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ELEMENTARY COORDINATE GEOMETRY
By C. V. DURELL, M.A.

LONDON
G. BELL AND SONS LTD
1960
PREFACE

This book provides a course of coordinate geometry which meets the requirements of the General Certificate of Education and examinations of similar standard. It is divided into two parts:

Part I, Advanced Level; Part II, Scholarship Level.

Part I is available separately.

The character of an introduction to analytical geometry which meets modern conditions has been discussed in great detail in the Report issued in 1953 by the Mathematical Association on the Teaching of Higher Geometry in Schools. Full use has been made of its recommendations; for example, the ellipse is introduced as the orthogonal projection of a circle, transforming the locus $x = a \cos \theta, y = a \sin \theta$ to the locus $x = a \cos \theta, y = b \sin \theta$ by shortening the ordinates in the ratio $b : a$, instead of by the focus-directrix definition, thus linking up properties of diameters and tangents of the ellipse with those of the circle; this is in fact the historical order of development.

Attention may be called to the following features:

(i) While keeping steadily in mind the fact that the primary function of coordinate geometry is its use as an instrument for geometrical investigation, it must be recognised that the pupil cannot handle this weapon successfully until he has acquired some facility in the fundamental processes, on which the use of coordinate methods depends. Accordingly in the early chapters there are numerous examples designed to enable the pupil to write down readily equations of lines satisfying given conditions, to express ratio and distance relations algebraically, and conversely to interpret algebraic relations in geometrical language. The quick-revision papers are intended to serve the same purpose; also the summary of formulas and equations should be used for revision as well as for reference.

(ii) Measurement of lengths of steps by directed numbers receives detailed discussion. By this means, formulas of a general nature are established so that it becomes unnecessary to consider the various ways in which a figure is situated with reference to the coordinate axes. It is then reasonable for a pupil to recall a formula by using a sketch in which the axes are taken in the simplest position.
PREFACE

(iii) Among the variety of methods and ideas to which the interest and value of the subject is due, the use of parameters on account of its novelty calls for special emphasis. This is illustrated extensively by properties of the parabola and rectangular hyperbola, and more effectively than by the geometry of the circle. For this reason, unless prevented by examination requirements, the systematic treatment of the circle in Chapter 5 may well be postponed till after the discussion of the loci \( x = a^2t, y = 2at \) and \( x = c, y = c/t \) in Chapters 7 and 8.

(iv) Some incidental revision of algebraic processes may be necessary, such as calculations by equal ratios, properties of symmetric functions of roots of equations and use of undetermined coefficients in factorisation. For most pupils, however, the algebra course at this stage has not included the use of determinants; consequently there is much to be said for delaying their introduction until the pupil realises that their use saves him trouble and lightens the burden on his memory, as for example in the discussion of the general equation of the second degree.

(v) Examples are given which compare geometrical and algebraic forms of proof; pupils should be encouraged to use whichever is the simpler in any particular problem or when appropriate combine the two lines of approach in a single solution.

(vi) The distinction between scholarship level and advanced level is indicated mainly by an increase in difficulty of applications. This should not involve the elaborate algebraic manipulation of the old-fashioned analytical conics but calls for the introduction of more general and more powerful methods, and these form the instructive subject matter of Part II as indicated by the details set out in the Table of Contents.

The author acknowledges with thanks permission to include questions set in the examinations for the General Certificate of Education by the following bodies: Cambridge Local Examinations Syndicate, Senate of the University of London, Joint Matriculation Board of the Northern Universities, Delegates of the Oxford Local Examinations, and the Oxford and Cambridge Schools Examinations Board; such questions are marked respectively C, L, N, O, OC in the text.

C. V. D.

July 1950

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ELEMENTARY FORMULAS AND EQUATIONS

Notation and Abbreviations. In the following list of formulas, 
\((x_1, y_1), (x_2, y_2), (x_3, y_3)\) denote the coordinates of \(P_1, P_2, P_3\).
The line \(lx + my + n = 0\) means the line whose equation is \(lx + my + n = 0\).
If a locus is given by parametric equations in terms of a variable \(t\), the
point \(t_1\) means the point \(P_1\) determined by \(t = t_1\), etc.; the chord \(t_1t_2\) means
the chord \(P_1P_2\) etc. Similarly for variable \(t\).
1. **Distance**: \(P_1P_2 = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}\)

   p. 12

2. **Step** from \((x_1, y_1)\) to \((x_1 + r \cos \theta, y_1 + r \sin \theta)\) is of directed
length \(r\) in direction making \(\theta\) with \(x'\) Ox.

   p. 17

3. **Gradient** of \(P_1P_2\) equals \((y_2-y_1)/(x_2-x_1)\).

   p. 24

4. **Polar Coordinates**: \(x = r \cos \theta, y = r \sin \theta; \quad r^2 = x^2 + y^2, \tan \theta = y/x)\).

   p. 18

5. **Forms of Equation of a Line**:
   (i) Parallel to \(Ox, y = b\); parallel to \(Oy, x = a\).

   p. 23
   (ii) Gradient \(m\), passing through \(P_1: y - y_1 = m(x - x_1)\).

   p. 25
   (iii) Gradient \(m\), intercept \(c\) on \(Ox: y = mx + c\).

   p. 26
   (iv) \(OP_1, x = x_1, y = y_1, P_2P_3, \quad x_2-x_1, y_2-y_1\).

   p. 26
   (v) Intercepts \(a, b\) on \(Ox, Oy: x/a + y/b = 1\).

   p. 26
   (vi) Perpendicular form: \(x \cos \alpha + y \sin \alpha = p = 0\).

   p. 35

6. \(y = mx + c, y = m_1x + c_1\) and \(ax + by + c = 0, ax_1 + by_1 + c_1 = 0\) are
parallel if \(m_1 = m\) and if \(a_1/a_2 = b_1/b_2 = c_1/c_2\), and are
perpendicular if \(m_1m_2 = -1\) and if \(a_1a_2 + b_1b_2 = 0\).

pp. 28-31

7. **Pair of Lines through Origin**: \(ax^2 + 2hxy + by^2 = 0, h^2 > ab\).

   p. 41
   Lines are at right angles if \(a + b = 0\).
   Lines are coincident if \(h^2 = ab\).
   If \(\theta^0\) is the angle between the lines,
   \(\tan \theta^0 = \pm (2 \sqrt{h^2 - ab})/(a + b)\).

   p. 41
   If the lines can be written, \(y = m_1x = 0, y = m_2x = 0\),
   \(m_1 + m_2 = -2h/b\) and \(m_1m_2 = a/b\).

   p. 41
   For angle \(\theta^0\) between the lines,
   \(\tan \theta^0 = \pm (m_1 - m_2)/(1 + m_1m_2)\).

   p. 32

8. **Length of Perpendicular** from \((x_1, y_1)\) to a line:
   Line, \(x \cos \alpha + y \sin \alpha = p = 0\), length is \(\pm (x_1 \cos \alpha + y_1 \sin \alpha - p)\).

   p. 36
   Line, \(ax + by + c = 0\), length is \(\pm (ax_1 + by_1 + c)/\sqrt{a^2 + b^2}\).

   p. 37

9. **Area** \(\Delta P_1P_2P_3 = \pm \frac{1}{2} \left| (x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|\).

   p. 39

10. **Ratio**. (i) Mid-point of \(P_1P_2\) is \(\{x_1 + x_2, \frac{1}{2} \left(y_1 + y_2\right)\}\)

   p. 47
   (ii) if \(P(x, y)\) divides \(P_1 \to P_2\) in ratio \(m_1 : m_2, m_1 + m_2 + 0, \quad x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \quad y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}\).

   p. 47
11. Forms of Equation of a Circle:

(i) Centre at origin, radius \( r \): \( x^2 + y^2 = r^2 \).

(ii) Centre at \((h, k)\), radius \( r \): \( (x-h)^2 + (y-k)^2 = r^2 \).

(iii) Diameter \( P_1P_2 \): \((x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0\).

(iv) \( x^2 + y^2 + 2gx + 2fy + c = 0 \), centre \((-g, -f)\), radius \(\sqrt{g^2 + f^2 - c}\).

(v) Point-circle, centre \((-g, -f)\): \((x+g)^2 + (y+f)^2 = 0\).

(vi) General equation, \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\), represents a circle if \(a = b \neq 0\) and \(h = 0\), \(g^2 + f^2 - ac > 0\).

12. (i) **Square of length of tangent** to \(x^2 + y^2 + 2gx + 2fy + c = 0\) from a point \(T(x', y')\) outside circle equals:

\[x'^2 + y'^2 + 2gx' + 2fy' + c = 0\]

(ii) **Power of point** \((x', y')\) with respect to \(x^2 + y^2 + 2gx + 2fy + c = 0\) equals \(x'^2 + y'^2 + 2gx' + 2fy' + c = 0\).

13. **Tangency.**

(i) The tangent to \(x^2 + y^2 + 2gx + 2fy + c = 0\) at \((x_1, y_1)\) and the chord of contact of tangents from \((x_1, y_1)\) is \(2x_1x + 2y_1y + x + y + c = 0\).

(ii) \(y = mx \pm \sqrt{1+m^2}\) touches \(x^2 + y^2 = r^2\).

(iii) Circles, centres \(C_1, C_2\), radii \(r_1, r_2\), touch one another if \(C_1C_2 = r_1 + r_2\) (external contact) or if \(C_1C_2 = r_1 - r_2\) (internal contact).

14. \(x^2 + y^2 + 2gx + 2fy + c = 0\), \(x^2 + y^2 + 2gx + 2fy + c = 0\) are orthogonal circles if \(2g' + 2f' = c + c'\).

15. **Coaxal Circles.** Let \(S_1^2 = x^2 + y^2 + 2gx + 2fy + c_1\) and \(S_2^2 = x^2 + y^2 + 2gx + 2fy + c_2\).

(i) \(S_1 - S_2 = 0\) is the radical axis of \(S_1 = 0\) and \(S_2 = 0\).

(ii) \(S_1 + S_2 = 0\), \(k = -1\), is a circle coaxal with \(S_1 = 0\), \(S_2 = 0\).

(iii) If \(L = ax + by + n = 0\) is the radical axis of \(S_1 = 0\), \(S_2 = 0\), any circle coaxal with \(S_1 = 0\), \(S_2 = 0\) can be denoted by \(S_1 + kL = 0\) or \(S_2 + kL = 0\).

(iv) Circles of a coaxal system can be denoted by \(x^2 + y^2 + 2gx + 2fy + c = 0\) where \(c\) has the same value for each circle of the system.

16. **Concurrent Lines.** Let \(L = px + qy + r\) and \(L' = px' + qy' + r'\). Any line through the point of intersection of \(L = 0\) and \(L' = 0\) can be denoted by \(L + kL' = 0\).

17. **Lines through Origin.** If \(lx + my = 1\) cuts \((x^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\) at \(P, Q\), the lines \(OP, OQ\) are given by the homogeneous equation, \((ax^2 + 2hxy + by^2)(lx + my) + c(lx + my)^2 = 0\).
32. Tangency. (i) The tangent to \( z^2/a^2 + y^2/b^2 = 1 \) at \((x_1, y_1)\) and the chord of contact of the tangents from \((x_1, y_1)\) is
\[ xx_1/a^2 + yy_1/b^2 = 1. \]

(ii) The tangent at \((a \cos \theta, b \sin \theta)\) is
\[ (x/a) \cos \theta + (y/b) \sin \theta = 1. \]

(iii) \( x + my + n = 0 \) is a tangent if \( a^2m^2 - b^2n^2 = n^2 \).

(iv) \( x \cos \alpha + y \sin \alpha = \pm \sqrt{(a^2 \cos \alpha + b^2 \sin \alpha)} \) is a tangent.

(v) \( y - mx \pm \sqrt{(a^2m^2 + b^2)} \) is a tangent.

(vi) Director Circle is \( x^2 + y^2 = a^2 + b^2 \).

33. Normal at \((x_1, y_1)\) is \( a^2x/x_1 - b^2y/y_1 = a^2 - b^2 \).

34. (i) Chord \(0, b\) is
\[ \left( x/a \right) \cos \beta + \left( y/b \right) \sin \beta = \cos \beta \left( 0 \right) + \sin \beta \left( b \right). \]

(ii) If \(0, b, 0\), chord is \( x/a \cos \beta + y/b \sin \beta = \cos \beta \left( 0 \right) + \sin \beta \left( b \right) \).

(iii) The chord of \( x^2/a^2 + y^2/b^2 = 1 \) whose mid-point is \( (b, h) \) is \( bx + qy - ph^2 + qh^2 = 0 \).

35. (i) \( y = mx, y = m'x \) are conjugate diameters if \( mm' = -b^2/a^2 \).

(ii) If the points whose eccentric angles are \( \theta_1, \theta_2 \) are the excentricities of conjugate diameters, \( \theta_1 - \theta_2 = 90^\circ \) or \( 270^\circ \).

36. (i) Eccentricity \( e \), \( 0 < e < 1 \); \( b^2 = a^2(1 - e^2) \), \( l = b^2/a \).

(ii) Foci: \( S(ae, 0), S'(-ae, 0) \); directrices: \( XYZ, x = a/e; X'Z', x = -ae \).

(iii) Focal Distances: \( SP = a - ex, S'P = a + ex \).

37. Parametric Equations for point \((a \cos \theta, b \sin \theta)\):
\[ x/a = \cos \theta, \quad y/b = \sin \theta \]

(i) Chord \( t_1, t_2 \) is \( (1 - t_1)x/(1 + t_1) + (1 + t_1)x/(1 - t_1) = 0 \).

(ii) Tangent at \( t \) is \( (1 - t)x/a + 2ty/b - (1 + t) = 0 \).

Hyperbola. \( x^2/a^2 - y^2/b^2 = 1 \) or \( x = a \sec \theta, y = b \tan \theta \).

38. Asymptotes: \( OE, x/a, y/b = 0 \); \( OP, x/a + y/b = 0 \).

39. Tangency. (i) The tangent to \( x^2/a^2 - y^2/b^2 = 1 \) at \((x_1, y_1)\) and the chord of contact of the tangents from \((x_1, y_1)\) is
\[ xx_1/a^2 - yy_1/b^2 = 1. \]

(ii) \( x + my + n = 0 \) is a tangent if \( a^2m^2 - b^2n^2 = n^2 \).

(iii) \( x \cos \alpha + y \sin \alpha = \pm \sqrt{(a^2 \cos \alpha - b^2 \sin \alpha)} \) is a tangent.

(iv) \( y - mx \pm \sqrt{(a^2m^2 - b^2)} \) is a tangent.

(v) Director circle, \( x^2 + y^2 = a^2 - b^2 \), where \( a > b > 0 \).

40. (i) Eccentricity \( e, e > 1 \); \( b^2 = a^2e^2 - 1 \), \( l = b^2/a \).

(ii) Foci, \( \pm ae, 0 \); directrices, \( x = \pm a/e \).

If \( b^2 = \tan \phi, e = \sec \phi \).

(iii) Focal Distances: \( SP = \pm (ex - a), S'P = \pm (ex + a) \).

41. (i) \( y = mx, y = m'x \) are conjugate diameters if \( mm' = -b^2/a^2 \).

(ii) \( P(a \tan \theta, b \tan \theta) \) on \( x^2/a^2 - y^2/b^2 = 1 \) and \( Q(a \tan \theta, b \tan \theta) \) on conjugate hyperbola \( x^2/a^2 + y^2/b^2 = -1 \) are ends of conjugate diameters.

42. Parametric Equations. \( x/a : y/b : 1 = (1 + t^2) : 2t : (1 - t^2) \).

Chord \( t_1, t_2 \) is \( (t_1^2 + 1)x/a - (t_1 + t_2)y/b + (t_1t_2 - 1) = 0 \).

CHAPTER 1

EQUATION OF A LOCUS

1.1. A New Weapon. Many problems in geometry can be solved more easily by using the methods of algebra and trigonometry than by a direct geometrical approach. Such methods are valuable, but become even more effective when the familiar work with graphs is extended and combined with the use of geometrical properties previously established. Descartes discovered, more than 300 years ago, how this can be done. The object of this book is to explain his method, often called Cartesian geometry, and so to give the reader a new weapon with which he can tackle successfully problems which might otherwise be beyond his powers to solve.

1.2. Coordinates. The method for describing the position of a point in a plane by means of coordinates is used in elementary algebra in connection with locus-graphs.
1.3. The numerical examples in 1.2 are included in the following general statement:

If a point $P$ is reached from the origin $O$ by a step $h$ units $x$-wards and then a step $k$ units $y$-wards, where $h$ and $k$ are directed (positive or negative) numbers,

$h$ is called the x-coordinate or abscissa of $P$,

and $k$ is called the y-coordinate or ordinate of $P$.

For convenience we often write $P(h, k)$ to denote the point $P$ whose coordinates are $(h, k)$, and say that $P$ is the point $(h, k)$.

The x-coordinate is always named first.

Thus the position of a point $P$ in the plane is fixed precisely when its coordinates $(h, k)$ referred to given rectangular Cartesian axes $x'Ox, y'Oy$ are known, where $h$ and $k$ are directed numbers, that is, positive or negative numbers or zero.

In Fig. 2, the point $N$ lies on $x'Ox$ and so its $y$-coordinate is zero; hence, with the data of Fig. 2, $N$ is the point $(h, 0)$. Similarly in Fig. 2, the point $M$ lies on $y'Oy$ and so its $x$-coordinate is zero; hence, with the data of Fig. 2, $M$ is the point $(0, k)$.

In particular, the origin $O$ is the point $(0, 0)$.

The letters $x, y$ are often used in place of $h, k$ to denote the $x$-coordinate and $y$-coordinate of any point $P$; we then speak of the point $P(x, y)$, where $x$ and $y$ are themselves directed numbers.

The axes $x'Ox, y'Oy$ divide the plane into four regions. The region in which a point $P$ lies when its abscissa and ordinate are both positive is called the first quadrant. The remaining regions, turning counterclockwise about $O$, are called in order the second, third and fourth quadrants as shown in Fig. 2.

\[\begin{array}{c|c|c|c|c}
 x & -2 & 0 & 2 & 4 \\
 y & 1 \frac{1}{2}x - 2 & -5 & -2 & 1 \\
\end{array}\]

Here the table values are constructed by calculating the value of $y$, corresponding to any chosen value of $x$, such that $y = 1 \frac{1}{2}x - 2$.

The table shows that the points $(-2, -5), (0, -2), (2, 1), (4, 4)$ lie on the graph, and other such points can be found as before.

Each point $P(x, y)$ whose coordinates satisfy the equation

\[y = 1 \frac{1}{2}x - 2\]

is a point of the graph.

Conversely, if $P(x, y)$ is any point of the graph, its coordinates satisfy this equation.

Thus the graph is the complete locus of a point $P(x, y)$ where $x$ and $y$ vary subject to the law, $y = 1 \frac{1}{2}x - 2$. For this reason, this equation is called the equation of the locus.

The equation may also be written in the form

\[3x - 2y - 4 = 0\]

and the graph in Fig. 3 is often called the graph of $y = 1 \frac{1}{2}x - 2$ or the graph of $3x - 2y - 4 = 0$.

The reader knows from experience that the graph of any function of $x$ of the first degree, $ax + b$, where $a, b$ are constants, is a line, and so a function of $x$ of the first degree is called linear. This important property will be proved in Chapter 3, but for the present it will be assumed. Hence $3x - 2y - 4 = 0$ is the equation of the line-locus in Fig. 3, which cuts $x'Ox, y'Oy$ at $A, B$.

Since $A$ lies on $x'Ox$, its y-coordinate is zero, and so $A$ may be taken as the point $(a, 0)$.

Since $A(a, 0)$ lies on the line whose equation is $3x - 2y - 4 = 0$,

\[3a - 0 - 4 = 0; \quad a = 1\frac{1}{3}; \quad A \text{ is the point } (1\frac{1}{3}, 0)\]

Similarly, $B$ may be taken as the point $(0, b)$, where $0 - 2b - 4 = 0$;

\[b = -2; \quad B \text{ is the point } (0, -2)\]

The lengths, $+1\frac{1}{3}, -2$, of $OA, OB$ are called the intercepts of the line $AB$ on the axes $Ox, Oy$. 

**Fig. 2**

![Diagram showing the x and y coordinates and the first quadrant.](image)

**Fig. 3**

![Diagram showing the graph of $y = 1\frac{1}{2}x - 2$.](image)
1.5. Quadratic Function. If \( y \) denotes the function \( \frac{1}{2}(x^2 - x - 6) \), the graph which shows the relation between corresponding values of \( x \) and \( y \) can be drawn by making a table of values as before. This gives a curve shaped as in Fig. 4: it is the locus of a point \( P(x, y) \) where \( x \) and \( y \) vary subject to the law

\[
y = \frac{1}{2}(x^2 - x - 6)
\]

For this reason, equation (1) is called the equation of the curve.

The equation may also be written in the form

\[
3y = x^2 - x - 6
\]

(2)

Fig. 4

The point or points, if any, where the curve meets \( x'Ox \) can be written in the form \((h, 0)\) since the \( y\)-coordinate is zero. But the point \((h, 0)\) lies on the curve whose equation is \(3y = x^2 - x - 6\) if, and only if,

\[
0 = h^2 - h - 6;
\]

\[
(h + 2)(h - 3) = 0; \quad h = -2 \text{ or } h = +3.
\]

Hence in Fig. 4, \( A \) is the point \((-2, 0)\) and \( B \) is the point \((3, 0)\).

Similarly, if the curve meets \( y'Oy \) at \( C \), the coordinates of \( C \) are of the form \((0, k)\); hence as before the equation of the curve \(3y = x^2 - x - 6\) is satisfied by the values \( x = 0, y = k; \)

\[
3k = 0 - 6, \quad \text{that is}, \quad k = -2.
\]

Thus in Fig. 4, \( C \) is the point \((0, -2)\).

Suppose in Fig. 4 that \( P \) is the point of the curve whose abscissa is equal to 2; then the ordinate of \( P \) is obtained by substituting 2 for \( x \) in the equation of the curve;

\[
3y = 2^2 - 2 - 6 = -4; \quad y = -1\frac{1}{3};
\]

\( P \) is the point \((2, -1\frac{1}{3})\).

Suppose in Fig. 4 that \( P_1 \) is a point of the curve whose ordinate is equal to 2; then the abscissa of \( P_1 \) is obtained by substituting 2 for \( y \) in the equation of the curve;

\[
6 = x^2 - x - 6, \quad \text{that is}, \quad x^2 - x - 12 = 0;
\]

\[
(x + 3)(x - 4) = 0; \quad x = -3 \text{ or } x = +4.
\]

Thus there are two points on the curve whose ordinates are equal to 2, namely \( P_1(4, 2) \) and \( P_2(-3, 2) \), see Fig. 4.

1.7. Notation. It is often more convenient to denote the particular positions of a variable point \( P \) of a locus by \( P_1, P_2, P_3, \) etc. than by different letters \( P, Q, R, \) etc.; their coordinates may then be denoted by \((x_1, y_1), (x_2, y_2), (x_3, y_3), \) etc.

This is called a suffix notation.

It is customary to replace the phrase, the curve whose equation is \( y = \frac{1}{2}(x^2 - x - 6) \), by the abbreviation, the curve \( y = \frac{1}{2}(x^2 - x - 6) \).

Similarly, we speak of the line, \( y = 1\frac{1}{2}x - 2 \), in place of the phrase, the line whose equation is \( y = 1\frac{1}{2}x - 2 \).

1.7. The Parabola. The curve whose equation is \( y = \frac{1}{2}(x^2 - x - 6) \) is a special case of the curve whose equation is \( y = ax^2 + bx + c \), where \( a, b, c \) are given constants; it is called a parabola.

All parabolas have the same general shape but can occupy various positions relative to the axes \( x'Ox, y'Oy \). For example, the curve \( y = \frac{1}{2}(x^2 - x - 6) \) in Fig. 4 and the curve \( y = (x - 2)(x - 7) \) in Fig. 8, p. 6, are both concave upwards, but the curve \( y = 10x - x^2 \) in Fig. 7, p. 6, is concave downwards.

EXERCISE 1 (Class Discussion)

The diagrams in this exercise are not drawn to scale.

1. Fig. 5 represents the line \( P_6ABP_4 \), \( y = x^2 + 3; \) \( N_1P_1, N_2P_2 \) are the ordinates of \( P_1(x_1, y_1), P_2(x_2, y_2) \).

(i) If \( ON_1 = 8 \), find \( y_1P_1 \).
(ii) If \( y_2 = 11 \), find \( x_2 \).
(iii) If \( P_2 \) is the point \((-3, y_2)\), find \( y_2 \).
(iv) If \( P_1 \) is the point \((x_1, -2)\), find \( x_1 \).
(v) Find the intercepts \( OA, OB \) on the axes.
(vi) Which of the points, \((60, 47), \) \((-20, -12), \) \((-40, -28), \) \((8, 8)\) lie on the line \( AB? \)

2. The line \( P_6ABP_4 \) whose equation is \( 4x - 5y + 100 = 0 \) cuts the axes \( x'Ox, y'Oy \) at \( A, B \).

(i) Find the intercepts \( OA, OB \) made by the line on the axes; hence show on a sketch the position of the line relative to \( x'Ox, y'Oy \).
(ii) If \( P_1 \) is the point \((2, y_1)\), find \( y_1 \).
(iii) If \( P_2 \) is the point \((x_1, 4)\), find \( x_1 \).
(iv) If \( P_3 \) is the point \((-5, y_3)\), find \( y_3 \).
(v) If \( P_4 \) is the point \((x_4, -6)\), find \( x_4 \).
(vi) Which of the points \((-10, 12), \) \((25, 40), \) \((-35, -10)\) lie on the line \( AB? \)
3. Fig. 6 represents the line \( P_1AP_2BP_3 \), \( 2x + 5y = 12 \); \( P_1, P_2, P_3 \) are the points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\).

(i) If \( x_1 = 3 \), find \( y_1 \).
(ii) If \( y_2 = 4 \), find \( x_2 \).
(iii) If \( P_1 \) is the point \((x_1, 1)\), find \( x_1 \).
(iv) If \( P_3 \) is the point \((0, y_3)\), find \( y_3 \).
(v) Find the intercepts \( OA, OB \) on the axes.
(vi) Which of the points \((-2, 3), (-9, 0), (26, -8)\) lie on \(AB\)?
(vii) If the point \((-2, c)\) lies on \(AB\), find the value of \(c\). 

4. Fig. 7 represents the parabola \( OP_1P_2P_3 \), \( y = 10x - x^2 \); \( N_1P_1, N_2P_2 \) are the ordinates of \(P_1, P_2\).

(i) If \( ON_1 = 2 \), find \( N_1P_1 \).
(ii) If \( ON_2 = 8 \), find \( N_2P_2 \).
(iii) If \( N_1P_1 = 24 \), find \( ON_1 \).
(iv) Find \( OA \).
(v) Find the conditions for the following points to lie on the parabola: \((11, a), (-1, b), (6, 21), (26, -24)\).

5. Fig. 8 represents the parabola \( CAP_1BP_2 \), \( y = (x - 2)(x - 7) \); \( P_1, P_2 \) are the points \((x_1, y_1), (x_2, y_2)\).

(i) If \( x_1 = 4 \), find \( y_1 \).
(ii) If \( x_2 = 9 \), find \( y_2 \).
(iii) If \( y_2 = 4 \), find \( x_2 \).
(iv) Find \( OA, OB, OC \).
(v) Find the conditions for the following points to lie on the parabola: \((-1, a), (10, b), (6, 6), (26, 21)\).

1.8. Example 1. State the conditions for the line \( y = 2x - c \) to pass through \( P(1, 3) \) and \( Q(-2, 5) \). Hence find the equation of the line \( PQ \).

The line \( y = 2x - c \) passes through \( P(1, 3) \) if, and only if, the equation \( y = 2x - c \) is satisfied by \( x = 1, y = 3 \), that is,

\[ 3 = 2 + c \quad \text{(1)} \]

Similarly, the line \( y = 2x - c \) passes through \( Q(-2, 5) \) if, and only if,

\[ 5 = -4 + c \quad \text{(2)} \]

Solve equations (1), (2) for \( b, c \); then \( 2b + 2 = -2 \), \( b = 1 \); \( c = 3 - 1b = 3 \).

\therefore the equation of \( PQ \) is \( y = 2x - 3 \),

that is,

\[ 4x + 5y = 17 \]

The reader should illustrate the result by a sketch.

EXERCISE 2

Illustrate the examples in this exercise by free-hand sketches.

1. Write down the conditions that the line \( y = 2x + c \) passes through \( P(-1, 3) \) and \( Q(2, 5) \). Hence find the equation of the line \( PQ \).

2. Write down the conditions that the parabola \( y = ax^2 + bx + c \) passes through the origin \( O \) and the points \( A(5, 0), P(1, 3) \). Hence find the equation of the parabola.

