle B = 90^\circ \), \( AB = 5 \) in., \( BC = 2 \) in. If the perpendicular bisector of \( AC \) cuts \( AB \) at \( Q \), find \( AQ \).

*44. A rectangular sheet of paper \( ABCD \) is folded so that \( D \) falls on \( B \); the crease cuts \( AB \) at \( Q \). If \( AB = 11 \) in., \( AD = 7 \) in., find \( AQ \).

*45. \( PQ \) is a chord of a circle 5 cm. long; the tangents at \( P \), \( Q \) meet at \( T \); \( PR \) is a chord parallel to \( TQ \). If \( PT = 8 \) cm., find \( PR \).

46. \( A, B, C \) are 3 points on level ground (see fig. 848); \( AB = 9 \) ft.; 
\( AP, BQ \) are vertical poles, each 8 ft. high; \( CR \) is a vertical post 8 ft. high. Straight lines \( PR, QR \) run from \( P, Q \) to \( R \) and are continued to meet the ground at \( Y, Z \). Find the length of \( YZ \).

*47. A cuboid, 2 in. by 3 in. by 4 in., rests on one of its largest faces on a table. A hollow cone, whose height is equal to its base-diameter, rests on the table covering the cuboid and touching its four upper corners. Find the height of the cone.

*48. Fig. 849 represents an object \( HK \) and its image \( PQ \) in a concave mirror, centre \( O \), focus \( F \); \( CH = x, CP = y, CF = FO = f \), 
\( HK = z, PQ = y \). Prove that (i) \( \frac{1}{x} = \frac{1}{u} + \frac{1}{v} \); (ii) \( y = \frac{z}{u} \).

49. Fig. 850 also represents an object \( HK \) and its image \( PQ \) in a convex mirror. What do you notice about the image?

With the notation and data of No. 48, prove that \( \frac{1}{u} = \frac{1}{v} - \frac{1}{f} \).

*50. Fig. 851 represents an object \( HK \) and its image \( PQ \) in a thin concave lens, centre \( O \), focus \( F \); \( OH = u, OP = v, OF = f \), 
\( HK = x, PQ = y \). Prove that (i) \( \frac{1}{u} - \frac{1}{v} = \frac{1}{f} \); (ii) \( y = \frac{rz}{u} \).

*51. Fig. 852 represents an object \( HK \) and its image \( PQ \) in a thin convex lens. With the notation of No. 50, prove that \( \frac{1}{u} = \frac{1}{v} + \frac{1}{f} \) and find \( y \) in terms of \( x, u, f \).
THEOREM 75

If two triangles are equiangular, their corresponding sides are proportional.

![Figure 383]

Given two triangles $ABC$, $XYZ$ in which

$\angle A = \angle X$,  $\angle B = \angle Y$,  $\angle C = \angle Z$.

To prove that $\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}$.

Construction.  From $AB$, $AC$ cut off $AH$, $AK$ equal to $XY$, $XZ$.

Join $HK$.

Proof.  In the $\triangle$S $AHK$, $XYZ$,

$AH = XY$  \text{const.},

$AK = XZ$  \text{const.},

$\angle A = \angle X$  \text{given},

$\therefore$ $\triangle$S $AHK$, $XYZ$ are congruent  \text{SAS},

$\therefore \angle AHK = \angle XYZ$,

but $\angle ABC = \angle XYZ$  \text{given},

$\therefore \angle AHK = \angle ABC$.

But these are corresponding angles,

$\therefore HK$ is parallel to $BC$;

$\therefore HK$ divides $AB$, $AC$ proportionally,

$\therefore AH = AK$.

But $AH - XY$, $AK = XZ$ \text{const.},

$AB$  $AC$  \text{constr.},

$XY = XZ$.

Similarly, by cutting off lengths from $BA$, $BC$ equal to $YX$, $YZ$, it can be proved that

$BA$  $BC$  \text{constr.},

$YX = YZ$.

$AB$  $AC$  $BC$  \text{constr.},

$XY = XZ = YZ$.

Abbreviation for reference: equiangular $\triangle$S.

Theorem 75 is sometimes stated in the form,

Equiangular triangles are similar.

It has already been pointed out, see p. 481, that quadrilaterals which are equiangular need not be similar. See also p. 532, Paper 84, No. 4 (ii), and p. 533, Paper 86, No. 4 (i).

Attention should be called to the part of the proof printed in thick type. An additional construction is needed for proving that $\frac{BC}{BA} = \frac{YZ}{YX}$.

The fact that $HK$ is parallel to $BC$ does not give a value for the ratio $BC$.  To say that $\frac{BC}{AB}$ is equivalent to assuming what has to be proved in this theorem.

The application of Theorem 75 to the principle of the Diagonal Scale should be pointed out and provides a useful opportunity for revision, and it is suggested that some oral examples similar to those on p. 190 should be taken.

Important Hint.  In rider work, when writing down the three equal ratios of pairs of sides of two similar triangles $AB$, $XZ$, it is better to take the ratio of each side of $\triangle ABC$ to the corresponding side of $\triangle XYZ$,

\[ \frac{BC}{CA} \frac{AB}{XY} \]

\[ \frac{AB}{XY} \frac{BC}{YZ} \frac{CA}{ZX} \]

\[ \frac{AB}{XY} \frac{BC}{YZ} \frac{CA}{ZX} \]

\[ \frac{BC}{CA} \frac{AB}{XY} \]

\[ \text{i.e.} \]

\[ \frac{BC}{CA} \frac{AB}{XY} \]

than to write $\frac{AB}{XY}$ and $\frac{BC}{YZ}$.

\[ \frac{AB}{XY} \frac{BC}{YZ} \frac{CA}{ZX} \]
THEOREM 76 (First Proof)

If the three sides of one triangle are proportional to the three sides of a second triangle, then the triangles are equiangular.

Given two triangles $ABC$, $XYZ$ in which

\[ \frac{AB}{BC} = \frac{CA}{YA} = \frac{ZX}{ZX} \]

To prove that $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.

Construction. On the side of $YZ$ opposite to $X$, draw $YK, ZK$ so that

$\angle KYZ = \angle B$ and $\angle KZY = \angle C$.

Proof. With the notation in the figure,

in $\triangle ABC$, $KYZ$,

\[ b_1 = b_2 \quad \text{constr.} \]
\[ c_1 = c_2 \quad \text{constr.} \]

\[ \therefore \angle A = \angle K \quad 3rd \angle s \text{ of } \triangle s. \]

\[ \therefore \triangle ABC \quad \text{are similar to equiangular } \triangle s. \]

But $\triangle ABC$ is given,

\[ \frac{AB}{BC} = \frac{KY}{YZ} \]
\[ \frac{AB}{BC} = \frac{XY}{YZ} \]
\[ \frac{AB}{AB} = \frac{KY}{XY} \]
\[ \frac{KY}{KY} = \frac{XY}{XY} \]
\[ \therefore \triangle KYZ \quad \text{are congruent \ SSS.} \]

Similarly, $\triangle XYZ$, $\triangle KYZ$,

\[ \frac{XY}{KY} = \frac{XZ}{KZ} \quad \text{proved,} \]
\[ \frac{XZ}{KZ} = \frac{YZ}{YZ} \quad \text{proved,} \]

\[ \therefore \triangle XYZ \quad \text{are congruent \ SSS.} \]

In $\triangle XYZ$, $\angle A = \angle X$ is $3rd \angle s$ of $\triangle s$. 3rd $\angle s$ of $\triangle s$.

Abbreviation for reference: 3 sides proportional.

Theorem 76 is sometimes stated in the form,

If the three sides of one triangle are proportional to the three sides of a second triangle, then the triangles are similar.
THEOREM 76 (Second Proof)

If the three sides of one triangle are proportional to the three sides of a second triangle, then the triangles are equiangular.

\[ \frac{AH}{XY} = \frac{BC}{AB} = \frac{HK}{XY} \]

But
\[ \frac{BC}{AB} = \frac{YZ}{XY} \]

\[ \therefore HK = YZ. \]

\[ \therefore \angle A = \angle X, \quad \angle B = \angle Y, \quad \angle C = \angle Z. \]

To prove that \( \angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z. \)

Construction. From \( AB, AC \) cut off \( AH, AK \) equal to \( XY, XZ \).

Join \( HK. \)

Proof.
\[ \frac{AB}{AC} = \frac{XY}{XZ} \]

But \( XY = AH \) and \( XZ = AK \)

\[ \therefore \frac{AB}{AC} = \frac{AH}{AK} \]

\[ \therefore HK \text{ divides } AB, AC \text{ proportionally,} \]

\[ \therefore HK \text{ is parallel to } BC, \]

\[ \therefore \angle ABC = \angle AKH \text{ and } \angle ACB = \angle AKH \text{ corr. } \angle s. \]

\[ \therefore \triangle ABC = \triangle AHK \text{ corr. } \angle s. \]

\[ \therefore \triangle ABC \text{ are equiangular,} \]

\[ \therefore \triangle AHK \text{ are equiangular,} \]

\[ \therefore \frac{BC}{AB} = \frac{HK}{AH} \text{ corr. sides proportional,} \]

but
\[ \frac{AH}{XY} = \frac{str.,}{BC} = \frac{AB}{str.,}{HK} = \frac{XY}{Y} \]

\[ \therefore \triangle AHK, XYZ \]

\[ \frac{AH}{XY} = \frac{str.,}{AK} = \frac{XZ}{str.,}{HK} = \frac{YZ}{proved}, \]

\[ \therefore \triangle AHK \text{ are congruent } \triangle XYZ. \]

\[ \therefore \angle A = \angle X \text{ and } \angle AHK = \angle Y \text{ and } \angle AKH = \angle Z, \]

but \( \angle AHK = \angle B \) and \( \angle AKH = \angle C \)

\[ \therefore \angle B = \angle Y \text{ and } \angle C = \angle Z. \]

Hints for Rider Work. In many riders it is required to prove that \( \frac{p}{r} = \frac{q}{s} \), where \( p, q, r, s \) are the lengths of lines in a given figure.

(i) To prove that \( \frac{p}{r} = \frac{q}{s} \), is equivalent to proving that \( \frac{p}{q} = \frac{r}{s} \).

(ii) See whether there are two triangles one of which has \( p, q \) as sides and the other \( r, s \) as sides, or one of which has \( p, r \) as sides and the other \( q, s \) as sides, and try to prove these triangles are similar.

(iii) See whether the ratio \( \frac{p}{q} \) or the ratio \( \frac{p}{r} \) can be replaced by a more convenient equal ratio by using parallels or similar triangles.
THEOREM 77

If two triangles have an angle of one equal to an angle of the other, and the sides about these equal angles proportional, the triangles are equiangular.

\[ \triangle A = \triangle X \quad \text{and} \quad \frac{AB}{AC} = \frac{XY}{XZ} \]

To prove that \( \angle B = \angle Y \) and \( \angle C = \angle Z \).

Construction. From \( AB, AC \) cut off \( AH, AK \) equal to \( XY, XZ \). Join \( HK \).

Proof. In \( \triangle AHK, XYZ \),

\[ AH = XY \quad \text{constr.} \]
\[ AK = XZ \quad \text{constr.} \]
\[ \angle A = \angle X \quad \text{given.} \]

\( \therefore \triangle AHK \) are congruent \( \triangle XYZ \).

\[ \therefore \angle AHK = \angle Y \] and \( \angle AKH = \angle Z \).

Also

\[ \frac{AB}{AC} = \frac{XY}{XZ} \]
\[ \text{given,} \]

and

\[ XY = AH, \quad XZ = AK \quad \text{constr.} \]

\[ \therefore \frac{AH}{AK} \]

\[ \therefore HK \text{ divides the sides } AB, AC \text{ proportionally,} \]

\[ \therefore HK \text{ is parallel to } BC. \]

\[ \therefore \angle AHK = \angle Y \] and \( \angle AKH = \angle Z \quad \text{corr.} \quad \angle s. \]

But \( \angle AHK = \angle Y \) and \( \angle AKH = \angle Z \) proved,

\[ \therefore \angle B = \angle Y \] and \( \angle C = \angle Z. \]

Abbreviation for reference: ratio of 2 sides, inc. \( \angle \).

EXERCISE 90

[Arrows indicate that lines are given parallel.]

Nos. 1-4 refer to fig. 857. Name two ratios equal to the ratios in Nos. 1-4:

1. \( \frac{HK}{AB} \) 
2. \( \frac{AB}{PQ} \)
3. \( \frac{CH}{CK} \)
4. \( \frac{AB}{BC} \)

Nos. 5-10 refer to fig. 858.

5. Prove that \( \frac{BC}{DA} = \frac{YZ}{DX} \)
6. Prove that \( \frac{XY}{AB} = \frac{YZ}{BC} \)
7. Prove that \( \frac{YP}{AX} = \frac{BC}{AD} \)
8. Prove that \( \frac{BP}{DZ} = \frac{XA}{XD} \)
9. If the line joining \( C \) to \( Y \) is parallel to \( DX \), prove that \( BC = XY = YZ = XZ \).
10. If \( XB \) is joined and produced to meet the line through \( D \) parallel to \( XZ \) at \( R \), prove that \( BP = DZ = DX = XR \).

[1] With the data of fig. 801, p. 491, prove that \( \frac{QK}{AH} = \frac{PC}{AB} \).
[12] With the data of fig. 812, p. 496, prove that \( \frac{CO}{PA = CR = RQ} \).
13. \( BE, CF \) are altitudes of \( \triangle ABC \). Prove that \( BE : CF = AB : AC \).
14. ABC is a triangle inscribed in a circle. The bisector of \( \angle BAC \) cuts BC at Q and cuts the circle again at P. Prove that AC : AP = AQ : AB and name a ratio equal to BQ : AB.

15. AB is a diameter of a circle AQP; PT is the perpendicular from P to the tangent at A. Prove that PT : PA = AP : AB.

16. In \( \triangle ABC \), \( \angle BAC = 90^\circ \); AD is an altitude. Prove that DC : AC = BC and name two ratios equal to CD : DA.

17. The medians BY, CZ of \( \triangle ABC \) intersect at G. Prove that GY = \( \frac{1}{2} \) BY. [Join YZ.]

Nos. 18–25 refer to fig. 859 in which XAD, AQZ, CQYX are straight lines.

18. Complete the relations:
   
   \[
   QC \quad BC \quad BY \quad QY \\
   QX = \cdots = \cdots.
   \]

19. Complete the relations:
   
   \[
   BZ \quad DQ \quad BY \\
   AD = \cdots = \cdots.
   \]

20. Prove that QC : QX = CY : CX.

21. Prove that QC : QD = AD : AB.

22. If BZ = 2ZC, find the values of (i) QD ; (ii) DB.

23. If DA = 3AX, find the values of (i) BQ ; (ii) ZC.

24. If AZ = BY, prove that XQ = 2QC.

25. Prove that \( \triangle CDX : \triangle BDX = \triangle BDA : \triangle BCY. \)

26. ABCDE is a regular pentagon; AX, AY are the perpendiculars from A to CD, CB produced. Prove that AX : AY = AD : AB.

27. The bisector of \( \angle BAC \) meets BC at D, X is a point on AD, produced if necessary, such that CX = CD. Prove that \( \triangle ABD, \angle ACX \) are similar and deduce that \( AB : AC = BD : DC. \)

28. M is the mid-point of AB; AXB, MYB are equilateral triangles on opposite sides of AB; XY cuts AB at Z. Prove that AZ = 2ZB.

29. BE, CF are altitudes of \( \triangle ABC \). Prove that EF : BC = AF : AC.

30. Prove that the common tangents of two non-intersecting circles divide, internally and externally, the line joining the centres in the ratio of the radii.

31. If \( \angle ABC \) are similar and if AP, AQ are medians, prove that \( \triangle BAP = \triangle YXQ. \) [Use Theorem 77.]

32. ABCD is a cyclic quadrilateral and AK is the perpendicular from A to BD. If AC : AB = CD : KB, prove that AC is a diameter.

33. In fig. 890, \( \angle ABK = \angle ACD \) and \( \angle AKB = \angle ADC \). Prove that \( \triangle ACB \) and \( \angle ADC \) are similar. [Use Theorem 77.]

34. A radius CB of a circle, centre C, is produced to D so that \( CB = BD \). If M is the mid-point of CB and if P is any point on the circle, prove that \( \angle CPD = \angle CDP \).

35. ABPQ, ABRS are two circles. If PAS, QAR are straight lines, prove that BP : BQ = BS : BR.

36. ABCD is a rectangle. Two perpendicular lines are drawn; one cuts AB, CD at E, F; the other cuts AD, BC at G, H. Prove that EF : GC = BD : AB.

37. P is a variable point on a given circle, centre A; O is a fixed point outside the circle; Q is a point on OP such that OQ = 3 OP. Prove that the locus of Q is a circle. [Draw QKII PA to cut OA at K.]

38. ABCD is a quadrilateral in which AC bisects \( \angle BAD \) and \( \angle ACD = \angle ABC \). If X, Y are the mid-points of BC, CD, prove that A, X, C, Y are concyclic. [Join AX, AY.]

39. ABC is a triangle inscribed in a circle; the tangent at C cuts the line through B parallel to AC at D. Prove that CD : AB = DB : CA.

40. ABCD is a quadrilateral in which AB is parallel to DC, and \( \angle D = \angle C < 90^\circ \). If P is a point on CD such that \( \angle APB = \angle C \), prove that DP : PC = PA : PB.