3. Write down the conditions that the line \( y = 2x + c \) passes through the origin \( O \) and \( P(-4, 7) \). Hence find the equation of \( OP \).

4. Write down the conditions that the line \( \frac{x}{a} + \frac{y}{b} = 1 \) passes through \( A(-3, 0), B(0, -5) \). Hence find the equation of \( AB \).

5. A variable point \( P(x, y) \) moves so that its abscissa is always equal to \( 2y \). Describe the locus of \( P \). Describe also the locus whose equation is \( x + 1 = 0 \).

6. The equation of the locus of a variable point \( P(x, y) \) is \( (i) y = 3 = 0 \), \( (ii) y = 0 \). Describe each locus.

7. Find the condition that the parabola \( x = ay^2 \) passes through the point \( (3, 6) \). Hence find the equation of this parabola.

8. The parabola whose equation is \( y = ax^2 + bx + c \) cuts the \( x \)-axis at the points \((-2, 0) \) and \((5, 0) \) and cuts the \( y \)-axis at the point \((0, 1) \). Find the values of \( a, b, c \).

9. The parabola whose equation is \( y = ax^2 + bx + c \) passes through the points \( P(-1, 0), Q(-2, 5), R(2, 0) \). Find the values of \( a, b, c \) and the coordinates of the second point \( P' \) at which the parabola cuts the \( x \)-axis.

10. State the conditions that the point \( P(h, k) \) lies on the line \( 3x + 2y = 5 \), \( (i) \) the line \( 5x + 2y = 6 \). Hence find the values of \( h \) and \( k \) if \( P(h, k) \) is the point of intersection of the two lines.

11. State the conditions that the point \((x_1, y_1) \) lies on the line \( 2x + y = 2 \), \( (i) \) the parabola \( y = x^2 - 3x - 4 \). Hence find the coordinates of the points of intersection of the line and parabola.

12. Find the condition that the curve, whose equation is \( x^2 + y^2 - c^2 \), passes through the point \((3, 4) \). Find in this case the coordinates of the points \( A, A' \) where the curve meets \( x' \)-axis and of the points \( B, B' \) where the curve meets \( y' \)-axis. Prove that the curve also passes through the points \((-3, 4), (-3, -4), (3, -4), \) and \( \pm 4, \pm 3 \). Show these twelve points on a sketch and state the name of the curve.

13. Find \( t \) in terms of \( m \) if the point \((2t, f) \) lies on the line whose equation in \( y = mx + 2 + \frac{1}{m^2} \).
CHAPTER 2

LENGTH AND DISTANCE

2.1. Units. In geometrical statements, about lengths of lines, a unit of length is always implied but usually is not stated explicitly because the choice of unit is unimportant; all that matters is that the same unit is chosen for all lines. For example,

In $\triangle ABC$, if $AB = AC$, then $\angle ACB = \angle ABC$.

Here $AB$ denotes the number of units of length between $A$ and $B$, $AC$ denotes the number of units of length between $A$ and $C$, and these two numbers are given equal. The theorem is true whatever unit is chosen, provided only that the same unit is used for graduating $AC$ as for graduating $AB$.

In drawing graphs of functions in elementary algebra, it is often convenient to use different units for graduating $x'Ox$, $y'Oy$; but in elementary geometry, if the positions of the points of a figure are given in terms of coordinates, the units chosen for the $x$-axis and $y$-axis are always taken to be the same. If it is stated that a figure contains the given points $P_1(x_1, y_1), P_2(x_2, y_2)$, it is implied that in considering their relative positions, for example the angle which $P_1P_2$ makes with $x'Ox$, the units for $x'Ox$ and $y'Oy$ are the same. Again, the statement that $A, B$ are the points $(c, 0), (0, c)$ implies that $A$ and $B$ are equidistant from the origin $O$; this would be untrue if different units were used for graduating $x'Ox$ and $y'Oy$.

2.2. Length of a Step. It is only the use of positive and negative numbers and zero which makes it possible to state the position of any point of a plane in terms of Cartesian coordinates. If $P, Q$ are any points on $x'Ox$, the step from $P$ to $Q$ is said to have a positive length if $P \rightarrow Q$ and $x' \rightarrow x$ have the same sense, see Fig. 9, and to have a negative length if $P \rightarrow Q$ and $x' \rightarrow x$ have reverse senses, see Fig. 10.

![Fig. 9](image1)

![Fig. 10](image2)

Signless numbers are sufficient for describing the distance between two points. The distance of $P$ from $Q$ is the same as the distance of $Q$ from $P$, but the step from $P$ to $Q$ is not the same as the step from $Q$ to $P$; if, say, the first step is eastwards, then the second step is westwards.

2.2.2. For all relative positions on $y'Oy$ of $O(0, 0), M_1(0, y_1), M_2(0, y_2)$, the length of step from $M_1$ to $M_2 = (M_1 \rightarrow M_2) = y_2 - y_1$.
The expression for the length of a step along the x-axis is important because it does not depend on how $N_1$ and $N_2$ are situated relative to each other and to $O$. There are six possible cases, since $O$ may lie between $N_1$ and $N_2$ or on $N_1N_2$ produced or on $N_2N_1$ produced and $N_1$ may be to the left or right of $N_2$, see Fig. 11.

**Example 1.** (Class Discussion) Verify the formula, $(N_1 \rightarrow N_2) = x_2 - x_1$, where $N_1$, $N_2$ are the points $(x_1, 0)$, $(x_2, 0)$ in the six cases in Fig. 11.

\[ \begin{align*}
0 &+2 &-2 &+3 &-2 &0 &+3 &-2 &0 &+3 &-2 &0 \\
0 &N_1 &N_2 &N_1 &N_2 &N_1 &N_2 &N_1 &N_2 &N_1 &N_2 &N_1 \\
x_1 &0 &x_2 &x_1 &x_2 &x_1 &x_2 &x_1 &x_2 &x_1 &x_2 &x_1 \\
0 &+2 &-2 &+3 &-2 &0 &+3 &-2 &0 &+3 &-2 &0
\end{align*} \]

**Fig. 11**

(i) $N_1(+2, 0)$, $N_2(+5, 0)$. Fig. 11 (i) shows that $N_3$ is 3 units to the right of $N_1$. $x_2 - x_1 = (+5) - (+2) = +3$.

(ii) $N_1(-2, 0)$, $N_2(+4, 0)$. Fig. 11 (ii) shows that $N_3$ is 6 units to the right of $N_1$. $x_2 - x_1 = (+4) - (-2) = +6$.

(iii) $N_1(-3, 0)$, $N_2(-1, 0)$. Fig. 11 (iii) shows that $N_4$ is 2 units to the right of $N_1$. $x_2 - x_1 = (-1) - (-3) = +2$.

The reader should now verify the formula for the data in Fig. 11 (iv)-(vi).

### 2.3. Displacement in a Plane

It is convenient to use the abbreviations, a step x-wards or an x-step, for the phrase, the length of a step along a line parallel to the x-axis; its length is measured by a positive or negative number according as its sense is that of $x' \rightarrow x$ or of $x \rightarrow x'$ respectively. A similar meaning is given to the abbreviations, a step y-wards or a y-step.

We can move from any point $P_1(x_1, y_1)$ to any other point $P_2(x_2, y_2)$ by taking a suitable step $P_1 \rightarrow R$ x-wards and then a step $R \rightarrow P_2$ y-wards, see Fig. 12. In general, no ambiguity is caused by omitting the brackets in the expression for the length of a step, and in future we shall do so.

Let $N_1$, $M_1$ and $N_2$, $M_2$ be the feet of the perpendiculars from $P_1$ and $P_2$ to $x'Ox$, $y'Oy$, see Fig. 13; then $P_1M_1 \equiv P_2N_2$ produced if necessary, at the required point $R$. By construction, $P_1R$ and $N_1N_2$ are equal and parallel and drawn in the same sense.

Similarly by 2.2.2, $R \rightarrow P_2 = M_1 \rightarrow M_2 = y_2 - y_1$.

The statement in 2.3.1, p. 11, follows from these results.

### 2.4. Displacement in a Plane

#### 2.3.1.

It is possible to move from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ by taking an x-step, $x_2 - x_1$, and then a y-step, $y_2 - y_1$.

Conversely, if we start from $P_1(x_1, y_1)$ and take an x-step of length $h$ and then a y-step of length $k$, where $h$ and $k$ may be positive or negative, then the point of arrival $P_2(x_2, y_2)$, see Fig. 13, is such that

\[ x_2 - x_1 = h \quad \text{and} \quad y_2 - y_1 = k. \]

\[ x_2 - x_1 = h \quad \text{and} \quad y_2 - y_1 = k. \]

This result may be stated as follows:

#### 2.3.2.

If, from a point of departure $P_1(x_1, y_1)$, an x-step of length $h$ and then a y-step of length $k$ are taken the point of arrival is $(x_1 + h, y_1 + k)$.

### 2.4. Generality of Formulas

Although Fig. 12 and Fig. 13 are drawn so that the coordinates of $P_1$ and $P_2$ and the lengths of the steps $P_1 \rightarrow R$ and $R \rightarrow P_2$ are all positive, the proofs which have been given are completely general because they are derived from the general theorems, 2.2.1 and 2.2.2, on p. 9; it is therefore unnecessary to consider all over again the various ways in which the figure may be situated with reference to the coordinate axes. If this were not so, coordinate proofs of geometrical properties would be intolerably long.

### EXERCISE 3

Illustrate the answers to the numerical examples by freeland sketches.

State the x-step and y-step when moving from $P_1$ to $P_2$, Nos. 1-15:

1. $P_1(0, 1)$; $P_2(0, 5)$.
2. $P_1(0, -4)$; $P_2(0, -1)$.
3. $P_1(0, -2)$; $P_2(0, 3)$.
4. $P_1(0, 8)$; $P_2(0, 2)$.
5. $P_1(0, -4)$; $P_2(0, 1)$.
6. $P_1(0, 4)$; $P_2(0, -1)$.
7. $P_1(4, 7)$; $P_2(10, 9)$.
8. $P_1(-3, -5)$; $P_2(5, 2)$.
9. $P_1(-4, 7)$; $P_2(1, -2)$.
10. $P_1(8, 2)$; $P_2(4, -1)$.
11. $P_1(-1, 0)$; $P_2(-5, 5)$.
12. $P_1(7, -2)$; $P_2(-6, 3)$.
13. $P_1(-a, b)$; $P_2(-a, b)$.
14. $P_1(-e, -2)$; $P_2(e, d)$.
15. $P_1(-h, b)$; $P_2(m, -n)$. 
State the coordinates of the point of arrival, Nos. 16-21:

16. Start from (3, 7); take x-step +2 and y-step -1.

17. Start from (-4, 5); take x-step +4 and y-step -2.

18. Start from (-7, -6); take x-step -3 and y-step +6.

19. Start from (2, -3); take x-step -4 and y-step -5.

20. Start from (a, b); take x-step -p and y-step -q.

21. Start from (x₁, y₁); take x-step r cos θ and y-step r sin θ.

2.5. The Distance $P₁P₂$ between $P₁(x₁, y₁)$ and $P₂(x₂, y₂)$ is

\[ \sqrt{(x₂-x₁)^2 + (y₂-y₁)^2}. \]

To move from $P₁$ to $P₂$, take an x-step $P₁ → R$ and a y-step $R → P₂$; then $P₁ → R = x₂-x₁$ and $R → P₂ = y₂-y₁$.

By Pythagoras' theorem, $P₁P₂^2 = P₁R^2 + R²P₂^2$.

Since $P₁ → R = x₂-x₁$, the distance between $P₁$ and $R$ is $x₂-x₁$ if $x₂ > x₁$ and is $x₁-x₂$ if $x₂ < x₁$; in either case, $P₁R^2 = (x₂-x₁)^2$.

Similarly, $RP₂² = (y₂-y₁)^2$ either if $y₂ > y₁$ or if $y₂ < y₁$.\[ P₁P₂ = \sqrt{(x₂-x₁)^2 + (y₂-y₁)^2}. \]

In particular, the distance of $P(x, y)$ from the origin $O(0, 0)$ is given by

\[ OP = \sqrt{x^2 + y^2}. \]

The distance-formula should be committed to memory, and the best way of doing so is either to visualise Fig. 14 or to sketch the right-angled triangle $P₁RP₂$ and mark on it the lengths of the x-step and y-step from $P₁$ to $P₂$.

For simplicity, Fig. 14 is drawn so that the coordinates of $P₁$, $P₂$, and the lengths of the steps from $P₁$ to $P₂$ are all positive, but the proof of the distance-formula which has been given is completely general.

Example 2. Find the distance between $P₁(-3, 1)$ and $P₂(3, -7)$.

By the distance-formula, $P₁P₂ = \sqrt{(-3 - 3)^2 + (1 - 7)^2}$;

\[ P₁P₂ = \sqrt{36 + 36} = \sqrt{72} = 6.0. \]

Example 3. Find the relation between $x₁$ and $y₁$ if $P₁(x₁, y₁)$ lies on the circle, centre the origin $O$, radius 3.

By the distance-formula, $OP₁² = x₁² + y₁²$; but $OP₁ = 3$;

\[ x₁² + y₁² = 9. \]

The equation, $x² + y² = 9$, is called the equation of the circle, centre $O$, radius 3, because a point $P(x, y)$ lies on this circle if and only if $x² + y² = 9$.

2.7. Equation of a Circle. Since a circle is the locus of a variable point whose distance from a given point is constant, the equation of a circle whose centre and radius are given can be formed by using the distance-formula.

Example 4. Find the equation of a circle, centre $O(-2, 3)$, radius 4.

If $P₁(x₁, y₁)$ lies on the given circle, $CP₁ = 4$.

By the distance-formula,

\[ CP₁² = (x₁ + 2)² + (y₁ - 3)² = 16, \]

\[ x₁² + y₁² + 4x₁ - 6y₁ - 3 = 0. \]

Thus, the coordinates of any point $P(x, y)$ on the given circle are connected by the equation

\[ x² + y² + 4x - 6y - 3 = 0. \]  

Conversely, if the coordinates $(x, y)$ of a point $P$ are connected by the equation (1), then working backwards $(x + 2)² + (y - 3)² = 16$;

\[ x₁² + y₁² = 4, \]

\[ x₁² - 2x₁ + 4x₁ - 12 + 6y₁ = 3. \]

Thus, the distance of $P(x, y)$ from $C(-2, 3)$ is equal to 4;

\[ P(x, y) \] lies on the circle, centre $C(-2, 3)$, radius 4.

Hence equation (1) is the equation of the given circle.

2.7. The Length of the Tangent from a Point to a Circle.

Example 5. Find the length of the tangent $TP₁$ to the circle, centre $C(-2, 3)$, radius 4, from a point $T(x', y')$ outside the circle.

By Pythagoras' theorem, since $CP₁T$ is a right-angled triangle $P₁PT$, and $OP₁T = 4$, then

\[ CP₁² + PT² = CT², \]

\[ TP₁² = CT² - OP₁². \]

By the distance-formula, $CT² = (x' + 2)² + (y' - 3)²$;

\[ TP₁² = (x' + 2)² + (y' - 3)² - 16, \]

\[ TP₁² = x'² + y'² - 4x' - 6y' - 3. \]

There is a significant resemblance between the form of the equation of the circle in Example 4 and the form of the expression for the square of the length of the tangent to this circle from an external point in Example 5.

If the equation of the circle is written in the form

\[ x² + y² + 4x - 6y - 3 = 0, \]

then the square of the length of the tangent from $(x', y')$ to this circle is

\[ x'² + y'² + 4x' - 6y' - 3. \]

A comparison of the working shows that the resemblance is not accidental.
2.8. The distance-formula can be used to interpret the locus of a
variable point \( P(x, y) \) which moves so that
\[
x^2 + y^2 + 2gx + 2fy + c = 0
\]
where \( g, f, c \) are constants.

Example 6. Interpret the locus whose equation is
\[
2x^2 + 2y^2 - 3x + 5y + 1 = 0.
\]
Divide by 2 so as to make the coefficients of \( x^2 \) and \( y^2 \) equal to 1,
\[
x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y + \frac{1}{2} = 0.
\]
This equation becomes more easily intelligible if we complete the squares for \( x^2 - \frac{3}{2}x \) and for \( y^2 + \frac{5}{2}y \);
\[
x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \left(x - \frac{3}{4}\right)^2,
\]
and
\[
y^2 + \frac{5}{2}y + \left(\frac{5}{4}\right)^2 = \left(y + \frac{5}{4}\right)^2.
\]
Hence the equation of the locus can be written
\[
\left(x - \frac{3}{4}\right)^2 + \left(y + \frac{5}{4}\right)^2 = \frac{1}{26}.
\]
that is,
\[
\left(x - \frac{3}{4}\right)^2 + 
\]
By the distance-formula,
\[
if \ C \ is \ the \ point \ \left(\frac{3}{4}, -\frac{5}{4}\right) \ and \ if \ P \ is \ the \ point \ (x, y),
\]
\[
C P^2 = \left(x - \frac{3}{4}\right)^2 + \left(y + \frac{5}{4}\right)^2.
\]
Hence, if \( P(x, y) \) is any point of the given locus,
\[
C P^2 = \frac{1}{26} \therefore \ C P = \frac{1}{\sqrt{26}}.
\]
Hence the locus is a circle, centre \( C\left(\frac{3}{4}, -\frac{5}{4}\right) \), radius \( \frac{1}{\sqrt{26}} \).

A locus-property involves a theorem and its converse. In most cases, the
converse follows by reversing the argument of the original theorem as
in Example 4, p. 13. In future, the converse property will be
assumed without special mention unless some restrictions are necessary.

2.8.1. The general equation, \( x^2 + y^2 + 2gx + 2fy + c = 0 \), can be
written
\[
(x + g)^2 + (y + f)^2 - f^2 + c = 0,
\]
that is,
\[
(x + g)^2 + (y + f)^2 = \sqrt{g^2 + f^2 - c}.
\]
Hence as in Example 6, provided that \( g^2 + f^2 - c > 0 \), the locus is a
circle, centre \( C(-g, -f) \), radius \( \sqrt{g^2 + f^2 - c} \).

2.8.2. If \( TP \) is the tangent from a point \( T(x', y') \) to the circle
\[
x^2 + y^2 + 2gx + 2fy + c = 0,
\]
then
\[
TP^2 = x'^2 + y'^2 + 2gx' + 2fy' + c.
\]
By 2.8.1, the centre \( C \) of the circle is \( (-g, -f) \) and the radius is
\[
\sqrt{g^2 + f^2 - c};
\]
\[
\therefore TP^2 = CT^2 - CP^2 = \{(x' + g)^2 + (y' + f)^2\} - \{(\sqrt{g^2 + f^2 - c})^2\};
\]
\[
\therefore TP^2 = x'^2 + 2gx' + g^2 + y'^2 + 2fy' + f^2 - (g^2 + f^2 - c);
\]
\[
\therefore TP^2 = x'^2 + y'^2 + 2gx' + 2fy' + c.
\]

Exercise 4

Find the distance between the given pair of points, Nos. 1-8:

1. \((0, 0); (3, -4)\) 2. \((2, 4); (6, 7)\) 3. \((2, -5); (10, 1)\)
4. \((-2, -3); (1, 1)\) 5. \((5, 7); (-7, 2)\) 6. \((-3, -1); (-3, 2)\)
7. \((x, y); (x + r \cos \theta, y + r \sin \theta)\) 8. \((a, b); (1/\sqrt{2}, -1/\sqrt{2})\)

9. Prove that the points \((-3, 2), (4, 3), (2, 7)\) are the vertices of an
isosceles triangle and find the length of the base.

10. \(O, P, Q, R\) are the points \((0, 0), (a, b), (b, a), (b, -a)\), where
\(a = h - k, b = h + k\). Prove \(\triangle OQK = 2\triangle PKK\) (ii) \(\triangle OQR\) is a square.

11. A variable point \(P\) is equidistant from \((1, -2, 3)\) and \(B(4, -3, 0)\).
Find the equation of the locus of \(P\) and illustrate by a sketch.

12. Centre \((0, 0); \) radius \(\frac{1}{2}\). 13. Centre \((0, -2); \) radius 2.
14. Centre \((-3, 4); \) radius 6. 15. Centre \((2, -5); \) radius 4.
16. Centre \((-\frac{3}{2}, -\frac{1}{2}); \) radius 2. 17. Centre \((-1, -\frac{1}{2}); \) radius 1\(\frac{1}{2}\).
18. Centre \((a, 0); \) radius \(a\). 19. Centre \((-b, -b); \) radius \(b\sqrt{2}\).
20. Centre \((2, -3)\) and passing through the point \((1, 1)\).
21. Centre \((-1, 2)\) and passing through the origin.

22. Prove that the points \((1, 8), (-6, 7), (6, -1), (-2, -1)\) lie on a
circle, centre \((-2, 4)\). Find the radius and equation of the circle.

23. \(x^2 + y^2 - 3 = 0\) 24. \(x^2 + 4y^2 = 25\) 25. \((x-1)^2 + y^2 = 0\).

26. \((x-2)^2 + (y + 1)^2 = 9\) 27. \((x + 3)^2 + (y - 4)^2 = 25\).

28. \(x^2 + y^2 + 10x - 24 = 0\) 29. \(x^2 + y^2 - 6y + 5 = 0\).
30. \(x^2 + y^2 - 3x - 8y + 12 = 0\) 31. \(x^2 + y^2 + 10x - 2y - 10 = 0\).
32. \(x^2 + y^2 + 4x + 3y = 0\) 33. \(x^2 + y^2 + x + 4y + 2 = 0\).
34. \(x^2 + 4y^2 - 8x + 8y = 1\) 35. \(2x^2 + 2y^2 + 12x - 4y = 5\).
36. \(x^2 + y^2 - 2xz = 0 (a > 0)\) 37. \(x^2 + y^2 + 2xy - y = 0 (b > 0)\).
38. Find length of tangent from \((4, 2)\) to circle, \(x^2 + y^2 + 2x - 8y = 0\).
39. Find length of tangent from \((5, 4)\) to circle, \(x^2 + y^2 - 4x + 2y = 13\).
40. Find the distance of \(P(7, 5)\) from the centre of the circle,
\(x^2 + y^2 - 6x - 4y = 23\). Is \(P\) inside or outside the circle?

41. Prove that, if \(g^2 + f^2 - c < 0\), there is no point \((x, y)\) whose
coordinates satisfy the equation, \(x^2 + y^2 + 2gx + 2fy + c = 0\).

42. If \(x^2 + y^2 + 2gx + 2fy + c\) is negative, prove that the point
\(P(x, y)\) lies inside the circle whose equation is
\(x^2 + y^2 + 2gx + 2fy + c = (x + g)^2 + (y + f)^2 - (g^2 + f^2 - c) = 0\).
2.9. Directed Lines. A convention was made, see p. 9, that the length of a step along \( x'Ox \) or any line parallel to it is positive if taken in the sense \( x' \rightarrow x \) and negative if taken in the reverse sense. We now proceed to make a sign convention for steps along any line.

![Figure 16](image)

Let any line, not parallel to \( x'Ox \), meet \( x'Ox \) at \( A \) and take points \( C', C \) on the line such that the angle \( 0^\circ \), measured counterclockwise from the direction \( x'Ax \) to the direction \( C'AC \) is between \( 0^\circ \) and \( 180^\circ \). We now make the following convention.

2.9.1. If \( P, Q \) are any points on the line \( C'C \), the step from \( P \) to \( Q \) is said to have a positive length if \( P \rightarrow Q \) and \( C' \rightarrow C \) have the same sense, see Fig. 17 (i), (iii), and to have a negative length if \( P \rightarrow Q \) and \( C' \rightarrow C \) have reverse senses, see Fig. 17 (ii), (iv).

If \( \theta = 90^\circ \), this is the same as the convention made on p. 9 for the lengths of steps along \( y'Oy \).

![Figure 17](image)

If the distance between the points \( P \) and \( Q \) is \( d \) units, \( d \) is a signless number; the statements \( PQ = d \) and \( QP = d \) mean the same thing.

But if the length of the step from \( P \) to \( Q \) along a directed line \( C'AC \) is \( r \) units, then \( r \) is a directed (positive or negative) number such that

\[
\text{if } P \rightarrow Q \text{ and } C' \rightarrow C \text{ have the same sense, then } r = +d
\]

and

\[
\text{if } P \rightarrow Q \text{ and } C' \rightarrow C \text{ have reverse senses, then } r = -d
\]

Thus \( r \) is positive in Fig. 17 (i), (iii) and is negative in Fig. 17 (ii), (iv).

2.9.2. If \( P(x, y) \) is a point on the directed line \( C'C \) which makes an angle \( \theta \) with \( x'Ox \), where \( \theta \) is between \( 0^\circ \) and \( 180^\circ \), and if a step \( P \rightarrow Q \) of length \( r \) is taken along the line, the point of arrival \( Q \) is

\[
(x + r \cos \theta, y + r \sin \theta)
\]

The proof just given of this important statement is completely general; this step-formula can therefore be used without considering the various possible positions of \( P \) and \( Q \) relative to each other and to the axes. The easiest way of remembering the formula is to draw the figure in a simple position as in Fig. 18.

Example 7. Start from \( P(1, -2) \) and take a step of length \(-4\) in direction making \( 60^\circ \) with \( x'Ox \). Find the point of arrival \( Q(x, y) \) and illustrate by a sketch.

By the step-formula, \( x = 1 + (-4) \cos 60^\circ = 1 - 2 = -1 \)

and \( y = (-2) + (-4) \sin 60^\circ = -2 - 2 \sqrt{3} \).

Thus \( Q \) is the point \((-1, -2 - 2 \sqrt{3})\), see Fig. 19.
2.10. Polar Coordinates. If we start from the origin $O$ and take a step of length $r$ along a directed line $OC$ which makes an angle $\theta^\circ$ with $x'Ox$, we can reach any given point $P(x, y)$ of the plane by choosing a suitable value of $\theta^\circ$ such that $0^\circ \leq \theta^\circ < 180^\circ$ and a suitable positive or negative value of $r$.

![Diagram](image1)

For example, in Fig. 20 (i), the value of $r$ is positive and in Fig. 20 (ii) the value of $r$ is negative.

By the step-formula, the point of arrival, starting from $O$, is

$$0 + r \cos \theta^\circ, 0 + r \sin \theta^\circ),$$

and this is the given point $P(x, y)$ if

$$r \cos \theta^\circ = x \quad \text{and} \quad r \sin \theta^\circ = y,$$

where $0^\circ \leq \theta^\circ < 180^\circ$.

For the point $P(0, y)$, $\theta^\circ = 90^\circ$ and $r = y$; ($y$ may be $+$ or $-$).

For the point $P(x, y)$, $x \neq 0$, $\tan \theta^\circ = \frac{y}{x}$ or $\cos \theta^\circ = \frac{x}{r}$.

This relation gives a unique value of $\theta^\circ$ between $0^\circ$ and $180^\circ$ and the values of $x$ and $y$ are given. (If $y = 0$, $\theta^\circ = 0^\circ$ and $r = x$.)

Further

$$x^2 + y^2 = r^2 \cos^2 \theta^\circ + \sin^2 \theta^\circ = r^2,$$

this relation gives a unique value of $r$ in direction making $\theta^\circ$ with $x'Ox$.

Also $r \cos \theta^\circ = x$ is positive for $0^\circ < \theta^\circ < 180^\circ$ and $y = r \sin \theta^\circ$, therefore $r$ has the same sign as $y$.

The unique values of $r$ and $\theta^\circ$ which are determined by

$$r \cos \theta^\circ = x, \quad r \sin \theta^\circ = y, \quad 0^\circ \leq \theta^\circ < 180^\circ,$$

are called the polar coordinates of the given point $P(x, y)$.

The origin $O$ is called the pole, the directed line $x'Ox$ from which the angle $\theta^\circ$ is measured is called the initial line, $OP$ is called the radius-vector and $\theta^\circ$ is called the vectorial angle of $P$.

Example 8. Find the polar coordinates of the point $P(-1, 2)$, taking $O$ as pole and $x'Ox$ as initial line.

By the transformation formulas, $r \cos \theta^\circ = 2$, $r \sin \theta^\circ = 2$, $\theta^\circ < 180^\circ$.

$$\tan \theta^\circ = -\frac{2}{1}, \quad 0^\circ < \theta^\circ < 180^\circ$$

Also $r^2 = 2^2 + 1^2 = 5$, but $\sin \theta^\circ$ is positive, and $r = -2 \sin \theta^\circ$.

$r$ is negative, so $r = \sqrt{5}$.

The polar coordinates of $P$ are given by $r = -\sqrt{5}, \theta^\circ = 180^\circ$.

2.11. Roots of a Quadratic. The following properties should, if necessary, be revised; frequent use is made of them.

If $x_1$ and $x_2$ are the roots of $ax^2 + bx + c = 0$,

$$x_1 + x_2 = -\frac{b}{a} \quad \text{and} \quad x_1x_2 = \frac{c}{a}.$$

The roots are equal if $b^2 - 4ac = 0$.

There are no roots if $b^2 - 4ac < 0$.

EXERCISE 5

Illustrate Nos. 1-14 by free-hand sketches.

In Nos. 1-6, find the coordinates of $Q$ if $P \rightarrow Q$ is the step of length $r$ in direction making $\theta^\circ$ with $x'Ox$.

1. $P(3, 5)$; $r = 2$, $\theta^\circ = 150^\circ$.
2. $P(-1, 2)$; $r = 4$, $\theta^\circ = 45^\circ$.
3. $P(2, -3)$; $r = -6$, $\theta^\circ = 150^\circ$.
4. $P(-2, 1)$; $r = 3$, $\theta^\circ = 190^\circ$.
5. $P(-1, -1)$; $\tan \theta^\circ = \frac{1}{3}$, $0^\circ < \theta^\circ < 90^\circ$; (i) $r = 1 + 0$, (ii) $r = -5$.
6. $P(-3, 2)$; $\tan \theta^\circ = -\frac{2}{3}$, $90^\circ < \theta^\circ < 180^\circ$; (i) $r = 4$, (ii) $r = -1$.

In Nos. 7-14, find the polar coordinates $r$ and $\theta^\circ$, $0^\circ < \theta^\circ < 180^\circ$, of the points whose Cartesian coordinates are given.

7. $(1, \sqrt{3})$. 8. $(-\sqrt{3}, 1)$. 9. $(1, -1)$. 10. $(1, -1)$.
11. $(3, -4)$. 12. $(\cos 100^\circ, \sin 100^\circ)$. 13. $(\cos 50^\circ, \sin 50^\circ)$.
14. $(-\sin 110^\circ, \cos 110^\circ)$.

15. In Fig. 21, $A \rightarrow P$ is the step, length $r$, in direction making $\theta^\circ$ with $x'Ox$, where $0^\circ < \theta^\circ < 180^\circ$, from $A(-4, 0)$ to a point $P(x, y)$ on the circle $x^2 + y^2 = 4$.

(i) Express $x, y$ in terms of $r, \theta^\circ$.
(ii) Find an equation connecting $r$ and $\theta^\circ$, and find the two values of $r$ if $\cos \theta^\circ = \frac{1}{2}$.
(iii) Illustrate by a sketch why there is no value of $r$ if $30^\circ < \theta^\circ < 180^\circ$.
(iv) Find $r$ if $\cos \theta^\circ = -\frac{1}{2}$.

16. In Fig. 21, $B \rightarrow Q$ is the step, length $r$, in direction making $\phi^\circ$ with $x'Ox$, where $0^\circ < \phi^\circ < 180^\circ$, from $B(1, 0)$ to a point $Q(x, y)$ on the circle $x^2 + y^2 = 4$.

Express $x, y$ in terms of $r, \phi^\circ$ and find an equation connecting $r$ and $\phi^\circ$. Find the values of $r$ if (i) $\cos \phi^\circ = \frac{1}{2}$, (ii) $\phi^\circ = 90^\circ$, (iii) $\cos \phi^\circ = -\frac{1}{2}$.

If $r_1, r_2$ are the values of $r$ for any given value of $\phi^\circ$, prove that $r_1 + r_2 = 4$. 

Illustrate by a sketch and interpret the result geometrically.
17. In Fig. 22, $A$ is the point $(4, 0)$ and $O \rightarrow P$ is a step of length $r$ in direction making $\theta^\circ$ with $x'Ox$ such that $\angle OPA = 90^\circ$. Sketch the complete locus of $P$ when $\theta^\circ$ varies from $0^\circ$ to $180^\circ$. Obtain the equation between $r$ and $\theta^\circ$ for all points of the locus and show on the sketch the point of the locus for which $\theta^\circ = 120^\circ$.

18. The polar coordinates $r, \theta^\circ$ of a variable point $P$ are connected by the equation $r = 4 \cos \theta^\circ$, $(0^\circ \leq \theta^\circ < 180^\circ)$. Explain why the complete locus of $P$ is the circle on $OA$ as diameter in Fig. 22. Draw this locus and use it to sketch the complete locus of the point $Q$ whose polar coordinates $r, \theta^\circ$ are connected by either of the equations, $r = 4 \cos \theta^\circ + 4$, $r = 4 \cos \theta^\circ - 4$. The locus of $Q$ is called a cardioid.

19. The length of the step from the origin $O$ to $P_1(x_1, y_1)$ is $r_1$ in direction making $\theta_1^\circ$ with $x'Ox$, and the length of the step from $P_1$ to $P_2(x_2, y_2)$ is $r_2$ in direction making $\theta_2^\circ$ with $x'Ox$. Write down the values of $x_1$, and $y_1$ in terms of $r_1$, $\theta_1^\circ$, and then the values of $x_2$ and $y_2$ in terms of $r_2$, $\theta_2^\circ$, $\theta_1^\circ$, $\theta_2^\circ$.

20. In Fig. 23, $O \rightarrow P$ is a step of length $p$ from the origin $O$ in direction making $\alpha^\circ$ with $x'Ox$; $P \rightarrow Q$ is a step of length $r$ in direction making $90^\circ - \alpha^\circ$ with $x'Ox$. If $Q$ is the point $(x, y)$, find the values of $x$ and $y$ in terms of $p$, $r$, $\sin \alpha^\circ$, $\cos \alpha^\circ$ by using the formulas $\cos(90^\circ - \alpha^\circ) = -\sin \alpha^\circ$, $\sin(90^\circ - \alpha^\circ) = \cos \alpha^\circ$. Hence prove that $x = r \cos \alpha^\circ + y \sin \alpha^\circ - p$. Sketch the locus of $Q$ represented by this equation (i) if $\alpha^\circ = 30^\circ$, $p = 3$, (ii) if $\alpha^\circ = 150^\circ$, $p = -3$.

2.12. Geometrical Applications. Most of the preceding examples are numerical and so refer to a figure of a particular size. To prove a general geometrical property, the constants must be expressed algebraically; but it is important to choose the axes so as to make the algebra as simple as possible, and advantage should be taken of any symmetry the figure possesses; see 2.12.1, 2.12.2.

2.12.1. Apollonius' Theorem. If $O$ is the mid-point of the base $BC$ of $\triangle ABC$, prove that $AB^2 + AC^2 = 2AO^2 + 2OB^2$.

Take $O$ as origin and the $x$-axis $x'Ox$ along $BC$, see Fig. 24; then $B, C$ are the points $(-d, 0), (d, 0)$ where $BC = 2d$.

2.12.2. Segments of a Chord. If a line $KP$ through a given point $K$ meets a circle, centre $O$, radius $a$, at $P_1$, $P_2$, prove that $KP_1 \cdot KP_2 = OK^2 - a^2$.

Take $O$ as origin and $x'Ox$ along $OK$, see Fig. 25 (i), (ii), then the equation of the circle is $x^2 + y^2 = a^2$ and $K$ may be taken as $(b, 0)$, where $OK = k$.

The directed line $KP$ makes angle $\theta^\circ$ with $x'Ox$.

A step of length $r$ along $KP$ from $K(b, 0)$ ends at the point $(k + r \cos \theta^\circ, r \sin \theta^\circ)$; this point lies on the circle if

$$(k + r \cos \theta^\circ)^2 + (r \sin \theta^\circ)^2 = a^2,$$

that is, $k^2 + 2kr \cos \theta^\circ + r^2(\cos^2 \theta^\circ + \sin^2 \theta^\circ) = a^2$; hence, if $K \rightarrow P_1 = r_1$ and $K \rightarrow P_2 = r_2$, $r_1$ and $r_2$ are the roots of $r^2 + 2kr \cos \theta^\circ + k^2 - a^2 = 0$; $r_1r_2 = k^2 - a^2 = OK^2 - a^2$.

In Fig. 25 (i), $r_1$ and $r_2$ have the same sign, $r_1r_2$ is positive; $KP_1 \cdot KP_2 = OK^2 - a^2$.

In Fig. 25 (ii), $r_1$ and $r_2$ have opposite signs, $r_1r_2$ is negative; $KP_1 \cdot KP_2 = a^2 - OK^2$.

where $KP_1, KP_2$ mean the distances between $K, P_1$ and between $K, P_2$.

Note. If in Fig. 25 (i), where $K$ is to the right of $O$, the directed line $KP$ makes an acute angle $\theta^\circ$ with $x'Ox, r_1$ and $r_2$ are each negative; but the proof is the same whether $K$ is right or left of $O$ and whether $\theta^\circ$ is acute or obtuse.
EXERCISE 6

1. **ABCD** is a rectangle. Take the origin *O* at the centre of **ABCD** and *x'Ox*, *y'Oy* parallel to **BA**, **DA**. If *A* is the point (*a*, *k*), write down the coordinates of **B**, **C**, **D** in terms of *a*, *k*. Hence prove that for any point *P*(*x*, *y*),

\[ PA^2 + PC^2 = PB^2 + PD^2. \]

2. If, with the data and axes in No. 1, **AQ**, **BE**, **DS** are the tangents from **A**, **B**, **D** to a given circle and if \( A Q^2 = B P^2 + D S^2 \), prove that **C** lies on the circle.

3. **OD** is the perpendicular bisector of the side **BC** of \( \triangle **ABC** \); **AM** is the perpendicular from **A** to **OD**. Take **BC** and **OD** as axes *x'Ox*, *y'Oy*. Find the equation of the locus of **P** for the given law and interpret the equation.

4. **PA^2 - PB^2 = k^2**. \[ PA^2 + PB^2 = k^2. \]

5. **PA^2 - PD^2 = k^2**. \[ PA^2 + PB^2 = k^2. \]

6. **PA = k . PD**.

7. In Fig. 26, **S** and **X** are the given points (*x*, *y*), (−*a*, 0); **O** is the origin. The curve is a **parabola** whose equation is \( y^2 = 4ax \) and **P**1 is any point on the curve.

   (i) If \( P_1 M_1 \) is the perpendicular from \( P_1 \) to the line \( XD \) through **X** perpendicular to **XS**, prove \( SP_1 = P_1 M_1 \).

   (ii) If a line through **S** making an angle \( 90^\circ \) with \( x'Ox \) meets the parabola in \( P_1 \) and \( P_2 \), prove that the lengths \( r_1, r_2 \) of the steps \( S \rightarrow P_1, S \rightarrow P_2 \) are the roots of the equation, \( r^2 \sin^2 90^\circ - 4ar \cos 90^\circ + 4a^2 = 0 \). Write down the value of \( r_1 r_2 \) in terms of *a*, *x* and deduce that if \( P_1 N_1, P_2 N_2 \) are the perpendiculars from \( P_1, P_2 \) to **XS**, then \( N_1 P_1 \cdot P_2 N_2 = 4a^2 \).

   (iii) Express the value of \( ON_1 \cdot ON_2 \) in terms of **OS**.

8. **A** is a fixed point on a given circle, centre **O**, radius *a*. Take **O** as origin and the axis **x'Ox** along **OA**. **C** and **D** are the points (**a*, *0*), (**a*, *0*) and **P** is a point (**x*, **y**) on the circle. Prove that \( x^2 + y^2 = a^2 \) and find expressions for **PD** and **PC** in terms of **a**, **x**. Hence prove that \( PD = 3PC \).

9. The centres of the circles, \( x^2 + y^2 - 2ax = a^2 \), \( x^2 + y^2 - 2ay = a^2 \), which meet at **H**, **K**, are **A**, **B**; **TP**, **TQ** are the tangents from **T**(*x*, *y*); **T** **N** is the perpendicular to **HK**. Find the coordinates of **A**, **B**, **H**, **K**; sketch the figure and prove that **TP** \( \sim **TQ** = 2**AB** . **TN**.

CHAPTER 3

LINES

3.1. **Lines parallel to an Axis**. If a point **P**(*x*, *y*) moves in a plane subject to a given law, its coordinates satisfy an equation, called the **equation of the locus**. The reader may at first be puzzled by statements such as the equation of the locus of **P**(*x*, *y*) is

\[ x + 4 = 0 \quad \text{or} \quad x = -4, \]

where the equation does not contain *y*.

It means that **P**(*x*, *y*) moves so that its *x*-coordinate is always −4; its *y*-coordinate can have any value.

Hence the locus is a line **AP** parallel to **y'Oy** through the point **A**(−4, 0), see Fig. 27.

Similarly, the statement that the equation of the locus of **P**(*x*, *y*) is

\[ y - 3 = 0 \quad \text{or} \quad y = 3, \]

means that **P**(*x*, *y*) moves so that its *y*-coordinate is always 3; its *x*-coordinate can have any value. Hence the locus is a line **BP** parallel to **x'Ox** through the point **B**(0, 3), see Fig. 27.

If the equation of the locus of a variable point **P**(*x*, *y*) is

\[ x = a, \]

where *a* is a constant, positive or negative or zero, **P** moves so that its *x*-coordinate is always equal to *a*; its *y*-coordinate can have any value.

Hence the locus is the line **AP** parallel to **y'Oy** through the point **A**(a, 0), see Fig. 28.

Similarly, if the equation of the locus of **P**(*x*, *y*) is

\[ y = b, \]

where *b* is a constant, the locus is the line **BP** parallel to **x'Ox** through the point **B**(0, *b*), see Fig. 28.

In particular, the equation of the **y-axis** is **x** = 0,

and the equation of the **x-axis** is **y** = 0.
3.2. Gradient of a Directed Line. Let \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) be any two points on a given directed line \( AP \) which meets \( x'Ox \) at \( A \) and makes an angle \( \theta \) with \( x'Ox \), where by the convention in 2.9, p. 16, \( 0 \leq \theta < 180^\circ \). The reader is reminded that the trigonometrical ratios of obtuse angles are defined as follows.

\[
\sin \theta = \sin (180^\circ - \theta), \quad \cos \theta = -\cos (180^\circ - \theta),
\]

and

\[
\tan \theta = \frac{\sin \theta}{\cos \theta}.
\]

Denote the length of the step from \( P_1 \) to \( P_2 \) by \( r \) and denote by \( P_1 \rightarrow R \) and \( R \rightarrow P_2 \) the lengths of the \( x \)-step and \( y \)-step from \( P_1 \) to \( P_2 \).

![Diagram showing gradient of a directed line](image)

The restriction \( 0^\circ \leq \theta < 180^\circ \) in the convention in 2.9, p. 16, means that a line is directed so that the length of a step \( P_1 \rightarrow P_2 \) along the line is positive or negative according as the \( y \)-coordinate increases, see Fig. 29 (i), or decreases, see Fig. 29 (ii), when the step is taken.

A line parallel to \( x'Ox \) is directed in the same sense as \( x'Ox \) and is said to make an angle \( 0^\circ \) with \( x'Ox \).

By 2.3.1, p. 11, and 2.9.1, p. 17, if \( \theta \) is acute or obtuse,

\[
x_2 - x_1 = P_1 \rightarrow R = r \cos \theta \quad \text{and} \quad y_2 - y_1 = P_2 \rightarrow R = r \sin \theta.
\]

\[
\therefore \quad \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta.
\]

The value of \( \tan \theta \) is called the gradient of the directed line \( AP \).

If \( \theta = 90^\circ \), \( AP \) is parallel to \( y'Oy \); the step from \( P_1 \) to \( P_2 \) is then a \( y \)-step and the gradient of \( AP \) is undefined.

**Example 1.** Find the equation of the line joining \( P_1(-4, +2) \), \( P_2(+6, -5) \).

Gradient of \( P_1P_2 \): 
\[
\frac{y}{x} = \frac{-5 - 2}{6 - (-4)} = \frac{-7}{10};
\]

Gradient of line joining \( P_1(-4, +2) \) to any point \( P(x, y) \) is 
\[
\frac{y - 2}{x + 4};
\]

\( P(x, y) \) lies on \( P_1P_2 \) if, and only if, 
\[
\frac{y - 2}{x + 4} = \frac{-7}{10};
\]

the equation of \( P_1P_2 \) is 
\[
7x + 10y + 8 = 0.
\]

The reader should illustrate this argument by a sketch.

3.3. Example 2. Prove that the locus whose equation is \( 3x + 5y = 12 \) is a line. Find its gradient and the angle it makes with \( x'Ox \).

Select any particular point \( P_1(x_1, y_1) \) of the locus, then 
\[
3x_1 + 5y_1 = 12.
\]

(For example, \( P_1 \) can be \((4, 0)\) or \((0, 2\frac{2}{3})\) or \((-1, 3)\), etc.)

Let \( P(x, y) \) be any other point of the locus, then 
\[
3x + 5y = 12;
\]

\( \therefore \) by subtraction, 
\[
3(x - x_1) + 5(y - y_1) = 0, \quad \text{that is}, \quad \frac{y - y_1}{x - x_1} = -\frac{3}{5}.
\]

\( \therefore \) the gradient of the line joining the fixed point \( P_1 \) of the locus to a variable point \( P \) of the locus is equal to \( -\frac{3}{5} \);

\( \therefore \) the locus is the line through \( P_1 \) whose gradient is \( -\frac{3}{5} \).

Further the line makes with \( x'Ox \) an angle \( \theta \) where \( \tan \theta = -\frac{3}{5} \);

\( \therefore \) from tangent-tables, \( \theta = 180^\circ - 31^\circ - 149^\circ \).

The reader should illustrate these results by a sketch.

3.3. Example 1 and Example 2 illustrate the two properties which form the foundation of the geometrical method invented by Descartes:

The equation of a line is of the first degree in \( x \) and \( y \).

Conversely, the locus whose equation is of the first degree in \( x \) and \( y \) is a line.

These statements are proved in 3.3.1 and 3.3.4.

3.3.1. The equation of a line is of the form 
\[
ax + by + c = 0.
\]

(i) If the line is parallel to \( y'Oy \), it has been proved, see p. 23, that its equation is of the form \( x = a = 0 \).

(ii) If the line is not parallel to \( y'Oy \), it may be taken as a line of given gradient \( \tan \theta \) passing through a given point \( P_1(x_1, y_1) \).

The gradient of the line joining \( P_1(x_1, y_1) \) to \( P(x, y) \) is 
\[
y - y_1 \quad \text{is}
\]

\( P(x, y) \) lies on the given line if, and only if, 
\[
\frac{y - y_1}{x - x_1} = \tan \theta;
\]

\( \therefore \) the equation of the line is 
\[
y - y_1 = (x - x_1) \tan \theta.
\]

3.3.2. The gradient of a line is often denoted by \( m \);

\( \therefore \) the equation of the line of gradient \( m \) through \( (x_1, y_1) \) is 
\[
y - y_1 = m(x - x_1),
\]

In particular, the equation of the line of gradient \( m \) through the origin is 
\[
y = mx,
\]

and the equation of the line of gradient \( m \) which meets \( y'Oy \) at \((0, c)\) is 
\[
y = mx + c.\]
3.3.3. The gradient of the line joining \(P_1(x_1, y_1), P_2(x_2, y_2)\), where \(x_1 \neq x_2\), is \(\frac{y_2 - y_1}{x_2 - x_1}\); therefore the equation of \(P_1P_2\) is

\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},
\]

or

\[
\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.
\]

3.3.4. The locus whose equation is

\[ax + by + c = 0\]

where \(a, b, c\) are constants and \(a, b\) are not both zero, is a line. Further, if \(b = 0\), the gradient of the line equals \(-a/b\).

(i) Suppose \(b = 0\); then by hypothesis \(a \neq 0\), and so the equation may be written \(x = -c/a\); hence by p. 23 the locus is the line parallel to \(y' y\) through the point \((-c/a, 0)\).

(ii) Suppose \(b \neq 0\). Select a point \(P_1(x_1, y_1)\) of the locus, for example \(P_1(0, -c/b)\); then \(ax_1 + by_1 + c = 0\).

Let \(P(x, y)\) be any other point of the locus, then

\[ax + by + c = 0.\]

\[\therefore \quad a(x - x_1) + b(y - y_1) = 0;\]

\[\therefore \quad \frac{y - y_1}{x - x_1} = \frac{-a}{b}.\]

Therefore the gradient of the line joining the fixed point \(P_1\) of the locus to a variable point \(P\) of the locus has the fixed value \(-a/b\); therefore the locus of \(P\) is the line through \(P_1\) of gradient \(-a/b\).

The proofs of the properties in 3.3.1 and 3.3.4 establish the fundamental theorem in 3.3. For this reason, an equation of the first degree in \(x\) and \(y\) is called a linear equation.

Linear equations are often expressed in forms which can be interpreted by inspection.

3.3.5. The equation \(\frac{x}{x_1} + \frac{y}{y_1} = 1\) is satisfied by \(x = x_1, y = y_1\), and by \(x = 0, y = 0\); it is therefore the equation of the line joining the origin \(O\) to the point \(P_1(x_1, y_1)\), see Fig. 30.

3.3.6. The equation \(\frac{x}{a} + \frac{y}{b} = 1\), is satisfied by \(x = a, y = 0\) and by \(x = 0, y = b\); it is therefore the equation of the line which cuts \(x' x\) at \(A(a, 0)\) and cuts \(y' y\) at \(B(0, b)\). \(a\) and \(b\) are called the intercepts made by \(AB\) on the axes.

In Fig. 31, \(a\) is negative and \(b\) positive.

---

3.3] Example 3. Find the equation of the line passing through the point \((2, -5)\) with gradient equal to \(-\frac{3}{2}\).

The equation of any line whose gradient is \(-\frac{3}{2}\) is of the form, \(y = -\frac{3}{2}x + c\), where \(c\) is a constant.

This may be written more simply in the form:

\[3x + 4y = k,\]

where \(k\) is a constant whose value is found from the condition that the line passes through the point \((2, -5)\); this gives

\[3(2) + 4(-5) = k, \text{ that is, } k = 6 - 20 = -14;\]

the equation of the required line is \(3x + 4y = -14\).

As soon as the method has become familiar, most of the working in Example 3 can be done mentally. It should be sufficient to write down:

\[y = -\frac{3}{2}x + \ldots; \quad 3x + 4y = \ldots = 3(2) + 4(-5) = 6 - 20 = -14.\]

---

EXERCISE 7

Nos. 1-7 refer to Fig. 32, which represents a rectangle \(ABCD\) with sides parallel to the axes.

1. If \(A\) is the point \((10, 3)\), write down the equations of \(AB, AB, OA, PS\).

2. If \(C\) is the point \((-4, -7)\), write down the equations of \(BC, CD, OC, QR\).

3. If \(B, D\) are the points, \((-3, 2)\), \((9, -5)\), write down (i) the coordinates of \(A, C\); (ii) the equations of \(AB, AD\); (iii) the equations of \(OA, OB, OC, OD, PQ, RS\).

Use tables to find the angles which \(OA, OB\) make with \(OX\).

4. If \(A, C\) are the points \((20, 8)\), \((-5, -6)\), write down (i) the coordinates of \(B, D\); (ii) the equations of \(BC, CD\); (iii) the equations of \(OA, OB, OC, OD, PS, QR\).

Use tables to find the angles which \(OA, OB\) make with \(OX\).

5. If the equations of \(PS, QR\) are \(\frac{x}{5} + \frac{y}{2} + 1 = 0, \frac{x}{3} + \frac{y}{4} + 1 = 0\), find the coordinates of \(A, C\) and the equations of \(PQ, RS\).

6. If the equations of \(PQ, RS\) are \(\frac{x}{3} - \frac{y}{7} + 1 = 0, \frac{x}{11} - \frac{y}{8} = 1\), find the coordinates of \(B, D\) and the equations of \(PS, QR\).

7. If the mid-point of \(PS\) is \((h, k)\), prove that the equation of \(PS\) is \(\frac{x}{2h} + \frac{y}{2k} = 1\).
3.4. Parallel Lines. The directed lines $A_1P_1, A_2P_2$ which make angles $\theta_1, \theta_2$ with $Ox$ are parallel if, and only if, $\tan \theta_1 = \tan \theta_2$, see Fig. 33. If the equations of $A_1P_1, A_2P_2$ are

$$y = m_1x + c_1, \quad x = m_2x + c_2,$$

then $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$, therefore the lines are parallel if $m_1 = m_2$.

3.4.1. If the lines $A_1P_1, A_2P_2$ are parallel, their equations can be expressed in the form

$$ax + by + c_1 = 0, \quad ax + by + c_2 = 0.$$

If $A_1P_1, A_2P_2$ are parallel to $y'Oy$, their equations are of the form,

$$ax + c_1 = 0, \quad ax + c_2 = 0.$$

If $A_1P_1$ is not parallel to $y'Oy$, its equation is of the form,

$$ax + by + c_1 = 0, \quad b \neq 0,$$

and so its gradient equals $-a/b$; hence the gradient of any line $A_1P_1$ parallel to $y'Oy$ equals $-a/b$; therefore the equation of $A_1P_1$ is of the form $ax + by + c_1 = 0$.

3.4.2. The lines, $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$, are coincident if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2},$$

and are parallel if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

(i) If $a_1b_2 - b_1a_2 = k$, say, then $a_1x + b_1y + c_1 = k(a_2x + b_2y + c_2)$.

(ii) If $c_1 = 0$, then $a_1 = a_2, b_2 = b_2, c_1 = c_2$.

The gradients of the lines, $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$, are equal.

Example 4. Find the equation of the line through $(1, 3)$ parallel to the line $2x + 5y + 3 = 0$.

The required equation is of the form, $2x + 5y = k$; this is satisfied by $-7 + 3 = -4$, if $2(-7) + 5(4) = k$.

The required equation is $2x + 5y = 6$.

The working should be shortened when the method has become familiar.

Example 5. Explain why the lines, $2x - 3y = 7, 3x + 2y = 8$, intersect and find the coordinates of their point of intersection $P_1(x_1, y_1)$.

The lines are not parallel because their gradients $\frac{2}{3}$ are unequal. $P_1(x_1, y_1)$ lies on the line $2x - 3y = 7$ if $2x_1 - 3y_1 = 7$, and lies on the line $3x_1 + 2y_1 = 8$ if $3x_1 + 2y_1 = 8$.

Therefore the values of $x_1, y_1$ are found by solving the equations of the two lines, $2x - 3y = 7, 3x + 2y = 8$, as simultaneous equations, this gives $x = 2, y = -1$; hence $P_1$ is the point $(2, -1)$. 
**Exercise 8**

Find the equation of the line passing through the given point and parallel to the given line, Nos. 1–8:

1. \( (3, 2); \ x - y = 10. \)
2. \( (1, -3); \ 3x + 2y = 0. \)
3. \( (-5, -2); \ \frac{1}{2}x - \frac{1}{2}y = 1. \)
4. \( (0, 0); \ 2x - \frac{2}{5}y = 5. \)
5. \( (-3, 2); \ 2y = 5(x - 1). \)
6. \( (1, -10); \ 2x + \frac{7}{2}y = 7. \)
7. \( (x_1, y_1); \ ax + by + c = 0. \)
8. \( (b, k); \ cx + ay + b = 1. \)

Find the point of intersection of the pair of lines, Nos. 9–13:

9. \( x - y + 3 = 0; \ 4x + 3y - 2 = 0. \)
10. \( x - 2y = 0; \ 3x - 5y - 7 = 0. \)
11. \( 7x + 2y - 1 = 0; \ x + 3y + 8 = 0. \)
12. \( y = mx; \ my + x = d. \)
13. \( x \cos \theta + y \sin \theta = p = 0; \ x \sin \theta - y \cos \theta = 0. \)
14. \( x \cos \theta + y \sin \theta = p = 0; \ x \sin \theta - y \cos \theta = 0. \)
15. \( \frac{x}{x_1} + \frac{y}{y_2} = 1; \ (x_1, y_2). \)
16. \( \frac{x}{x_1} + \frac{y}{y_2} = 1; \ (x_1, y_2). \)
17. \( \frac{x}{x} + y = 2; \ 4x + 5y = 1; \ one \ vertex \ is \ (3, 5). \)
18. \( Find \ a, b \ if \ ax - 10y = 9; \ 3x + by = 12. \)

**3.5. Gradients of Perpendicular Lines.** In Fig. 34, the lines \( OP, OP' \) make angles \( \theta, \theta + 90' \) with \( x'Ox; \ OP = OP' \) and \( PN, P'N' \) are drawn perpendicular to \( x'Ox; \)

\[ \angle N'OP = 90' - \theta \]

hence

\[ \angle N'OP = \triangle NPO; \]

\[ \therefore \ if \ P \ is \ the \ point \ (k, k), \ P' \ is \ the \ point \ (-k, k). \]

Denote the gradients of \( OP, OP' \) by \( m, m'; \)

\[ m = k/k, \ m' = -k/k; \]

\[ \therefore \ mm' = -1. \]

Hence, if \( y = mx, \ y = m'x \) are the equations of a pair of perpendicular lines through the origin, \( OP, OP' \), then

\[ mm' = -1. \]

Alternatively, we may say the gradients of the perpendicular lines \( OP, OP' \) are \( \tan \theta, \tan (\theta + 90') \); but

\[ \tan (\theta + 90') = -\cot \theta = -1/\tan \theta; \]

therefore the product of the gradients of \( OP, OP' \) is \(-1\).
3.6. The angle which OP, \( y = m_1x \), makes with OP, \( y = m_2x \), when \( m_1, m_2, m_1 \neq -1 \), is given by
\[
\tan \theta = \frac{m_2 - m_1}{1 + m_1m_2}.
\]
Let \( OP_1, OP_2 \) make angles \( \alpha, \beta \) with \( x'z \), then \( \tan \alpha = \frac{m_1}{2}, \tan \beta = \frac{m_2}{2, m_2}, \) where \( \theta = \theta, \theta + 90^\circ \).
\[
\tan \theta = \tan (\theta, \theta, \theta) = \frac{m_2 - m_1}{1 + m_1m_2}.
\]

3.6.1. If \( QR_1, QR_2 \) are the lines \( a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0, \)
where \( b_1b_2 \neq 0 \) and \( a_1a_2 + b_1b_2 = 0, \)
then
\[
\tan R_1R_2 = \frac{a_1b_2 - a_2b_1}{a_1b_2 + a_2b_1}.
\]

Where \( QR_1 \) and \( QR_2 \) are parallel to \( OP_1, y = m_1x, \) and \( OP_2, y = m_2x, \), if
\[
m_1 = -\frac{a_2}{b_1} \quad \text{and} \quad m_2 = -\frac{a_1}{b_2}, \quad \text{since} \quad b_1b_2 \neq 0.
\]
Also \( m_1m_2 \neq -1 \) since \( a_1a_2 + b_1b_2 = 0, \quad \therefore \tan R_1R_2 + 90^\circ = 0.
\]
\[
\tan R_1R_2 = \tan R_1R_2 = \frac{a_2b_1 - a_1b_2}{a_1b_2 + a_2b_1}.
\]

By 3.5.2., if \( a_1a_2 + b_1b_2 = 0, \), \( QR_2 \) is parallel to \( QR_1 \).
If \( b_1 = 0, \) \( QR_1 \) is parallel to \( y/Oy. \quad \therefore \tan R_1R_2 - 90^\circ \).