41. AB, DC are the parallel sides of a trapezium ABCD; any line parallel to AB cuts CA, CB at H, K; DH, DK cut AB and AD produced at X, Y. Prove that AB = XY.

42. In \( \triangle ABC \), \( \angle BAC = 90^\circ \); AXB, AY are squares outside \( \triangle ABC \). If BZ, CX cut AC, AB at K, H, prove that AH = AK.
Ratios and Areas

The distinction between the meanings of \( AB : XY \) and \( AB \cdot XY \) needs emphasis.

The ratio \( AB : XY \) is represented by the fraction \( \frac{AB}{XY} \); but \( AB \cdot XY \) denotes the area of a rectangle whose adjacent sides are equal to \( AB \) and \( XY \) and is measured by the product of the numbers of units of length in \( AB \) and \( XY \).

If \( \frac{c}{d} = \frac{p}{q} \), then \( cq = dp \); and if \( \frac{AB}{XY} = \frac{CD}{ZW} \), then \( AB \cdot ZW = CD \cdot XY \).

Thus if \( AB : XY = CD : ZW \), the rectangle contained by \( AB \) and \( ZW \) is equal in area to the rectangle contained by \( CD \) and \( XY \).

Conversely, if the rectangles \( ABCD, APQR \) are equal in area, \( AB \cdot AD = AP \cdot AR \); and it then follows that \( \frac{AB}{AP} = \frac{AD}{AR} \).

The geometrical proof of this algebraic argument forms a useful exercise.

Example for Oral Discussion

In fig. 861, \( BAP, DAR \) are perpendicular lines such that the rectangles \( BADC, PARQ \) are equal in area.

Prove that \( AB : AP = AR : AD \).

Join \( BR, DP, BD, PR \).

(i) Prove that \( \triangle BDR = \triangle BPR \).

(ii) What can you now say about the lines \( BR, DP \)?

Hint on Rider Work. If it is required to prove a rectangle-property, it is often best to convert it into a statement about equal ratios.
Examples for Oral Discussion
1. Fig. 863 represents two chords $AB$, $CD$ of a circle intersecting (when produced if necessary) at $X$. Prove that $XA \cdot XB = XC \cdot XD$.
   Join $AD$, $BC$. What ratios must you prove equal?
   Explain why $\triangle X\triangle AXD$ are similar in each figure.

2. In fig. 864, a chord $AB$ of a circle $ABT$ meets, when produced, at $X$ the tangent to the circle at $T$. Prove that $XA \cdot XB = X^2$.
   Join $TA$, $TB$. What ratios must you prove equal?
   Explain why $\triangle AXT$ are similar.

3. In fig. 863 (i), if $AX = 4$ cm., $XB = 5$ cm., $CX = 3$ cm., find $CD$.
4. In fig. 863 (ii), if $AB = 6$ cm., $BX = 4$ cm., $DX = 5$ cm., find $CD$.
5. In fig. 864, if $XB = 4$ cm., $XT = 6$ cm., find $AB$.

6. Fig. 865 represents two straight lines $AB$, $CD$ which are divided both internally, or both externally, at the same point $X$. If $XA \cdot XB = XC \cdot XD$, prove that $A, B, C, D$ are concyclic.
   Join $AD$, $BC$.
   Express the data in the form of two equal ratios and then explain why $\triangle AXD$ are similar in each figure.

NUMERICAL EXAMPLES

Exercise 92

[For additional examples, see Exercise 78, p. 423.]

1. If in fig. 863 (i), p. 502, $AB = 9$ cm., $AX = 4$ cm., $CX = 2.5$ cm., find $CD$.
2. If in fig. 863 (ii), p. 502, $AB = 7$ cm., $BX = 3$ cm., $DX = 4$ cm., $AD = 9$ cm., find $CD$ and $BC$.

Nos. 3–6 refer to fig. 864, p. 502.

[3] If $AB = 9$ cm., $BX = 3$ cm., find $XT$.
[4] If $BX = 6$ cm., $TX = 12$ cm., find $AB$.

7. $PN$ is the perpendicular from a point $P$ on a circle to a diameter $AB$. If $AN = 5$ cm., $BP = 7.3$ cm., find $PN$.

[8] In $\triangle ABC$, $\angle BAC = 1$ rt. $\angle$; $AD$ is an altitude. If $AB = 5$ in. and $AC = 12$ in., find $BD$.

[9] In $\triangle ABC$, $AB = 8$ cm., $AC = 12$ cm.; a circle through $B, C$ cuts $AB$, $AC$ at $P, Q$. If $BP = 5$ cm., find $CQ$.

[10] $ABC$ is a triangle inscribed in a circle; $AB = AC = 10$ in., $BC = 12$ in.; the perpendicular $AD$ from $A$ to $BC$ is produced to meet the circle at $E$. Find $DE$ and the radius of the circle.
THEOREM 78

Given a triangle ABC in which \( \angle A \) is a right angle, and the perpendicular AD from A to BC.

To prove that \( \triangle ABC, DBA, DAC \) are similar.

Proof. In \( \triangle ABC, DBA, \)

\[ \angle BAC = \angle BDA \text{ rt. } \angle s, \text{ given,} \]
\[ \angle ABC = \angle DBA \text{ same angle,} \]

\( \therefore \) the third angles ACB, DAB are equal.

\[ \therefore \triangle ABC \text{ and } DBA \text{ are equiangular.} \]

In the same way it may be proved that

\[ \triangle ABC \text{ and } DAC \text{ are equiangular.} \]

But triangles which are equiangular are also similar,

\( \therefore \triangle ABC, DBA, DAC \) are similar.

Corollary 1. The square on the perpendicular to the hypotenuse is equal to the rectangle contained by the segments of the hypotenuse, that is

\[ AD^2 = BD \cdot DC. \]

Since \( \triangle DBA \) are similar,

\[ DA = DC, \]
\[ DB = DA', \]

\( \therefore \) \[ DA^2 = DB \cdot DC. \]

Corollary 2. The square on either of the sides containing the right angle is equal to the rectangle contained by the hypotenuse and the segment of the hypotenuse adjacent to that side, that is

\[ BA^2 = BD \cdot BC \text{ and } CA^2 = CD \cdot CB. \]

Since \( \triangle ABC \) are similar,

\[ \frac{BA}{BC} = \frac{BD}{BA}, \]

\( \therefore \) \[ BA^2 = BD \cdot BC. \]

Similarly, \[ CA^2 = CD \cdot CB. \]

Alternative methods of proof of these Corollaries have been given on p. 429. The deduction of Corollary 2 from the ordinary proof of Pythagoras' theorem was given on p. 223.

On the other hand, the proof of Corollary 2 by means of similar triangles provides an alternative method for proving Pythagoras' theorem and is usually adopted in French text-books.

The results given in these Corollaries may also be stated as follows, see the definition on p. 454.

If AD is an altitude of \( \triangle ABC \) and if \( \angle BAC \) is a right angle,

(i) AD is a mean proportional between BD and DC;
(ii) BA is a mean proportional between BD and BC;
CA is a mean proportional between CD and CB.
THEOREM 79

If two chords of a circle, produced if necessary, cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Given two chords $AB$, $CD$ of a circle intersecting at $X$, either inside, fig. 867 (i), or outside, fig. 867 (ii), the circle.

To prove that $XA \cdot XB = XC \cdot XD$.

Construction. Join $BC$, $AD$.

Proof. In $\triangle AXD$, $CXB$,

$\angle A = \angle C$ \hspace{1cm} \text{same segment},

$\angle AXD = \angle CXB$ \hspace{1cm} \text{vert. opp. } \angle s$, fig. (i).

Same $L$, fig. (ii),

$\therefore$ the third angles $ADX$, $CBX$ are equal,

$\therefore \triangle AXD \cong \triangle CXB$ \hspace{1cm} \text{are equiangular,}

$\therefore \frac{XA}{XD} = \frac{XC}{XB}$ \hspace{1cm} \text{corr. sides proportional,}

$\therefore XA \cdot XB = XC \cdot XD$.

Abbreviation for reference: intersecting chords.

THEOREM 80

If from any point outside a circle, a secant and a tangent are drawn, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.

Given the tangent $XT$ to a circle from a point $X$ outside the circle and a straight line $XBA$ cutting the circle at $B$, $A$.

To prove that $XA \cdot XB = XT^2$.


Proof. In the $\triangle AXT$, $TXB$,

$\angle TAX = \angle BXT$ \hspace{1cm} \text{alt. segment,}$

$\angle AXT = \angle TXB$ \hspace{1cm} \text{same } L$,

$\therefore$ the third angles $AXT$, $TXB$ are equal,

$\therefore \triangle AXT \cong \triangle TXB$ \hspace{1cm} \text{are equiangular,}

$\therefore \frac{XA}{XT} = \frac{XT}{XB}$ \hspace{1cm} \text{corr. sides proportional,}$

$\therefore XA \cdot XB = XT^2$.

Abbreviation for reference: tangent property.
THEOREM 81

If two straight lines $AB$ and $CD$ are divided both internally or both externally at the same point $X$ such that $XA \cdot XB = XC \cdot XD$, the four points $A$, $B$, $C$, $D$ are concyclic.

Construction. Join $AD$, $BC$.

Proof. In $\triangle AXD$, $CXB$,

$\angle AXD = \angle CXB$ \hspace{1cm} \text{vert. opp. } \angle s, \text{fig. (i)},$

$\angle XD = \angle XB$ \hspace{1cm} \text{same } \angle, \text{fig. (ii)},$

and $\frac{XA}{XC} = \frac{XD}{XB}$ \hspace{1cm} \text{because } $XA \cdot XB = XC \cdot XD$.

$\therefore \triangle AXD \sim \triangle CXB$ \hspace{1cm} \text{ratio of 2 sides, inc. } \angle$.

$\therefore \angle DAX = \angle BCX$.

that is, $\angle DAB = \angle DCB$.

$\therefore DB$ subtends equal angles at points $A$, $C$ on the same side of $DB$.

$\therefore D$, $B$, $A$, $C$ lie on a circle.

NOTE. In Theorems 81, 82, it is unnecessary to start by stating what is given and what is to be proved, because this is merely a repetition of the enumeration which is here given in terms of the letters to be used.

THEOREM 82

If the straight line $AB$ is divided externally at $X$, and if $C$ is a point, not on $AB$, such that $XA \cdot XB = XC^2$, the circle $ABC$ touches $XC$ at $C$.

Construction. Join $CA$, $CB$.

Proof. In $\triangle AXC$, $CXB$,

$\angle AXC = \angle CXB$ \hspace{1cm} \text{same } \angle,$

and $\frac{XA}{XC} = \frac{XD}{XB}$ \hspace{1cm} \text{because } $XA \cdot XB =XC^2$.

$\therefore \triangle AXC \sim \triangle CXB$ \hspace{1cm} \text{ratio of 2 sides, inc. } \angle$.

$\therefore \angle CAX = \angle BCX$.

$\therefore CX$ touches the circle $ABC$ at $C$ \hspace{1cm} \text{conv. alt. segment}.$

NOTE. Alternative proofs of Theorems 79–82 have been given in Theorems 67–69, pp. 420–428.

Theorem 80 may be deduced from Theorem 79 by taking the limiting case when $D$ coincides with $C$ in fig. 867 (ii), so that $XDC$ becomes a tangent.

Similarly, Theorem 82 may be deduced from Theorem 81.
NUMERICAL EXAMPLES

EXERCISE 93

1. AB, CD are two perpendicular chords of a circle, centre O; AX = 6 cm., CX = 10 cm., DX = 12 cm.; find OX and OA.

2. The diagonals of the quadrilateral ABCD cut at K. If KA = 4 cm., KB = 6 cm., AC = 16 cm., BD = 14 cm., prove that ABCD is cyclic.

[3] H and K are points on the sides AB, AC respectively of \( \triangle ABC \). If AH = 4 cm., HB = 11 cm., AK = 5 cm., KC = 7 cm, prove that HKCB is a cyclic quadrilateral.

4. In \( \triangle ABC \), AB = 18 cm., BC = 12 cm., CA = 13-5 cm.; K is a point on AB such that AK = 10 cm. Prove that BC is a tangent to the circle AKC and find the length of CK.

5. In \( \triangle ABC \), \( \angle B = 90^\circ \), AB = 3 in., BC = 4 in.; the circle which touches BC at C and passes through A is drawn. If it cuts BA produced at P, find AP and the radius of the circle.

6. If in fig. 864, p. 502, \( \triangle X = 2XT \), prove that AB = 3BX.

7. If in fig. 864, p. 502, \( XT = \frac{3}{2}XA \), find the ratio \( XB : BA \).

8. In fig. 864, p. 502, BT = p cm., TA = q cm., AB = r cm. Prove that \( \frac{XA}{q^2} = \frac{XB}{p^2} \).

9. X is the mid-point of a line TY of length 2 in.; TZ is drawn so that \( \angle ZTX = 45^\circ \). A circle is drawn through X and Y and to touch TZ at P. Prove that \( \angle TXP = 90^\circ \), and find the radius of the circle.

10. (Sphereometer.) The vertices A, B, C of an equilateral triangle lie on a sphere; the diameter PQ of the sphere perpendicular to the plane ABC cuts it at Q. If PQ = 1 cm. and AB = 6 in., find the diameter of the sphere.

11. A sphere of diameter 6 in. rests in a 5-in. diameter hole in a table. A thin square plate of side 10 in. with a central hole of diameter 2 in. rests with one edge AB on the table and the circumference of the hole lying on the sphere. Find the distance of AB from the centre of the 5-in. hole.

Examples for Oral Discussion

1. In fig. 871, AE is a diameter of the circumcircle of \( \triangle ABC \). If AD is an altitude of \( \triangle ABC \), prove that \( \frac{AB}{AC} = \frac{AD}{AE} \).

Join BE.

(i) What ratios must you prove equal?

(ii) Prove that \( \triangle ABE, ADC \) are similar.

2. Ptolemy's Theorem. If ABCD is a cyclic quadrilateral, prove that

\[ BC \cdot AD + AB \cdot CD = AC \cdot BD. \]

Take a point P on BD so that \( \angle DAP = \angle BAC \).

(i) Prove that \( BC \cdot AD = AC \cdot DP \).

(ii) Prove that \( AB \cdot CD = AC \cdot BP \).

3. What special result can be deduced from Ptolemy's theorem? If, in fig. 872, \( \triangle ABC \) is equilateral?

EXERCISE 94

[For additional examples, see Exercises 79, p. 430.]

1. The diagonals of a cyclic quadrilateral ABCD intersect at K. Prove that \( AD \cdot KC = BC \cdot KD \).

2. Two lines \( \triangle XAB, XCD \) cut a circle at A, B, C, D. Prove that \( XA \cdot BC = XC \cdot AD \).

3. If in fig. 874, p. 512, \( XB = BT \), prove that \( XA \cdot XB = TA \).

4. The sides \( BA, CA \) of \( \triangle ABC \) are produced to \( N, E \) respectively so that \( \angle AED = \angle ABC \). Prove that

\[ AB \cdot AD = AC \cdot AE. \]

5. In \( \triangle ABC \), \( \angle B = 90^\circ \); P is any point on AB. If the circle, diameter AP, cuts AC at Q, prove that \( AP \cdot AB = AQ \cdot AC \).

6. In \( \triangle ABC \), \( \angle C = 90^\circ \); a circle is drawn to touch AC at C and cuts AB at P, Q. If CN is an altitude of \( \triangle ABC \), prove that \( AN : AP = AQ : AB \).
7. PQ, RS are chords of the circles ABQP, ABRS, which meet, when produced if necessary, at a point on AB or AB produced. Prove that PQ, SQ, RS are concurrent.

8. In fig. 873, AP, AQ are tangents to the circles ABQP, ABP. Prove that AB = BP, BQ.

9. AC is a chord of a circle. From a point P on the minor arc AC, lines PD, PE are drawn parallel to the tangents at A, C cutting AC at D, E respectively. Prove that AD = PD, CE = PE.

10. In \( \triangle ABC \), AB = AC; D is a point on AC such that BD = BC. Prove that BC = AC = BD.

11. Two chords AB, CD of a circle intersect at X. If D is the midpoint of the arc AB, prove that CA \cdot CB = CX \cdot CD.

12. AB is a diameter of a circle, centre O; AP, PQ are equal chords. Prove that AP \cdot PB = AQ \cdot QP.

13. TA, TB are the tangents from T to a circle; TPQ is a straight line cutting the circle at P and Q. Prove that AP \cdot AQ = TP \cdot TA; AP \cdot AQ = BP \cdot BQ.

14. E is a point inside the quadrilateral ABCD such that \( \angle EAD = \angle BAC \) and \( \angle EDA = \angle BCA \). Prove that EB \cdot AC = AB \cdot DC.

15. ABC is a triangle. D is a point on the line bisecting \( \angle BAC \) such that \( AD = AB \cdot AC \). Prove that BD touches the circle ACD.

16. A line PQ is divided at R so that PR = PQ, QR. TQR is a triangle such that TQ = TR = QR. Prove that PT = PQ.

17. If in fig. 874, the line through B parallel to AT cuts the tangent XT at K, prove that XB = XA, XT.

18. In fig. 874, prove that XA = TA, XB = TB.

19. AB is a diameter of a circle, centre O. The tangents at A, B meet any other tangent at H, K. Prove that AH \cdot BK = AO^2.

[20] In \( \triangle ABC \), \( \angle A = 90^\circ \); E is a point on BC such that AE = AB. Prove that BE = BC = 2AE.
[Draw the perpendicular AN from A to BC.]