EXERCISE 9

Write down the equation of the line through the origin perpendicular to the given line, Nos. 1-6:
1. \( y = 2x - 1. \)
2. \( x = 2y + 2. \)
3. \( 5x - 4y = 10. \)
4. \( x/a + y/b = 1. \)
5. \( y = mx + c. \)
6. \( x \sec 60^\circ - y \cos 60^\circ = 0. \)

Write down the equation of the line through the given point perpendicular to the given line, Nos. 7-18:
7. \( (1, 2); \) \( 2x - y = 8. \)
8. \( (-2, 3); \) \( x + 2y = 5. \)
9. \( (4, -1); \) \( 2x + 3y - 7 = 0. \)
10. \( (6, -4); \) \( 4x - 7y = 2 = 0. \)
11. \( (0, 0); \) \( y = \frac{1}{2}(x + 4). \)
12. \( (0, 0); \) \( y = \frac{1}{2}(y + 1). \)
13. \( (1, 4); \) \( x - y = 6 = 1. \)
14. \( (6, -3); \) \( 2x - 4y = 3 = 5 = 1. \)
15. \( (x, y); \) \( x \cos \alpha^2 + y \sin \alpha^2 = -9. \)
16. \( (b, b); \) \( y = mx + c. \)
17. \( (a^2, 2a); \) \( 6y = x + 2a^2. \)
18. \( (a \cos \theta^4, a \sin \theta^6); \) \( x \cos \theta^2 + y \sin \theta^2 = a. \)
19. Find \( m \) if \( y = mx + c \) is perpendicular to \( 7x + 3y = 8 = 0. \)
20. Find the condition that \( y = x \tan \alpha \) is perpendicular to \( y = x \tan \beta. \)
21. Find \( m \) if \( 7x + (1 + m)y = p \) is perpendicular to \( 9x + (1 - a)y = q. \)

3.6. [22] Find the condition that \( x/a + y/b = 1 \) and \( x/c + y/d = 1 \) are perpendicular lines.

Find the acute angle between the pairs of lines, Nos. 23-26:
23. \( x - 2y = 5, x - 3y = 1. \)
24. \( x - 2y = 1, x + 3y + 4 = 0. \)
25. \( 3x - 2y = 10, 2x - 5y = 3. \)
26. \( 2x - y = 4, 5x + 3y + 1 = 0. \)

27. If the lines \( 3x - 2y = 3, 2x + by = 7 \) meet at the point \( (1, 1), \) prove that they cut at right angles.

28. Two opposite vertices of a rectangle are \( (-1, 2), (3, 5), \) and one side is parallel to \( 2x + 3y = 0. \) Find the equations of the four sides and the coordinates of the other two vertices.

29. [29] Find the coordinates of the feet \( H, K \) of the perpendiculars from the point \( (5, 1) \) to the lines \( 4x - y = 2, x - 3y = 12, \) and find the equation of \( HK. \)

30. Find the equation of the line through the origin perpendicular to \( x + 3y = 25, \) and the coordinates of the point where they meet; then find the length of the perpendicular from the origin to the line \( 4x + 3y = 25. \)

Interpret with respect to the point \( P(x, y) \) the gradient relations, Nos. 31-33:
31. \( \frac{y - 3}{x - 4} = \frac{y - 10}{x - 7}. \)
[32] \( \frac{y - 1}{x - 2} = \frac{y - 7}{x - 5} = -1. \)
33. \( \frac{y - 3}{x - 1} = \frac{y + 1}{x - 4} = -1. \)

34. [34] Find the equation of the line through the point \( (h, k) \) perpendicular to the line joining the points \( (x_1, y_1), (x_2, y_2). \)

35. The vertices of a triangle are \( A(1, 3), B(9, 7), C(4, 12). \) Find the point \( H(f, g) \) such that \( BH, OH \) are perpendicular to \( CA, AB \) respectively and prove \( AH \) is perpendicular to \( BC. \)

36. [36] Find the orthocentre of the triangle whose vertices are \( (-4, 1), (-1, 10), (1, 0). \)

37. [37] The origin \( O \) is the orthocentre of the triangle whose vertices are \( (0, f), (4, g), (3, 2). \) Prove that: (i) \( g = 2f; \) (ii) the circumcentre of the triangle is \( (3, 4, 0). \)

38. [38] Two opposite vertices of a rectangle are the points \( (2, 5), (6, 8). \) If the other vertices lie on the line \( x = 4, \) find their coordinates and the equations of the four sides.

39. Find the coordinates of the foot \( N \) of the perpendicular from \( A(1, 0) \) to the line \( y = mx + b. \) If \( m \) varies, prove that the equation of the locus of \( N \) is \( y^2 = 1 - x. \) [OC]

40. A pair of perpendicular lines through the given point \( C(0, a) \) meet \( x'tx \) at \( P, Q; \) \( PQR \) is an equilateral triangle. If the gradient of \( CP \) is \( m, \) find in terms of \( a, m \) the coordinates of \( P, Q \) and the equations of \( PQ, QR. \) Hence prove that if \( m \) varies the equation of the locus of \( R \) is \( y^2 = 3x^2 + 2a^2. \)
3.7. If \(P_1, P_2\) are the given points \((x_1, y_1), (x_2, y_2)\), the equation of the circle on \(P_1P_2\) as diameter is

\[
(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.
\]

If \(P(x, y)\) is a point on the circle, diameter \(P_1P_2\), then

\[
\angle P_1PP_2 = 1 \text{ rt. } \angle;
\]

but the gradients of \(PP_1\) and \(PP_2\) are \(\frac{y - y_1}{x - x_1}\) and \(\frac{y - y_2}{x - x_2}\).

\[
\therefore \frac{y - y_1}{x - x_1} \cdot \frac{y - y_2}{x - x_2} = -1; \quad \therefore \frac{(y - y_1)(y - y_2) - (x - x_1)(x - x_2)}{x - x_1} = 0.
\]

Observe the coordinates of each point \(P(x, y)\) on the circle are connected by the equation, \((x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0\).

**EXERCISE 10**

1. \(A, B, P\) are the points \((a, 0), (0, b), (x, y)\). Find the equation connecting \(x\) and \(y\) if: (i) \(\angle BAP = 1 \text{ rt. } \angle\); (ii) \(\angle APB = 1 \text{ rt. } \angle\). Interpret each equation.

2. \(H, K\) are the points \((-2, -1), (4, 7)\). Find the coordinates of \(P, Q\) on \(x' Ox\) such that \(\angle HPK = \angle HQK = 1 \text{ rt. } \angle\).

3. State the condition that \(P(x_1, y_1)\) lies on the circle \(x^2 + y^2 = a^2\), whose centre is the origin \(O\). Find in the simplest form the equation of the line \(P, T\) perpendicular to \(OP\). Illustrate by a sketch.

4. Prove that \(P(a \cos \theta, b \sin \theta)\) lies on the line \(\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1\).

Express the equation of the perpendicular at \(P\) to \(L\) in the form \(lx + my = n\), where \(n\) does not depend on \(\theta\).

5. \(PQO, ROS\) are perpendicular diameters of the circle \(x^2 + y^2 = a^2\).

If the equation of \(PQ\) is \(y = x\), then \(x = 0\), (i) the equation of \(ROS\), (ii) the coordinates of \(P, Q\) and of \(R, S\).

6. The line \(OP, x = \frac{1}{2}my\), meets the curve \(y^2 = 4ax\) at the origin \(O\) and the point \(P\). Find the coordinates of \(P\), and of \(R, S\). If the line \(OQ\) perpendicular to \(OP\) meets the curve again at \(Q\), find: (i) the coordinates of \(Q\), (ii) the equation of \(PQ\). Prove that, when \(m\) varies, the line \(PQ\) meets \(x' Ox\) at a fixed point.

7. Prove that the foot of the perpendicular from \((-a, 0)\) to the line \(m'x + n'y = a'\) lies on the line \(x + a = 0\).

8. Prove that \(my = x + am^2\) meets the curve \(y^2 = 4ax\) at only one point \(P\). Find the equation of the perpendicular at \(P\) to \(my = x + am^2\).

9. A variable line passes through the given point \((x_1, y_1)\) and cuts \(x' Ox, y'Oy\) at \(P, Q\); \(POQ\) is a rectangle. Find the equation of: (i) the locus of \(R\); (ii) the locus of the mid-point of \(PQ\).

10. Find the coordinates of the points \(P, Q\) where \(y = kx^2\) cuts the curve \(xy = c^2\). Prove that the line through \(P\) whose gradient is \(-\frac{k}{2}\) meets the curve at no point except \(P\). Illustrate by a sketch.

3.8. **Perpendicular to a Line**

Let \(O\) be the foot of the perpendicular from the origin \(O\) to a given line \(AB\) and let the polar coordinates of \(O\), with origin \(O\) and initial line \(x' Ox\), be \((p, a^2)\). By the conventions on p. 18, \(0^0 \leq a^2 < 180^0\) and \(p\) has a suitable positive or negative value; in Fig. 35, (i), (ii), \(p\) is positive and in Fig. 35 (iii), (iv), \(p\) is negative.

![Fig. 35](image_url)

In all cases, see 2.10, p. 18, the Cartesian coordinates of \(R\) are \((p \cos a^2, p \sin a^2)\),

\[
\therefore OR = x(\cos a^2) - y(\sin a^2), \text{ that is, } x\cos a^2 - y\sin a^2 = 0.
\]

But \(AB\) is the line through \(R\) \((p \cos a^2, p \sin a^2)\) perpendicular to \(OR\),

\[
\therefore AB = x \cos a^2 + y \sin a^2 - (p \cos a^2) \cos a^2 + (p \sin a^2) \sin a^2;
\]

but \(\cos a^2 \sin a^2 = 1\), \(\therefore AB = x \cos a^2 + y \sin a^2 - p = 0\),

where \(\sin a^2 > 0\) since \(0^0 \leq a^2 < 180^0\).

This is called the **perpendicular form** of the equation of a line.

**Example 7**. Express the equation \(3x - 2y - 5 = 0\) in perpendicular form.

Write \(\cos a^2 = \frac{3}{\sqrt{13}}, \sin a^2 = \frac{2}{\sqrt{13}}, \tan a^2 = \frac{2}{3}\),

\[
\therefore \cos a^2 - \frac{3}{\sqrt{13}} \sin a^2 = \frac{2}{\sqrt{13}} 
\]

\[
\therefore \cos a^2 = \frac{180^0 - 30^0}{13} = 146^0 18^0;
\]

\[
\therefore \cos a^2 + y \sin a^2 - p = (3x + 2y + 5)/\sqrt{13}, \text{ if } p = -5/\sqrt{13};
\]

the perpendicular form is \(-\frac{3}{\sqrt{13}} x + \frac{2}{\sqrt{13}} y + \frac{5}{\sqrt{13}} = 0\).

Since \(a^2\) is obtuse and \(p\) is negative, the line is situated as in Fig. 35 (iv).
3.8.1. Length of Perpendicular. If the perpendicular form of the equation of $AB$ is
\[ x \cos \alpha + y \sin \alpha = p \quad (0 \leq \alpha < 180^\circ), \]
the number of units of length of the perpendicular from $P_1(x_1, y_1)$ to $AB$ is the numerical value of
\[ \pm (x_1 \cos \alpha + y_1 \sin \alpha - p). \]

![Diagram](image)

The directed line $OK$ perpendicular to $AB$ makes the angle $\alpha$ with $x'Ox$; therefore the directed line $P_1K_1$ parallel to $OK$ makes $\alpha$ with $x'Ox$ and meets $AB$ at right angles at $R_1$, say; and so $P_1R_1$ is the perpendicular from $P_1$ to $AB$.

Let $r$ be the length of the step, $P_1 \rightarrow R_1$, along the directed line $P_1K_1$ from $P_1$ to $R_1$; in Fig. 36 (i), $r$ is positive; in Fig. 36 (ii), $r$ is negative. In all cases, by 2.9.2, p. 17, the coordinates of $R_1$ are
\[ (x_1 + r \cos \alpha, y_1 + r \sin \alpha); \]
but $R_1$ lies on the line $x \cos \alpha + y \sin \alpha = p$,
\[ \therefore (x_1 + r \cos \alpha) \cos \alpha + (y_1 + r \sin \alpha) \sin \alpha = p, \]
\[ r(\cos^2 \alpha + \sin^2 \alpha) + (x_1 \cos \alpha + y_1 \sin \alpha) = p, \]
\[ \therefore \text{step,} \quad P_1 \rightarrow R_1 = r = (x_1 \cos \alpha + y_1 \sin \alpha - p) \]
\[ = p - x_1 \cos \alpha - y_1 \sin \alpha. \]
Hence if the perpendicular distance of $P_1$ from $AB$ is $d$ units,
\[ r = +d \text{ if } r \text{ is positive and } r = -d \text{ if } r \text{ is negative}; \]
we therefore say that the perpendicular distance of $P_1$ from $AB$ is the numerical value of
\[ \pm (x_1 \cos \alpha + y_1 \sin \alpha - p). \]

If we regard the directed line $y'Oy$ as pointing vertically upwards, $r$ is positive if $P_1$ lies "below" $AB$, Fig. 36 (i), and is negative if $P_1$ lies "above" $AB$, Fig. 36 (ii), but there is no need to remember any rule of signs. It is, however, easy and important to remember that, for any points $P_1(x_1, y_1), P_2(x_2, y_2)$, the expressions
\[ x_2 \cos \alpha + y_2 \sin \alpha = p, \]
\[ x_1 \cos \alpha + y_1 \sin \alpha = p \]
have the same sign if $P_1, P_2$ lie on the same side of the line $x \cos \alpha + y \sin \alpha - p = 0$ and have opposite signs if $P_1, P_2$ lie on opposite sides of the line.

3.8.2. The number of units of length of the perpendicular from $P_1(x_1, y_1)$ to the line $AB$, $ax + by + c = 0$, is the numerical value of
\[ \pm \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}. \]
The equation $x \cos \alpha + y \sin \alpha = p$ represents the same line as the equation $ax + by + c = 0$ if $a, b, p$ are chosen so that
\[ \frac{\cos \alpha}{a} = \frac{\sin \alpha}{b} = \frac{-p}{c}, \quad k \text{ say}; \]
\[ \cos \alpha = ak, \quad \sin \alpha = bk, \quad p = -ck, \]
where $a^2k^2 + b^2k^2 = \cos^2 \alpha + \sin^2 \alpha = 1$.
\[ k(a^2 + b^2) = 1 \]
\[ k = \pm 1/\sqrt{a^2 + b^2}. \]
The perpendicular distance of $P_1(x_1, y_1)$ from $ax + by + c = 0$ is the numerical value of
\[ \pm (ax_1 + by_1 + c)/\sqrt{a^2 + b^2}, \]
where
\[ ax_1 + by_1 = akx_1 + bky_1 + ck = k(ax_1 + by_1 + c); \]
the distance is the numerical value of
\[ \pm (ax_1 + by_1 + c)/\sqrt{a^2 + b^2}. \]
There is no ambiguity of sign in the value of $k$ which must be chosen to give the perpendicular form: If $0^\circ < \alpha < 180^\circ$, $\sin \theta - bk > 0$; therefore $k$ must be chosen so that its value has the same sign as $b$. Hence since the function $x \cos \alpha + y \sin \alpha = p$ takes values of opposite signs for two points $P_1(x_1, y_1), P_2(x_2, y_2)$ on opposite sides of the line $x \cos \alpha + y \sin \alpha = p$, 0, the function $ax + by + c$ takes values of opposite signs for points $P_1, P_2$ on opposite sides of the line
\[ ax + by + c = 0. \]

Example 8. Find whether the point $P(4, 3)$ and the origin $O$ are on the same side or on opposite sides of the line $2x - 3y - 5 = 0$. Find the lengths of the perpendiculars $PN, OR$ from $P, O$ to this line.
If $x = 4$ and $y = 3$, $3x - 2y - 5 = 12 - 5 = 7 = +1$; if $x = 0$ and $y = 0$, $3x - 2y - 5 = -5 = -5$; $c$ is the value of the function $3x - 2y - 5$ at $P$ and at $O$ have opposite signs and so $P$ and $O$ lie on opposite sides.

The lengths of the perpendiculars $PN, OR$ are the numerical values of
\[ \pm \frac{12 - 5}{\sqrt{3^2 + 2^2}} \quad \text{and of} \quad \pm \frac{0 - 5}{\sqrt{3^2 + 2^2}}; \]
\[ PN = \frac{1}{\sqrt{13}} \quad \text{and} \quad OR = \frac{5}{\sqrt{13}}. \]
Example 9. Find the equations of the lines bisecting the angles between the lines $y = -7x + 9, \ x + y = -1$.

Let the point $P(x_1, y_1)$ lie on one of the angle-bisectors, then $P_1$ is equidistant from the two lines:

$$
\frac{7x_1 - y_1 + 9}{\sqrt{50}} = \frac{x_1 + y_1 - 1}{\sqrt{2}}
$$

or the equations of the two angle-bisectors are

$$
\frac{7x - y + 9}{\sqrt{50}} = \frac{x + y - 1}{\sqrt{2}} \quad \text{and} \quad \frac{7x - y + 9}{\sqrt{50}} = \frac{x + y - 1}{\sqrt{2}}
$$

but $\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$, and so the equations may be written

$$
(7x - y + 9) - 5(x + y - 1) = 0 \quad \text{and} \quad (7x - y + 9) + 5(x + y - 1) = 0
$$

these reduce to $x - 3y = 0$ and $3x + y = 10$.

Notice as a check that the equations represent perpendicular lines through the point of intersection $(-1, 2)$ of the given lines.

The reader should illustrate the results by a sketch.

EXERCISE 11

Write the following equations in perpendicular form and show on sketches the values of $p$ and $\theta$, Nos. 1–7:

1. $x \cos 20^\circ - y \sin 20^\circ + 4 = 0$
2. $x \cos 50^\circ - y \sin 50^\circ - 2 = 0$
3. $x \sin 70^\circ + y \cos 70^\circ + 5 = 0$
4. $x \sin 50^\circ - y \cos 50^\circ - 3 = 0$
5. $3x - 4y + 7 = 0$ [6] $x \sqrt{3} - y - 3 = 0$
6. $x + y \sqrt{3} + 5 = 0$

Find the distance of the given point from the given line, Nos. 8–19:

8. $(0, 0); 6x + 8y - 5 = 0$ [9] $(0, 0); 5x - 12y + 6 = 0$
10. $(0, 0); 4x - 2y - 5 = 0$ [11] $(0, 0); x - 3y + 2 = 0$
12. $(1, 2); 3x + 4y = 1$ [13] $(4, 1); 4x - 3y = 3$
14. $(3, -9); 5x - 12y + 4 = 0$ [15] $(1, -2); x - y = 5$
16. $(-1, 4); 5y - 3x = 8$$[17] (-7, 2); 2x + 5y = 1$
18. $(0, k); y = x \tan \theta + c$
19. $(0, 0); y = mx + c$

Find whether the given pair of points lie on the same side or on opposite sides of the given line, Nos. 20–23:

20. $(0, 0); (3, 7); 5x + 1 = 2y$ [21] $(10, 7); (5, 4); 4x = 3y$
22. $(-2, 5); (3, -7); 11x + 4y = 8$
23. $(2, -7); (3, -2); 9x - 10y = 50$
24. Find if $(3, 8)$ is at unit distance from $4y = 3x + 1$ on the side of the line opposite to that of the origin.

25. Find the equation of the locus of a point $P$ at unit distance from the origin side of the line and of a point $P$ at distance 2 units from the origin side of the line.

3.9] AREA OF TRIANGLE

[26] Find the equations of the complete locus of a point $P$ at unit distance from the line $3x + 12y - 8$.
27. Find if the origin is equidistant from the lines, $3x + 4y = 1 + k, 4x - 5y = 7 - 2k$. Illustrate by a sketch.
28. Prove that the origin lies on one angle-bisector of $y = 7x + 20, y = x - 4$, and find the equation of the other angle-bisector.
29. Find the equations of the angle-bisectors of $3x - 4y = 1, 12x + 5y = 3$.
30. Explain why the line $x \cos \alpha + y \sin \alpha - a = 0$ touches the circle $x^2 + y^2 = a^2$ and write down the coordinates of the point of contact.
31. Prove that the circle $x^2 + y^2 = 9$ touches each of the lines, $4x - 3y = 15, 5x + 12y = 39, y = x + 3\sqrt{2}, y = 2x - 3\sqrt{5}, x + 3y = 6$.
32. Find if the circle $x^2 + y^2 = 9$ touches each of the lines, $x^2 + y^2 = 9$.

3.9. The area of the triangle $ABC$, whose vertices are $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ is the numerical value of

$$
\pm \frac{1}{2}(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)).
$$

Let $AD$ be the perpendicular from $A$ to $BC$, then the area of triangle $ABC = \frac{1}{2} AD \cdot BC$.

By the distance-formula, $BC = \sqrt{(x_1 - x_3)^2 + (y_2 - y_3)^2}$.
Since the gradient of $BC$ is $y_2 - y_3)/(x_2 - x_3)$, the equation of $BC$ is

$$
(x_1 - x_2) - y(y_2 - y_3) = \ldots = y_2(y_3 - y_1) - y_3(y_1 - y_2) = x_2 y_2 - x_3 y_3;
$$

therefore the length of $AD$ is the numerical value of

$$
\frac{1}{2}(y_3 - y_1)(y_2 - y_3) - (x_2 y_2 - x_3 y_3) - \sqrt{(y_2 - y_3)^2 + (x_1 - x_2)^2}.
$$

Hence the area of triangle $ABC$, equals the numerical value of

$$
\pm \frac{1}{2}(y_3(y_2 - y_1) - y_1(y_3 - y_2) - (x_1 y_2 - x_2 y_3))
$$

that is,

$$
\pm \frac{1}{2}(y_3(y_2 - y_1) + x_2(y_3 - y_1) + x_1(y_1 - y_2)).
$$

Example 10. Find the area of the quadrilateral $ABCD$ whose vertices are $A(2, 1), B(1, 10), C(5, 8), D(8, 6)$.

A sketch shows that area of quad. $ABCD$ equals sum of areas of $\triangle ABC, \triangle ADC$

$$
\frac{1}{2}(5 - 2)^2 \cdot (8 - 1)^2 = \sqrt{58}.
$$

Gradients of $BC = (10 - 1)/(5 - 2) = 3/3$.

$\alpha$ equation of $AC$ is $7x - 3y = \ldots = -14 - 3 = 11$.

$\alpha$ lengths of perpendiculars from $B, D$ to $AC$ are the numerical values of

$$
\frac{7 - 30 - 30}{\sqrt{52 + 33}} = \frac{42 - 15 - 11}{\sqrt{52 + 33}},
$$

that is, $\frac{34}{\sqrt{58}}$ and $\frac{16}{\sqrt{58}}$.

$$
\text{area } ABCD = \frac{1}{2} \cdot \frac{34}{\sqrt{58}} \cdot \sqrt{58} + \frac{16}{\sqrt{58}} \cdot \sqrt{58} = 17 + 8 = 25.
$$
EXERCISE 12

Find the area of $\triangle ABC$, whose vertices are given, Nos. 1-4:

1. $A(3, 1); B(7, 2); C(5, 8).$ $[2] A(-1, -3); B(1, -1); C(-3, 2).$
2. $A(-2, 3); B(2, 4); C(-1, -4).$
3. $A(-2, 3); B(2, 0); C(4, -1).$

Find the area of quad. $ABCD$, whose vertices are given, Nos. 5-7:

4. $A(1, 4); B(6, 9); C(8, 7); D(3, 2).$
5. $A(3, 1); B(5, 6); C(10, 8); D(8, 4).$
6. $A(-3, 8); B(-7, 9); C(-8, 3); D(-4, 4).$

7. Prove that the points $A(2, 7), B(3, 9), C(8, 5), D(7, 3)$ are the vertices of a parallelogram and find its area.

Find the area of $\triangle ABC$, whose sides are given, Nos. 9-11:

8. $BC, 2x-3y+2=0; CA, x+y=9; AB, 4x-y=6.$
9. $BC, x-y-3=0; CA, 3x+2y-15=0; AB, 17x-3y+47=0.$
10. $BC, x+y=1; CA, y=m_{1}x; AB, y=m_{2}x.$
11. $A, B$ are the fixed points $(a, 0); (b, 0).$ Find the equations of the complete locus of a variable point $P$ which moves so that the area of $\triangle ABP$ is the numerical value of $ab$.

12. Find the area of $\triangle AOP$, if the polar coordinates of $P_{1}, P_{2}$ are $(r_{1}, \theta_{1}); (r_{2}, \theta_{2}); r_{1} < r_{2}$, and if $O$ is the pole. Deduce area $\triangle O P_{1} P_{2}$ if $P_{1}, P_{2}$ are the points $(x_{1}, y_{1}); (x_{2}, y_{2})$, origin $O$.

[Use the formula $A=\frac{1}{2}r_{1}r_{2}sin(\theta_{2}-\theta_{1})$ as $A=\frac{1}{2}AB \times h$ in $\triangle O P_{1} P_{2}.$]

13. $P, Q, R$ are the points $(x, y); (c, d), (e, f)$; find area $\triangle PQR$ and interpret the answer if $x+y=0$.

14. $P, Q, R$ are the points $(x, y); (c, d), (x, y)$; find area $\triangle PQR$ and interpret the answer if $x+y=0$.

3.10 Pair of Lines. The locus whose equation is $5x^{2}+7xy-6y^{2}=0$ is a pair of lines through the origin.

If $P_{1}(x_{1}, y_{1})$ is a point of the locus, then

$5x_{1}^{2}+7x_{1}y_{1}-6y_{1}^{2}=(5x_{1}-3y_{1})(x_{1}+2y_{1})=0$.

.: either $5x_{1}-3y_{1}=0$ or $x_{1}+2y_{1}=0$.

.: the point $P_{1}(x_{1}, y_{1})$ lies on one or other of the two lines $5x=3y=0, x+2y=0$.

.: the locus of $P_{1}$ is a pair of lines through the origin.

Example 11. Examine the locus given by $3x^{2}-2xy+2y^{2}=0$.

$3x^{2}-2xy+2y^{2}$ cannot be factorised at sight; we then write the equation so that the coefficient of $x^{2}$ is the square of an integer and 'complete the square':

$3x^{2}-2xy+2y^{2}=0$.