[21] ABCD is a parallelogram; any line through C cuts DB, AB, DA produced, at Q, Y, X respectively. Prove that QX \cdot QY = QC^2.

[22] The tangent at a point C on a circle PDE is parallel to a chord DE and cuts two other chords PD, PE, when produced, at A, B. Prove that AC = CB = AD = BE.

[23] Two chords AB, AC of a circle are produced to P, Q so that AB = BP and AC = CQ. If PQ cuts the circle at R, prove that AR^2 = PR \cdot RQ. [Let BC cut AR at X.]

[24] The tangent at A to a circle ABC cuts the line through B, parallel to AC, at P; the line through C parallel to AB cuts AP at Q. Prove that AP = AQ = AB^2 = AC^2.

[25] An exterior common tangent to two circles touches them at A and C; a variable line parallel to AC cuts one circle at P and the other at Q. Prove that the ratio AP : CQ is constant. [Let PQ cut the diameters AB, CD of the circles at H, K. What do you know about AP : CQ?]

[26] N is a point on the diameter AB of a circle AEB such that AN = AB; E is the mid-point of the arc AB. If the perpendicular at N to AB cuts the arc AEB at K and cuts AE produced at Z, prove that ZK = KN.

[27] AB is a chord of a circle APB; the tangents at A, B meet at T; PH, PK, PZ are the perpendiculars to TA, TB, AB. Prove that PH \cdot PK = PZ^2.

[28] AC, CB are two sides of a regular decagon inscribed in a circle, centre O; M is the midpoint of AC; OM cuts AB at K. Prove that

(i) \( \angle BOK = 54^\circ \); \( \angle ACK = 18^\circ \); 
(ii) BK, BA = BO; AK, AB = AC; 
(iii) AC = AB = 2BC.

Therefore if p and d are the lengths of the sides of a regular pentagon and regular decagon inscribed in a circle, radius r,

\[ p^2 = d^2 + r^2. \]
Construct a mean proportional to two given lines

\[ \frac{b}{a} = \frac{x}{b} \]

Given two lines of lengths \( a \), \( b \) units.

To Construct a line of length \( x \) units such that \( x^2 = ab \).

**METHOD 1**

**Construction.** Take a point \( C \) on a straight line and cut off from the line on opposite sides of \( C \), parts \( CA \), \( CB \) of lengths \( a \), \( b \) units; fig. 875 (i).

On \( AB \) as diameter, describe a circle.

Draw \( CP \) perpendicular to \( AB \) to cut the circle at \( P \).
Then \( CP \) is the required mean proportional.

**Proof.** Produce \( PC \) to meet the circle at \( Q \).

\( PQ \) is a chord perpendicular to the diameter \( AB \).

\[ \therefore \quad PC = CQ. \]

But \[ PC.CQ = AC.CB \]

\[ \therefore \quad CP^2 = AC.CB \]

\[ = ab. \]

**METHOD 2**

**Construction.** Take a point \( C \) on a straight line and cut off from the line on the same side of \( C \), parts \( CA \), \( CB \) of lengths \( a \), \( b \) units; fig. 875 (ii).

On \( CA \) as diameter, describe a circle.

Draw \( BC \) perpendicular to \( CA \) to cut the circle at \( Q \).
Then \( CQ \) is the required mean proportional.

**EXERCISE 95**

1. Construct a mean proportional between two lines of lengths 8 in. and 12 in. Measure it.
2. Construct a line of length \( \sqrt{38} \) cm. and measure it. [Select values of \( a \), \( b \), for which \( ab = 38 \), which are fairly close together, e.g. 5 and 38 + 5, not 2 and 19.]
3. Find graphically an approximate value of \( \sqrt{31} \).
4. Draw a rectangle of sides 4 cm., 7 cm. and construct a square of equal area. Measure its side.
5. Draw a rectangle of sides 2.6 in., 1.8 in. and construct a square of equal area. Measure its side.
6. Construct a square equal in area to a rhombus \( ABCD \) in which \( AB = 6 \) cm., \( \angle A = 60^\circ \). Measure its side.
7. Construct a square equal in area to an equilateral triangle of side 6 cm. Measure its side.
8. Solve graphically the equation, \( (x - 3)^2 = 19 \).
9. Construct a square equal in area to a quadrilateral \( ABCD \) in which \( AB = BC = 4 \) cm., \( CD = CA = 6 \) cm., \( AD = 7 \) cm. Measure its side.
10. Construct a square equal in area to a regular pentagon, side 4 cm. Measure its side.
11. Draw a straight line $ABC$ such that $AB = 3$ cm., $BC = 5$ cm. Draw $AK$ perpendicular to $AB$ and construct a point $P$ on $AK$ such that $AP = AB : AC$. Construct a circle to pass through $B$, $C$ and to touch $AK$. Measure its radius.

[12] Draw an angle $AOB$ equal to $52^\circ$. Take two points $H, K$ on $OA$ such that $OH = 1$ cm., $OK = 2$ cm. and construct a circle to pass through $H, K$ and touch $OB$. Measure its radius.

**Ratio of Areas of Similar Figures**

**Examples for Oral Discussion**

1. (i) Draw a triangle $ABC$ in which $AB = 9$ cm., $BC = 7$ cm., $CA = 6$ cm. How many triangles whose sides are 3 cm., 2 cm., 2 cm. can be cut out of it?

(ii) $XYZ$ is a triangle such that $XY = 15$ cm., $YZ = 12$ cm., $ZX = 10$ cm. How many triangles whose sides are 3 cm., 2 cm., 2 cm. can be cut out of it?

(iii) What is the ratio of (i) corresponding sides, (ii) areas of $\triangle ABC$, $\triangle XYZ$?

Nos. 2-4 refer to fig. 877 in which $AH$, $BK$ are altitudes of the similar triangles $ABC$, $XYZ$.

2. If $BC = 10$ cm., $YZ = 7$ cm., $AH = 8$ cm., find $BK$ and find the ratio $\triangle ABC : \triangle XYZ$.

3. If $BC : YZ = 5 : 4$, find the ratio $\triangle ABC : \triangle XYZ$.

(i) What is the value of $AB : XY$?

(ii) Explain why $\triangle ABC$, $\triangle XYZ$ are similar.

(iii) Complete the relation, $\triangle ABC : \triangle XYZ = \frac{1}{2} BK . YZ$.

4. If $\triangle ABC$ are similar, prove that $\triangle ABC : \triangle XYZ = BC^2 : YZ^2$ and state the result in words.

Use the method indicated for No. 3.

5. If the polygons $ABCDE$ are similar, prove that $\text{area } ABCDE : \text{area } PQRST = AB^2 : PQ^2$.

**Fig. 878**

It is given that $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, $\angle D = \angle S$, $\angle E = \angle T$, and that


Join $AC$, $AD$, $PR$, $PS$.

(i) Prove in succession that $\triangle ABC \sim \triangle ACD \sim \triangle ADE \sim \triangle PQR \sim \triangle PRS \sim \triangle PST$ are similar.

(ii) Prove that the ratio of the areas of these pairs of triangles is in each case equal to $AB^2 : PQ^2$. 

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**NEW GEOMETRY**

517 **AREAS OF SIMILAR FIGURES**
The result established in No. 5 may be expressed as follows: 

The ratio of the areas of two similar polygons is equal to the ratio of the squares on corresponding sides.

Hence it follows that 

The ratio of the areas of the surfaces of similar solids is equal to the ratio of the squares of corresponding lengths. 

And it can be proved that 

The ratio of the volumes of similar solids is equal to the ratio of the cubes of corresponding lengths.

**NUMERICAL EXAMPLES**

**EXERCISE 96**

1. Find the ratio of the areas of the triangles in fig. 837, p. 485.

Find the ratio of the area of the quadrilateral to that of the smaller triangle in the following figures, Nos. 2–4:

2. Fig. 834, p. 485. [3] Fig. 835, p. 485. 4. Fig. 839, p. 485.

3. Find three integers to which the areas of the three triangles in fig. 836, p. 488, are proportional.

[6] In what ratio does HK divide ABC in fig. 833, p. 485, (i) if BH = 3 in.; (ii) if HK = 3 in.

7. With the data of fig. 844 (ii), p. 487, write down the value of (i) \( \frac{\triangle KAC}{\triangle KBD} \), (ii) \( \frac{\triangle KAC}{\text{quad.} ABDC} \).

8. The lengths of the sides of \( \triangle ABC \) are 10, 15, 20 cm, and the lengths of the sides of \( \triangle PQR \) are 6, 9, 12 cm. Find the ratio of (i) their perimeters, (ii) their areas.

9. The areas of two similar triangles are 18 sq. in., 32 sq. in. The largest side of the first triangle is 9 in., find the largest side of the second triangle.

10. A triangle \( \triangle ABC \) is divided by a line HK parallel to BC into two parts AHK, HKCB of areas 9 sq. cm., 16 sq. cm. respectively. If BC = 7 cm., find HK.

11. A screen 6 ft. high, not necessarily rectangular, requires 27 sq. ft. of material for covering. How much is needed for a screen of the same shape, 4 ft. high?

[12] On a map, scale 6 in. to the mile, a plot of a ground is represented by a quadrilateral of area 21 sq. in. Find the area of the plot in acres.

13. A light is 12 feet above the ground. Find the area of the shadow on the ground of the top of an oval table 4 ft. high, and 45 sq. ft. in area.

14. The area of the top of an oval table, 3 ft. high, is 20 sq. ft. and the area of its shadow on the floor cast by a lamp is 45 sq. ft. Find the height of the lamp above the floor.

15. If it costs £3 to gild a sphere of radius 3 ft., what will it cost to gild a sphere of radius 4 ft.?

16. How many times can a cylindrical tumbler 4 in. high and 3 in. in diameter be filled from a cylindrical cask 40 in. high and 30 in. in diameter?

17. Two hot-water cans are the same shape; the smaller is 9 in. high and holds a quart; the larger is 15 in. high, how much will it hold?

18. A solid metal sphere, radius 3 in., weighs 8 lb. Find the weight of a solid sphere of the same metal 1 ft. in radius.

19. A cylindrical tin 5 in. high holds \( \frac{1}{2} \) lb. of tobacco. How much will a tin of the same shape 8 in. high hold?

20. Two models of the same statue are made of the same material, both being solid. One is 3 in. high and weighs 8 oz.; the other weighs 4 lb., find its height.

21. A lodger pays 8 pence for a scuttle of coal, the scuttle being 20 in. deep. What would be the pay for a scuttle of coal of the same shape, 2\( \frac{1}{2} \) feet deep?

22. A tap can fill half of a spherical vessel, radius 1\( \frac{1}{2} \) feet, in 2 minutes. How long will two such taps take to fill one-quarter of a spherical vessel of radius 4 feet?

23. Two leaden cylinders of equal lengths and diameters 3 in., 4 in., are melted and recast as a single cylinder of the same length. Find its diameter.

24. ABC is a triangle in which \( AB = AC = 2BC; \) D is a point on AC such that \( \angle BDC = \angle BAC; \) a line through D parallel to BC cuts AB in E. Find the value of the ratio,

\[ \frac{\triangle ABC}{\triangle BCD} = \frac{\triangle BDE}{\triangle EDA}. \]

25. Find the ratios of the areas of the 5 small triangles into which fig. 841, p. 486, is divided.
THEOREM 83

The ratio of the areas of two similar triangles is equal to the ratio of the areas of the squares on corresponding sides.

Given two similar triangles \( \Delta ABC, XYZ \).

To prove that \( \frac{\Delta ABC}{\Delta XYZ} = \frac{BC^2}{YZ^2} \)

Construction. Draw the altitudes \( AH, XK \).

Proof. Since \( AH, XK \) are altitudes,

\[
\frac{\Delta ABC}{\Delta XYZ} = \frac{AH}{XY}, \frac{BC}{YZ}, \frac{AH}{XY}, \frac{BC}{YX}
\]

and

\[
\frac{\Delta ABC}{\Delta XYZ} = \frac{AH}{XY}, \frac{BC}{YX}
\]

\( \therefore \) \( \Delta XYZ = \frac{AH}{XY}, \frac{BC}{YX} \).

But in the \( \Delta s \) \( ABH, XKY \),

\( \angle B = \angle Y \)

\( \angle AHB = \angle XKY \) \( rt. \angle s, constr. \)

\( \therefore \) \( \Delta BAH = \Delta YKX \) \( 3rd \ angle \ of \ \Delta s \).

\( \therefore \) \( \Delta s \) \( ABH, XKY \) are similar \( \text{equiangular} \) \( \Delta s \),

\( \therefore \) \( AH \parallel AB \)

\( XK \parallel XY \) \( \text{corr. sides proportional} \).

But

\( \Delta ABC = \frac{AH}{XY}, \frac{BC}{YX} \)

\( \therefore \) \( \Delta ABC = \frac{BC^2}{YZ^2} \)

\( \therefore \) \( \Delta XYZ = \frac{YZ}{YZ} \) \( given, \)

\( \therefore \) \( \Delta ABC, AC^2 \) \( given, \)

\( \therefore \) \( \Delta XYZ = \frac{YZ}{YZ^2} \).

This argument also shows that

\( \text{If two triangles} \ ABC, XYZ \ \text{are such that} \ \angle A = \angle X, \)

\( \Delta ABC \ AB AC \)

\( \Delta XYZ \ XY XZ \)

This result may also be proved by the method of Theorem 83, by drawing the altitudes BN,YP.

EXERCISE 97

1. Two chords \( AB, CD \) of a circle, cut at \( X \). Complete the relation, \( \Delta AXD : \Delta BXD = AX : XD \).

2. From a point \( X \) outside a circle, two lines \( XA, XD \) are drawn cutting the circle at \( A, B, C, D \). Complete the relations,

\( \Delta AXD : \Delta BXD : \Delta AXC = \Delta ABD : \Delta BXC : \Delta XBD \).

3. A line parallel to the side \( AB \) of \( \Delta ABC \) meets \( AC \) produced at \( P, Q \). Complete the relations,

\( \Delta CAB : \Delta CAQ = \Delta AB : \Delta BC \).

4. \( \Delta ABC \) is a square; \( ABP, AQ \) are equilateral triangles. Find the ratio \( \Delta APB : \Delta AQ \).

5. \( \Delta ABC \) is a triangle such that

\( BC : CA : AB = 3 : 4 : 5 \).

If \( BPC, PQA, ARP \) are equilateral triangles, prove that

\( \Delta BPC + \Delta QPA = \Delta ARP \).

6. The sides of a \( \Delta ABC \) are trisected as shown in fig. 880. Prove that the area of \( PQR \) is \( \frac{1}{2} \Delta ABC \).
NEW GEOMETRY

[7] ABCD is a parallelogram; P, Q are the mid-points of CB, CD. Prove \( \triangle APQ = \frac{1}{2} \) parallelogram ABCD.

8. If in the \( \triangle ABC \), \( \angle A = \angle X \), prove that \( \triangle ABC : \triangle XYZ = AB : AC : XY : XZ \).

9. P, Q, R are points on the sides BC, CA, AB of \( \triangle ABC \) such that \( BP : PC = CQ : QA = AR : RB = 1 : 2 \). Prove that \( \triangle PQR = \frac{1}{6} \triangle ABC \).

[10] In \( \triangle ABC \), \( \angle A = 90^\circ \) and AD is an altitude. Prove that \( AB^2 : AC^2 = BD : DC \).

[11] In \( \triangle ABC \), \( \angle A = 90^\circ \) and AD is an altitude; DE is the perpendicular from D to AB. Prove that \( BE : EA = BA^2 : BC^2 \).

[12] Two circles cut at O. Straight lines AOP, BOS, COT cut one circle at A, B, C and cut the other at P, S, T. Prove that \( \triangle ABC : \triangle PST = AB^2 : PS^2 \).

13. ABC is a triangle inscribed in a circle. The tangent at C meets AB produced at T. Prove that
   (i) \( \triangle CAT : \triangle CBT = CA^2 : CB^2 \);
   (ii) \( CA^2 : CB^2 = AT : BT \). (Fig. 881)

14. In fig. 881, DPC is parallel to AB and \( \angle ADC = \angle BCD = \angle APB \). Prove that \( DP : PC = PA^2 : PB^2 \).

15. F is the mid-point of the side AB of \( \triangle ABC \); P is a point on AB such that \( AP^2 = AF \cdot AB \). Prove that the line through P parallel to BC bisects the area of \( \triangle ABC \).

16. In \( \triangle ABC \), \( \angle A = 90^\circ \); BCX, CAY, ABZ are similar triangles in which X, Y, Z are corresponding points. Prove that \( \triangle CAY + \triangle ABZ = \triangle BCX \).

17. AB is a diameter of a circle APB; AH, BK are the perpendiculars from A, B to the tangent at P. Prove that \( \triangle APB = \frac{1}{2} \) quadr. AHKB.

18. AB is a diameter of a circle APB; BK is the perpendicular from B to the tangent at P; C is a point on AB such that \( AC = AP \). If the line through C parallel to BP cuts AP at D, prove that \( \triangle BCP = \text{quadr.} BCDP \).

19. D, E, F are points on the sides BC, CA, AB of \( \triangle ABC \) such that DEF is a parallelogram. If \( BD : DC = x : y \), prove that the ratio of the areas of DEF to the area of ABC is equal to \( 2xy : (x + y)^2 \).

MISCELLANEOUS CONSTRUCTIONS

CONSTRUCTION 22

Construct a polygon similar to a given polygon ABCDE such that corresponding sides are in the given ratio XY : XZ.

In fig. 882 (i), AB is divided at B' so that \( AB' : AB = XY : XZ \) and arrows indicate that lines are drawn parallel. In fig. 882 (ii), O is any point; OA is divided at A' so that \( OA' : OA = XY : XZ \) and arrows indicate that lines are drawn parallel. In fig. 882 (i), A'B'C'D'E' is the required polygon. In fig. 882 (ii), A'B'C'D'E' is the required polygon. The reader should perform the construction by each method and prove that it is correct.

Similar Sections of a Pyramid

Two planes are called parallel, if they never meet one another. The reader should prove that if two parallel planes intersect a third plane, the lines of intersection are parallel. If fig. 882 (i) represents a pyramid, vertex O, base ABCDE, and if a plane parallel to the base cuts the edges at A', B', C', D', E', then the section A'B'C'D'E' is similar to the base ABCDE. Also area A'B'C'D'E' : area ABCDE = A'B'^2 : AB'^2 = OA'^2 : OA^2. The proof is left to the reader.
CONSTRUCTION 23

Construct a triangle similar to a given triangle $\triangle ABC$ and equal to a given fraction $\frac{XY}{XZ}$ of $\triangle ABC$.

![Diagram of construction 23]

(i) Divide $AB$ at $N$ so that $AN : AB = XY : XZ$.

(ii) Draw $NP$ perpendicular to $AB$ and let it meet the circle on $AB$ as diameter at $P$. Join $AP$.

(iii) From $AB$ cut off $AB'$ equal to the mean proportional $AP$ between $AN$, $AB$ (see p. 514).

(iv) Draw $B'C'$ parallel to $BC$.

Then $AB'C'$ is the required triangle.

**Proof.**

\[
\frac{\triangle AB'C'}{\triangle ABC} = \frac{AB'^2}{AB^2} = \frac{AN, AB}{AB^2} = \frac{AN}{AB} = \frac{XY}{XZ}
\]

**Note.** A similar method may be used for any polygon.

CONSTRUCTION 24

Construct a quadrilateral similar to a given quadrilateral $\square ABCD$ and equal in area to a given rectangle $\square XYZW$.

![Diagram of construction 24]

(i) Construct rectangle $\square ABHK$ equal in area to $\square ABCD$.

(ii) Construct rectangle $\square APQK$ equal in area to rect. $\square XYZW$.

(iii) Construct the mean proportional $AB'$ between $AP$ and $AB$.

(iv) Construct quad. $AB'C'D'$ similar to $\square ABCD$.

Then $AB'C'D'$ is the required quadrilateral.

For (i), start by reducing $\square ABCD$ to the equivalent triangle $\triangle ABE$ (p. 263).

For (ii), construct the fourth proportional $AP$ to $AK$, $XY$, $XW$ (p. 479).

**Proof.**

\[
\frac{\text{quad. } AB'C'D'}{\text{quad. } ABCD} = \frac{AB'^2}{AB^2} = \frac{AP, AB}{AB^2} = \frac{AP \text{ rect. } APQK}{AB \text{ rect. } ABHK} = \frac{\text{rect. } XYZW}{\text{quad. } ABCD}.
\]

\[\therefore \text{ quad. } AB'C'D' = \text{rect. } XYZW.\]
CONSTRUCTION 25
Divide a given line $AB$ at $P$ so that $AB \cdot PB = AP^2$.

![Fig. 885](image)

(i) Draw $BQ$ perpendicular to $BA$ and equal to $\frac{1}{3}AB$.
(ii) Join $QA$ and from it cut off $QR$ equal to $QB$.
(iii) From $AB$ cut off $AP$ equal to $AR$.

Then $P$ is the required point.

The proof provides a useful exercise for the reader.

Let $AB = 2l$ units, and $AP = AR = x$ units; explain why

$$(x + l)^2 = (2l)^2 + 2l^2.$$

Hence

$$AP^2 = x^2 - 2l(2l - x) = AB \cdot PB.$$

The line $AB$ is said to be divided at $P$ in medial section.

The construction for dividing a line $AB$ at $P$ so that $AB \cdot PB = AP^2$ is equivalent to the solution of the equation in $x$,

$$2l(2l - x) = x^2.$$

Therefore Construction 25 gives a geometrical solution of the quadratic equation,

$$x^2 + 2lx = 4l^2.$$

The reader should solve this equation by completing the square. This gives

$$x = l(\sqrt{5} - 1).$$

Regular Pentagon and Decagon

In Construction 26, the angles of $\triangle ABC$ are $36^\circ, 72^\circ, 72^\circ$. Therefore $BC$ is the side of a regular decagon inscribed in the circle, centre $A$, radius $AB$.

If we use Construction 26 to inscribe a regular decagon in the circle, we obtain a regular pentagon by joining alternate vertices. It is, however, quicker to use another method, see p. 528.

If $AB = 2l$ units, $AP = l(\sqrt{5} - 1)$ units, see p. 526. But $BC = AP$; therefore the length of the sides of a regular decagon inscribed in a circle, radius $2l$ units, is $l(\sqrt{5} - 1)$ units.

By drawing a perpendicular from $A$ to $BC$, we see that

$$\sin 18^\circ = \cos 72^\circ = \frac{1}{2}(\sqrt{5} - 1).$$
Construction of Regular Pentagon

The quickest formal method for inscribing a regular pentagon in a given circle depends on the following property:—

If \( p \) and \( d \) are the lengths of the sides of a regular pentagon and regular decagon inscribed in a circle, radius \( r \), then

\[
p^2 = d^2 + r^2.
\]

A method for proving this fact is indicated on p. 513, see No. 26.

Draw two perpendicular diameters \( AO, OB \), \( CO, OD \) of the given circle.

Bisect \( OC \) at \( E \).

With centre \( E \), radius \( EA \), draw a circle to cut \( ED \) at \( P \).

Then \( AP \) is equal to the sides of a regular pentagon inscribed in the circle.

(i) If the radius of the circle is 2 units, prove that \( EA = \sqrt{5} \) units.

(ii) Prove that \( OP \) is equal to the side of a regular decagon inscribed in the circle (see p. 527), and complete the proof.

EXERCISE 98

1. Given a triangle \( ABC \), construct a point \( P \) on \( BC \) such that the ratio of the perpendiculars from \( P \) to \( AB, AC \) is equal to 2 : 3.

2. \( ABC \) is an equilateral triangle, side 5 cm.; construct a point \( P \) inside \( \triangle ABC \) such that the perpendiculars from \( P \) to \( BC, CA, AB \) are proportional to 1, 2, 3. Measure \( AP \).

3. Construct an equilateral triangle \( ABC \) such that the length of the line joining \( A \) to a point of trisection of \( BC \) is 2 in. Measure \( BC \).

4. Construct a square \( ABCD \) such that the length of the line joining \( A \) to the midpoint of \( BC \) is 3 in. Measure \( AB \).

5. Using a protractor, construct a regular pentagon \( ABCDE \) such that the distance of \( A \) from \( CD \) is 7 cm. Measure \( CD \).

6. Given a quadrilateral \( ABCD \), construct a similar quadrilateral \( PQRS \) such that \( PQ : AB = 3 : 5 \).

7. Given a triangle \( ABC \), construct a square \( PQRS \) such that \( P, Q \) lie on \( AB, AC \) and \( RS \) lies along \( BC \). [Start by drawing the square \( BCHK \); join \( AH, AK \).]

The square \( PQRS \) is said to be inscribed in \( \triangle ABC \).

8. Inscribe in a given triangle a rectangle such that its height is half its base.

9. Inscribe in a given triangle a triangle whose sides are parallel to the sides of another given triangle.

10. Given two radii \( OA, OB \) of a circle, construct a square \( PQRS \) such that \( P \) lies on \( OA, Q \) lies on \( OB \) and \( R, S \) lie on the arc \( AB \).

11. \( ABC \) is an equilateral triangle, side 5 cm. Construct a line outside \( \triangle ABC \) such that the lengths of the perpendiculars to it from \( A, B, C \) are proportional to 2, 3, 4. Measure the perpendicular from \( C \).

12. Given a triangle \( ABC \), construct a line parallel to \( BC \) cutting \( AB, AC \) at \( P, Q \) such that \( \angle APQ = \frac{1}{2} \angle ABC \).

13. Given a square \( ABCD \), construct two lines parallel to \( AC \) which divide \( ABCD \) into three parts of equal area.

14. Given two equilateral triangles, construct an equilateral triangle whose area is the sum of their areas.

15. Given two squares \( ABCD, PQRS \) and a line \( XY \), construct a line \( ZW \) such that area \( ABCD : area PQRS = XY : ZW \).

16. Construct an equilateral triangle \( ABC \) equal in area to a square, side 5 cm. Measure \( BC \).

17. Construct a triangle \( ABC \) such that \( BC = CA : AB = 6 : 5 : 4 \) and of area equal to a rectangle 4 cm. long, 3 cm. high. Measure \( BC \).

18. Draw a quadrilateral \( ABCD \) in which \( \angle A = 90^\circ, AB = 4 \text{ cm.}, \ BC = 6 \text{ cm.}, CD = 5 \text{ cm.}, DA = 3 \text{ cm.} \) Construct a quadrilateral \( PQRS \) similar to \( ABCD \) and equal in area to a rectangle 4 cm. long, 3-5 cm. high. Measure \( QR \).
[19] Using a protractor, construct a regular pentagon $ABCDE$ equal in area to a square, side 6 cm. Measure $AB$.

*20. Construct an angle of $18^\circ$.

*21. Construct a circle to pass through two given points $A$, $B$ and to touch a given circle $S$. [Draw any circle through $A$, $B$ and cutting $S$ at $H$, $K$ say; let $AB$, $HK$ meet at $T$; draw the tangents $TP$, $TQ$ from $T$ to $S$.]

*22. Construct a circle to pass through a given point $A$ and touch a given circle $S$ and have its centre on a given line $BC$. [Take the image $A'$ of $A$ in $BC$.]

*23. Given four points $A$, $B$, $C$, $D$ in order on a straight line, construct a point $P$ on $BC$ such that $PA : PB = PC : PD$.

*24. Construct a circle to touch two given lines $AB$, $AC$ and touch a given circle, centre $C$, radius $a$. [Draw two lines parallel to $AB$, $AC$ at distance $a$ from them; construct a circle to touch these lines and pass through $O$.]

*25. Draw a circle of radius 5 cm. and take a point $A$ 3 cm. from the centre. Construct a chord $PQ$ of the circle passing through $A$ such that $PA = \frac{1}{2} AQ$.

**REVISION PAPERS 81-88 (Theorems 1-77)**

**Including Similar Triangles**

Arrows indicate that lines are given parallel.

81

1. (i) How many angles each greater than $170^\circ$ is it possible for a ten-sided convex polygon to have?

(ii) What is the least number of obtuse angles a ten-sided convex polygon must have?

2. The tangent at $C$ to a circle $ABC$ and a chord $DE$ of an intersecting circle $ABDE$ meet when produced at $T$. If $CAE$ is a straight line, prove that the circle through $T$, $C$, $D$ passes through $B$.

**REVISION PAPERS 81-88 (Theorems 1-77)**

**Including Similar Triangles**

Arrows indicate that lines are given parallel.

81

1. (i) A line $AB$, 8 cm. long, is divided internally and externally in the ratio 3:1 at $P$ and $Q$. Find the ratio $PQ : AB$.

(ii) Find the unknown marked lengths in fig. 887, the unit of length being 1 cm.

4. If in fig. 888,

\[ \angle ADC = \angle BEA = \angle CFB \]

and if $AD$, $BE$, $CF$ intersect at $Y$, $Z$, $X$ as shown, prove that

\[ YZ : BC = ZX : CA. \]

4. If in fig. 888,

\[ \angle ADC = \angle BEA = \angle CFB \]

and if $AD$, $BE$, $CF$ intersect at $Y$, $Z$, $X$ as shown, prove that

\[ YZ : BC = ZX : CA. \]

82

1. Fig. 889 is a plan of a tennis court, measurements in feet. The only corner-markers that are visible are those at $P$ and $Q$. Give the least necessary calculations you must make to find the exact position of $A$ with the aid only of two tape measures.

2. $AB$, $BC$, $CD$ are three equal chords of a circle. If the tangent at $D$ meets $BC$ produced at $T$, prove that

\[ \angle BAD = \frac{1}{2} \angle TDA. \]

3. Two intersecting lines $AODF$, $BOCE$ are cut by three parallel lines $AB$, $CD$, $EF$. If $AD = 7$ in., $DF = 3$ in., $CE = 4$ in., $EF = 2$ in., $AB = 3$ in., find $BC$ and $CD$.

(ii) Find the unknown marked lengths in fig. 890, the unit of length being 1 cm.

4. $ABP$, $AQB$ are two circles. If $PAQ$ is a straight line prove that

\[ BP : BQ = \frac{1}{2} \text{ the ratio of the diameters.} \]
1. ABCD is a square in a horizontal plane; DK is a vertical post; P, Q are the mid-points of BA, BC; PQ cuts BD at R. If AB = 8 ft., DK = 7 ft., find the length of KR.

2. ABC is a given triangle. P is a variable point inside \( \triangle ABC \) such that \( \angle PBA = \angle PCB \). Find the precise locus of P.

3. Draw an equilateral triangle ABC, side 5 cm. Without making any more measurements, construct (i) a rectangle BCPQ equal to \( \triangle ABC \), (ii) a rectangle QAXY equal to the rectangle BCPQ. State shortly your method.

4. AB is a diameter of the circle APB; Q is a point on the chord AP such that the perpendicular QN from Q to AB is equal to QP. Prove that \( AN : AP = QN : NB \). [Join BQ, BP.]

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84

1. K is any point on the diameter AB of a circle APB; P is the mid-point of the arc AB. Prove that \( AK^2 + BK^2 = 2PK^2 \).

2. ABCD is a parallelogram; P is any point on AC. If the circles PFD, PBC cut again at Q, prove that BQD is a straight line.

3. (i) Find the unknown marked lengths in fig. 891, the unit of length being 1 cm.

(ii) A light is placed 4 ft. in front of a circular hole, diameter 3 in., in a partition. Find the diameter of the illuminated part of a wall 5 ft. behind the partition and parallel to it.

4. (i) AB, DC are parallel sides of the trapezium ABCD; AC cuts DB at K; the line through K parallel to AB cuts AD, BC at P, Q. Prove that PK = QK.

(ii) ABCD, PQRS are quadrilaterals in which \( \angle A = \angle P \), \( \angle B = \angle Q \), \( \angle C = \angle R \), and \( AB : BC = PQ : QR \); prove that ABCD is similar to PQRS.
87

1. In $\triangle ABC$, $AB = AC$ and $\angle BAC = 120^\circ$. If the perpendicular bisector of $AB$ cuts $BC$ at $X$, prove that $BC = 3BX$.

2. In fig. 893, $PAQ$, $QKB$, $PMK$, $QN$ are straight lines. Prove that $QM = QN$. [Join $AB$, $AN$.]

3. (i) The diagonals $AC$, $BD$ of the quadrilateral $ABCD$ cut at $K$. The line through $K$ parallel to $AB$ cuts $AD$, $BC$ at $P$, $Q$ respectively. If $AP = 10$ cm, $PD = 6$ cm, $PK = 9$ cm, $KQ = 4$ cm, $BQ = 9$ cm, find $AB$ and $BC$.

(ii) A straight rod $AB$, 3 ft. 9 in. long, is fixed under water with $A$ 2 ft. 6 in. and $B$ 9 in. below the surface. Find the depth of a point $C$ on the rod where $AC = 1$ ft.

4. $ABCD$ is a quadrilateral; a line $AE$ parallel to $BC$ meets $BD$ at $F$; a line $BE$ parallel to $AD$ meets $AC$ at $E$. Prove that $EF$ is parallel to $CD$.

88

1. Two spheres, radii 6 in., 4 in., have their centres at distance 5 in. apart. Find the radius of the circle which is their curve of intersection and the distances of their centres from the planes of this circle.

2. The circles $APQ$, $AHK$ touch one another at $A$. If $PAH$ is a straight line and if the chords $QP$, $HK$ meet, when produced, at a point $T$ on the tangent at $A$, prove that $T$, $A$, $Q$, $K$ are concyclic.

3. (i) A line $HK$ parallel to $AC$ meets $AB$, $BC$ at $H$, $K$ respectively and the bisector of $\angle ABC$ meets $HK$ at $N$. If $AN = 6$ cm, $HK = 15$ cm, $KC = 4$ cm, find $HN$.

(ii) $P$, $Q$, $R$ are points on the sides $BC$, $CA$, $AB$ respectively such that

\[ BP : PC = 4 : 5, \quad CQ : QA = 3 : 1, \quad AR : RB = 3 : 7. \]

Find the ratio $\triangle PQN : \triangle ABC$.

4. $ABCD$ is a straight line; $O$ is a point outside it; a line through $B$ parallel to $OD$ cuts $OA$, $OC$ produced at $P$, $Q$. If $PB = BQ$, prove that $AB : BC = AD : CD$.

89

1. $AB$ is a diameter of a circle $APB$; the tangent at $A$ meets $BP$ produced at $Q$. Prove that the tangent at $P$ bisects $AQ$.

2. A circular cone is made from a sector of a circle of radius 6 in. and angle $240^\circ$. Find the height of the cone.

3. $XAY$ is a diameter of a circle, centre $A$; $Z$ is the middle point of $AY$. If a circle is drawn on $XZ$ as diameter, prove that the length of the tangent to this circle from any point $P$ on the outer circle is equal to $\frac{1}{2} PX$. [Let $PX$ cut the inner circle at $Q$; join $QZ$, $PY$.]