.: $(3x-4y)^{2}=y(3y-2x)$.

.: the locus is the pair of lines through the origin.

$3x-y(4+\sqrt{10})=0, 3x-y(4-\sqrt{10})=0$. 

3.10.2 If $ax^{2}+2hxy+by^{2}=0$ is the pair of lines

$y-m_{1}x=0, y-m_{2}x=0$.

then (i) $m_{1}+m_{2}=\frac{-2h}{b}, m_{1}m_{2}=-\frac{a}{b};$ (ii) $h^{2}>ab$.

(iii) the lines are at right angles if $a=b=0$.

(iv) if the angle between the lines is $\theta$, where $\theta=90^\circ$, then $tan \theta=\pm \sqrt{|h^{2}-ab|}$.

(i) $ax^{2}+2hxy+by^{2}=b(y-m_{1}x)(y-m_{2}x), b \neq 0$.

.: $y^{2}+2hxy+2y^{2}=by^{2}-(m_{1}+m_{2})y^{2}+m_{1}m_{2}x^{2}$.

Equate coefficients, then $2h/b=m_{1}+m_{2}$ and $m_{1}m_{2}=b$. Hence $h^{2} > ab$.

(ii) $(m_{1}-m_{2})^{2}=(m_{1}+m_{2})^{2}-4m_{1}m_{2}=(h^{2}-ab)/b^{2}$

but $(m_{1}-m_{2})^{2}$ cannot be negative.

.: $h^{2} > ab$ and $m_{1}-m_{2}=\pm 2\sqrt{(h^{2}<ab)}/b$.

(iii) if $a=b=0$, then $a=b, m_{1}=m_{2}, a=m_{1}a/m_{2}$.

.: the lines $y=m_{1}x, y=m_{2}x$ are at right angles.

(iv) tan $\theta=m_{1}-m_{2}$ where $m_{1}=\pm 2\sqrt{(h^{2}-ab)}/b$.

and $m_{1}+m_{2}=(b+a)/b$.

.: tan $\theta=\pm 2\sqrt{(h^{2}-ab)}(a+b)$. 

PAIR OF LINES

Example 12. Examine the loci:

(i) $4x^{2}+20xy+25y^{2}=0$;

(ii) $x^{2}-xy+y^{2}=0$.

The locus is therefore said to consist of two coincident lines, coincident with the line through the origin, $2x+5y=0$.

(ii) Complete the square for $x^{2}-xy+y^{2}=0$.

.: $x^{2}-xy+y^{2}=(x-\frac{1}{2}y)^{2}+\frac{3}{4}y^{2}$.

.: $x^{2}-xy+y^{2}$ is the sum of two squares:

.: $x^{2}-xy+y^{2}$ has no linear factors.

Since a perfect square cannot be negative, the only values of $x, y$ which satisfy $(x-\frac{1}{2}y)^{2}+\frac{3}{4}y^{2}=0$ must be such that $x-\frac{1}{2}y=0$ and $y=0$, that is, $x=0, y=0$. Thus the locus consists of only one point $(0, 0)$. 

3.10.1 The homogeneous equation of the second degree in $x, y,$

$ax^{2}+2hxy+by^{2}=0$;

represents a pair of lines through the origin if $h^{2}>ab$.

The statement is true if $a=0$, because the equation then becomes

$2hx+by^{2}=y(2hx+by)=0$;

this represents the pair of lines through the origin, $y=0, 2hx+by=0$.

If $a=0$, the equation is equivalent to

$ax^{2}+2hxy+by^{2}=0$.

that is,

$ax+by=0, x^{2}+h^{2}y^{2}=0$.

If $h^{2}>ab$, the equation represents the lines through the origin,

$ax+by=0, x^{2}+h^{2}y^{2}=0$.

These lines are coincident if $h^{2}=ab$.

If $h^{2}<ab$, the locus $ax^{2}+2hxy+by^{2}=0$ consists of only one point $(0, 0)$, as in Example 12 (ii).
3.11. It is often interesting to use Cartesian methods for a property proved previously by the methods of elementary geometry.

Example 13. The altitudes of a triangle are concurrent.

Take as axes Ox, Oy, the base AB and the altitude CO of \( \triangle ABC \); then \( A, B, C \) may be taken as \((a, 0), (b, 0), (0, c)\).

By the intercept formula, the equations of 
\\( BC, AC \) are
\\[ \frac{x}{b} + \frac{y}{c} = 1, \quad \frac{x}{a} + \frac{y}{c} = 1; \]
and the equation of the altitude \( AD \) is
\\[ bx - cy = \ldots = kb; \]
and the equation of the altitude \( BE \) is
\\[ ax - cy = \ldots = ab; \]

\( AD \) meets \( BE \) at \((0, -ab/c)\), a point on the altitude \( CO, x = 0 \).

EXERCISE 13

Find the separate equations of the line-pairs, Nos. 1-7:

1. \( x^2 - 4y^2 = 0 \).  
2. \( x^2 - 2xy - y^2 = 0 \).  
3. \( xy = 0 \).

4. \( 6x^2 - 5xy - 6y^2 = 0 \).  
5. \( 4x^2 + 4xy + y^2 = 0 \).

6. \( xy - ax + by - ab = 0 \).

Examine the loci whose equations are given, Nos. 8, 9:

8. (i) \( 9x^2 - 12xy + 4y^2 = 0 \); (ii) \( 9x^2 - 12xy + 4y^2 = 1 \).

9. (i) \( x^2 + y^2 = 1 \); (ii) \( x^2 + y^2 = 0 \); (iii) \( x^2 + y^2 = 1 \).

Find the values of \( m_1 + m_2, \frac{m_1}{m_2} \), \( m_1\), \( m_2 \); if \( y = m_1x, y = m_2x \) are factors of the given functions, Nos. 10-13:

10. \( 6x^2 - 7xy - 3y^2 = 0 \).  
11. \( 5x^2 - 13xy - 6y^2 = 0 \).  
12. \( 2x^2 - xy - y^2 = 0 \).  
13. \( 3x^2 - 20xy + 5y^2 = 0 \).

Find the tangent of the acute angle between each pair of lines in Nos. 14-19:

14. \( x = 0, 4x - 3y = 0 \).
15. \( x - 2y = 0, 3x - y = 0 \).
16. \( 2x^2 + 5xy - 3y^2 = 0 \).
17. \( 2x^2 + xy - y^2 = 0 \).
18. \( 3x^2 - 2xy + 3y^2 = 0 \).
19. \( 7x^2 + 8xy + 2y^2 = 0 \).

20. Find the gradient of each line of the line-pair given by the equation \( abx^2 + (a^2 + b^2)xy - aby^2 = 0 \) and verify that the lines are at right angles.

21. If the line \( x = 1 \) cuts the pair of lines \( 2x^2 - 5xy - y^2 = 0 \) at the points \( P, Q \); find the length of \( PQ \) and the area of the triangle \( OPQ \), where \( O \) is the origin.

PAIR OF LINES

22. If \( 2a^2 + 5xy - 3y^2 = x - 11y + c = 0 \) represents a pair of lines, explain why their equations are of the forms \( 2x - y + p = 0, x + 3y + q = 0 \), and find the values of \( x, y, c \).

23. One of the points \((1, 3), (3, 3), (3, 4)\) lies inside the triangle whose vertices are \((0, 0), (5, 1), (2, 8)\). Which is it?

24. Find the coordinates of the centre of the circle inscribed in the triangle formed by \( 3x + 4y + 5 = 0, 3x + 4y - 23, 5x - 12y = 29 \). (C)

25. If \( a^2 + b^2 + 1 \) is positive when \( t = 1 \) and when \( t = -3 \) and is negative when \( t = 2 \), prove that the point \((a, b)\) referred to given axes \( Ox, Oy \) lies in a certain region. Sketch this region. (N)

26. If \((x_1, y_1)\) lies on the line \( HK, lx + my + n = 0 \), and if \( lx + mb + n > 0 \), prove that the points \((a, b)\) and \((x_1 + l, y_1 + m)\) lie on the same side of \( HK \). (N)

27. One line of the pair \( ax^2 + 2bxy + by^2 = 0 \) passes through \((2, 3)\) and the other passes through \((4, 1)\). Find the ratios, \( a : b : c \).

28. \( SR \) is the perpendicular from \( S(a, 0) \) to the line through \( P(a^2, 2am) \) with gradient \( 1/m \); prove \( SR^2 = a \cdot SP \).

29. \( P, Q, R \) are points on the curve \( y = x(x + 1) \); their \( x \)-coordinates are \( k - 1, k, k + 1 \). Prove that the area of \( \triangle PQR \) remains constant when \( k \) varies. (N)

30. Prove that the product of the distances of \((c, 0), (-c, 0)\) from \((x/a) \cos \theta + (y/b) \sin \theta = 1 \) is independent of \( \theta \) if \( a^2 + b^2 = 4 \).

31. The line \( x + y = b = 1 \) meets \( x(2x), y(2y) \) at \( A, B \); \( H, K \) are the feet of the perpendiculars to \( y = mx \) from \( A, B \); lines through \( H, K \) parallel to \( Ox, Oy \) meet at \( R \). Prove that \( R \) lies on \( AB \).

32. The lines \( x - y + ap^2 = 0, x - y + a^2 = 0 \), \( x - y + ar^2 = 0 \) are the sides \( QR, RP, PQ \) of \( \triangle PQR \). Find the equation of the perpendicular \( PD \) from \( P \) to \( QR \) and the coordinates of the point where \( PD \) cuts \( x + y = 0 \). What are the coordinates of the orthocentre of \( \triangle PQR \)?

33. \( P, Q, R, S \) are points on the curve \( xy = c^2 \); their \( x \)-coordinates are \( cp, cq, cs \). Find:
(i) the equation of \( PQ \);
(ii) the condition that \( PQ \) is perpendicular to \( RS \).

If \( PQ \) is perpendicular to \( RS \), prove \( PS \) is perpendicular to \( QR \). Find in terms of \( p, q, r, \) the coordinates of the orthocentre of \( \triangle PQR \).

34. Prove \( px - y = cr(p + t), \quad rpx - y = cr(p + t), \quad px - y = cr(p + t) \) meet at \((p + q + r), cr(t) \).

35. Lines parallel to the sides of a rectangle \( OABC \) meet \( O, A, B, C \) at \( P, Q, R, S \). Explain how axes can be chosen so that \( A, C, P, S \) can be taken as \((a, 0), (0, c), (p, 0), (0, e)\). Find the equations of \( PQ \) and \( RS \). Prove that \( PQ \) and \( RS \) meet on \( OB \).
CHAPTER 4

RATIO

4.1. Internal and external Division. If \( P_1, P_2 \) are given points on a given directed line, \( P_1P_2 \) is called a \textit{segment} of the line.

If \( P \) is any other point of the line, the segment \( P_1P_2 \) is said to be divided at \( P \) in the ratio

\[
\text{step } P_1 \rightarrow P : \text{step } P \rightarrow P_2
\]

and this is represented by the fraction, \( \frac{\text{step } P_1 \rightarrow P}{\text{step } P \rightarrow P_2} \).

\[ P_1P_2 \text{ is said to be divided internally at } P \text{ if } P \text{ is inside the segment } P_1P_2, \text{ see Fig. 39 (i), and is said to be divided externally at } P \text{ if } P \text{ is outside the segment } P_1P_2, \text{ see Fig. 39 (ii), (iii).} \]

If \( P_1P_2 \) is divided internally at \( P \), the steps \( P_1 \rightarrow P, P \rightarrow P_2 \) are in the same sense, and so the ratio of their lengths is positive.

If \( P_1P_2 \) is divided externally at \( P \), either as in Fig. 39 (ii) or as in Fig. 39 (iii), the steps \( P_1 \rightarrow P, P \rightarrow P_2 \) are in opposite senses, and so the ratio of their lengths is negative.

If \( (P_1 \rightarrow P) : (P \rightarrow P_2) = k \), the value of \( k \) in Fig. 39 (ii) lies between 0 and \(-1\) because \( PP_1 \) is less than \( PP_2 \), and in Fig. 39 (iii) \( k \) is less than \(-1\) because \( P_1P \) is greater than \( P_2P \). Thus, if \(-1 < k < 0 \), \( P \) lies on \( P_2P_1 \), produced and, if \( k < -1 \), \( P \) lies on \( P_1P_2 \), produced.

\( P_1P_2 \) is said to be divided internally at \( Q \) and externally at \( R \) in the same ratio if

\[
\text{step } P_1 \rightarrow R : \text{step } R \rightarrow P_2 = -(\text{step } P_1 \rightarrow Q : \text{step } Q \rightarrow P_2) < 0.
\]

It is important to notice that in finding the ratio in which a segment \( P_1P_2 \) is divided by any point \( P \) on the line \( P_1P_2 \), \( P_1 \) is the point of departure and \( P_2 \) is the point of arrival after a first step from \( P_1 \) to \( P \) and then a second step from \( P \) to \( P_2 \).

\[ \text{Fig. 39} \]

4.1] INTERNAL AND EXTERNAL DIVISION

Example 1. Fig. 40 shows the \( x \)-coordinates of points on \( x'Ox \). In what ratio is \( P_1P_2 \) divided by: (i) \( A \); (ii) \( B \); (iii) \( C \)?

\[ \text{Fig. 40} \]

(i) \( \text{step } P_1 \rightarrow A = \frac{8-2}{2} = 3; \text{ ratio } 2:1. \)

(ii) \( \text{step } P_1 \rightarrow B = \frac{11-2}{15} = \frac{2}{5}; \text{ ratio } 2:5. \)

(iii) \( \text{step } P_1 \rightarrow C = \frac{17-2}{15} = \frac{5}{15} = \frac{5}{5}; \text{ ratio } 5:2. \)

Example 2. \( Q_1, Q_2 \) are the points \( (0, -3), (0, 7) \) on \( y'Oy \). Find the coordinates of the points \( E, F, G, H \), which divide \( Q_1Q_2 \) in the ratios \( 3:2, 1:4, -4:9, -6:1. \)

\[ \text{Fig. 41} \]

For any point \( Q(0, y) \),

\( \text{step } Q_1 \rightarrow Q = -y \)

\( \text{step } Q \rightarrow Q_2 = -y \)

\( \therefore \text{ if } Q_1Q_2 \text{ is divided at } Q \text{ in the ratio } k : 1, \)

\( y + 3 \)

\( 7 - y \)

\( \vdots y + 3 = 7k - ky \)

\( \vdots y(k + 1) = 7k - 3 \)

\( \vdots y = \frac{7k - 3}{k + 1} \) if \( k + 1 \neq 0 \).

\( \text{the point which divides } Q_1Q_2 \text{ in the ratio } k : 1 \text{ is } \left(0, \frac{7k - 3}{k + 1}\right). \)

\( \text{If } k = \frac{3}{2}, \left(7k - 3, \frac{7k - 3}{2}\right) = \left(\frac{7k - 3}{2}, \frac{5}{2}\right) = \left(\frac{3}{2}, 5\right) \).

\( \therefore F \text{ is the point } (0, 3). \)

\[ \text{Similarly the points } P, Q, H \text{ are given by } \varepsilon = \frac{1}{2}, k = -\frac{1}{2}, k = -6; \text{ the reader should verify the results: } P(0, -1); Q(0, -11); H(0, 9). \]

\( \text{If } k = 1, \text{ there is no value of } y \text{ for which } y(k + 1) = 7k - 3; \text{ this means that there is no point } Q \text{ which divides } Q_1Q_2 \text{ in the ratio } -1:1. \text{ This is obvious from a figure because if } Q \text{ is outside the segment } Q_1Q_2, Q_1Q \text{ and } Q_1Q_2 \text{ cannot be equal.} \)
EXERCISE 14

Illustrate each answer by a sketch.

1. \( P_1, P_2 \) are the points \( P_1(4, 0), P_2(20, 0) \). Find the ratio in which \( P_1P_2 \) is divided by each of the points: \( A(10, 0); B(16, 0); C(26, 0); D(2, 0); E(-8, 0); O(0, 0) \).

2. \( Q_1, Q_2 \) are the points \( Q_1(0, -10), Q_2(0, 14) \). Find the ratio in which \( Q_1Q_2 \) is divided by each of the points: \( A(0, 4); B(0, -4); C(0, 30); D(0, -16); O(0, 0) \).

3. \( P_1, P_2 \) are the points \( P_1(3, 0), P_2(9, 0) \). Find the coordinates of the points \( A, B, C, D, E \) which divide \( P_1P_2 \) in the ratios: \( 1 : 4; 4 : 1; -1 : 4; -4 : 1; -1 : 3 \).

4. \( Q_1, Q_2 \) are the points \( Q_1(0, -10), Q_2(0, 14) \). Find the coordinates of the points \( A, B, C, D, E \) which divide \( Q_1Q_2 \) in the ratios: \( 2 : 3; 1 : 1; -2 : 7; -5 : 2; 4 : 1 \).

5. Draw a line and mark two points \( H, K \) on it. Show approximately the positions of points which divide \( HK \) in the ratios, \( A, -3 : 1; B, -1 : 4; C, 2 : 1 \) and which divide \( HK \) in the ratios: \( D, -5 : 1; E, -1 : 6; F, 3 : 1 \).

6. \( P_1, P_2 \) are the points \( P_1(-6, 0), P_2(8, 0) \). The point \( A \) divides \( P_1P_2 \) in the ratio \( 5 : 2 \); (i) Find the coordinates of \( A \). (ii) Find the coordinates of the point \( B \) which divides \( P_1A \) in the ratio \( 2 : 3 \). (iii) Find the ratio in which \( B \) divides \( P_2A \).

7. \( A, B \) are the points \( A(0, 7), B(0, -5) \). The point \( P \) divides \( AB \) in the ratio \( 2 : 1 \); the point \( Q \) divides \( AB \) in the ratio \( -2 : 5 \); the point \( R \) divides \( AB \) in the ratio \( -5 : 4 \). (i) Find the coordinates of \( P, Q, R \). (ii) Find the ratio in which \( Q \) divides \( PR \). (iii) Find the ratio in which \( R \) divides \( PQ \).

8. \( AB \) is divided internally at \( P \) and externally at \( Q \) in the same ratio. If \( (A \rightarrow P): (P \rightarrow B) = 3 : 5 \), find the ratio in which \( PQ \) is divided by \( A \) and by \( B \).

9. Find the coordinates of the mid-point of the segment of the line: (i) from \( P_1(x_1, y_1) \) to \( P_2(x_2, 0) \); (ii) from \( Q_1(0, y_2) \) to \( Q_2(0, y_4) \).

10. \( A \) is the mid-point of the line joining \( P_1(x_1, 0) \) to \( P_2(x_2, 0) \); \( B \) is the mid-point of \( P_1A \). Find the coordinates of the mid-points of \( P_1B, BP_2 \).

11. \( AB \) is divided internally at \( Q \) and externally at \( R \) in the same ratio. If \( (A \rightarrow Q): (Q \rightarrow B) = k : 1 \), find the coordinates of the mid-points of the line segments: \( A \rightarrow Q \); \( Q \rightarrow B \); \( B \rightarrow R \); \( D \rightarrow H \).

12. \( P_1, P_2 \) are the points \( (x_1, 0), (x_2, 0) \). Find the coordinates of the point \( P \) which divides \( P_1P_2 \) in the ratio \( m : n \).

DIVISION IN A GIVEN RATIO

4.2. Ratio-formula. If \( P_1, P_2 \) are the points \( P_1(x_1, y_1), P_2(x_2, y_2) \) and if \( P(x, y) \) divides \( P_1P_2 \) in the ratio \( m_1 : m_2 \), then the coordinates of \( P \) are given by

\[
x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \quad y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}.
\]

Let \( M_1, M_2, M_3 \) be the feet of the perpendiculars from \( P_1, P, P_2 \) to \( x \) \( y \), see Fig. 42 (i).

By 2.2.1, p. 9, since \( P_1M_2 \), \( PM_1 \), \( P_2M_2 \) are parallel,

step \( P_1 \rightarrow P \)

step \( M_1 \rightarrow M \)

step \( P_2 \rightarrow M \)

\[
\begin{align*}
x - x_1 &= \frac{m_2}{m_1 + m_2} x_2 - m_2x_1, \\
x_2 - x &= \frac{m_1}{m_1 + m_2} x_1 - m_1x_2.
\end{align*}
\]

Similarly, by drawing the perpendiculars \( P_1N_1 \), \( PN \), \( P_2N_2 \) to \( y \) \( x \), see Fig. 42 (ii), it can be proved that

\[
y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}.
\]

The reader should work this out for himself.

4.2.1. Mid-point of \( P_1P_2 \). If \( m_2 : m_1 = 1 : 1 \), \( P \) is the mid-point of \( P_1P_2 \). Hence the coordinates of the mid-point of the line joining \( (x_1, y_1) \) to \( (x_2, y_2) \) are given by

\[
\begin{align*}
x &= \frac{1}{2}(x_1 + x_2), \\
y &= \frac{1}{2}(y_1 + y_2)
\end{align*}
\]

To remember this result, notice that the coordinates of the mid-points of \( P_1P_2 \) are the averages of those of \( P_1 \) and \( P_2 \).

4.2.2. Image. If \( N \) is the foot of the perpendicular from \( P \) to a line \( HK \) and if \( PN \) is produced to \( Q \) so that \( PN = NQ \), then \( Q \) is called the image or reflection of \( P \) in \( HK \). Further, if a line \( AP \) meets \( HK \) at \( A \), then the image of each point on \( AP \) in \( HK \) is a point on \( AQ \); and the line \( AQ \) is called the image of the line \( AP \) in \( HK \).
4.3. Positive and Negative Ratios. Fig. 42 on p. 47 which illustrates the ratio-formula is drawn so that P lies between $P_1$ and $P_2$ and so that the coordinates of $P_1$ and $P_2$ are all positive; but the same proof applies if P divides $P_1P_2$ externally or if any of the coordinates are negative because the formula in 2.2.1, p. 9, on which the proof depends was established for all relative positions of $M_1$, $M_2$, $M$, O. It is therefore unnecessary to consider separately the various cases that can arise or to draw the different corresponding figures. A figure illustrates a proof, it is not itself a part of the proof.

The ratio-formula gives uniquely the coordinates of the point P which divides $P_1P_2$ in any given ratio $m_2:m_1$, provided that $m_1 + m_2 ≠ 0$, that is, provided that $m_1 : m_2 = -1 : 1$. This exception can never occur because, if P is inside the segment $P_1P_2$, the value of the ratio is positive, and, if P lies outside the segment $P_1P_2$, then $P_2P + P_1P$ and so $P_1P : P_2P$ cannot be equal to $-1 : 1$.

In applying the ratio-formula, the reader must be careful to remember that the value of $m_2:m_1$ is negative when P lies on $P_1P_2$ produced or on $P_2P_1$ produced.

4.4. Centre of Mass. The reader may think it strange to use the notation $m_2:m_1$ instead of $m_1:m_2$ for the ratio in which P divides $P_1P_2$. The reason for doing so is to associate the ratio-formula with the corresponding formula for the centre of mass in Statics.

![Fig. 43](image)

The point of balance or the centre of mass of the system formed by a particle of mass $m_1$ at $P_1(x_1, y_1)$ and a particle of mass $m_2$ at $P_2(x_2, y_2)$ is the point $O$ on $P_1P_2$ such that

$m_1 \cdot P_1O = m_2 \cdot OP_2$.

that is, $P_1G = m_2 \cdot GP_2$.

Thus the centre of mass $O(x, y)$ of mass $m_1$ at $P_1(x_1, y_1)$ and mass $m_2$ at $P_2(x_2, y_2)$ is given by

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \quad y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}.$$

Alternatively, the coordinates of P given by the ratio-formula may be regarded as weighted averages of the corresponding coordinates of $P_1$ and $P_2$ with weights $m_1, m_2$ attached to the coordinates of $P_1, P_2$ respectively. The reader will find it easy to quote correctly the general ratio-formula if he uses the argument given in Example 3.

![Fig. 44](image)

4.4. Ratio Formula

The arrow used to denote the length of a step may be omitted if the context shows that a directed length is implied, as for example on p. 48; but the arrow-notation, $P \rightarrow Q$ or $P \vec{Q}$, should be retained if its absence causes ambiguity, or if emphasis is desirable.

**Example 3.** $A$ is the point $(-3, 4)$, $E$ is the point $(6, -2)$. Find the coordinates of the points $P$, $Q$, $R$ such that

(i) $P$ divides $AB$ in the ratio $4:5$;
(ii) $Q$ divides $AB$ in the ratio $-1:7$;
(iii) $R$ divides $AB$ in the ratio $-11:2$.

![Fig. 44](image)

The arrow used to denote the length of a step may be omitted if the context shows that a directed length is implied, as for example on p. 48; but the arrow-notation, $P \rightarrow Q$ or $P \vec{Q}$, should be retained if its absence causes ambiguity, or if emphasis is desirable.

(i) $P\quad \frac{4}{5}$

(ii) $Q\quad \frac{4}{5}$

(iii) $R\quad \frac{4}{5}$

that is,

$\begin{align*}
\frac{-5(3) + 4(6)}{5 + 4} &= \frac{-5(-3) + 4(-6)}{5 + 4} \\
&= \left(\frac{-15 + 24}{9}, \frac{-20 - 8}{9}\right)
\end{align*}$

(i) $P$ is the point $(1, 1)_{2}$.  
(ii) $AQ = 7 \quad 7 \cdot AQ = (-1) \cdot QB$;  
(iii) $AR = (11), RB$;  

that is, $\left(\frac{7(-3) + 1(-4)}{7 + 1}, \frac{7(-3) + 1(-4)}{7 + 1}\right)$, that is, $\left(-\frac{21 - 6}{8}, \frac{28 - 12}{8}\right)$.

$Q$ is the point $(-4, 5)$.

(iii) $AR = (11), RB$;

that is, $\left(\frac{2(-3) + 1(-11)}{2 + (-11)}, \frac{2(-3) + 1(-11)}{2 + (-11)}\right)$, that is, $\left(-\frac{6 - 26}{8}, \frac{8 - 22}{8}\right)$.

$R$ is the point $(3, -3)_{2}$.

Note. $Q$ lies on $BA$ produced, see Fig. 44, because $-\frac{1}{7}$, corresponding to $-1:7$, is between 0 and $-1$; $R$ lies on $AB$ produced because $-\frac{1}{11}$, corresponding to $-11:2$, is less than $-1$, see 4.1, p. 44.
Example 4. Find the ratio in which the line joining \( P_1(2, 3) \) to \( P_2(4, 2) \) is divided by the line \( 3x - 5y - 24 = 0 \).

The point \( P \) which divides \( P_1P_2 \) in the ratio \( m_2 : m_3 \) is given by:

\[
\left( \frac{2m_1 + 4m_2}{m_1 + m_2}, \frac{3m_1 + 2m_2}{m_1 + m_2} \right)
\]

Thus, \( P \) lies on the line if:

\[
3\left( \frac{2m_1 + 4m_2}{m_1 + m_2} \right) - 5\left( \frac{3m_1 + 2m_2}{m_1 + m_2} \right) - 24 = 0;
\]

this gives:

\[
6m_1 + 12m_2 - 15m_1 - 10m_2 - 24m_1 - 24m_2 = 0;
\]

or:

\[
-33m_1 - 32m_2 = 0;
\]

and:

\[
m_1 = m_2 = -3 : 2.
\]

Note. Since \( m_2 : m_3 \) is negative and is less than \(-1\), \( P \) lies on \( P_1P_2 \) produced, this means that \( P_1 \) and \( P_2 \) are on the same side of \( 3x - 5y - 24 = 0 \). The reader should illustrate this by a sketch.

4.5. The Centroid of a Triangle. The ratio-formula gives an alternative proof of a familiar theorem.

The medians \( AD, BE, CF \) of a triangle \( ABC \) meet at a point which divides each median, measured from a vertex, in the ratio \( 2 : 1 \).

Denote the coordinates of \( A, B, C \) by \((x_1, y_1), (x_2, y_2), (x_3, y_3)\); then the mid-point \( D \) of \( BC \) is

\[
\left( \frac{1}{2}(x_2 + x_3), \frac{1}{2}(y_2 + y_3) \right).
\]

Let \( G \) be the point on \( AD \) such that

\[
AG : GD = 2 : 1,
\]

that is,

\[
1 : AG = 2 : GD;
\]

then \( G \) is given by weight \( 1 \) for coordinates of \( A \) and weight \( 2 \) for coordinates of \( D \), and so the coordinates of \( G \) are

\[
\left( \frac{1}{2}x_2 + \frac{1}{2}x_3, \frac{1}{2}y_2 + \frac{1}{2}y_3 \right).
\]

The symmetry of this result shows that the point which divides \( BE \) in the ratio \( 2 : 1 \) is the same point \( G \); so also is the point which divides \( CF \) in the ratio \( 2 : 1 \). Therefore \( AD, BE, CF \) meet at \( G \), which is called the centroid of the triangle; its coordinates are the averages of those of the vertices of the triangle.