4. In fig. 894, $ABC$ is an equilateral triangle and $\angle PAQ = 120^\circ$; $PBCQ$ is a straight line. Prove that

(i) $PB : CQ = BC$;

(ii) $PB : CQ = AP : AQ$.

90

1. In $\triangle ABC$, $BC = 24$ in., $CA = 13$ in., $AB = 17$ in. If $BC$ is trisected at $P$, $Q$, find the lengths of $AP$, $AQ$.

2. In fig. 895, the circles touch at $A$; $QP$ produced cuts $AB$ at right angles at the centre of the larger circle. If $PQ = 3$ in., $BC = 5$ in., find the radius of each circle.

3. In fig. 896, prove that

(i) $PQ : BC = NR : NC$;

(ii) $AN : BP = AB : BQ$.

If $AX = 10$ cm, $XP = 5$ cm, $PB = 10$ cm, $AN = AC = 20$ cm, prove that $AR$ bisects $\angle PAQ$.

4. A chord $AD$ is parallel to a diameter $BC$ of a circle; the tangent at $C$ meets $AD$ produced at $E$. Prove that $BC : AE = BD$.
1. **PQRST** is a variable pentagon. If the mid-points of the sides **PQ**, **QR**, **RS**, **ST** are fixed, prove that the side **PT** is of constant length and is fixed in direction.

2. **AB**, **DC** are the parallel sides of a trapezium **ABCD**; **AC** cuts **BD** at **K**. If the areas of \( \triangle AKB, \triangle AKD \) are 3 sq. in. respectively, find the area of \( \triangle DKC \).

(i) With the data of fig. 897, find the values of \( m \) and \( p \).
(ii) Two straight lines **XAB**, **XCD** meet a circle at **A**, **B**, **C**, **D**. Prove that \( \frac{XA}{XD} = \frac{XB}{XC} = \frac{AD}{BC} \).

3. **P** is any point on the circle **ABCD**; **PH**, **PX**, **PK**, **PY** are the perpendiculars from **P** to **AB**, **BC**, **CD**, **DA**, produced if necessary. Prove that
   (i) \( \triangle XPK, \triangle HPY \) are similar; (ii) \( PH \cdot PK = PX \cdot PY \).
   [Join **PC**, **PA**.]

4. **ABCD** is a rectangle; **AB** is produced to **P** so that **AP = AC**;
   **AN** is the perpendicular from **A** to **PD**. Prove that **PN = 2ND**.

5. The centre **C** of a circle **ABP** lies on a circle **AQBC**. If **PAQ** is a straight line and if **QC** produced cuts **PB** at **R**, prove that \( \angle PRQ = \angle KPA \).
   [Join **CA**, **CB**, **CP**, **QB**.]

6. **ABC** is a triangle inscribed in a circle; **P** is any point on the minor arc **BC**; **L**, **M**, **N** are the feet of the perpendiculars from **P** to **BC**, **CA**, **AB**. Join **PB**, **PC** and prove that (i) **P**, **B**, **L**, **N** and **P**, **C**, **L**, **M** are concyclic, (ii) \( \triangle PLN = \triangle PCA \), (iii) **L**, **M**, **N** are collinear.

7. The straight line on which **L**, **M**, **N** lie is called the pedal line (or Simson line) of **P** with respect to **\( \triangle ABC \)**.

8. **AB** is a diameter of the circle **APB**, centre **O**; the chord **BP**, when produced, cuts **T** the tangent at **A**; **OT** meets the circle at **Q**. If **AT = 4 cm**, **QT = 2 cm**, find **OQ** and **PT**.

9. The diagonals of the quadrilateral **ABCD** cut at **K**. State with reasons what angle in the figure equals \( \angle DAC \) if (a) **AK, KC = BK, KD**, (b) **AK, KB = OK, KD**.

10. **AB** is a given line, length 8 cm.; **O** is the mid-point of **AB**; **P** is a variable point on a circle, centre **O**, radius 6 cm. If **PO** produced meets the circle **PAB** at **Q**, find the locus of **Q**.

11. In **\( \triangle ABC \)**, \( \angle A = 90^\circ \); **AD** is an altitude of **\( \triangle ABC \)**. If the bisector of \( \angle ABC \) meets **AD** at **X** and if the bisector of \( \angle DAC \) meets **BC** at **Y**, prove that **XY** is parallel to **AC**.
1. Two chords PQ, RS of a circle intersect at H; K is a point such that $\angle KQP$ and $\angle KRS$ are right angles. Prove that HK is perpendicular to QS.

2. (i) A brick ABCD, 9 in. by 3 in., rests on the ground, and an equal brick PQRS is propped up against it as in fig. 899. If AP = 2 in., find the height of Q, R, S above the ground.

(ii) A line HK parallel to BC cuts AB, AC at H, K; the distance between HK and BC is 5 cm. If the areas of $\triangle AHK$ and $\triangle HKC$ a 9 sq. cm., 40 sq. cm., find the length of HK.

3. ABCD is a cyclic quadrilateral; BA, CD, when produced, meet at X; the line through X parallel to BC meets AD produced at E. Prove that $EX = EA \cdot ED$.

4. P, Q are points on the sides AC, AB of $\triangle ABC$ such that $\angle ACP = \angle BQC$; BP cuts CQ at K; X, Y are points on AC, AB such that $\angle KXK = \angle YKX$ is a parallelogram. Prove that (i) $AX \cdot XC = AY \cdot YB$; (ii) the centre of the circle ABC is equidistant from X and Y.

$95^*$

1. ABCD is a quadrilateral such that $\angle ABD = \angle BDC = \angle ADC = 45^\circ$. Prove that $\angle ADB = \frac{1}{2} \angle ACB$. [Draw CQ bisecting $\angle ACB$ and cutting BD at Q; join AQ. Prove A, D, C, Q are concyclic.]

2. ABCD is a parallelogram; a line through A cuts BD, CD, BC produced, at P, Q, R respectively. Prove that $PQ : PR = PD : PB$.

3. (i) In $\triangle ABC$, AB = AC; the bisector of $\angle ABC$ meets AC at K. If the circle BAK cuts BC or BC produced again at D, prove that AK = CD.

(ii) Two circles, centres A and B, intersect at C, D; P is any point on CD. HPK is the chord of the circle, centre A, which is perpendicular to PA, and XPY is the chord of the circle, centre B, which is perpendicular to PB. Prove that HK = XY.

4. A is a fixed point on a given circle; AP, AQ are variable chords such that AP, AQ is constant. Prove that PQ touches a fixed circle, centre A.

APPENDIX

(t) Proofs of Fundamental Theorems . . . 540

(ii) Limits and Tangents . . . . 550

(iii) Summary of Theorems . . . . 563

(iv) Summary of Constructions . . . . 568
THEOREM 1

If a straight line stands on another straight line, the sum of the adjacent angles so formed is equal to two right angles.

![Diagram](image)

**Fig. 900**

**Given** a straight line CE meeting a straight line ACB.

**To prove that** \(\angle ACE + \angle BCE = \) 2 rt. \(\angle\)s.

**Case 1.** If \(\angle ACE = \angle BCE\), each is by definition a right angle.

\[\therefore \angle ACE + \angle BCE = 2 \text{ rt. } \angle\text{s}.\]

**Case 2.** If \(\angle ACE\) is not equal to \(\angle CEB\), suppose \(\angle ACE\) is the greater.

**Construction.** Draw CN perpendicular to ACB.

**Proof.**

\[\angle ACE + \angle BCE = \angle ACN + \angle NCE + \angle ECB\]

\[= \angle ACN + \angle NCB\]

\[= 2 \text{ rt. } \angle\text{s} \text{ constr.}\]

If \(\angle ECB\) is the greater, the proof is the same, except that A and B are interchanged.

---

**THEOREM 2**

If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.

![Diagram](image)

**Fig. 901**

**Given** two adjacent angles COA, COB such that \(\angle COA + \angle COB = 2\) rt. \(\angle\)s.

**To prove that** AOB is a straight line.

**Construction.** Produce AO to D.

**Proof.**

\(\angle COA + \angle COD = 2\) rt. \(\angle\)s \(\text{adj. } \angle\)s on st. line,

but \(\angle COA + \angle COB = 2\) rt. \(\angle\)s \(\text{given},\)

\[\therefore \angle COA + \angle COD = \angle COA + \angle COB.\]

From these equals, take away the common \(\angle COA\),

\[\therefore \angle COD = \angle COB.\]

But \(\angle COB, \angle COD\) are on the same side of OC,

\[\therefore \text{OD is the same straight line as OB.}\]

But \(\angle AOD\) is a straight line \text{ constr.},

\[\therefore \text{AOB is a straight line.}\]
**THEOREM 4**

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles included by those sides equal, the triangles are congruent.

Given two triangles $ABC$, $PQR$ such that

- $AB = PQ$, $AC = PR$, $\angle BAC = \angle QPR$.

To prove that $\triangle ABC \cong \triangle PQR$.

**Proof.** Apply the triangle $ABC$ to the triangle $PQR$ so that $A$ falls on $P$ and the line $AB$ falls along the line $PQ$ and $C$ falls on the same side of $PQ$ as $R$.

Since $AB = PQ$ \textit{given}, $B$ falls on $Q$.

Since $AB$ falls along $PQ$ and $\angle BAC = \angle QPR$ \textit{given}, $AC$ falls along $PR$.

Since $AC = PR$ \textit{given}, $C$ falls on $R$.

Since $B$ falls on $Q$ and $C$ falls on $R$, $BC$ coincides with $QR$,

$\therefore \triangle ABC$ coincides with $\triangle PQR$,

$\therefore \triangle ABC \cong \triangle PQR$ are congruent.

---

**INEQUALITY THEOREM**

If one side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.

Given a triangle $ABC$ with $BC$ produced to $D$.

To prove that $\angle ACD > \angle ABC$ and $\angle ACD > \angle ACB$.

**Construction.** Bisect $AC$ at $F$.

Join $BF$ and produce it to $G$ so that $BF = FG$.

Join $CG$.

**Proof.** In the triangles $CFG$, $AFB$,

- $CF = AF$ \textit{const.},
- $GF = BF$ \textit{const.},

$\angle CFG = \angle AFB$ \textit{vert. opp. $\angle s$},

$\therefore \triangle CFG \cong \triangle AFB$ S.A.S.

$\therefore \angle FCG > \angle BAF$.

But $\angle ACD$ is greater than its part $\angle FCG$,

$\therefore \angle ACD > \angle BAF$ or $\angle BAC$.

Similarly, if $BC$ is bisected at $H$ and if $AH$ is produced to $K$ so that $AH = HK$, it can be proved that $\angle BKE > \angle ABC$.

But $\angle ACD = \angle BKE$ \textit{vert. opp. $\angle s$},

$\therefore \angle ACD > \angle ABC$.

**Note.** This theorem may be used to prove Theorem 5.
THEOREM 5

Two straight lines are parallel if a transversal makes a pair of alternate angles equal.

Given a straight line $LHKM$ cutting two straight lines $PHQ$, $RKS$ such that $\angle PHK = \text{alternate } \angle HKS$.

To prove that $PQ$ is parallel to $RS$.

Construction and Proof. If $PQ$, $RS$ are not parallel, they will meet when produced, either towards $Q$ and $S$ or towards $P$ and $R$. Suppose if possible that $PQ$ and $RS$, when produced towards $Q$ and $S$, meet at $T$.

Take a point $X$ on $TH$ produced so that $HX = KT$.

Join $KX$.

In $\triangle XHK, TKH$, $\begin{align*}
XH &= TK \\
KH &= HK
\end{align*}$ given,

$. \because \angle XHK \text{ are congruent } \angle TKH$ SAS.

But $a_2 = a_3$ given, $\because c + a_3 = b + a_1$.

But $b + a_1 = 2 \text{ rt. } \angle s$ adj. $\angle s$ on $st. \ line$, $\because c + a_3 = 2 \text{ rt. } \angle s$.

But these are adjacent angles. $\because XKT$ is a straight line. $\therefore$ the straight lines $XHT, KXT$ coincide, and this is contrary to what is given.

$. \therefore PQ, RS$ cannot meet when produced towards $Q, S$.

Similarly, it can be proved that $QP, SR$ cannot meet when produced towards $P, R$.

$. \therefore PQ$ is parallel to $RS$.

THEOREM 6

If a transversal cuts two parallel straight lines, then:
(i) alternate angles are equal,
(ii) corresponding angles are equal,
(iii) interior angles on the same side of the transversal are supplementary.

Given two parallel lines $PHQ$, $RKS$ and a transversal $LHKM$.

To prove that
(i) $\angle PHK = \angle HKS$,
(ii) $\angle LHQ = \angle HKS$,
(iii) $\angle QHK + \angle HKS = 2 \text{ rt. } \angle s$.

(i) Construction. If $\angle PHK$ is unequal to $\angle HKS$, draw $HX$ so that $\angle XHK$ is equal to the alternate $\angle HKS$.

Proof. $\angle XHK = \text{alternate } \angle HKS$ constr., $\because XH$ is parallel to $KS$ given,

but $PH$ is parallel to $KS$ given,

$. \therefore$ two intersecting lines $XH, PH$ are both parallel to $KS$, but this is impossible by Playfair's Axiom.

$. \therefore \angle PHK$ cannot be unequal to $\angle HKS$,

$. \therefore \angle PHK = \angle HKS$.

(ii) $\angle LHQ = \angle PHK$ vert. opp. $\angle s$ proved, $\therefore \angle HKS$.

(iii) $\angle QHK + \angle HKS = 2 \text{ rt. } \angle s$ adj. $\angle s$ on $st. \ line$, $\therefore \angle PHK = \angle HKS$ proved,

$. \therefore \angle QHK + \angle HKS = 2 \text{ rt. } \angle s$. 
THEOREM 10

If two triangles have two angles of the one equal to two angles of the other, each to each, and also a side of one equal to the corresponding side of the other, the triangles are congruent.

\[ \triangle ABC \sim \triangle XRP; \]

but \( \angle ACB = \angle PRQ \) given or proved,

\[ \therefore \triangle PRQ \sim \triangle XRP; \]

but this is impossible because one of these angles is a part of the other.

\[ \therefore QP \text{ cannot be unequal to } BA. \]

\[ \therefore QP = BA. \]

\[ \therefore \text{in } \triangle ABC, PQR, \]

\[ \angle A = \angle P \text{ proved,} \]

\[ \angle C = \angle R \text{ given,} \]

\[ \therefore \triangle ABC \text{ and } PQR \text{ are congruent SAPS.} \]

Given two triangles \( ABC, PQR \) such that

\[ BC = QR \]

and the angles of two of the pairs

\[ \angle A, \angle P; \angle B, \angle Q; \angle C, \angle R \]

are equal.

To prove that \( \triangle ABC \text{ and } PQR \) are congruent.

Construction and Proof. Since the sum of the angles of a triangle is two right angles, the angles of the third pair are also equal.

Suppose, if possible, that \( QP \) is not equal to \( BA \), then there is a point \( X \) on \( QP \) or \( QP \) produced such that \( QX = BA \).

Join \( RX \).

In \( \triangle ABC, XQR \),

\[ BC = QR \text{ given,} \]

\[ BA = QX \text{ constr.} \]

\[ \angle B = \angle Q \text{ given,} \]

\[ \therefore \triangle ABC \text{ and } XQR \text{ are congruent SAPS.} \]
THEOREM 54

In equal circles, equal angles at the centres and equal angles at the circumferences stand on equal arcs.

(i) Given two equal circles AXBP, CYDQ, centres H, K, and two arcs AXB, CYD which subtend equal angles AHB, CKD at the centres.

To prove that \( \text{arc } AXB = \text{arc } CYD \).

Proof. Apply the circle AXB to the circle CYD so that the centre H falls on the centre K and HA falls along KC and HB falls on the same side of KC as KD.
Since the circles are equal, A falls on C and the circumferences coincide.
Since \( \angle AHB = \angle CKD \) given, HB falls along KD and B falls on D.
\( \therefore \) the arcs AXB, CYD coincide.
\( \therefore \text{arc } AXB = \text{arc } CYD \).

(ii) Given two equal circles ABP, CDQ, centres H, K, and two arcs AXB, CYD which subtend equal angles APB, CQD at the circumferences.

To prove that \( \text{arc } AXB = \text{arc } CYD \).

Proof. \( \angle AHB = 2 \angle APB \) \( \angle \text{at centre} = \text{twice } \angle \text{at } O^\circ \),
\( \angle CKD = 2 \angle CQD \) \( \angle \text{at centre} = \text{twice } \angle \text{at } O^\circ \),
but \( \angle APB = \angle CQD \) given, \( \therefore \angle AHB = \angle CKD \).
\( \therefore \text{arc } AXB = \text{arc } CYD \).

THEOREM 55

In equal circles, equal arcs subtend equal angles at the centres and equal angles at the circumferences.

Given two equal circles AXBP, CYDQ, centres H, K, and two equal arcs AXB, CYD.

To prove that (i) \( \angle AHB = \angle CKD \),
(ii) \( \angle APB = \angle CQD \).

(i) Proof. Apply the circle AXB to the circle CYD so that the centre H falls on the centre K and HA falls along KC and HB falls on the same side of KC as KD.
Since the circles are equal, A falls on C and the circumferences coincide.
Since \( \text{arc } AXB = \text{arc } CYD \) given, B falls on D and HB falls on KD,
\( \therefore \angle AHB \) coincides with \( \angle CKD \),
\( \therefore \angle AHB = \angle CKD \).

(ii) Proof. \( \angle AHB = 2 \angle APD \) \( \angle \text{at centre} = \text{twice } \angle \text{at } O^\circ \),
\( \angle CKD = 2 \angle CQD \) \( \angle \text{at centre} = \text{twice } \angle \text{at } O^\circ \),
But \( \angle AHB = \angle CKD \) proved,
\( \therefore \angle APB = \angle CQD \).
(π) THE TANGENT AS A LIMITING CHORD

A is a given point on a circle. Any line AQ through A cuts the circle at P.

Bisect arc AP at P₁; bisect arc AP₁ at P₂; bisect arc AP₂ at P₃, etc. We can repeat this bisection process as often as we like, and thus obtain a succession of lines AP, Q₁, AP₁Q₁, AP₂Q₂, AP₃Q₃, etc., which cut off from the circle arcs of continually decreasing lengths.

![Fig. 509](image)

Fig. 509

However often we repeat the process we cannot obtain a line cutting off an arc of zero length; but by repeating it sufficiently often we can obtain a line which cuts off an arc as short as we please, and all further lines obtained will cut off still shorter arcs.

The limiting position AR of this series of lines is called the tangent at A and is a line drawn through A cutting off an arc of zero length.

If the process of repeated bisection is performed on the other side of the chord AP, we obtain the limiting position AR of this series of lines, which is equally by our definition the tangent at A. It is necessary to prove that AR and AS are in one straight line, in order to show that there is only one tangent to a circle at any point A. This is done by showing that each is at right angles to the radius through A.

---

THEOREM 59

The tangent to a circle is perpendicular to the radius drawn through the point of contact.

![Fig. 910](image)

Fig. 910

Given a tangent XAY at a point A on a circle, centre O.

To prove that \( \angle OAY \) is a right angle.

Construction. Through A draw any line X'AP'Y' cutting the circle again at P'.

Join OP'.

Proof. \( OA = OP' \) radii,

\[ \angle OAP' = \angle OP'A \] base \( \angle s, \) isos. \( \Delta. \)

Since X'AP'Y' is a straight line, \( \angle OAX', \angle OP'Y' \) are the supplements of \( \angle OAP', \angle OP'A, \)

\[ \angle OAX' = \angle OP'Y'. \]

Now the tangent XAY at A is the limiting position of the line X'AP'Y' when the arc AP' is decreased without limit so that P' coincides with A.

\[ \angle OAY = \angle OAY. \]

But these are adjacent angles on a straight line,

\[ \angle OAY - 1\text{ rt. } \angle. \]
THEOREM 61

If a straight line touches a circle and from the point of contact a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments of the circle.

Fig. 911

Given a tangent XAY at a point A on a circle and a chord AD forming the two segments AHD, AKD.

To prove that
(i) \( \angle DAX = \angle AKD \) in alternate segment AKD,
(ii) \( \angle DAX = \angle AHD \) in alternate segment AHD.

Construction. Through A draw any line X'AP'Y' cutting the arc AHD at P'.

Join DP'.

Proof. \( \angle DP'Y' = \angle AKD \) ext. \( \angle \) of cyclic quad. = int. opp. \( \angle \).

Now the tangent at A is the limiting position of the line X'AP'Y' when the arc AP' is decreased without limit so that P' coincides with A.

But the limiting position of \( \angle DP'Y' \) is \( \angle DAY \).

\( \therefore \) when X'AP'Y' becomes the tangent XAY at A, \( \angle DAY = \angle AKD \).

Similarly, it may be proved that \( \angle DAX = \angle AHD \).

(iii) SUMMARY OF THEOREMS

Theorem 1 (pp. 86, 540)
If \( \triangle ABC \) is a straight line,\[ a + b = 2 \text{ rt. } \angle s. \]

Theorem 2 (pp. 87, 541)
If \( a + b = 2 \text{ rt. } \angle s \),\[ \triangle AOB \] is a straight line.

Theorem 3 (pp. 28, 87)
If two straight lines intersect, the vertically opposite angles are equal.\[ a = b \text{ and } p = q. \]

Theorem 4 (p. 542)
If \( AB = PQ, AC = PR, \angle A = \angle P \),\( \triangle ABC \) and \( \triangle PQR \) are congruent SAS.

Theorem 5 (pp. 95, 544)
If \( PQ \) and \( RS \) are parallel if\[ \text{either } a = b \text{ alt. } \angle s, \]
or \( c = b \) corr. \( \angle s \),
or \( b + d = 2 \text{ rt. } \angle s \) int. \( \angle s \).

Theorem 6 (p. 545)
If \( PQ \) and \( RS \) are parallel,\[ a = b \text{ alt. } \angle s, \]
\[ c = b \text{ corr. } \angle s, \]
\[ b + d = 2 \text{ rt. } \angle s \text{ int. } \angle s. \]

† The figures in this Summary are small reproductions of the figures in the main text; reference is given in each case to the page on which the larger-scale figure will be found.
Theorem 7 (pp. 39, 96)
If \( PQ \) and \( RS \) are each parallel to \( XY \), then \( PQ \) is parallel to \( RS \).

Theorem 8 (p. 102)
(i) If the side \( BC \) of \( \triangle ABC \) is produced to \( D \),
\[
\angle ACD = \angle A + \angle B.
\]
(ii) \( \angle A + \angle B + \angle ACB = 2 \text{ rt. } \angle s \).

Theorem 9 (p. 112)
(i) The sum of the interior angles of a convex polygon with \( n \) sides is \((2n - 4) \text{ rt. } \angle s\).
(ii) The sum of the exterior angles of a convex polygon with \( n \) sides formed by producing the sides in order is \( 4 \text{ rt. } \angle s \).

Theorem 10 (p. 546)
If \( \angle B = \angle Q \) and \( \angle C = \angle R \)
and if either \( BC = QR \)
or \( AB = PQ \) or \( AC = PR \),
\[
\triangle ABC \text{ are congruent } \triangle PQR \text{ ASA, AAS.}
\]

Theorem 11 (p. 119)
If \( AB = AC \), then \( \angle B = \angle C \).

Theorem 12 (p. 120)
If \( \angle B = \angle C \), then \( AB = AC \).

Theorem 13 (p. 128)
If \( AB = PQ \), \( BC = QR \), \( CA = RP \),
\[
\triangle ABC \text{ are congruent } \triangle PQR \text{ SSS.}
\]

Theorem 14 (p. 130)
If \( AC = XZ \), \( AB = XY \), and if \( \angle B, \angle Y \text{ are right angles}, \)
\[
\triangle ABC \text{ are congruent } \triangle XYZ \text{ RHS.}
\]

Theorem 15 (p. 150)
If \( ABCD \) is a parallelogram,
(i) \( AB = DC \) and \( AD = BC \);
(ii) \( \angle A = \angle C \) and \( \angle ABC = \angle ADC \);
(iii) \( BD \text{ bisects area } ABCD \).

Theorem 16 (p. 151)
If \( ABCD \) is a parallelogram whose diagonals cut at \( K \),
\( AK = KC \) and \( BK = KD \).

Theorem 17 (p. 152)
If \( ABCD \) is a quadrilateral in which \( AB \) is equal and parallel to \( DC \), then \( ABCD \) is a parallelogram.
Theorem 18 (p. 153)
If $ABCD$ is a quadrilateral in which
\[ \angle A = \angle C \quad \text{and} \quad \angle B = \angle D, \]
then $ABCD$ is a parallelogram.

Fig. 322

Theorem 19 (p. 154)
If $ABCD$ is a quadrilateral in which
\[ AB = DC \quad \text{and} \quad AD = BC, \]
then $ABCD$ is a parallelogram.

Fig. 353

Theorem 20 (p. 155)
If the diagonals of a quadrilateral $ABCD$ cut at $K$ and if $AK = KC$ and $BK = KD$, then $ABCD$ is a parallelogram.

Fig. 354

Theorem 21 (p. 167)
If in $\triangle ABC$, $AC > AB$,
then $\angle ABC > \angle ACB$.

Fig. 371

Theorem 22 (pp. 168, 169)
If in $\triangle ABC$, $\angle B > \angle C$,
then $AC > AB$.

Fig. 372

Theorem 23 (p. 170)
If $CN$ is the perpendicular from $C$ to a straight line $ANPB$,
then $CN < CP$.

Fig. 376

Theorem 24 (p. 172)
If $ABC$ is any triangle,
\[ BA + AC > BC. \]

Fig. 377

Theorem 25 (p. 180)
If $H$, $K$ are the mid-points of $AB$, $AC$,
then
(i) $HK$ is parallel to $BC$,
(ii) $HK = \frac{1}{2} BC$.

Fig. 397

Theorem 26 (p. 181)
If $H$ is the mid-point of $AB$, and if the line through $H$ parallel to $BC$ cuts $AC$ at $K$,
then $AK = KC$.

Fig. 398

Theorem 27 (p. 182)
If two transversals $ABCD$, $PQRST$ are cut by the parallel lines $BQ$, $CR$, $DS$, $ET$,
and if $BC = CD = DE$,
then $QR = RS = ST$.

Fig. 399

Theorem 28 (p. 192)
The medians $AD$, $BE$, $CF$ of $\triangle ABC$ concur at a point $G$, such that
\[ DG = \frac{1}{2} DA, \quad EG = \frac{1}{2} EB, \quad FG = \frac{1}{2} FC. \]

Fig. 416
Theorem 29 (pp. 200, 201)

The locus of a point equidistant from two given points A and B is the perpendicular bisector of AB.

Theorem 30 (p. 204)

The perpendicular bisectors of the three sides of a triangle are concurrent. The point at which they concur is the circumcentre of the triangle.

Theorem 31 (p. 205)

The altitudes of a triangle are concurrent. The point at which the altitudes concur is called the orthocentre of the triangle.

Theorem 32 (pp. 208, 209)

The locus of a point which is equidistant from two given intersecting straight lines is the pair of lines which bisect the angles between the given lines.

Theorem 33 (p. 213)

The internal bisectors of the three angles of a triangle are concurrent. The point at which they concur is the in-centre of the triangle.

Theorem 34 (p. 235)

The area of a rectangle is measured by the product of the measures of two adjacent sides.

Theorem 35 (p. 252)

The area of a parallelogram ABCD is equal to the area of a rectangle ABHK on the same base AB and between the same parallels AB, KHDC.

Corollary 1. Parallelograms on the same base and between the same parallels are equal in area.

Corollary 2. Area of parallelogram = base × height.

Corollary 3. Parallelograms on equal bases and between the same parallels are equal in area.

Theorem 36 (p. 254)

The area of a triangle ABC is equal to half the area of a rectangle PQBC on the same base BC and between the same parallels BC, QPA.

Corollary 1. Area of triangle = 1/2 base × height.

Corollary 2. Triangles on the same or equal bases and of equal altitudes are equal in area.

Corollary 3. If triangles of the same area have the same or equal bases, their altitudes are equal.

Theorem 37 (p. 256)

If AD is parallel to BC,
\[ \triangle ABC = \triangle DBC. \]

Theorem 38 (p. 257)

If \( \triangle ABC = \triangle DBC \) and if A, D are on the same side of BC, then AD is parallel to BC.
Theorem 39 (p. 288)
If a triangle $ABC$ and a parallelogram $PQBC$ are on the same base $BC$ and between the same parallels $BC$, $AQP$, then $\triangle ABC = \frac{1}{2}$ parallelogram $PQBC$.

Theorem 40 (p. 282)
If $\angle BAC = 1$ rt. $\angle$, and if $AX$ is the perpendicular from $A$ to $BC$, then
(i) $BA^2 = BX \cdot BC$ and $CA^2 = CX \cdot CB$,
(ii) $AB^2 + AC^2 = BC^2$.

Theorem 41 (p. 285)
If in $\triangle ABC$, $AB^2 + BC^2 = AC^2$, then $\angle ABC = 1$ rt. $\angle$.

Theorem 42 (p. 300)
If $M$ is the mid-point of a chord $AB$ of a circle, then $\angle OMA = 1$ rt. $\angle$.

Theorem 43 (p. 301)
If $ON$ is the perpendicular from the centre $O$ of a circle to a chord $AB$, then $AN = NB$.

Theorem 44 (p. 302)
If the chords $AB$ and $CD$ of a circle are equal, they are equidistant from the centre.

Theorem 45 (p. 303)
If the chords $AB$ and $CD$ of a circle are equidistant from the centre, then $AB = CD$.

Theorem 46 (p. 304)
There is one circle, and only one circle which passes through three given points $A$, $B$, $C$, not in the same straight line. The perpendicular bisectors of $BA$, $BC$ meet at the centre $O$ of the circle.

Theorem 47 (p. 313)
The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference. If $O$ is the centre of the circle $APB$, then $\angle AOB = 2 \angle APB$.

Theorem 48 (p. 320)
If $APQB$ is a circle, then $\angle APB = \angle AQB$.

Theorem 49 (p. 321)
If $AB$ is a diameter of the circle $APB$, then $\angle APB = 1$ rt. $\angle$. 
Theorem 50 (p. 322)
(i) If \(ABCD\) is a circle,
\[\angle B + \angle D = 2\text{ rt. }\angle\text{s.}\]
(ii) If \(ABCD\) is a circle and if the chord \(AD\) is produced to \(E,\)
\[\angle CDE = \angle ABC.\]

Theorem 51 (p. 331)
If \(\angle BAC = 1\text{ rt. }\angle,\)
the circle on \(BC\) as diameter passes through \(A.\)

Theorem 52 (p. 332)
If \(\angle APB = \angle AQB,\) and if \(P, Q\)
are on the same side of \(AB,\)
\(A, B, P, Q\) are concyclic.

Theorem 53 (p. 334)
If \(ABCD\) is a quadrilateral in which \(\angle ABC + \angle ADC = 2\text{ rt. }\angle\text{s,}\)
\(A, B, C, D\) are concyclic.

Theorem 54 (p. 548)
If \(H, K\) are the centres of two equal circles \(APBX, CQDY,\)
and if \(\angle AHB = \angle CKD,\)
or if \(\angle APB = \angle CQD,\)
then \(\text{arc } AXB = \text{arc } CYD.\)

Theorem 55 (p. 549)
If \(H, K\) are the centres of two equal circles \(APBX, CQDY,\)
and if \(\text{arc } AXB = \text{arc } CYD,\)
then \(\angle AHB = \angle CKD\)
and the angles at the circumferences standing on the equal arcs \(AXB, CYD\) are equal.

Theorem 56 (p. 347)
If \(AB\) and \(CD\) are equal chords of equal circles,
then \(\text{minor arc } AB = \text{minor arc } CD.\)

Theorem 57 (p. 348)
If \(AB\) and \(CD\) are equal arcs of equal circles,
then \(\text{chord } AB = \text{chord } CD.\)

Theorem 58 (p. 358)
If \(OA\) is a radius of a circle, centre \(O,\) and if \(BAC\) is a straight line perpendicular to \(OA,\)
\(BAC\) is a tangent to the circle.

Theorem 59 (pp. 359, 551)
If \(BAC\) is the tangent at \(A\) to a circle, centre \(O,\)
\[\angle OAB = 1\text{ rt. }\angle.\]

Theorem 60 (p. 360)
If \(TP, TQ\) are the tangents from \(T\) to a circle, centre \(O, P\) and \(Q\) being the points of contact,
(i) \(TP = TQ;\)
(ii) \(\angle TOP = \angle TOQ;\)
(iii) \(OT\) bisects \(\angle PTQ.\)
**Theorem 61** (pp. 368, 552)

If $BAC$ is the tangent at $A$ to a circle and if $AD$ is any chord,

$$\angle DAC = \angle APD$$

in alternate segment.

$$\angle DAB = \angle AQD$$

in alternate segment.

**Fig. 680**

**Theorem 62** (p. 370)

If $P$ and $C$ are points on opposite sides of $AD$ such that $\angle DAC = \angle APD$,

$AC$ touches at $A$ the circle $APD$.

**Fig. 681**

**Theorem 63** (p. 379)

If two circles, centres $A$, $B$, touch one another at $P$, then $A$, $P$, $B$ are collinear.

If the contact is external,

$$AB = \text{sum of radii}.$$  

If the contact is internal, $\text{AB = difference of radii}.$

**Theorem 64** (p. 417)

If $CN$ is an altitude of $\triangle ABC$, and if $\angle BAC$ is obtuse,

$$BC^2 = BA^2 + CA^2 + 2BA \cdot AN.$$  

**Fig. 749**

**Theorem 65** (p. 418)

If $CN$ is an altitude of $\triangle ABC$, and if $\angle BAC$ is acute,

$$BC^2 = BA^2 + CA^2 - 2BA \cdot AN.$$  

**Fig. 750**

**Theorem 66** (p. 419)

If $AD$ is a median of $\triangle ABC$,

$$AB^2 + AC^2 = 2AD^2 + 2BD^2.$$  

**Fig. 751 (ii)**

**Theorem 67** (p. 426)

If the chords $AB$, $CD$ of a circle, centre $O$, radius $r$, intersect at a point $X$ inside the circle,

$$XA \cdot XB = XC \cdot XD = r^2 - OX^2.$$  

**Fig. 761**

**Theorem 68** (p. 427)

If the chords $AB$, $CD$ of a circle, centre $O$, radius $r$, intersect at a point $X$ outside the circle, and if $XT$ is the tangent from $X$ to the circle,

$$XA \cdot XB = XC \cdot XD = XT^2 = OX^2 - r^2.$$  

**Fig. 762**

**Theorem 69** (p. 428)

(i) If $AB$ and $CD$ intersect at $X$ or meet, when both are produced, at $X$, and if $XA \cdot XB = XC \cdot XD$,

then $A$, $B$, $C$, $D$ are concyclic.