EXERCISE 15

Write down the coordinates of the mid-point of the line joining the given pair of points, Nos. 1-8:

1. \((1, 2), (7, 4)\); 2. \((-2, 5), (1, -1)\); 3. \((4, -3), (6, 1)\); 4. \((0, 0), (-3, -5)\); 5. \((-7, 3), (7, -3)\); 6. \((5, -8), (-9, 6)\); 7. \((3, 5), (10, 12)\); 8. \((2, 4), (-2, -4)\); 9. \((5, 3), (-3, -1)\); 10. \((1, 0), (2, -3)\); 11. \((-2, 5), (4, 6)\); 12. \((-7, -2), (9, -5)\); 13. \((-5, 4), (4, 1)\); 14. \((5, 1), (3, 7)\); 15. \((-2, -3), (6, 1)\); 16. \((-3, 4), (1, 2)\); 17. \((-1, 5), (5, 1)\); 18. \((-2, 3), (2, -3)\); 19. \((-1, 4), (2, -3)\).
52. **RATIO**

25. A line meets the lines \( x+2y=1 \), \( 2x+y=5 \) at \( A, B \). If the
mid-point of \( AB \) is \( (3, \frac{1}{2}) \), find the equation of \( AB \).

[26] The point \( (x, y) \) divides the line joining \((5, 1)\) to \((2, 4)\) in the ratio
\((1-k):(1+k)\). Find \( x \) and \( y \) in terms of \( k \).

27. A line meets the lines \( 3x+y=3 \), \( 5x+2y=4 \) at \( H, K \); \( P(1, 4) \)
divides \( HK \) in the ratio \(-2:3\). If \( H \) is the point \((x_1, y_1)\), find
the coordinates of \( K \) in terms of \( x_1, y_1 \). Hence find \( x_1 \) and \( y_1 \)
and the equation of \( HK \).

28. The vertices of a quadrilateral are \( P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3), S(x_4, y_4) \); \( A, B, C, D, H, K \) are the mid-points of \( PQ, QR, RS, SP, PR, QS \). Write down the coordinates (i) of \( A, B, C, D \), (ii) of the mid-points of \( AC, BD, HK \). What follows? Illustrate by a sketch.

29. \( A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \) are the vertices of a triangle; \( BC, \)
\( CA, AB \) meet \( z'Qx \) in \( P, Q, R \). Find the values of \( BP/PQ, CQ/QA, AR/PR \)
and prove that their product is equal to \(-1\).

This is called Menelaus’ theorem.

30. The perpendicular from \( P(-1, 2) \) to \( HK, 3x+2y=14 \), meets \( HK \)
at \( N \) and is produced to \( Q \) so that \( NQ=2PN \). Prove that \( Q \) is the point
\((8, 8)\).

[31] Find \( m \) if the line through \( P(2, 3) \) parallel to \( y=mx \) cuts \( x-y=2, \)
\( 2x-3y+2=0 \) at \( Q, R \) such that \( PQ=PR \).

32. Find the image of \((5, 1)\) in the line \( x-3y+8=0 \). \( (L) \)

[33] Prove that the equation of the image of the line \( x-y=4 \) in the line \( 3x+y=1 \) is \( x-3y=18 \).

34. Prove that the image \((x_2, y_2)\) of \((x_1, y_1)\) in \( bx+my+n=0 \) is given
by \( x_2=x_1-2m(2x_1+my_1+n)/(b^2+m^2), y_2=y_1-2n(2x_1+my_1+n)/(b^2+m^2) \).

35. \( Q \) is the fixed point \((a, b)\); \( O \) is the origin; \( P \) is a variable
point \((x, y)\); \( E \) and \( S \) divide \( PQ \) in the fixed ratios, \( m:n \) and \(-m:n \).
If \( OR \) is perpendicular to \( OS \), prove that the equation of the locus of \( P \) is
\( m^2(x^2+y^2)-n^2(a^2+b^2) \). \( (OC) \)

36. \( A, B \) are the given points \((a, 0), (-a, 0); AB \) is divided at \( C, D \)
in the fixed ratios, \( k:1, -k:1 \); \( P(x, y) \) is a variable point such that
\( \angle COP=1 \) rt. \( \angle \), find the equation of the locus of \( P \).

37. \( A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) \) are the vertices of \( \triangle ABC \). Prove that
\( P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \) lies on the perpendicular bisector of \( BC \) at distance \( k \). \( BC \) from \( BC \); \( Q \) and \( R \) are points related to \( CA \) and \( AB \) in the same way as \( P \) is related to \( BC \). Prove \( \triangle PQR, \triangle ABC \)
have the same centroid.

**CHAPTER 5**

5.1. **Equation of a Circle.** Examples were given in Chapter 2
to show how the equation of a circle can be deduced from the
distance-formula, see pp. 13, 14. On account of their importance,
the results which were obtained are repeated below.

5.1.1. If the centre of a circle, radius \( r \), is taken as origin, the equation of the circle is
\[ x^2+y^2=r^2. \]

5.1.2. If the centre \( C \) of a circle, radius \( r \), is the point \( C(h, k) \), the
equation of the circle is
\[ (x-h)^2+(y-k)^2=r^2. \]

If \( P(x, y) \) is a point on the circle, centre \( C(h, k) \), radius \( r \),
then \( CP^2=r^2 \) and \( CP^2=(x-h)^2+(y-k)^2 \),
\[ (x-h)^2+(y-k)^2=r^2. \]

Conversely, this equation represents the locus of a point \( P(x, y) \)
which moves so that \( CP^2=r^2 \), and this is a circle, centre \( C \), radius \( r \).

5.1.3. If the points \( P_1(x_1, y_1), P_2(x_2, y_2) \) are the extremities of a
diameter of a circle, the equation of the circle is
\[ (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0. \]

The proof was deduced in 3-7, p. 34, from the property that if \( P \) is a
point on the circle, \( \angle PP_1P_2=1 \) rt. \( \angle \), the angle in a semi-circle.

5.1.4. The locus of a point \( P(x, y) \), whose equation is
\[ x^2+y^2+2gx+2fy+c=0, \]
is a circle, centre \( C(-g, -f) \), radius \( \sqrt{(g^2+f^2-c)} \),
provided that
\[ g^2+f^2-c>0. \]
By the method of ‘completing the square’, see 2.8.1, p. 14, the equation of the
locus can be written in the form
\[ (x+y)^2+(y+f)^2=g^2+f^2-c. \]
If \( g^2+f^2-c \leq 0 \), there are no values of \( x, y \) which satisfy the equation,
and so there is no locus corresponding to this equation.
If \( g^2+f^2-c=0 \), the locus may be regarded as a circle,
centre \( C(-g, -f) \), radius zero, and so the locus consists of the
single point \( (-g, -f) \) and is called the point-circle \( C \).
5.1.5. For any choice of origin and rectangular axes, the equation of any circle can be written in the form,
\[ x^2 + y^2 + 2gx + 2fy + c = 0. \]
Whatever rectangular axes are chosen, the equation of a circle can be expressed in the form
\[ (x - h)^2 + (y - k)^2 = r^2, \]
that is,
\[ x^2 + y^2 - 2hx - 2ky + (h^2 + k^2 - r^2) = 0, \]
and this is an equation of the stated standard form.

5.1.6. The choice of letters for the constants in the standard form needs explanation:
In the homogeneous quadratic function of \(x, y, z\), it is natural to associate \(a, b, c\) with \(x^2, y^2, z^2\); the letters \(f, g, h\) are associated with \(xy, xz, yz\); the letter \(e\) is not used here because later it is given a special geometrical meaning. Also to assist in 'completing the squares' it is convenient to take the coefficients as \(2f, 2g, 2h\). For these reasons, the standard form of the homogeneous quadratic function is
\[ ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy. \]
If we put \(z = 1\), we obtain the general function of \(x\) and \(y\) of the second degree in the form
\[ ax^2 + by^2 + c + 2fxy + 2gx + 2hxy, \]
and so the general equation of the second degree is written
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \]

5.1.7. Tests for a Circle. It has been shown that for any choice of rectangular axes the equation of a circle can be reduced to the form,
\[ x^2 + y^2 + 2gx + 2fy + c = 0. \]
The general equation of the second degree
\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \]
can be reduced to this form if, and only if,
\[ a = b + 0 \quad \text{and} \quad h = 0. \]
Division by \(a\) then gives
\[ x^2 + y^2 + 2(g/a)x + 2(f/a)y + c/a = 0. \]
This equation represents a circle (or point-circle) provided that
\[ g^2/a^2 + f^2/a^2 - c/a > 0, \]
that is,
\[ g^2 + f^2 - ac > 0. \]

Example 1. Find the equation of the circle having the point \((2, -3)\) as centre and touching the line \(3x + y = 11\).

The radius of the circle equals the length of the perpendicular from \((2, -3)\) to \(3x + y - 11 = 0\), that is, \(\pm (6 + 3 - 11)/\sqrt{3^2 + 1^2} \approx \pm 8/\sqrt{10}\); hence, the equation of the circle is \((x - 2)^2 + (y + 3)^2 = (8/\sqrt{10})^2 = 32/5\), that is,
\[ 5x^2 + 5y^2 - 20x + 30y + 33 = 0. \]

EQUATION OF A CIRCLE

Example 2. Find the equation of the circle which passes through the points \(A(1, 2), B(2, -1), C(-1, 3)\).

\(A, B, C\) lie on the circle \(x^2 + y^2 + 2gx + 2fy + c = 0\) if
\[ 5 + 2g + 4f + c = 0 \quad \text{(i)} \]
\[ 5 + 4g - 2f + c = 0 \quad \text{(ii)} \]
and
\[ 10 - 2g + 6f + c = 0 \quad \text{(iii)} \]
From (i), (ii), by subtraction,
\[ 2g - 6f = 0; \]
From (i), (iii), by subtraction,
\[ 5 - 4g + 2f = 0; \]
hence
\[ f = \frac{3}{2}, g = -1, \]
and so \(c = -5 - 3 - 2 = -10\).

\(\therefore\) the equation of the circle \(ABC\) is
\[ x^2 + y^2 + 3x + y - 10 = 0. \]

EXERCISE 16

1. Find the equations of the circles having \((1, -2)\) as centre and touching (i) \(Ox\), (ii) \(Oy\).

2. Find the distance of the point \(P(3, -2)\) from the centre of the circle \(x^2 + y^2 + 4x + 2y - 29 = 0\). Is \(P\) inside or outside the circle?

3. Find the equation of the diameter of \(x^2 + y^2 - 3x - 4y = 0\) which passes through the point \((1, 2)\).

4. Find the equation of the circle which passes through the points \((2, 4), (3, -1), (-6, 5)\).

5. Find the equation of the circle through \((2, -1)\) and concentric with the circle \(x^2 + y^2 - 7x + 3y + 2 = 0\).

6. Find the general equation of a circle which passes through the origin and has its centre on \(y = mx\).

7. Find the equation of the circle, centre \((-2, -1)\), touching \(x + 2y = 1\).

8. If \(x^2 + y^2 + 3x - 6y + c = 0\) represents the circle on \(PQ\) as diameter and if \(P\) is \((-2, 7)\), find \(c\) and coordinates of \(Q\).

9. Interpret the loci:
(i) \(x^2 + y^2 - 2ax - 2ay + a^2 = 0\); (ii) \(2x^2 + 2y^2 - 3xy = 2a^2\).

10. Find \(k\) if \((y - 3x)(y + 3x + 2) + k(x - y) = 0\) represents a circle and then find the coordinates of the centre and the radius. (OC)

11. Find the equation of the circle through the points \((0, 4), (3, -2)\), whose centre lies on \(Oy\).

12. If \(a, b, k\) are constant and if \(\theta^o\) varies, find the equation of the locus of the point \((a + k \cos \theta^o, b + k \sin \theta^o)\).

13. \(A, B\) are the points \((2, 1), (6, 4)\); \(O\) is the origin; \(OA\) meets the circle, diameter \(AB\), again at \(C\). Find length of \(AC\). (OC)

14. Find the equations of the two circles through the origin which touch each of the lines \(x + 2 = 0\) and \(x - 6 = 0\).
5.2. Equation of a Tangent. The simplest way of finding the equation of the tangent to a circle at any given point on the circle is to use the geometrical property that the tangent is perpendicular to the radius through the point of contact.

5.2.1. The equation of the tangent \( P_1T \) to the circle \( x^2+y^2=r^2 \) at the point \( P_1(x_1, y_1) \) on the circle is

\[
xx_1+yy_1=r^2.
\]

The centre of \( x^2+y^2=r^2 \) is the origin \( O \),

: gradient of radius \( OP_1 \) is \( y_1/x_1 \),

: gradient of tangent \( P_1T \) is \( -x_1/y_1 \),

: the equation of \( P_1T \) is

\[
xx_1+yy_1=\ldots=x_1^2+y_1^2,
\]

since \( P_1(x_1, y_1) \) lies on \( P_1T \).

But \( P_1(x_1, y_1) \) lies on \( x^2+y^2=r^2 \),

: \( x_1^2+y_1^2=r^2 \),

: the equation of the tangent \( P_1T \) is \( xx_1+yy_1=r^2 \).

5.2.2. The equation of the tangent \( P_1T \) at \( P_1(x_1, y_1) \) to the circle

\[
x^2+y^2+2gx+2fy+c=0
\]

is

\[
xx_1+yy_1+g(x-x_1)+f(y-y_1)+c=0.
\]

The centre \( C \) of the circle is the point \( (-g, -f) \),

: gradient of radius \( CP_1 \) is \( y_1+f/x_1+g \),

: gradient of tangent \( P_1T \) is \( -x_1/y_1+g' \),

also \( P_1(x_1, y_1) \) lies on \( P_1T \); therefore the equation of \( P_1T \) is

\[
x(x_1+g)+y(y_1+f)=\ldots=x_1(x_1+g)+y_1(y_1+f),
\]

that is,

\[
x(x_1+g)+y(y_1+f)=x_1^2+y_1^2+gx_1+fy_1.
\]

But \( P_1(x_1, y_1) \) lies on \( x^2+y^2+2gx+2fy+c=0 \),

: \( x_1^2+y_1^2=-2gx_1-2fy_1-c \),

: the equation of the tangent \( P_1T \) can be written

\[
x(x_1+g)+y(y_1+f)=-gx_1-fy_1-c,
\]

that is,

\[
xx_1+yy_1+g(x-x_1)+f(y-y_1)+c=0.
\]

5.2.3. Mnemonic for Equation of Tangent. It is easy to remember the equation of the tangent to a circle at a given point \( (x_1, y_1) \) by associating it with the equation of the circle.

Modify the equation of the circle as follows:

(i) Replace \( x^2 \) by \( xx_1 \) and \( y^2 \) by \( yy_1 \).

(ii) Replace \( 2x \) by \( x+x_1 \) and \( 2y \) by \( y+y_1 \).

Thus the equation

\[
x^2+y^2+2gx+2fy+c=0
\]

is modified to read

\[
xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.
\]
5.3. If the line \( y = mx + c \) touches the circle \( x^2 + y^2 = r^2 \), then
\[
\pm r = \pm r \sqrt{(1 + m^2)}.
\]

**First Method.** A condition for tangency is that the length of the
perpendicular from centre \((0, 0)\) to \( mx - y + c = 0 \) equals radius \( r \);
\[
\pm \frac{c}{\sqrt{1 + m^2}} = r; \quad c = \pm r \sqrt{(1 + m^2)}.
\]

**Second Method.** A condition for tangency is that \( y = mx + c \) meets
\( x^2 + y^2 = r^2 \) at one, and only one, point.
At any point of intersection, \( x^2 + (mx + c)^2 = r^2 \),
that is,
\[
x^2(1 + m^2) + 2mcx + c^2 - r^2 = 0.
\]
For tangency, the roots of this quadratic are equal,
\[
\therefore c^2 = (1 + m^2)(c^2 - r^2); \quad c^2 = r^2(1 + m^2).
\]

**Example 3.** Find the gradients of the tangents from the point \((h, k)\) to the
circle \( x^2 + y^2 = r^2 \).
If the gradient of a tangent is \( m \), its equation is \( y = mx \pm r \sqrt{(1 + m^2)} \); this
tangent passes through \((h, k)\) if \( k = mh \pm r \sqrt{(1 + m^2)} \);
\[
\therefore (k - mh)^2 = r^2(1 + m^2);
\]
\[
\therefore m^2(h^2 - r^2) + 2hkm - k^2 - r^2 = 0.
\]
The gradients of the tangents from \((h, k)\) are the roots \( m_1, m_2 \) of this
quadratic equation, and can therefore be written down separately, but in
any problem involving their gradients it is in general more convenient to
work with the symmetric relations
\[
m_1 + m_2 = \frac{2hkm}{h^2 - r^2}; \quad m_1m_2 = \frac{(k^2 - r^2)(h^2 - r^2)}{h^2 - r^2}.
\]

5.3.1. Find the condition for the line \( lx + my + n = 0 \) to touch the
circle \( x^2 + y^2 = 2gx + 2fy + c = 0 \).
For tangency, the length of the perpendicular from centre \((-g, -f)\) to \( lx + my + n = 0 \) is equal to the radius \( \sqrt{g^2 + f^2 - c} \);
\[
\therefore \pm \frac{(gl + fm - n)}{\sqrt{(l^2 + m^2)}} = \sqrt{g^2 + f^2 - c}.
\]
Hence, squaring each side, the condition can be written
\[
(gl + fm - n)^2 = (g^2 + f^2 - c)(l^2 + m^2).
\]

**Example 4.** Find the equations of the tangents from the point \((11, -3)\) to the
circle \( x^2 + y^2 + 6x - 4y - 180 = 0 \).
The centre of the circle is \((-3, 2)\) and its radius is
\[
\sqrt{(3^2 + 2^2 + 180)} = \sqrt{209}.
\]
The line \( x - 11 + k(y + 3) = 0 \) which passes through \((11, -3)\) touches the
circle if \( k \) is chosen so that
\[
\pm \frac{(11 - 11) + k(2 + 3)}{\sqrt{(1 + k^2)}} = \sqrt{209},
\]
that is, \( 200(1 + k^2) = (6k - 15k)^2 \).
Hence \( 8k^2 = 25k^2 - 15k + 9 \), \( 7k^2 + 25k - 1 = 0 \);
\[
\therefore (k + 1)(7k - 1) = 0; \quad k = -1 \text{ or } \frac{1}{7}.
\]
\( k = -1 \) gives \( x - y - 14 = 0 \); \( k = \frac{1}{7} \) gives \( 7x + y - 74 = 0 \).
Example 5. If the line \(x + 3y - 10 = 0\) meets the circle \(x^2 + y^2 = 50\) at \(P, Q\) and if the tangents at \(P, Q\) meet at \(T(x_1, y_1)\), find \(x_1, y_1\).

The equation of the chord of contact \(PQ\) is \(x_1x + y_1y = 50\); but by hypothesis the equation of \(PQ\) is \(x + 3y = 10\);

\[
\frac{x_1 + y_1}{1} = \frac{50}{10} \Rightarrow x_1 - 5, y_1 = -15.
\]

Example 6. The centres \(C_1, C_2\) of two circles are the points \((h_1, k_1), (h_2, k_2)\) and their radii are \(r_1, r_2\). Find the coordinates of the point of intersection \(T\) of the exterior common tangents of the circles.

By symmetry, \(T\) lies on \(C_1C_2\) produced or \(C_2C_1\) produced.

If \(P_1, P_2\) are the points of contact of the common tangent \(P_1P_2\), by similar triangles, \(C_1T \cdot C_2T = C_1P_1 \cdot C_2P_2 = r_1 \cdot r_2\).

\(T\) divides the line joining \(C_1(h_1, k_1)\) to \(C_2(h_2, k_2)\) in the ratio \(-r_1 : r_2\).

\(T\) is the point \((r_1h_2 - r_2h_1)/(r_1 - r_2), (r_1k_2 - r_2k_1)/(r_1 - r_2))\).

Note: The method of Example 4 may now be used to find the equations of the exterior common tangents.

Example 7. Find the equation of the chord \(PQ\) of the circle \(x^2 + y^2 + 2px + 2qy + c = 0\), if \(N(x_1, y_1)\) is the mid-point of \(PQ\).

The centre of the circle is \((-p, -q)\).

\(\frac{y_1 + f}{x_1 + g} = \frac{y_1 + f}{x_1 + g}\).

The equation of \(PNQ\), which passes through \(N(x_1, y_1)\), is

\[(x + g)x + (y + f)y = (x_1 + g)x_1 + (y_1 + f)y_1\].

EXERCISE 17

Find, without using the formula, the equation of the tangent to the given circle, Nos. 1–9:

1. \(x^2 + y^2 = 25\); \((3, -4)\).
2. \(x^2 + y^2 = 125\); \((-2, -11)\).
3. \(2x^2 + 2y^2 = 5\); \((-\frac{1}{2}, -\frac{1}{2})\).
4. \(9x^2 + 9y^2 = 40\); \((-\frac{1}{2}, 2)\).
5. \((x - 2)^2 + (y - 3)^2 = 100\); \((3, -5)\); \((-6, 9)\).
6. \((x + 5)^2 + (y + 2)^2 = 10\); \((-4, 1)\); \((-2, -3)\).
7. \(x^2 + y^2 = 2x + 8y - 17 = 0\), at the points given by \(y = 1\).
8. \(x^2 + y^2 + 4x + 6y = 28 = 0\), at the points given by \(x = -6\).

Write down the equation of the tangent at \(P(x_1, y_1)\), Nos. 9–16:

9. \(x^2 + y^2 = 6x + 7 = 0\).
10. \(x^2 + y^2 + 6y - 4 = 0\).
11. \(x^2 + y^2 + y - 3 = 0\).
12. \(x^2 + y^2 = 3x + 2 = 0\).
13. \(x^2 + y^2 + 8x - 4y = 0\).
14. \(x^2 + y^2 + 6x - 3y + 10 = 0\).
15. \(3x^2 + 8y^2 - x - y = 4\).
16. \(4x^2 + 4y^2 - 5x - 4y = 1\).

Find the equation of the chord of contact of the tangents from the given point to the given circle, Nos. 17–20:

17. \((3, -1)\); \(x^2 + y^2 = 4\).
18. \((0, 0)\); \(x^2 + y^2 - 3x + 5y + 4 = 0\).
19. \((-1, -2)\); \(x^2 + y^2 - 4x - 3y = 5\).
20. \((1, 3)\); \(x^2 + y^2 - 3x - y = 1\).

5.4 TANGENTS

21. Find the equation of the tangent to \(5x^2 + 5y^2 + 3x + y = 0\) at \((0, 0)\).

22. \(3x + 4y + 10 = 0\); \(x^2 + y^2 = 23\).
23. \(2x - 3y = 0\); \(x^2 + y^2 = 9\).
24. \(x \cos \alpha + y \sin \alpha = r; \quad x^2 + y^2 = a^2\), where \(a > p > 0\).
25. \(x - 8y - 22; \quad x^2 + y^2 - 2x + 2y - 3 = 0\).
26. \(5x - y + 3 = 0; \quad 2x^2 + 2y^2 + 2x + 6y - 3 = 0\).
27. Prove that \(x + 3y = 0\) meets \(x^2 + y^2 = 50\) at \((7, 1)\), \((-5, 5)\), find the point of intersection of the tangents at \(Q\) and \(R\).
28. If the origin is the mid-point of the chord \(PQ\) of the circle \(x^2 + y^2 + 2px + 2qy + c = 0\); \((c, 0)\), find the equation of \(PQ\) and the length of \(PQ\).
29. Find the equation of the chord \(PQ\) of \(2x^2 + 2y^2 - 2x - 6y = 3\), if its mid-point is \((1, 2)\); find the length of \(PQ\).
30. The line \(ax + by + c = 0\) meets \(x^2 + y^2 = r^2\) at \(P, Q\). Find the coordinates of the point of intersection of the tangents at \(P, Q\).
31. The lines \(x + 2y = 0, \quad y = mx\) touch a circle, centre \((1, -3)\); find \(m\).
32. The line \(y = b\) meets the circle \(x^2 + y^2 = ax + by\) at \(P, Q\); the tangents at \(P, Q\) meet at \(T\). Prove \((b/a) \cdot PT\) equals the radius.
33. Find the equations of the two circles through \((1, 8)\) which touch \(Ox, Oy\). Find the equations of the tangents to these circles at \((1, 8)\) and the equation of the common chord. (L)
34. Find the equation of the circle through \((0, 1), (0, 4), (2, 5)\). Prove \(Ox\) is a tangent and find the equation of the other tangent from \(O\). (N)
35. \(O\) is the origin; \(A, B\) are the points \((a, 0), (0, b)\); find the equations of the circles, diameters \(OA, OB\) if the circles cut again at \(C\), find the coordinates of \(C\) and verify that \(C\) lies on \(AB\).
36. \(x + a = 0\) is the side \(BC\) of an equilateral triangle \(ABC\), circumscribing \(x^2 + y^2 = a^2\). Find the equations of \(AB, AC\) and of the circle circumscribing \(\triangle ABC\). (L)
37. Find the equations of the tangents from the point \((-1, 2)\) to the circle \(x^2 + y^2 - 4x - 6y + 5 = 0\).
38. Prove that the equation of the chord of contact of the tangents from \((x_1, y_1)\) to the circle \((x - h)^2 + (y - k)^2 = r^2\) can be written in the form,

\[
(x - h)(x_1 - h) + (y - k)(y_1 - k) = r^2.
\]
39. If the exterior common tangents to the circles \((x - h)^2 + (y - k)^2 = r^2\) touch the first circle at \(P; Q\), prove equation of \(P; Q\), \(PQ = (x - h_1)(x_1 - h) + (y - k_1)(y_1 - k) = r_1(x_1 - x_1)\).
40. Find the point \(P\) on \(x^2 + y^2 - 2x - 4y - 4\) which is nearest to \((-2, -2)\). Find also the equation of the tangent to the circle at \(P\). (L)
41. Find the condition that \((x_1, y_1)\) lies on the chord of contact, when produced, of the tangents from \(T(x_1, y_1)\) to \(x^2+y^2=r^2\). Prove that in this case the chord of contact of tangents from \(R\) passes through \(T\).

42. Prove that the points \(T(-2, 5), R(-7, 0)\) are such that the chord of contact of the tangents from either to \(x^2+y^2=16\) passes through the other. Illustrate by a sketch.

43. Find the equations of two circles through \((0, 4)\) and \((8, 20)\) and touching \(Ox\). Prove that the line of centres of the circles meets \(Ox\) at \((38, 0)\). Find the equation of the other common tangent. (L)

5.5. Angle of Intersection. If two curves cut at \(C\) and if the tangents \(CH, CK\) to \(C\) if the curves contain an angle \(\theta\), the curves are said to cut at angle \(\theta\) at \(C\). In particular, if \(\theta=90^\circ\), the curves are said to cut orthogonally or cut at right angles at \(C\).

5.5.1. If two circles, centres \(A, B\), cut at \(C\), then one of the angles between the tangents \(HCH', KCK'\) is equal to \(\angle ACB\).

Since a tangent is perpendicular to the radius through the point of contact, if \(HCH', KCK'\) are rotated counterclockwise about \(C\) through \(90^\circ\), they will lie along the radii \(CA, CB\), see Fig. 48;

one of the angles between \(HCH', KCK'\) equals \(\angle ACB\).

Hence if the radii \(CA, CB\) and the distance \(AB\) between the centres are given, the angle at which the two circles cut is found from \(\Delta ABC\) by using the cosine formula.

5.5.2. If two circles, centres \(A, B\), radii \(a, b\), intersect orthogonally at \(C\), then: (i) the tangent at \(C\) to either circle passes through the centre of the other circle; (ii) \(AB^2=a^2+b^2\).

(i) Let \(CH, CK\) be the tangents at \(C\) to the circles, centres \(A, B\), see Fig. 49.

Since \(CK\) is perpendicular to the tangent \(CH\) to the circle centre \(A, CK\) passes through \(A\).

Similarly, the tangent \(CH\) passes through \(B\).

(ii) Since \(A, B\) lie on \(CK, CH, \angle ACB = \angle KCH = 1\ rt. \angle; \therefore AB^2 = CA^2 + CB^2 = a^2 + b^2\).

5.5.3. If two circles, centres \(A, B\), radii \(a, b\), intersect at \(C\), the circles are orthogonal if

(i) \(\angle ACB = 1\ rt. \angle; \)

or (ii) \(AB^2 = a^2 + b^2; \)

or (iii) \(AC\) touches circle, centre \(B\).

(i) Since \(\angle ACB = 1\ rt. \angle, \) \(CB\) is perpendicular to radius \(CA; \therefore \) \(CB\) touches the circle, centre \(A\).