(ii) If $AB$ is produced to $X$ and if $C$ is a point not on $AB$ such that

$$XA \cdot XB = XC^2,$$

then the circle $ABC$ touches $XC$ at $C$.

**Theorem 70** (p. 462)

If the altitudes $AH$, $XK$ of $\triangle ABC$, $\triangle XYZ$ are equal,

$$\frac{\triangle ABC}{BC} = \frac{\triangle XYZ}{YX}.$$  

**Fig. 806**
Theorem 71 (p. 464).
If a line parallel to BC cuts AB, AC at X, Y,
then \[ \frac{AX}{XB} = \frac{AY}{YC} \]

Theorem 72 (p. 465).
If X, Y are points on AB, AC or on AB, AC produced such that \[ \frac{AX}{XB} = \frac{AY}{YC} \]
then XY is parallel to BC.

Theorem 73 (p. 472).
If the internal or external bisector of \( \angle BAC \) meets BC or BC produced at D,
then \[ \frac{BD}{DC} = \frac{AB}{AC} \]

Theorem 74 (p. 473).
If the base BC of \( \triangle ABC \) is divided internally or externally at D so that \[ \frac{BD}{DC} = \frac{AB}{AC} \]
then \( AD \) is an internal or external bisector of \( \angle BAC \).

Theorem 75 (p. 490).
If in \( \triangle ABC, XYZ \),
\[ \angle A = \angle X, \quad \angle B = \angle Y, \quad \angle C = \angle Z, \]
then \[ \frac{AB}{BC} = \frac{CA}{XY} = \frac{Z}{ZX} \]

Theorem 76 (p. 492).
If in \( \triangle ABC, XYZ \),
\[ \frac{BC}{CA} = \frac{AB}{YZ} = \frac{ZX}{XY} \]
then \[ \angle A = \angle X, \quad \angle B = \angle Y, \quad \angle C = \angle Z. \]
SUMMARY OF CONSTRUCTIONS

2. Construction of a square; construction with numerical data of parallelogram, trapezium, etc. Pages 162, 163.
3. Division of a line into equal parts. Page 188.
4. Reduction of a quadrilateral and a polygon to an equivalent triangle. Page 263.
5. Construction of circle through three points. Page 304.
10. Construction for inscribing a triangle in a given circle and circumscribing a triangle about a given circle, equiangular to a given triangle. Page 392.
11. Construction of circles through given points or touching given lines or circles. Pages 392–394, 434.
12. Construction of a square equivalent to a rectangle or polygon. Pages 432, 433.
16. Construction of a polygon similar to a given polygon and with (i) the sides, (ii) the areas, in a given ratio. Pages 523, 524.
17. Construction of a quadrilateral similar to a given quadrilateral and equivalent to a given rectangle. Page 525.

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ANSWERS
STAGE A

Page 3
EXERCISE 1
1. 3, 4, 2. 2, 3, 4. 3 GBD; GBDF; CDH; ABH; CBE.
5. 6; 10; [4]. 6. Cylindrical; sphere; cylinder; cone; cuboid.
9. Cylinder; prism; cone and cylinder.
8. 7; 15; 10.
11. Frustum of pyramid.
13. (i) 6, 8, 12; (ii) 5, 6, 9;
(iii) 4, 4, 6; (iv) 8, 12, 18; (v) 6, 8, 12; (vi) $n + 2$, 2$n$, 3$n$.
17. 12; 6. 18; 30; 20. 19. 30; 12. 20. $\frac{n(n-1)}{2}$.

Page 8
EXERCISE 2
1. 2.69, 2.77, 3.02, in.; 6.82, 7.05, 7.68, cm.; 8.48 in.; 21.55 cm.
2. 2.60, 3.43, 4.12, in.; 6.06, 6.17, 10.49, cm.; 9.15 in.; 23.26 cm.
3. 1.31, 2.92, 1.68, 2.35, in.; 8.56 in.
4. 3.25, 5.49, 3.10, 3.74, cm.; 15.58 cm.
5. 3.15 cm., 5.68 cm., 3.21 in. 6. 4.16 cm., 4.12 cm., 3.98 in.
7. 3.34, 3.56, 5.02, 1.88, cm.; 6.90, 6.90, cm. 8. 2.28 cm.
9. 1 P, 2.36 in.; EF, 2.43 in. 10. AB. 11. 3.94 in.; 0.304 in.
12. 12.7 cm., 2.54 cm. 14. 2, 5, 9. 15. 4.
16. 35, 8; $\frac{n(n-3)}{2}$, $n - 2$.

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EXERCISE 3
1. 8 cm.; 7.94 cm. 2. 2.5 in. 3. 3, 8, 3, cm. 4. 4 in.
5. 4.5 in., 9.5 in. 6. 1 in. 7. D, E, A, F, Y, N, C.
8. 7 cm., 1 cm. 9. 2.5, 2, 1.7, in. 10. 4.8 cm.
17. 0.12 cm. 18. 8.71 cm. 19. 6.66 cm. 23. 6 cm.
24. 11 cm. 26. 4 cm. 29. 2.7 in. or 6.85 cm.

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EXERCISE 4
5. 2.5 in. 6. 6.40 cm. 9. 4.95 cm.
STAGE A
EXERCISE 6
[The unit for angle-measurements is 1 right angle]
1. 1. 2. 3. 3. 4. 3. 5. 1\frac{1}{2}. 6. 2\frac{1}{2}.
7. 2. 8. 3. 9. 1\frac{1}{2}. 10. 5. 11. W. 12. W.
24. 2. 1\frac{1}{2}. 1\frac{3}{4}. 25. 1\frac{1}{2}. 26. 14. 1\frac{1}{4}.

EXERCISE 7
[The unit for angle-measurements is 1 right angle]
1. 1. 2. 3. 3. 1\frac{1}{2}. 4. a + b, a + b - c, a + 2b + c, c.
5. \angle POR, \angle QOS, \angle POS. 6. b = c; a = c; a + b = 1\text{rt. \angle}.
7. 1\frac{1}{2}. 8. 3. 9. 1\frac{1}{3}. 10. 3. 11. 1\frac{1}{3}. 12. 2\frac{1}{4}.
13. 1\frac{1}{4}. 14. 1\frac{1}{4}. 15. 1\frac{1}{4}. 1\frac{1}{2}. 16. 1\frac{1}{4}.
17. a, b (acute), c (obtuse), c, c (reflex).
18. 2\frac{1}{4}.
19. 2\frac{1}{4}. 3\frac{1}{2}.
20. (i) yes, (ii) no, (iii) yes.

EXERCISE 8
1. 180°, 45°, 60°, 126°.
2. 2\frac{1}{2}°, 22\frac{1}{2}°, 120°, 234°.
3. 3\frac{1}{3}, 1\frac{1}{3}, 3\frac{1}{3}, rt. \angle.
4. 1\frac{1}{2}, 1\frac{1}{3}, 2\frac{1}{2}, 3\frac{1}{3}, \text{rt. \angle}.
5. 30°, 5\frac{1}{2}, 22\frac{1}{2}, 75°.
6. 90°, 30°, 90°, 30°.
7. 200°.
8. 120°.
9. 117°, 63°.
10. 83°, 97°.
11. 33\frac{1}{3}, 55\frac{1}{3}°.
12. A 57\frac{1}{2}°, B 67\frac{1}{2}°, C 55°.
13. D 33\frac{1}{3}, E 110°, F 36\frac{1}{3}°.
14. L 90°, M 102°, N 63°, P 105°.
15. 297°.
16. 240°.

EXERCISE 9
1. N. 70° E.
2. S. 42° W.
3. S. 20° W.
4. N. 40° E.
5. N. 38° W.
6. S. 30° E.
7. N. 75° E.
8. N. 30° E.
9. S. 5° W.
10. S. 65° E.
11. 222°; 40°; 302°.
12. 25°, 13. 120°, 14. 140°, 15. 120°, 16. S. 30° E.; N. 10° W.
18. 22\frac{1}{2}, 157\frac{1}{2}, 45°.
19. 67\frac{1}{2}°, 67\frac{1}{5}°, 112\frac{1}{3}°.
21. 1\frac{1}{4}, 56\frac{1}{2}, 33\frac{1}{2}°.
22. 56\frac{1}{2}, 67\frac{1}{2}°, 86°.

EXERCISE 10

EXERCISE 11

EXERCISE 12

EXERCISE 13

EXERCISE 14
STAGE A

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Exercise 22

1. 0.290.
2. 0.391.
3. 0.339.
4. 0.848.
5. 30°.
6. 25° 50′.
7. 41° 49′.
8. 4.19, 2.72, cm.
9. 2.46, 1.72, in.
10. 9.45, 3.26, cm.
11. 66° 56′.
12. 53° 8′.
13. 39° 48′.
14. 24° 9′.
15. 19-0, 6-18, ft.
16. 73° 44′.
17. 201, 223, yd.
18. 399, 3-01, mi.
19. N. 51° 4′ E.
20. 662 ft.
21. 3-18, 5-09, cm.
22. 92-7 ft.
23. 5-49 in.
24. 2-17 in.

ANSWERS

STAGE B

PART I

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Exercise 24

1. 160°, 30°, 88°.
2. 36°.
3. 22°.
4. 144°.
5. 6, 11, 22.
6. 180° 50′.
7. 270° 10′.
8. 150° 100′.
9. 60°.
10. 240°.
11. 45°.
12. 157° 5′.
13. 112° 3′.
14. 135°.
15. 157°.
16. 130°.
17. 140°.
18. 157°.
19. 130°.
20. 120°.
22. 45°.
23. 20°.
24. 15°.
25. 24°.
26. 95°.
27. 50°.
28. 120°.
29. 135°.
30. 135°.
31. 105°.
32. 40°.
33. x = y + z = 180°.
34. 120°, 60°.
35. PAS, QAT.
36. 110°; KAM, HAL.

Exercise 25

1. a + b, b + 2c + d.
2. b + c + d, c.
3. ∠ROT, ∠POS.
4. c = d, a + b + c + d = 180°.
5. b = d, b + c = 90°.

Exercise 26

1. 24°.
2. 55°.
3. 20°.
4. 36°, 72°.
5. 22°, 57°.
6. 30°.
7. 80°.
8. 120°.
9. 80°.
11. AQ, BY; AP, BZ.

Exercise 28

1. 55°, 80°.
2. 65°, 50°.
3. 35°, 75°.
4. 47°.
5. 35°; 90° - x°.
6. 147°.
7. 78°.
8. 68°.
9. 45°.
10. 80°.
11. 90° - 4c.
12. 90° - A.
13. 90° - A.
14. 20°.
15. 40°.
16. 110°.
17. 110°.
18. 40°.
19. x = 36°, z = 72°.
20. 25°.
21. 38°.
22. 20°.
23. 130°.
24. 92°.
27. 53°, 70°, 57°.
28. 60°.
29. 84°.

Exercise 29

8. b + c - a.
12. p - a + b.
13. c - a - b.
14. f - d - e.
15. ∠B.

Exercise 30

p + q + r - 4t = 4 rt. ∠s.

Oral Examples

1. 8 rt. ∠s.
2. 10 rt. ∠s.
3. 16, 196, 2n - 4, rt. ∠s.
4. 4 rt. ∠s.
5. 4 rt. ∠s.
6. 4, 4, rt. ∠s.
7. 4 rt. ∠s.
8. 8 + q + r - 4t = 4 rt. ∠s.

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Exercise 30

1. 70°.
2. 122°.
3. 80°.
4. 18°.
5. 54°.
6. 80°.
7. 36°.
8. 56°, 78°, rt. ∠s.
9. 45°, 36°.
NEW GEOMETRY

Page 118  Exercise 32
1. ABC, SAS.  No. 3. ABC, ASA.  No.
5. ABC, SAS.  No. 7. No. 8. ∠X, CA.
18. ∆BAP, ∆CAQ.  20. ∆QXC, ∆XDP, ∆XDP; ∆XDP.

Page 122  Exercise 33
1. 35°, 2. 56°. 3. x, 180 - 2x or 90 - x, 90 - x, degrees.
4. 72°, 72°, 30°. 5. 36°, 36°, 90°. 6. 30°. 7. 571°.
8. 35°, 120°, 20°. 9. 38°. 11. 38°. 15. 360 - 2y.
17. (45 - ½) degrees. 18. 3x - 180. 19. 4(90 - x) degrees. 22. 36°.

Page 149  Exercise 37
1. 58°. 2. 23°. 3. 62°. 4. 110°. 5. 55°, 35°. 6. 32°.
7. 125°. 8. 18°, 27°. 9. 30°, 30°. 11. 224°. 12. 54°.
17. 67°.
18. 224°, 135°, 224°.

Page 160  Exercise 39
1. 363°; impossible. 2. 259. 3. 293. 4. 479; impossible.
5. 11-3. 6. 6-68; 5-66, 3-53; impossible. 7. 8-37. 8. 8-41.
9. 1041°. 10. 4-96. 11. 6-70. 12. 651°. 13. 5-18. 14. 3-82.
15. 62°.
16. 5-23 cm. 17. 194°. 18. 824°, 81°.

Page 182  Oral Examples
1. 6-78 cm. 2. 7-36 cm. 3. 3-55 cm.

Page 184  Exercises 40
1. 4-77 cm. 2. 477°. 3. 2-55 cm. 4. 3-54 cm.
5. 7-13, 3-63 cm. 6. 1061°. 7. 8-74 cm. 8. 4-08 in.
9. 5-41 cm. 10. 3-39 in. 11. 6-09 cm. 12. 25°. 13. 8-25.
14. 5-34. 15. 6-21. 16. 6. 17. 8-64 cm. 19. 621°.
20. 1173°. 21. 3-68 cm. 22. 8-24 cm. 25. 4-96 cm.
27. 4-26. 28. 9-28 or 3-72. 29. 10-69 or 2-92. 30. 8-07.
31. 4-78. 32. 7-82. 33. 8-71. 34. 6-22 cm.
NEW GEOMETRY

Page 230
Paper 6. 1. 6 ft. rt. L s. 2. 108. 3. 90 - x - y, x - y, degrees.
Paper 7. 1. 60°. 3. 90°.

Page 231
Paper 8. 1. 36, 28°. 2. 50°, 72°.
3. y = \frac{x}{8 - x}; x = 4, 5, 6, 7; y = 6, 10, 18, 42.
Paper 9. 1. 68°.

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Paper 10. 1. 72°. 2. a + b - c. 3. 36°.
Paper 11. 1. 110°.
Paper 12. 2. 120°; AE, CD; ED, BC.

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Paper 13. 1. 180 - 4x, \frac{x}{x}, degrees. 2. a + b - c.
Paper 14. 1. 72°. 2. 26 rt. L s.
Paper 15. 1. 20°.

Page 234
Paper 16. 2. 108°.
Paper 17. 2. 54°.

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Paper 18. 1. 65°. 2. 3:26 in.
Paper 19. 1. 85°.
Paper 20. 2. 54°.

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Paper 21. 1. 67\frac{1}{3}°. 2. 45\frac{1}{3}°.
Paper 22. 2. 4:10 cm.

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Paper 23. 1. 18°.
Paper 24. 1. 1:57 in.

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Paper 25. 3. RS; SP.
Paper 27. 2. 7:62 cm. 3. QP.

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Paper 29. 2. 30°.

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Paper 31. 3. BA, AQ, QB.
Paper 32. 2. 3:73 cm.
NEW GEOMETRY

Page 264  
Exercise 52

1. $37^\circ$ or $142^\circ$.
2. 5-74 cm.
3. 5-89 cm.
4. $47^\circ$.
5. 51$^\circ$.
6. 38$^\circ$.
7. 6 cm.; 41-6 sq. cm.
8. 5-98 cm.
9. 2-05, 4-03 in.
11. 18 sq. cm.
12. 29-1 sq. cm.
18. 4-8 cm.
19. 1-66 in.
20. 2-90 in.
21. 46$^\circ$.
22. 5 cm.

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Oral Examples

(i) No; (ii) Yes.

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Oral Examples

1. 16, 16, 9, 9, sq. in.; 9-2, 1-8, 3, in.
2. 36, 36, 64, 64, sq. in.; 3-6, 6-4, 8, in.
3. 144, 25, 144, 25, sq. in.; 13 in.
4. 5 sq. in.; 2-24 in.
5. 3-16.
6. 2-65.
7. 25 sq. in.; 8$\frac{1}{8}$, 5$\frac{1}{4}$, 6$\frac{3}{4}$, in.

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Exercise 53

1. 17. 2. 13.
3. 3-61.
4. 20.
5. 3-32.
6. 2-83.
7. 3, 5-20.
8. 8, 6-93.
9. 6-46 in.
10. 16 ft.
11. 19-7 ft.
12. 15-8 mi.
13. 8-60 in.
14. 5-83 cm.
15. 162 mi.
16. 10 cm.
17. 6 in.
18. 17 cm.; 114 sq. cm.
19. 8-66 in.
20. 13 in.
21. 40 sq. cm.
22. 4-77 in.
23. 60 sq. in.
24. 48 sq. in.
25. 4-47 cm.
26. 15,000 yd.
27. 3-61 in.
28. 3-52 in.
29. 5 in.
30. 8-91 units.
31. 5 units.
32. 210 sq. cm.
33. 244 sq. in.
34. Obtuse-angled.
35. Acute-angled.
37. 13 in.
38. 4$\frac{3}{4}$ in.
39. 8$\frac{1}{4}$ in.
40. 3-67 in.
41. 7 in.
42. 3$\frac{1}{2}$.
43. 8 ft. 10 in.
44. $x^4(x^2 + y^2)$.
45. area = $ab(a + b)$.
46. 16 ft.
47. 9 in.
48. 9 ft.

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Oral Examples

1. 485, 506; 23-8 ft.
2. 9-11 cm.
3. 119 sq. in.; 10-5 in.
4. 27 sq. cm.; 12 sq. cm.; 7-21 cm.

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Exercise 54

1. 26-8 ft.
2. 8-66 in.
3. 8 ft.
4. 4-24 in.
5. 18 ft.
6. 7$\frac{1}{2}$ sq. ft.
7. 260 sq. ft.
8. 13 cm.
9. 9-16 in.
10. 8-04 in.
11. 5 in.
12. 4 cm.
13. 60 sq. in.; 11$\frac{7}{8}$.
14. 12 cm.
15. 8-616 in.
16. 3 cm.
17. 8 in.
18. 8 in.
19. 2-24 in.
20. 9-90, 7-07, 8-77, in.
21. 7-34 in.

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Revision Papers, 35-50

Page 289  
Papers 37, 3. 3-54 in.

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Papers 39, 1. 36$^\circ$.
3. 4 in.

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Papers 40. 1. KC, BK.
3. 3-77 sq. in.

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Papers 43. 1. 15$^\circ$.
3. 3-45 sq. in.

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Papers 44. 1. 17$\frac{1}{2}$ sq. in., 2$\frac{1}{2}$ in.

Page 294  
Papers 49. 1. (i) $pq(p^2 - q^2)$ in.
(ii) 17-3 sq. in.; 4-33, 7-31 in.

Page 295  
Papers 50. 1. 10, 17-3, cm.
2. 7-5 sq. cm., 5-83 cm.
ACT II (Section 2)

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ORAL EXAMPLES
1. 3 cm.  2. 9 in.  4. $\frac{9}{2}$ cm.

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EXERCISE 56
1. 13 cm.  2. 4.47 cm.  3. 11.5 cm.  4. Circle, radius 6 cm.
5. 8 cm.  6. 8.58, 0.883, cm.  7. 13.0 in.  8. 11.3 cm.
9. 8 in.  10. 5.83 cm.  11. 84 in.  12. 8.94 in.  13. 9 in.
14. 8$\frac{1}{2}$ cm., 11 cm.  15. 7$\frac{3}{4}$ in.  16. 3.46 cm.  17. 5 in.
19. 5.38 in.  20. 6.5 cm.  21. 4.8 in.; 3.6, 0.6, in.  22. $\frac{1}{2}$ in.
23. 2, 2$\frac{1}{4}$ in.

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EXERCISE 58
1. 55°.  2. 37°.  3. 65°.  4. 60°.  5. 107°.  6. 105°.
7. 72°.  8. 40°.  9. 128°.  10. 30°.  11. 54°, 99°.
12. 180° or $\frac{1}{2}$°.  13. 110°.  14. 25°.  15. 70°.  17. 124°.
18. 105° or 5°.  19. 100°, 110°.  21. 54°.  23. 38°.
25. 78°.  98°.  132°, 124°, 110°.  26. 8 cm.  27. 5-29 cm.
28. 8.04, 4.47, cm.  29. 2-12 in.  30. 2-45 in.

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EXERCISE 60
1. (i) No, (ii) yes.  2. (i) Yes, (ii) no.  3. 35°.  4. 45°.
5. 40°.  6. $\angle B D C = 50^\circ$, $\angle D C B = 65^\circ$.  8. 25°.
10. 60°, 70°, 50°.  13. 2 in.

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EXERCISE 62
1. 18°.  2. 12°.  3. 105°.  4. 135°.  5. $\frac{1}{2}$.  6. 5.
7. 3:2.  8. 35°.  9. 5.  11. 3:2.  12. 53°.  13. 60°.
14. 50°.  15. 10-5, 6-98, cm.  16. 2-9 cm.

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EXERCISE 63
1. 3-146.  2. 25-1 in.  3. 22-0 cm.  4. 628 yd.
5. 1-78(5) in.  6. 1-4 cm.  7. 70 yd.  8. 6-05 ft.
9. 9-0 in.  10. 3-49 cm.  11. 10-8 cm.  12. 40°.
13. 4-3 cm.  14. 3-14.  15. 314 sq. in.  16. 50-0 sq. cm.
17. 154 sq. ft.  18. 14 in.  19. 3-9 cm.  20. 22-0 sq. in.
21. 5-89 sq. cm.  22. 4-57 sq. in.  23. 42-9 sq. in.
24. 45°, 105°, 50°.  25. 72°, 72°, 36°.
26. 71°, 22$\frac{1}{2}$°, 106° or 127°, 22$\frac{1}{2}$°, 30°.
27. 45°, 75°, 60°.
28. 46°, 37°.  29. 87°, 108°.  31. 3:1.  33. 16°.  34. 5.
35. 77-4 sq. cm.  36. 2-5 cm.  37. 7 in.  38. 5 in.

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EXERCISE 65
1. 65°, 50°.  2. 35°.  3. 8 cm.  4. 13 cm.  5. 110°.
6. 68°.  7. 72°.  8. 117°.  9. 62°, 38°.  10. 42°.
11. 155°.  12. 3 in.  13. 8 cm.  14. 60°, 65°, 55°.
15. 128°; 44°, 52°, 84°.  16. 5 cm.  17. 2-6 in.  18. 120°.
19. 77°, 90°, 103°, 90°.  20. 8, 2, cm.  21. 6, 1, in.
22. 12 cm.  23. 17 cm.  24. 2, in.  26. 16 in.

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ORAL EXAMPLES
6. 1-56 in.

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EXERCISE 67
1. 56°.  2. 36°.  3. 68°, 65°, 47°.  4. 65°, 75°, 40°.
5. 58°, 84°.  7. 100°.  8. 63°, 54°, 63°.  9. 94°, 8°.
10. 5° or 99°.  11. 79°, 114°, 101°, 66°.  12. 2s + y = 90°.
13. $\angle A B P = \angle T P K = 75°$.  14. 72°, 65°.  15. 53°, 28°.

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ORAL EXAMPLES
4. 5-5, 6-5, 7, cm.  5. 1-8, 1-4, 0-8, in.  6. 2, 1-5, 1-2, in.
9. 2-1 cm.  10. 1-75, 5-29, cm.

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EXERCISE 69
1. 3 cm.  2. 6 cm.  3. 7$\frac{1}{2}$, 4, in.  4. 2-5, 1-5, 4-5, in.
5. 4-5, 3-5, 2-5, cm.  6. 8, 4, 3, in.  7. 5-3, 3-6, 4-5, cm.
12. 13 in.  13. 2 cm.  14. 32, 8, cm.  15. 1-5 cm.
16. 4-45, 11-125, in.  17. 15 cm.  18. 5 - $3\sqrt{2}$ = 0-75, ft.  19. 2.7.
NEW GEOMETRY

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Exercise 71
1. 4 cm. 9. 12 cm. 10. 19-1, 12, cm. 11. 5-45 cm.
12. 5-59 cm. 14. \( \sqrt{d^2 - (a+b)^2} \), \( \sqrt{d^2 - (a-b)^2} \), in.
16. 7, 1, cm.

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Exercise 72
1. 3-49 cm. 3. 6-93 cm. 4. 6-93 cm. 6. 3-11 cm.
9. 9 < \( \frac{1}{2}(d-a) \). 14. 0-65, 5-81, 1-94, 1-16, cm. 15. 4-61 cm.
16. Yes. 17. 1-46 cm. 18. 2-67 cm. 30. 2-66 cm.
31. 1-56 in. 32. 5-80 cm. 33. 8-13 cm. 34. 5-00, 2-14, cm.
35. 6-07, 4-02, cm. 38. 4-16 cm. 40. 3-2 cm. 41. 1-80 cm.

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Exercise 73
5. 20°.

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Exercise 74
3. 6-65 or 1-35, cm. 4. 2-63 cm.; yes. 6. Yes. 8. 4-47 cm.
15. 3-20 cm. 25. 5-87, 2-23, cm.

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Oral Examples
7. 2-299; 4-561. 8. 4-081; 11-46. 9. 82° 49'; 41° 24'.
10. 101° 32'; 34° 3'.

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Exercise 76
1. Obtuse. 2. Acute. 3. Obtuse. 4. Obtuse. 5. 19, 10, in.
7. 5-85, 6-84, in.; 34-2 sq. in. 8. 1-375, 2-07, cm.; 5-33 sq. cm.
9. 11, 6-93, in.; 34-6 sq. in. 10. 3, 8-48(5), cm.; 42-4 sq. cm.
11. 1-2 cm. 13. 6-63 in. 14. 9³ ft. 15. 4 cm.
16. 6-5, 8-5, in. 17. 5-45(5), 6-52, 7-97, cm. 18. 10 in.
19. 12-7 cm. 20. 9-16(5) cm. 21. 11, 9, cm. 22. 12-2 cm.
25. (3a+b) in. 26. 4³ in. 28. 2³ in.

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Exercise 78
1. 4 in. 2. 10 cm. 3. 12 cm. 4. 4 in. 5. 24 sq. in.
6. 13 in. 7. 6 cm. 8. 7³ in. 9. 56 sq. cm. 10. 8 in.
11. 3-2 cm. 12. 1 cm. 16. 8, 7, 12, cm. 17. 5, 3, cm.
18. 6, 5-6, 15, cm. 19. 2, 4, 7-8, cm. 20. 2-25 cm.
21. 20 ft. 22. 10³, 5³, cm. 23. 3960 mi. 24. 3-5¾, 6-52, in.
25. 2³ ft. 26. 84 in. 27. 6-5 cm. 28. 3, 3, mi.

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Exercise 80
1. 6-12 cm. 2. 5-57 cm. 3. 1-97 in. 4. 8-06 cm. 5. 4-58 cm.
6. 8, 3. 7. 1-87, 1-93. 10. AB < 2BC.

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Exercise 82
2. \( \triangle RBC = y \); \( \triangle C D Q = z \); \( \triangle C D Q = \angle B D R = \angle B A C \).
3. \( \angle B D C \). 4. \( m + n \). 6. No, BR || CQ. 7. No, PD || CQ.
8. \( \angle R B C = 3 \angle R C B \). 9. 3\( \angle C D P = \angle B D P = 180° \).
10. PQ || BA; PR || CA; CQ = QA.
11. \( \angle B D A - \frac{1}{2} \angle B C A = 90° \). 12. \( \frac{1}{2}(\angle A P B - \angle A B P) \).
39. AB || DC; 60° - \( \frac{1}{2} \theta \); 120° - \( \frac{3}{2} \theta \); \( \frac{3}{2} \theta - 60° \). 40. AG.
41. ABCD is rectangle.

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Revision Papers, 51-80
Paper 51. 2. 55-2 sq. in. 3. 14 cm.
Paper 52. 3. 4 in.
Paper 53. 2. 2-9 cm. 3. 162°.

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Paper 54. 2. 4-57 cm.
Paper 55. 2. 12 sq. cm.; 4-8 cm. 3. 47°.
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Paper 56. 2. 3-25 cm.
Paper 57. 2. 29-1 sq. cm. 3 37°.

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Paper 58. 3. 3-6 in.
Paper 59. 2. 17-14 in.

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Paper 61. 2. 22°.
Paper 62. 2. 55°, 40°.

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Paper 63. 2. 5, 7, in.

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Paper 65. 1. 13 in. 3. 36°.
Paper 66. 2. $b + \frac{a^2}{4b}$. 3. 60°.
Paper 67. 2. 15°.

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Paper 68. 3. 5-66, 8-48(2), cm.
Paper 69. 2. 31°.

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Paper 72. 2. (90 - 2z) degrees.

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Paper 73. 3. 43-2 in.
Paper 74. 3. 5 in.
Paper 75. 4. 8 in.

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Paper 76. 3. 18-75 cm.
Paper 77. 1. 3. 2. 6-43 cm. 4. 12 cm.
Paper 78. 3. 7 cm.

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Paper 79. 1. (4, 3); (6, 5).
Paper 80. 3. 14, 16, cm. 4. 12-65, 6-92, in.

PART III

Exercise 83
1. 3:8. 2. 9:4. 3. 2:3. 4. 7:5. 5. 12:6.
10. $AB : PQ = QR : BC$. 11. 3-2 in. 12. 14 in. 13. 2-4 in.
14. 14-1, 14-7, cm. 16. 16 : 1. 17. 2 : 5 externally, 1 : 2 internally.
26. 2bc. 2bc.
13. $e + d = d = d$ in. 19. $(x - y) : 2(x + y)$.
20. $d : a, b : d ; a : c$. 21. $(e + d) : d ; c : (c + d)$.
22. $a + e + a$. 26. $d - f$. 27. $2b + 7d - 5f$.
23. 1 : 11; 4 : 7; 5 : 3; 1 : 1. 11; 1 : 6 in.
29. $p : (p + q + r + s + t); (q + r + s) : (r + s + t)$;
$(p + q) : (p + q + r + s + t), (r + s) : t; \frac{x(p + q + r + s)}{q + r}$ in.
30. 1 : $(a - 1)$. 31. $2xy : (x^2 - y^2); (x - y) : (x + y)$.
32. $AY ; XY ; BD$.

Exercise 84
1. 3 : 8. 2. 7 : 11, 7 : 11.
3. 14, 16; $\frac{7}{8} ; \frac{7}{15} ; \frac{7}{15}$.
5. $P_1, \frac{P}{P + q}$.

Exercise 86
1. 24, 114°, in. 2. 3, 15, cm. 3. 8 in. 4. 3-35 in.
6. $\frac{ac}{b + c}$ units. 7. 12 in. 8. 9 sq. cm. 10. 1 in.
11. 12 sq. in. 12. 3 sq. in. 13. $(b + c) : a$. 14. 3\frac{2}{3} cm. 15. 3 : 2.
NEW GEOMETRY

EXERCISE 87

18. 14 1/4 cm.

EXERCISE 88

7. 7 1/2 cm. 8. 7 1/2 cm. 9. 4 2/3. 10. 4 4/9, 6 2/3.

EXERCISE 89

4. (i) Δs ABC, MNL, RPQ; (ii) 7 1/2, 6, 15, 22 1/2; (iii) Δs ABC, FED. 6. (i) Δs AKC, DKB; (ii) 6.

EXERCISE 90

1. 3 1/2, in. 2. 3 1/2, 2 1/4, in. 3. 6 1/2, in. 4. 3 1/4, 3 1/2, in.

EXERCISE 91

1. 7 1/2, 13 1/4, cm. 2. 4, in. 3. 2 1/4, 11/2, in. 4. 7 1/2, 5 1/2, 11/2, 4.

EXERCISE 92

1. 10 1/2, cm. 2. 3 1/2, 6 7/8, cm. 3. 6, in. 4. 13 1/4, cm.

EXERCISE 93

1. 7 1/2, 13 1/4, cm. 2. 4, in. 3. 2 1/4, 11/2, in. 4. 7 1/2, 5 1/2, 11/2, 4.

EXERCISE 94

1. 6 3/4, 7 1/2, in. 2. 6 1/2, 7 1/2, in. 3. 5 1/2, 5 1/2, 4.

EXERCISE 95

1. 6 3/8, cm. 2. 6 1/4, 6 1/8, cm. 3. 0 1/8, in. 4. 7 1/2, 5 1/2, 11/2, 4.

EXERCISE 96

1. 4 1/2, 2. 2 1/2, 3. 15 1/2, in. 4. 3 1/2, 4.

EXERCISE 97

1. 4, in. 2. 6 1/2, 6 1/2, in. 3. 5, 3.

EXERCISE 98

2. 3 1/2, cm. 3. 2 1/2, 2 7/8, in. 4. 2 1/2, 2 1/2, in. 5. 4 1/2, 4 1/2, cm.

ORAL EXAMPLES

1. 11 cm. 2. 3 cm. 3. 5 cm.
Revision Papers, 81-96

Page 530

Paper 81. 1. 8, 7. 3. 3 : 4; 5½, 4½ cm.

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Paper 82. 1. QA = 60 ft. 2 in., PA = 25 ft. 5½ in.
3. 9½, ½ in.; 14, 40, cm.

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Paper 83. 1. 11 ft.
Paper 84. 3. 15, 9, 7½, 22½, cm.; 6½ in.

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Paper 85. 3. 2 : 3, 4 : 1, 1 : 3.
Paper 86. 1. 7 cm., 6·49(5) sq. cm.
3. (i) 1 : 2, 3 : 1, 1 : 12; (ii) 5 : 4, 3½ cm.

Page 534

Paper 87. 3. 24, 10·8, cm.; 24·4 in.
Paper 88. 1. 3·97, 4·5, 0·5, in. 3. 9 cm.; 71 : 360.

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Paper 89. 2. 4·47 in.
Paper 90. 1. 11, 9, in. 2. 9, 6½, in.

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Paper 91. 1. 5½ sq. in. 3. 3½, 3½.
Paper 92. 3. 5½, 2½ in.; 1½ in.

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Paper 93. 3. 17, 7, 11.
Paper 94. 2. 3, 2·22, cm. 3. Circle, radius 2½ cm.

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Paper 95. 2. 7·49, 9·16, 1·66, in.; 4·8 cm.