Similarly, \(CA\) touches circle, centre \(B\).

\(\therefore\) tangents at \(C\) are at right angles.

(ii) Since \(AB^2 = a^2 + b^2 = AC^2 + BC^2\), it follows that \(\angle ACB = 1\ rt. \angle; \therefore\) by (i), the circles are orthogonal.

(iii) Since \(AC\) touches circle, centre \(B\), it follows that \(AC\) is perpendicular to radius \(CB\), that is, \(\angle ACB = 1\ rt. \angle; \therefore\) by (i), the circles are orthogonal.

The properties in 5.5.2 and 5.5.3 give necessary and sufficient conditions for orthogonal circles.

If two circles, centres \(A, B\), intersect at \(C\) and \(D\), symmetry about the line of centres \(AB\) shows that if the circles cut orthogonally at \(C\), then they also cut orthogonally at \(D\).

5.5.4. The circles,

\[x^2+y^2+2gx+2fy+c=0,\]
\[2gg'+2ff'=c+c'.\]

The centres \(A, B\) of the circles are \((-g, -f), (-g', -f')\) and the radii are \(\sqrt{(g^2+f^2-c)}, \sqrt{(g'^2+f'^2-c')};\)

\[AB^2 = (g-g')^2 + (f-f')^2;\]

a necessary and sufficient test for the circles to be orthogonal is

\[g^2+2gg'+g'^2+f^2+2ff' = g^2+f^2-c^2+g'^2+f'^2-c'^2;\]

\[2gg'+2ff' = c+c'.\]

5.6. Contact of Circles. If two circles touch one another, then the distance between the centres equals the sum of the radii, and conversely.

If \(r_1, r_2\) are the radii of two circles, centres \(C_1, C_2\), which touch at \(A\), then

\[C_1C_2 = r_1 + r_2, \text{ external contact,}\]

\[C_1C_2 = r_1 - r_2, \text{ internal contact.}\]

Conversely, \(C_1C_2 = r_1 + r_2, C_1C_2 = r_1 - r_2\), are sufficient conditions for contact.

Fig. 50

Fig. 49

Fig. 48

Fig. 51
5.7 Inverse Points. If P is a point inside a circle, centre A, radius a, and if AP is produced to Q so that AP/AQ = a², then P and Q are called inverse points with respect to the circle, see Fig. 52.

Example 9. A circle cuts orthogonally at C, D a circle, centre A, radius a, and cuts a diameter HAK at P and Q. Prove that P and Q are inverse points with respect to the circle, centre A, radius a.

Fig. 52

First method. Take A as origin and HAK as x-axis; then the circles may be taken as

\[ x^2 + y^2 + 2px + 2fy + c = 0, \]
\[ x^2 + y^2 - a^2 = 0, \]

where by 5.5.4, since the circles are orthogonal, \( c - a^2 = 0 \).

P and Q are given by \( y = 0, \) \( x^2 + y^2 + 2px + 2fy + c = 0; \)
\[ \therefore \ AP, AQ \ are \ the \ roots \ of \ the \ quadratic, \ x^2 + 2px + c = 0; \]
\[ \therefore \ AP . AQ = c - a^2. \]

Second method. By 5.5.2, since the circles are orthogonal, AC is the tangent from A to the circle CPQ at C.

\[ \therefore \ AP . AQ = AC^2 = a^2. \]

EXERCISE 18

Prove the pairs of circles are orthogonal, Nos. 1–4:

1. \( x^2 + y^2 - 6x + 4y + 9 = 0, \) \( x^2 + y^2 + 2x - 6y - 9 = 0. \)
2. \( x^2 + y^2 - 2x - 4y + 4 = 0, \) \( 2x^2 + y^2 - 10x + 2y + 10 = 0. \)
3. \( x^2 + y^2 + 2hx + 2ky + c = 0, \) \( x^2 + y^2 - 2hx + 2ky - c = 0. \)
4. \( x^2 + y^2 + 2px + c = 0, \) \( x^2 + y^2 + 2fy - c = 0. \)
5.8. Segments of a Chord. The rectangle property of segments of intersecting chords of a circle is proved in elementary geometry by the use of similar triangles; a coordinate-proof was given on p. 21. The property may be stated as follows:

5.8.1. If a variable line drawn from a given point \( T(x', y') \) meets at \( P_1, P_2 \) a circle, centre \( C, radius \ r \), whose equation is
\[
x^2 + y^2 + 2gx + 2fy + c = 0,
\]
and if \( TP_1, TP_2 \) denote directed lengths of steps from \( T \) to \( P_1, P_2 \), then
\[
TP_1 \cdot TP_2 = CT^2 - r^2 = (x' - g)^2 + (y' - f)^2 - (g^2 + f^2 - c);
\]
\[
\therefore CT^2 - r^2 = x'^2 + y'^2 + 2gx' + 2fy' + c.
\]

A proof that \( TP_1 \cdot TP_2 = CT^2 - r^2 \) was given on p. 21.

Further, since \( C \) is the point \((-g, -f)\) and \( r^2 = g^2 + f^2 - c \),
\[
CT^2 - r^2 = (x' + g)^2 + (y' + f)^2 - (g^2 + f^2 - c); \quad \therefore CT^2 - r^2 = x'^2 + y'^2 + 2gx' + 2fy' + c.
\]

If \( T \) lies outside the circle, \( CT > r \), \( \therefore CT^2 - r^2 \) is positive; if \( T \) lies inside the circle, \( CT < r \), \( \therefore CT^2 - r^2 \) is negative; hence \( T(x', y') \) lies outside or inside the circle according as
\[
x^2 + y^2 + 2gx + 2fy + c
\]
is positive or negative. The expression is zero if \( T \) lies on the circle.

If \( T \) lies outside the circle, a line \( TP \) can be drawn to touch the circle at \( P \); then, as proved on p. 14, since \( \angle CTP = 1 \) rt. \( \angle \),
\[
TP^2 = CT^2 - CP^2 = CT^2 - r^2 = x'^2 + y'^2 + 2gx' + 2fy' + c.
\]

The value of \( TP_1 \cdot TP_2 \) or \( CT^2 - r^2 \) is called the power of \( T \) with respect to the circle.

Thus, for all positions of \( T(x', y') \), the power of \( T \) with respect to the circle,
\[
x^2 + y^2 + 2gx + 2fy + c = 0,
\]
is equal to
\[
x^2 + y^2 + 2gx + 2fy + c.
\]

The power of \( T \) is positive and equal to the square of the length of the tangent from \( T \) if \( T \) lies outside the circle; it is negative if \( T \) lies inside the circle; it is zero if \( T \) lies on the circle.

5.8.2. The locus of a point whose powers with respect to two circles are equal is a straight line perpendicular to the line joining the centres of the circles.

Let the equations of the two circles be
\[
x^2 + y^2 + 2gx + 2fy + c_1 = 0, \quad x^2 + y^2 + 2gx + 2fy + c_2 = 0.
\]
The powers of \( T(x', y') \) with respect to the circles are equal if, and only if,
\[
x^2 + y^2 + 2gx' + 2fy' + c_1 = x^2 + y^2 + 2gx + 2fy + c_2;
\]
that is,
\[
2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0.
\]

\( \therefore \) the locus of \( T \) is the straight line
\[
2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0.
\]

This line is perpendicular to the line joining the centres \( C_1(-g_1, -f_1), C_2(-g_2, -f_2) \) of the circles because its gradient is \(-\frac{(g_1 - g_2)}{(f_1 - f_2)}\); and the gradient of \( C_1C_2 \) is \( (f_1 - f_2)/(g_1 - g_2) \), and so the product of their gradients is equal to \(-1\).

5.8.3. The locus of a point whose powers with respect to two circles are equal is called the radical axis of the circles.

Hence by 5.8.2 if the equations of two circles are denoted by
\[
S_1 = x^2 + y^2 + 2gx + 2fy + c_1, \quad S_2 = x^2 + y^2 + 2gx + 2fy + c_2,
\]
written so that the coefficients of \( x^2 \) and \( y^2 \) are all unity, the equation of the radical axis is
\[
S_1 - S_2 = 0.
\]

If the coordinates of a point satisfy each of the equations, \( S_1 = 0 \) and \( S_2 = 0 \), they also satisfy \( S_1 - S_2 = 0 \); therefore, if two circles intersect at \( A \) and \( B \), their radical axis passes through \( A \) and through \( B \) and so is the common chord \( AB \) of the circles. Similarly, if the radical axis \( S_1 - S_2 = 0 \) cuts the circle \( S_2 = 0 \) at \( A \), then \( A \) also lies on \( S_2 = 0 \); hence if two circles do not intersect, their radical axis does not cut either circle.

5.8.4. If a problem involves two circles, it will often be found best for the sake of symmetry to choose the line of centres as \( x \)-axis and the radical axis as \( y \)-axis; these axes are at right angles because the radical axis is perpendicular to the line of centres. With these axes, the \( y \)-coordinate of the centre of each circle is zero; therefore their equations are of the form
\[
x^2 + y^2 + 2gx + c_1 = 0, \quad x^2 + y^2 + 2gx + c_2 = 0.
\]

Since the origin lies on their radical axis, the powers of \( (0, 0) \) with respect to the circles are equal,
\[
\therefore c_1 = c_2.
\]

Hence the equations of the two circles can be written
\[
x^2 + y^2 + 2gx + c = 0, \quad x^2 + y^2 + 2gx + c = 0.
\]
5.9. Coaxal Circles. If a given line is the radical axis of each pair of a system of circles, the system is called coaxal.

5.9.1. The centres $C_1, C_2, C_3, C_4, \ldots$ of the circles of a coaxal system are collinear because the lines $C_1C_2, C_2C_3, C_3C_4, \ldots$ are all perpendicular to the radical axis.

Take the line of centres as $x$-axis and the common radical axis as $y$-axis; then the equation of each circle of the coaxal system is of the form

$$x^2 + y^2 + 2gx + c = 0,$$

where $c$ has the same value for each circle of the system.

The radical axis $x = 0$ meets $x^2 + y^2 + 2gx + c = 0$ if there is a value of $y$ for which $y^2 = -c$.

Hence if $c < 0$, each circle of the system cuts the radical axis at the points $0, \sqrt{(-c)}$ and $0, -\sqrt{(-c)}$, as in Fig. 55.

If $c > 0$, the radical axis does not meet any circle of the system, and so no two circles intersect another, as in Fig. 54.

Example 10. Sketch the circles of the coaxal system,

$$x^2 + y^2 + 2gx + 100 = 0,$$

corresponding to the values $\pm 10, \pm 12, \pm 14, \pm 16$ of $g$.

The equation can be written $(x + g)^2 + y^2 = g^2 - 100$. The centre of this circle is $(-g, 0)$ and its radius is $\sqrt{g^2 - 100}$, therefore there is no circle of the system when $-10 < g < 10$.

Corresponding to $g = 10$ and $g = -10$, the equations of the circles are

$$(x + 10)^2 + y^2 = 0 \quad \text{and} \quad (x - 10)^2 + y^2 = 0;$$

these equations represent two circles, centres $L_1(10, 0), L(-10, 0)$, each of radius zero, called the point-circles or limiting points, $L', L$, of the system.

Corresponding to $g = \pm 12, \pm 14, \pm 16$, there are circles of the system with centres $(12, 0), (14, 0), (16, 0)$ and radii $\sqrt{44}, \sqrt{96}, \sqrt{156}$, respectively, see Fig. 54.


\[5.9\]

COAXAL CIRCLES

Any point $T$ on the radical axis, $x = 0$, of the coaxal system in Example 10 may be denoted by $(0, t)$ and the power of $T(0, t)$ with respect to each circle of the system equals $t^2 + 100$. Since the power of $T$ is positive, $T$ lies outside each circle of the system. If a line drawn from $T$ meets any circle of the system at $Q, R$ and if $TP$ is a tangent from $T$ to this circle, see Fig. 54, then

$$TP^2 = TQ \cdot TR = t^2 + 100 > 0.$$

Note. The directed lines $TQ, TR$ are drawn in the same sense and so the product of the directed lengths $TQ, TR$ is positive.

Example 11. Sketch the circles of the coaxal system,

$$x^2 + y^2 + 2gx - 100 = 0,$$

corresponding to the values $0, \pm 4, \pm 8$ of $g$.

The equation can be written $(x + g)^2 + y^2 = g^2 - 100$. The centre of this circle is $(-g, 0)$ and its radius is $\sqrt{g^2 - 100}$, therefore $g$ may take any positive or negative value, but no circle has a radius less than 10.

Corresponding to $g = 0$, the centre is $(0, 0)$ and the radius is 10. Each circle of the system cuts the radical axis $x = 0$ where $y^2 = 100$.

If, each circle passes through the fixed points $A(0, 10), B(0, -10)$ on the radical axis.

![Fig. 55](image)

Corresponding to $g = \pm 4, \pm 8$, there are circles of the system with centres $(\mp 4, 0), (\pm 8, 0)$ and radii $\sqrt{116}, \sqrt{164}$, respectively, see Fig. 55.

If $K(0, k)$ is any point on the radical axis between $A$ and $B$, then $-10 < k < 10$, see Fig. 55, and so the power of $K$ with respect to each circle of the system, $k^2 - 100$, is negative. If $KQ$ meets a circle of the system at $Q, R$, then $KQ \cdot KR = KA \cdot KB = k^2 - 100 < 0$.

If $T(0, t)$ lies on $AB$ produced or on $BA$ produced, $t < -10$ or $t > 10$, and so the power of $T$, $t^2 - 100$, is positive. If $TM$ meets a circle of the system at $M, N$ and if $TP$ is a tangent, see Fig. 55, then

$$TM \cdot TN = TA \cdot TB = t^2 - 100 > 0.$$
EXERCISE 19

Sketch the circles of the coaxal systems, Nos. 1-3:
1. \( x^2 + y^2 + 2px + 3 = 0 \), corresponding to \( p = -3, \pm 4, \pm 5 \).
2. \( x^2 + y^2 + 2gx - 9 = 0 \), corresponding to \( g = 0, \pm 2, \pm 3, \pm 4 \).
3. \( x^2 + y^2 + 2px = 0 \), corresponding to \( p = 0, \pm 1, \pm 2, \pm 3 \).
4. State the geometrical meaning of the values of \( k \) for which the expression \( x^2 + y^2 - 16 \) has solutions: (i) \( x = 0, y = 3 \); (ii) \( x = 1, y = 2 \); (iii) \( x = 5, y = 0 \); (iv) \( x = 3, y = 5 \).

5. Find the condition that the point \( (k + r \cos \theta, k + r \sin \theta) \) lies on the circle \( x^2 + y^2 = a^2 \). If \( r_1, r_2 \) are the roots of this quadratic in \( r \), state the value of \( r_1 r_2 \) and interpret the result.

6. Interpret (i) if \( k = 1 \), (ii) if \( k = 4 \), the locus given by \( x^2 + y^2 - 4x + 3 = k(x^2 + y^2 + 6y + 5) \).

7. Interpret (i) if \( k = 1 \), (ii) if \( k = 4 \), the locus given by \( (x+1)^2 + (y+2)^2 = k(x^2 + y^2) \).

8. \( x^2 + y^2 - 1, (x-5)^2 + y^2 = 4 \), \( x^2 + y^2 = 1, x^2 + (y+1)^2 = 9 \).
9. \( x^2 + y^2 = 2x - 4y + 1 = 0, x^2 + y^2 + 4x - 2y + 4 = 0 \).
10. \( x^2 + y^2 - 6x + 8 = 0, x^2 + y^2 - 2y - 3 = 0 \).
11. \( x^2 + (y+2)^2 = 0, (x-3)^2 + (y+2)^2 = 0 \).
12. \( x^2 + y^2 = 4x^2 + 2y^2 + 8x - 7 = 0 \).
13. \( 2x^2 + 2y^2 + x^2 - 4y = 0, 3x^2 + 3y^2 - 2x^2 - 2y - 5 = 0 \).

Find the equation of the radical axis of the given pair of circles and illustrate by a sketch, Nos. 8-14:

5.10. Properties of Coaxal Circles.

5.10.1. The system of circles given by
\( (x^2 + y^2 + 2gx + 2fy + c) + k(lx + my + n) = 0 \),
where \( k \) varies and \( g, f, l, m, n \) remain constant, is coaxal and the equation of the radical axis is \( lx + my + n = 0 \).

By 5.8.3, the radical axis of the pair of circles given by \( k = k_1, k = k_2 \)
is \( (k_1 - k_2)(lx + my + n) = 0 \), that is, \( lx + my + n = 0 \);
\( \therefore \) each pair of circles of the system has as radical axis the line \( lx + my + n = 0 \).

5.10.2. If the equations of two circles are given by
\( S_1 = x^2 + y^2 + 2gx + 2fy + c_1 = 0, S_2 = x^2 + y^2 + 2gx + 2fy + c_2 = 0 \),
then
\( S_1 + kS_2 = 0 \), where \( k = -1 \),
is the equation of a circle coaxal with \( S_1, S_2 \).

The coefficients of \( x^2 \) and \( y^2 \) in \( S_1 + kS_2 \) are each equal to \( 1 + k \), where \( 1 + k = 0 \), and there is no term in \( xy \); \( S_1 + kS_2 = 0 \) represents a circle.
Divide throughout by \( 1 + k \), then the coefficients of \( x^2 \) and \( y^2 \) become \( 1 + k \);
by 5.8.3, the equation of the radical axis of this circle and the circle \( S_2 = 0 \) is \( (S_1 + kS_2)/(1 + k) - S_2 = 0 \),
that is, \( S_1 + kS_2 - (1 + k)S_2 = 0 \), that is, \( S_1 - S_2 = 0 \);
this represents the radical axis of \( S_1 = 0, S_2 = 0 \).

5.10.3. Through any given point outside the radical axis of a coaxal system, there passes just one circle of the system.

With the notation of 5.10.2, \( S_1 + kS_2 = 0 \), where \( k = -1 \), represents a circle which passes through the given point \( (x', y') \) if, and only if,
\( x^2 + y^2 + 2gx' + 2fy' + c_1 + k(x^2 + y^2 + 2gx' + 2fy' + c_2) = 0 \); this equation determines \( k \) uniquely and such that \( k = -1 \) since \( (x', y') \) does not lie on the radical axis.

Example 12. The equations of three circles \( S_1, S_2, S_3 \) are
\( x^2 + y^2 - x - 2y - 3 = 0, 2x^2 + 2y^2 + 2x + y - 1 = 0, 3x^2 + 3y^2 - 6x + 4y - 1 = 0 \).
Find the equations of the radical axes of the pairs \( S_1, S_2 \), \( S_1, S_3 \), and the coordinates of their point of intersection \( T \). Prove that the radical axis of the pair \( S_2, S_3 \) passes through \( T \).

The powers of any point \( (x, y) \) with respect to \( S_1, S_2, S_3 \) are
\( x^2 + y^2 - x + 2y - 3, \frac{1}{2}(2x^2 + 2y^2 + 2x + y - 11), \frac{1}{2}(3x^2 + 3y^2 - 6x + 4y - 1) \);
the power of the radical axis of the pair \( S_1, S_3 \) is
\( x^2 + y^2 - x - 2y - 3 = \frac{1}{2}(2x^2 + 2y^2 + 2x + y - 11) \), that is,
\( 4x - 3y = 5 \).

Similarly, the equation of the radical axis of the pair \( S_2, S_3 \) is
\( x^2 + y^2 - x - 2y - 3 = \frac{1}{2}(3x^2 + 3y^2 - 6x + 4y - 1) \), that is,
\( 3x - 2y = 8 \).

Hence \( T \) is the point \( (2, 1) \).

Since the powers of \( T \) with respect to \( S_2 \) and with respect to \( S_3 \) are each equal to the power of \( T \) with respect to \( S_1 \), they are equal to each other; therefore \( T \) lies on the radical axis of the pair \( S_2, S_3 \). The reader may check this by showing that the radical axis of the pair \( S_2, S_3 \) is \( 15x - 15y = 31 \), which passes through \( T(2, 1) \).

If the circles \( S_2, S_3, S_4 \) are not coaxal and if \( T \) is a point whose powers with respect to \( S_2, S_3, S_4 \) are equal, \( T \) is called the radical centre of \( S_2, S_3, S_4 \).
Example 13. Find the equation of the circle passing through (3, 2) and coaxial with \(x^2+y^2-2x+7y-1=0\), \(x^2+y^2+4x-5y+2=0\).

The equation of the radical axis of the given circles is
\[(x^2+y^2-2x+7y-1)-(x^2+y^2+4x-5y+2)=0,\]
that is,
\[-6x+12y-3=0 \text{ or } 2x-4y+1=0.\]

Hence any circle coaxial with the given circles is of the form
\[(x^2+y^2-2x+7y-1)+k(2x-4y+1)=0;\]
this circle passes through (3, 2) if \((9+4-6+14-1)+k(6-8+1)=0\), that is, if \(k=20\). Hence the equation of the required circle is
\[x^2+y^2+38x-73y+19=0.\]

Example 14. \(S_1, S_2, S_3\) are three given coaxial circles; \(TP, TQ\) are the tangents from a variable point \(T\) on \(S_1\) to the circles \(S_2, S_3\). Prove that \(TP: TQ\) is constant.

Take the line of centres as \(x\)-axis and the radical axis as \(y\)-axis; then the equations of \(S_1, S_2, S_3\) can be written
\[x^2+y^2+2gx+c=0, \ x^2+y^2+2gy+x=0, \ x^2+y^2+2gy+c=0.\]

Let \(T\) be the point \((x_1, y_1)\).

Since \(T\) lies on \(S_1\), \(x_1^2+y_1^2+2gx_1+c=0;\)
\[TP^2=x_1^2+y_1^2+2gx_1+c=2y_1.\]

Similarly, \(TQ^2=2y_1\).

\[TP^2:TQ^2=(y_1^2+y_1^2):\ (y_1^2-y_1^2)=\text{constant}.\]

Exercise 20

1. The radical axis of a coaxal system is \(2x+y-15=0\); one circle of the system is \(x^2+y^2-2x+4y-45=0.\) Find the equation of the circle of the system which passes through \(7, 6)\).

2. Find the equation of the circle through \((-1, 1)\) and coaxal with the circles \(x^2+y^2-3x+y=0, \ x^2+y^2+3x+4y=0.\)

3. Find the equation of the circle through the origin and through the points of intersection of \(x-4=0\) with \(x^2+y^2=25.\)

4. Find the equations of two circles of radius zero coaxal with \(x^2+y^2+4x+2y=3, \ x^2+y^2+8x+6y+11=0.\)

Interpret these equations.

Find the coordinates of the radical centre of the circles, Nos. 5-7:

5. \(x^2+y^2=6, \ x^2+y^2-2x-2y=6, \ x^2+y^2+4x+2y=5.\)

6. \(x^2+y^2=1, \ x^2+y^2-6x-3y=1, \ x^2+y^2-x-3y=7.\)

7. \(x^2+y^2=x, \ x^2+y^2-2x-y=2, \ x^2+y^2-2x-2y=5.\)

8. Prove that the system of circles, \(x^2+y^2-4x+2y+3=0, \) where \(f\) varies, has two points in common. Find: (i) the circle of the system through \((4, 3)\); (ii) the circles of the system which touch the \(y\)-axis; (iii) the circles of the system which touch \(x+y=5.\)

5.10] COAXAL CIRCLES

9. \(A\) is the point \((3, 0)\); \(TP\) is the tangent from a variable point \(T\) to the circle \(x^2+y^2-6y=0.\) If \(TP=2TA, \) find the equation of the locus of \(T.\) Prove it is a circle coaxal with the given circle and limiting point \(A.\)

10. Prove that the tangents from \(P(0, h)\) to the circles \(x^2+y^2+2ax+k=0, \ x^2+y^2+2ax+k=0\) are equal. Find the equation of the circle, centre \(P\), with radius equal to the common length of these tangents and prove that it is orthogonal to the given circles and that when \(A\) varies it passes through two fixed points. \(O(0)\)

11. Two circles \(S, S'\) are respectively concentric with and enclosed by two intersecting circles \(T, T'\) and have the same radical axis as \(T, T'.\) The circles \(S, T\) lie on the opposite sides of the radical axis from the circles \(S', T'.\) Draw figures showing the cases when \(S\) and \(S'\) intersect and when they do not intersect. Prove the length of the tangent from any point on \(T\) to \(S\) equals the length of the tangent from any point on \(T'\) to \(S'.\) \((N)\)

12. \(O\) is the origin, \(B\) is the fixed point \((0, b), b=0, \) and \(k, k'\) are constants. \(P(x, y)\) and \(P'(x', y')\) are variable points such that \(OP=kx\) and \(RP'^2=k'x'.\) Prove that the loci of \(P\) and \(P'\) are circles which touch one another if \(kk'=b^2.\)

13. \(TP, TQ\) are the tangents from \(T\) to two circles, centres \(A, B,\) and \(TN\) is the perpendicular from \(T\) to their radical axis. Prove that
\[TP^2:TQ^2=2AB.\]

14. \(AB\) is the radical axis of two circles \(S_1, S_2; TP\) is the tangent to \(S_2\) from a variable point \(T\) on \(S_1;\) \(TN\) is the perpendicular from \(T\) to \(AB.\)

Prove that \(TP^2:TN^2\) is constant.

15. \(PQ\) is the chord of contact of the tangents from a fixed point \(C\) to a variable circle belonging to a given coaxal system. Prove that \(PQ\) passes through a fixed point \(D\) such that the mid-point of \(CD\) lies on the radical axis.

16. \(TP_1, TP_2, TP_3\) are the tangents from a point \(T\) to three coaxal circles, centres \(C_1, C_2, C_3,\) if \(C_2\) lies between \(C_1\) and \(C_3,\) prove that
\[TP_1^2=C_1C_2+TP_2^2; \ (C_2C_3+TP_3^2). \ C_1C_2.\]

17. Three circles have two common points, the origin and the point \(A(a, 0); a\) line through \(O\) which makes an angle \(\theta\) with \(OX\) meets the circles again at \(P_1, P_2, P_3.\) If the lengths of \(OP_1, OP_2, OP_3\) are \(r_1, r_2, r_3,\) prove that the value of the ratio, \((r_1-r_2):(r_2-r_3),\) does not depend on \(\theta.\)

18. A variable line through the origin \(O\) meets the given circles \(x^2+y^2-2ax, x^2+y^2-2fy\) at \(P, Q.\) Prove that the locus of the mid-point of \(PQ\) is a circle coaxal with the given circles.
6.1. Intersecting Lines and Curves. The locus represented by the general equation of the second degree, see 5.16, p. 44, is called a curve of the second degree; it is best at first to use special forms of this equation to illustrate coordinate methods.

6.1.1. The coordinates of any point of intersection of a line with a curve of the second degree are found by solving two simultaneous equations, one linear and one quadratic in \( x \) and \( y \). If the elimination of \( x \) or of \( y \) gives a quadratic equation, this equation has either no roots or two unequal roots or two equal roots. If there are no roots, the line does not intersect the curve; but if there are two unequal roots, the line meets the curve at two distinct points which are the extremities of a chord of the curve. If, however, there are two equal roots, there is just one point of intersection of the line with the curve and the nature of the intersection is described by saying that the line meets the curve at 'two coincident points' and the line is then called a tangent to the curve.

6.1.2. If a variable line drawn through a fixed point \( P \) on the curve meets the curve again at a variable point \( Q \), the points \( P \) and \( Q \) are determined by the unequal roots of a quadratic equation. If now \( Q \) moves along the curve, in the limiting position when \( Q \) moves up to \( P \), the roots of the corresponding quadratic equation tend to equality and the limiting position of the chord \( PQ \) is the tangent at \( P \) to the curve. Hence the test for tangency is associated with equality of roots.

For example, the line \( y - 6x + 9 = 0 \) meets the curve \( y - x^2 = 0 \) where \( x^2 - 6x + 9 = 0 \), that is, \( (x-3)^2 = 0 \).

Thus there is just one point of intersection, \( x = 3, y = 9 \), and so the line \( y - 6x + 9 = 0 \) is the tangent at \( P(3, 9) \) to the curve \( y - x^2 = 0 \). The test that there are two equal roots is necessary for tangency. For example, the line \( x - 3 = 0 \) meets the curve \( y - x^2 = 0 \) at just one point \( P(3, 9) \); but the line \( x - 3 = 0 \) is not the tangent at \( P(3, 9) \) to \( y - x^2 = 0 \). Elimination of \( x \) from the equations, \( y - x^2 = 0, x - 3 = 0 \), gives an equation of the first degree in \( y \), namely \( y - 9 = 0 \); there is one root, not two equal roots. The reader should make a sketch showing the curve \( y - x^2 = 0 \) and the two lines, \( y - 6x + 9 = 0 \) and \( x - 3 = 0 \).
5. Prove that \( y = mx + c \) meets \( y^2 = 4ax \) \((a > 0)\), at two distinct points if \( mc < a \) and does not meet it at any point if \( mc > a \). Illustrate by a sketch.

6. (i) State the coordinates of the points where \( y = (x + 2)(x - 0) \) cuts \( x'0x \) and \( y'0y \). Sketch the curve. (ii) The line \( y = -7 \) meets this curve at \( P, Q \). Find the equations of \( OP, OQ \). (iii) Find the coordinates of the points where the curve is met by the lines \( y + 16 = 0 \), for \(-y - 20 = 0, 4x - y - 28 = 0\).

7. Explain why the line \( x + t'y = 2at \) is a tangent to the curve \( xy = a^2 \) and find the coordinates of the point of contact.

8. Find the coordinates of the points of intersection with \( y^2 = 4ax \) of the line \( x = 4a/x^2 = a(t^2 - 1) \).

9. Find the coordinates of the points of intersection with \( xy = a^2 \) of the line \( x = 4a/x^2 = a(t^2 - 1) \).

6.2. Parametric Equations. For all values of \( t \), the point \((at^2, 2at)\) lies on the parabola \( y^2 = 4ax \) because the values of \( x \) and \( y \) given by \( x = at^2 \), \( y = 2at \) satisfy the equation \( y^2 = 4ax \).

Conversely, if \( P(x, y) \) is a point on \( y^2 = 4ax \), there is just one value \( t \) of \( t \), which may be positive or negative or zero, such that \( 2at = y \), and for this value \( t \) of \( t \), \( x = y^2/4at \).

Hence \((at^2, 2at)\) can be called without ambiguity ‘the point \( t \)’ of the parabola \( y^2 = 4ax \); we then say that the coordinates of any point of the parabola are given in terms of a parameter \( t \) by means of the equations

\[
\begin{align*}
x &= at^2, \\
y &= 2at.
\end{align*}
\]

These relations are also called parametric equations of the parabola \( y^2 = 4ax \), and for some purposes it is convenient to express them in ratio form,

\[
\frac{x}{y} = \frac{t^2}{2t} : 1.
\]

The result obtained in Example 2, p. 75, indicates one geometrical interpretation of the meaning of \( t \): the slope of the line, \( 2x - ty = 0 \), joining the point \( t \), where \( t = 0 \), to the origin is \( 2/t \). In the particular case \( t = 0 \), the point \( t \) is the origin which lies on the curve and the equation of the tangent at \((0, 0)\) is \( x = 0 \).

Example 4. Find the \((x, y)\) equation of the locus determined by the parametric equations, \( x = a \cos t \), \( y = a \sin t \).

For all values of \( t \), \( \cos^2 t + \sin^2 t = 1 \),

\[
\begin{align*}
x^2 + y^2 &= a^2(\cos^2 t + \sin^2 t) \\
x^2 + y^2 &= a^2.
\end{align*}
\]

Example 5. A small object is projected from the ground upwards and forwards. The highest point \( O \) of its path (called its trajectory) is taken as origin and the axes \( x'0x, y'0y \) are taken horizontally forwards and vertically downwards respectively. At time \( t \) seconds after passing \( O \), where \( t \) takes both positive and negative values, the position of the object is given by the parametric equations,

\[
\begin{align*}
x &= 20t, \\
y &= 16t^2,
\end{align*}
\]

where distances are measured in feet. Draw the trajectory from \( t = -2 \) to \( t = +2 \) and mark on it the position of the point \( t \) at half-second intervals.

Construct a table of values:

<table>
<thead>
<tr>
<th>( t )</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>( y )</td>
<td>64</td>
<td>36</td>
<td>16</td>
<td>4</td>
<td>0</td>
<td>4</td>
<td>16</td>
<td>36</td>
<td>64</td>
</tr>
</tbody>
</table>

A small-scale diagram is shown in Fig. 56. The reader should use the table to draw the graph on the scale, 1 in. = 30 ft., on each axis and mark on it the points \( t = -2, t = 1.5,\) etc.

The \((x, y)\) equation of the trajectory is found by eliminating \( t \) from \( x = 20t, y = 16t^2\).

\[
t = x/20, \quad t^2 = x^2/400; \quad y = 16x^2/400 = x^2/25;
\]

\(t\): the trajectory is the parabola whose equation is \( x^2 - 25y = 0 \).

Example 6. Express in terms of a parameter \( t \) the coordinates of any point \( P \) on the curve \( x = at^2, y = at^3 \), by finding where the line \( y = at^3 \) meets the curve. 

Find the values of \( t \) which give the points where the line \( 7x - 2y = 9a \) meets the curve.

\[
y = ax, \quad x = at^2\], \( y = at^3\), that is, \( x = 0 \) or \( x = at^2\); but \( y = ax \), \( x = 0, y = 0 \) or \( x = at^2, y = at^3\).

Since the form \( x = at^2, y = at^3 \) includes the point \( x = 0, y = 0 \) by taking the value \( t = 0 \), it follows that any point \( P \) on the curve can be denoted by \( (at^2, at^3) \).

The point \( P(at^2, at^3) \) also lies on the line \( 7x - 2y = 9a \) if the value of \( t \) satisfies the equation \( 7at^2 - 2at^3 = 9a \).

\(t\): the required values of \( t \) are the roots of \( 2t^2 - 7t^3 + 9 = 0 \), that is, \((t + 1)(2t^2 - 9t + 9) = (t + 1)(t - 3)(t - 3) = 0; \)

\(t\): the points of intersection are given by \( t = -1, t = 1, t = 3 \).
63. Cubic Curves. The equation \( x^3 - ay^2 = 0 \) of the curve for which parametric equations were obtained in Example 6, p. 77, is of the third degree in \( x \) and \( y \). For this reason it is called a cubic curve. The example shows that a line may meet a cubic curve at three distinct points determined by the roots of a cubic equation in the parameter \( t \).

If the cubic equation in \( t \), whose roots determine the points where a line \( tx + my + n = 0 \) meets the cubic curve given by parametric equations, is of the form

\[
(t-t_1)(t-t_2)(t-t_3) = 0,
\]

the roots are \( t = t_1 \) (repeated) and \( t = t_2 \).

Hence if \( t = t_1 \), \( t = t_2 \) determine the points \( P_1, P_2 \), respectively, the line is said to meet the curve at two coincident points which coincide at \( P_1 \) and to meet the curve again at \( P_2 \); the line is then said to touch the curve at \( P_1 \) and to cut it again at \( P_2 \).

Similarly, if the cubic equation in \( t \), whose roots determine the intersection of a line \( tx + my + n = 0 \) with the cubic curve is

\[
(t-t_1)(t-t_2)(t-t_3) = 0, \quad q + 0 = 0,
\]

there is only one root, \( t = t_1 \), because there is no value of \( t \) for which \( (t+p)^2 + q^2 = 0 \). Hence if \( t = t_1 \) determines the point \( P_1 \), the line cuts the curve at just one point \( P_1 \); the line is not a tangent at \( P_1 \) because \( t = t_1 \) is not a repeated root.

Example 7. Examine the nature of the intersections of the curves, \( x = at^2, \ y = at^3 \), with the lines (i) \( 3x - 2y = a \), (ii) \( x + y = 2a \).

(i) The point \( P(at^2, \ at^3) \) lies on the line \( 3x - 2y = a \) if \( t \) is a root of the equation, \( 3at^2 - 2at^3 = a \), that is, \( 2t^2 - 3at^3 = 1 \).

but \( 2t^2 - 3at^3 = 0 \), \( t^2(2t - 3at^2) = 0 \);

and \( t = \frac{a}{2} \), \( t = \frac{3a}{2} \).

Hence the line \( 3x - 2y = a \) touches the curve at the point \( t = 1, P(a, a) \), and cuts it again at the point \( t = -\frac{a}{2}, \frac{3a}{2} \).

(ii) The point \( P(at^2, \ at^3) \) lies on the line \( x + y = 2a \) if \( t \) is a root of the equation, \( at^2 + at^3 = 2a \), that is, \( t^2 + t^3 = 2 \);

but \( t^2 + t^3 = 0 \), \( t^2(1 + t^2) = 0 \);

and \( t = 1, 1 + t^2 \) is a root for which \( (t+1)^2 = 0 \).

Hence the line \( x + y = 2a \) touches the curve at only one point, \( t = 1, P(a, a) \). This line does not touch the curve at \( P \) because \( t = 1 \) is not a repeated root.

In fact, it was shown in (ii) that the tangent at \( P(a, a) \) is \( 3x - 2y = a \).

63. PARAMETRIC EQUATIONS

EXERCISE 22

1. Draw the locus determined by the parametric equations, \( x = \frac{1}{t^2}, \ y = \frac{1}{t} \), for values of \( t \) from \(-4 \) to \( +4 \). Mark on the graph the positions of the point \( t = -4, -3, -2 \), etc.

2. The parametric equations of the locus \( P(x, y) \) are \( x = 4t - 5, y = 3t + 2 \); prove that the locus is a line and find the values of \( t \) for which \( P \) lies on: (i) \( x \) \( Oy \); (ii) \( y \) \( Oy \); (iii) \( y^2 + 25x = 0 \).

3. \( P_1(x_1, y_1), P_2(x_2, y_2) \) are given points; the parametric equations of the locus of \( P(x, y) \) are \( x = (1-t)x_1 + tx_2, y = (1-t)y_1 + ty_2 \). Interpret geometrically the meaning of \( t \); hence state the locus of \( P \).

4. A variable point \( P(x, y) \) moves so that \( x = 3t^2, y = 6t \), where \( t \) varies. Find the \( (x, y) \) equation of the locus. Find also the values of \( t \) for which \( P \) lies on \( x + 2y = 36 \).

5. If \( x = 3 + t, y = 3t + 2t^2 \), are the parametric equations of the locus of \( P(x, y) \), find the \( (x, y) \) equation of the locus. Sketch the locus and interpret geometrically the meaning of \( t \).

6. For what values of \( t \) does the locus given by the parametric equations, \( x = at, y = ct + d \), where \( a, b, c, d \) are constant, meet: (i) \( x \) \( Oy \); (ii) \( y \) \( Oy \); (iii) \( x + my + n = 0 \)? Find the condition that the origin lies on the locus.

7. For what values of \( t \) does the locus, \( x = at^2, y = 2at \), where \( a \) is constant, meet the line \( 2x + 3y = 8a \)? Find the values of \( l, m \) if the line \( lx + my + a = 0 \) meets the locus at the points \( t = -1, t = 2 \).

8. Find the \( (x, y) \) equation of the curve given by the parametric equations, \( x = 1 + t^2, y = \frac{a}{1 + t^2} \).

9. Express in terms of a parameter \( t \) the coordinates of any point \( P \) on the curve \( x^2y = a^5 \) by finding where the line \( y = tx \) meets the curve. Find the condition for the line \( lx + my + na = 0 \) to meet the curve at the point \( t \); find the values of \( t \) which give the points where (i) the line \( 3x + y = 7a \), (ii) the line \( 2x + y = 3a \), meets the curve. Interpret the answers and illustrate them by a sketch.

10. Prove that there are values of \( l, m \) such that the line \( lx + my + a = 0 \) meets the curve given by the parametric equations, \( x = at^2, y = ct \), at the points \( t = 1, t = -1, t = -2 \).

Find the values of \( t \) for the points where the curve is met by the lines: (i) \( x - 3y + 2a = 0 \); (ii) \( x - 2y - 4a = 0 \). Interpret the answers and illustrate them by a sketch.
6.4. Roots of an Equation. Examples were given in Chapter 2, see pp. 19, 21, of the importance of working with symmetric functions of the roots of a quadratic equation. It is still more important to do so with equations of higher degree, see Example 9, p. 81.

6.4.1. Quadratic Equation. If \( t_1, t_2 \) are the roots of 
\[
a t^2 + b t + c = 0,
\]
this equation is equivalent to 
\[
(t - t_1)(t - t_2) = t^2 - (t_1 + t_2)t + t_1 t_2 = 0;
\]
\[
\therefore \quad 1 : (t_1 + t_2) : t_1 t_2 = a : -b : c.
\]

6.4.2. Cubic Equation. If \( t_1, t_2, t_3 \) are the roots of 
\[
a t^3 + b t^2 + c t + d = 0,
\]
this equation is equivalent to 
\[
(t - t_1)(t - t_2)(t - t_3) = \beta - (\Sigma t_i)t^2 + (\Sigma t_i t_j)t - \Sigma t_i t_j t_k = 0,
\]
where 
\[
\Sigma t_i \equiv t_1 + t_2 + t_3 \quad \text{and} \quad \Sigma t_i t_j \equiv t_1 t_2 + t_2 t_3 + t_3 t_1;
\]
\[
\therefore \quad 1 : \Sigma t_i : \Sigma t_i t_j : t_1 t_2 t_3 = a : -b : c : -d.
\]
or 
\[
\Sigma t_i = -b/a, \quad \Sigma t_i t_j = c/a, \quad t_1 t_2 t_3 = -d/a.
\]

6.4.3. Quartic Equation. If \( t_1, t_2, t_3, t_4 \) are the roots of 
\[
a t^4 + b t^3 + c t^2 + d t + e = 0,
\]
this equation is equivalent to 
\[
(t - t_1)(t - t_2)(t - t_3)(t - t_4) = (t - t_1)(t - t_2)(t - t_3)(t - t_4) = 0;\]
\[
\therefore \quad 1 : \Sigma t_i : \Sigma t_i t_j : \Sigma t_i t_j t_k : t_1 t_2 t_3 t_4 = a : -b : c : -d : e.
\]
or 
\[
\Sigma t_i = -b/a, \quad \Sigma t_i t_j = c/a, \quad \Sigma t_i t_j t_k = -d/a, \quad t_1 t_2 t_3 t_4 = e/a.
\]

Example 8. Find the equation of the chord \( P_1 P_2 \) of the parabola,
\[x = at^2, \quad y = 2at,\]
if \( P_1, P_2 \) are the points given by \( t = t_1, t = t_2 \)

Denote the equation of \( P_1 P_2 \) by \( lx + my + na = 0; \)
\[
\therefore \quad P_1 P_2 \quad \text{meets the parabola} \quad x = at^2, \quad y = 2at, \quad \text{where} \;
\]
\[
at^2 + 2mt + na = 0;
\]
\[
\therefore \quad t_1, t_2 \quad \text{are the roots of} \;
\]
\[
l t^2 + 2mt + n = 0;\]
\[
\therefore \quad \text{this equation is equivalent to} \quad (t - t_1)(t - t_2) = t^2 - (t_1 + t_2)t + t_1 t_2 = 0;
\]
\[
\therefore \quad 1 : (t_1 + t_2) : t_1 t_2 = a : -b : c;
\]
\[
\therefore \quad l : m : n = 1 : -2(t_1 + t_2) : t_1 t_2;
\]
\[
\therefore \quad \text{the equation of} \quad P_1 P_2 \quad \text{is} \;
\]
\[
x - \frac{1}{2}(t_1 + t_2)y + a t_1 t_2 = 0.
\]

This parametric method for finding the equation of a line is important; a more elementary method for finding the equation of the line joining \( P_1(at_1^2, 2at_1) \) to \( P_2(at_2^2, 2at_2) \) was given in 3.3.3., p. 26.

Note. The equation of \( P_1 P_2 \) could have been denoted by 
\[
x = \frac{1}{2}(t_1 + t_2)y + at_1 t_2 = 0;
\]
but it is a little simpler to use the form homogeneous in \( x, y, a. \)

Exercise 23

[Use the method of Example 8, p. 80, in this exercise.]

1. Find the equation of the chord \( PQ \) of the hyperbola, \( x = at, \quad y = c/t, \) where \( P, Q \) are the points \( t = p, t = q; \)

If \( R, H \) are the points \( t = r, t = h, \) find the condition that \( RH \) is perpendicular to \( PQ \) and, if this condition is satisfied, prove that \( PH \) is perpendicular to \( QR. \)

If \( P, Q, R \) are any three points on the hyperbola \( x = at, \quad y = c/t, \) that is, the orthocentre of \( \Delta PQR \) lies on the hyperbola.

[2] Find \( l : m : n \) in terms of \( t, \) if the line \( lx + my + na = 0 \) meets the parabola, \( x = at^2, \quad y = 2at, \) at the point \( P_1, t = t_1 \) (repeated). Hence find the equation of the tangent to the parabola at \( P_1. \)

3. Find the equation of the chord \( P_1 P_2 \) of the curve given by 
\[x = t^2 + t, \quad y = t + 1,\] when \( P_1, P_2 \) are the points \( t = t_1, t = t_2. \) Find also the equation of the tangent at \( P_1. \)

4. Find the equation of the chord \( PQ \) of the curve given by 
\[x = t^2 + 1/t, \quad y = t - 1/t, \] when \( P, Q \) are the points \( t = p, t = q. \) Find also the equation of the tangent at \( P. \)

5. Find \( l : m : n \) in terms of \( t, \) if the line \( lx + my + na = 0 \) meets the hyperbola, \( x = at, \quad y = c/t, \) at the point \( P_1, t = t_1 \) (repeated). Hence find the equation of the tangent to the hyperbola at \( P_1. \)

If the tangent at \( P_1 \) meets \( x^2/a^2 + y^2/b^2 = 1, \) prove that \( P_1 \) is the midpoint of \( QR. \)

[6] Prove that the \( (x, y) \) equation of the ellipse given by 
\[
x/a : y/b : 1 = (1-t^2) : 2t : (1+t^2)
\]
is \( x^2/a^2 + y^2/b^2 = 1. \) Find the equations of:

(i) the chord joining the points \( t = t_1, t = t_2; \)
(ii) the tangent at the point \( t_1; \)
(iii) the tangent at the point \( (x_1, y_1). \)
PARAMETRIC NOTATION

7. If the circle \(x^2 + y^2 + 2gx + 2fy + h = 0\) cuts the hyperbola, \(x = kt, y = kt\), at the four points \(A, B, C, D\) given by \(t = a, t = b, b = c, t = d\), prove that \(abcd = 1\). Find the gradient of \(AB\) and prove that if \(D'\) is the point given by \(t = d\), then \(OD'\) is perpendicular to \(AB\). Hence prove that the orthocentre of \(ABC\) is the point \(D'\) of the curve such that the mid-point of \(DD'\) is the origin.

8. A circle cuts the parabola \(x = at^2, y = 2at\) at the origin and at the points \(P, Q, R\). Prove that the centroid of the triangle \(PQR\) lies on the \(x\)-axis. (See 4.5, p. 60.)

9. Find the values of \(t\) which determine the points of intersection of the parabola \(x = at^2, y = 2at\), with the line \(y = px + qy + r\). Hence find the condition that the lines \(y = px + qy + r\), \(y = qx - 2aq + ap^2\), meet the parabola.

10. Sketch the curve given by the parametric equations, \(x = at^2, y = 2at\). Find the values of \(t\) which determine the points of intersection of the curve \(x = -7y + 6 = 0\) with the curve.

If the line \(lx + my + n = 0\) meets the curve at the points, \(t = t_1, t = t_2, t = t_3\), prove that \(t_1 + t_2 + t_3 = 0\). If \(A, B, C, D\) are the points \(t = a, t = b, b = c, t = d\), on the curve and if no three are collinear, find the condition that \(AB\) and \(CD\) intersect at a point on the curve.

11. The points of intersection \(P, Q\) of the line \(lx + my + n = 0\) with the curve, \(x = at^2, y = 2at\), are given by \(t = p\) (repeated) and \(t = q\).

(i) Find \(q\) in terms of \(p\).

(ii) Find the ratios \(l : m : n\) in terms of \(p\) and then write down the equation of the tangent at \(P_t = p\).

(iii) If \(t_1 + t_2 + t_3 = 0\), prove that the line whose equation is \(x = p_1t_1 + p_2t_2 + p_3t_3\) and \(y = q_1t_1 + q_2t_2 + q_3t_3\) meets the curve at the points \(t = t_1, t = t_2, t = t_3\).

Hence prove that if the tangents at \(P_1, P_2, P_3\) meet the curve again at \(Q_1, Q_2, Q_3\), then the points \(Q_1, Q_2, Q_3\) are collinear.

12. Sketch the curve given by the parametric equations, \(x = at^2, y = 2at\).

(i) If the line \(lx + my + n = 0\) meets the curve at the points \(P, Q, R\), given by \(t = p, t = q, t = r\), prove that \(gr + rp + pg = 0\) and find the ratios \(l : m : n\) in terms of \(p, q, r\).

(ii) If the line \(lx + my + n = 0\) meets the curve at the points given by \(t = p\) (repeated), \(t = r\), find \(r\) in terms of \(p\), and the equation of the tangent at \(P_t = p\).

(iii) If \(gr + rp + pg = 0\), prove that the line \(x = y(pq + r + r) + x = 2aq + ap^2\) meets the curve at the points \(t = p, t = q, t = r\). Hence prove that if the tangents at \(P, P, Q\), and \(Q, Q, R\), and \(P, P, R\), meet the curve again at \(P', Q', R'\), then \(P', Q', R'\) are collinear.

6.5. It has been shown that, with the notation of 5.10.2, p. 71, any given circle \(S_1 = 0\), coaxal with the circles \(S_2 = 0, S_3 = 0\), can be represented by the equation \(S_1 + kS_2 = 0\), where the value of the parameter \(k\) is chosen so that the coordinates of one selected point on the given circle satisfy the equation \(S_1 + kS_2 = 0\). Thus any particular member of a system of coaxal circles can be identified by a value of the parameter \(k\) in \(S_1 + kS_2 = 0\). A similar use can be made of a parameter \(k\) to identify a particular member of a system of concurrent lines.

6.5.1. If \(px + qy + r = 0\), \(px + p'y + r' = 0\), are the equations of the given intersecting lines \(CA, CA'\), then, for each value of \(k\),
\[(px + qy + r) + k(p'x + p'y + r') = 0\]
represents a line through the point of intersection \(C\) of \(CA, CA'\).

The equation represents a line because it is of the first degree in \(x\) and \(y\).

Further, since the coordinates of \(C\) satisfy each of the equations \(px + qy + r = 0\) and \(px + p'y + r' = 0\), they also satisfy for any value of \(k\) the equation
\[(px + qy + r) + k(p'x + p'y + r') = 0\]
therefore this equation represents a line passing through \(C\).

6.5.2. The equation of any given line \(CP\) through the point of intersection \(C\) of the given lines \(CA, CA'\), whose equations are \(px + qy + r = 0, p'x + p'y + r' = 0\), can be expressed in the form
\[(px + qy + r) + k(p'x + p'y + r') = 0\]
by a suitable choice of the value of the parameter \(k\).

By 6.5.1, the equation represents a line through \(C\). Denote by \((u, v)\) the coordinates of one selected point \(P\) on \(CP\). Choose the value of \(k\) so that
\[(pu + qv + r) + k(p'u + q'v + r') = 0\]
then for this value of \(k\) the equation represents the line \(CP\).

6.5.3. Notation. It is an important labour-saving device to use here for lines an abbreviated notation such as was used in 5.8.3, p. 67, for circles.

We write \(px + qy + r = L(x, y) = L\), and \(p'x + p'y + r' = L'(x, y) = L'\);
then, for each value of the parameter \(k\),
\[L + kl' = 0\]
represents a line through the point of intersection of the intersecting lines, \(L = 0, L' = 0\).

Conversely, any given line concurrent with the lines \(L = 0, L' = 0\), can by a suitable choice of the value of \(k\) be represented by \(L + kl' = 0\).
Example 10. The equation of a variable line is
\[ x(1 + 2k) + y(1 - 2k) - (10 + k) = 0. \]
(i) Prove the line passes through a fixed point A and find its coordinates.
(ii) Find the equation of the line joining A to P (1, 3).
(iii) The equation of the variable line can be written
\[ (x + y - 10) + k(3x - 2y - 1) = 0. \]
The coordinates of a point which satisfies each of the equations
\[ x + y - 10 = 0 \quad \text{and} \quad 3x - 2y - 1 = 0 \]
also satisfy the equation of the variable line for any value of k.
Solving these equations, we find \( x = 3, \ y = 7 \).
\[
\therefore \text{the variable line passes through the fixed point } A(3, 7).
\]
(ii) The line \( (x + y - 10) + k(3x - 2y - 1) = 0 \) passes through \( P(1, 3) \) if
\[ (1 + 3 - 10) + k(3 - 2 - 1) = 0, \]
that is, \(-6 - 2k = 0\), giving \( k = -3 \).
Put \( k = -3 \), then the equation of \( AP \) is
\[ -14x + 7y + 7 = 0, \]
that is, \( 2x - y + 1 = 0 \).

Example 11. Find the equation of the perpendicular from the point of intersection of \( 2x + 5y = 3, 3x - 2y = 1 \) to \( 4x + 8y = 7 \).
The line is of the form \( (2x + 5y - 3) + k(3x - 2y - 1) = 0 \),
that is,
\[ (2 + 3k)x + (5 - 2k)y - (3 + k) = 0. \]
This line is perpendicular to the line \( 4x + 8y = 7 \) if
\[ 4(2 + 3k) + 8(5 - 2k) = 0, \]
that is, \( 2 + 3k + 10 - 8k = 0 \);
\[ \therefore k = 12; \]
this gives
\[ 38x - 19y - 15 = 0. \]

Example 12. Show by inspection that the lines,
\[ 5x - 2y - 9 = 0, 4x - 8y + 5 = 0, 9x - 3y - 4 = 0, \]
are concurrent.
The pair of values of \( x \) and \( y \), which satisfy
\[ 5x - 2y - 9 = 0 \quad \text{and} \quad 4x - 3y + 5 = 0, \]
also satisfy
\[ (5x - 2y - 9) + (4x - 3y + 5) = 9x - 5y - 4 = 0; \]
\[ \therefore \text{the line } 9x - 5y - 4 = 0 \text{ passes through the point of intersection of the lines}\]
\[ 5x - 2y - 9 = 0, 4x - 3y + 5 = 0. \]

6.6. The \( L + kL' \) theorem illustrates one use that is made of the property that the common points of two given loci belong to the locus whose equation is formed by any combination of their equations; there are other important applications of this property.

If the line \( lx + my = 1 \) meets the locus represented by
\[ ax^2 + 2kxy + by^2 + 2gx + 2fy + c = 0 \]
at \( P \) and \( Q \), the locus whose equation is formed by combining these equations to give a homogeneous equation in \( x \) and \( y \), namely
\[ (ax^2 + 2kxy + by^2) + (2gx + 2fy)(lx + my) + c(lx + my)^2 = 0, \]
represents a pair of lines through the origin \( O \) and passing through \( P \) and \( Q \), that is, the pair of lines \( OP, OQ \), see Example 13.

Example 13. If \( P \) and \( Q \) are the points of intersection of the line
\[ x + 3y = 2 = 0 \]
and the circle \( x^2 + y^2 - 2x + 4y - 4 = 0 \), prove that the circle on \( PQ \) as diameter passes through the origin \( O \).

It is sufficient to prove that \( \angle POQ \) is a right angle. A homogeneous equation of the second degree in \( x \) and \( y \) represents a pair of lines through the origin. Further, since the coordinates of \( P \) and \( Q \) satisfy the equation of the circle and the equation of the line, they also satisfy any combination of these two equations. If then the equation of the circle and line are combined so as to give a homogeneous equation of the second degree in \( x \) and \( y \), the equation is satisfied by the coordinates of \( P \) and \( Q \) and also represents a pair of lines through \( O \); it therefore represents the pair of lines \( OP, OQ \).

The equation of the line can be written, \( \frac{1}{2}(x + 3y) = 1 \).
 Modify the equation of the circle by multiplying the terms of first degree by \( \frac{1}{2}(x + 3y) \) and the constant term by \( \left( \frac{1}{2}(x + 3y) \right)^2 \) this gives
\[ x^2 + y^2 - (2x - 6y)(\frac{1}{2}(x + 3y)) - 4(\frac{1}{2}(x + 3y))^2 = 0, \]
that is,
\[ x^2 + y^2 - x^2 + 9y^2 - x^2 - 6xy - 9y^2 = 0; \]
\[ \therefore \text{the equation of the pair of lines } OP, OQ \text{ is} \]
\[ x^2 + 6xy - y^2 = 0. \]
This equation represents a pair of perpendicular lines because the sum of the coefficients of \( x^2 \) and \( y^2 \) is zero, see 3.16.3, p. 43;
\[ \therefore \angle POQ \text{ is a right angle.} \]

Example 14. Find the values of \( m \) and \( n \) if the equation
\[ (x - 7y + 8)(x - 2y - 2) + m(x - 2y - 2)(3x - y + 4) + n(3x - y + 4)(x - 7y + 8) = 0 \]
represents a circle and then identify this circle.

The equation represents a circle if the coefficients of \( x^2 \) and \( y^2 \) are equal and if the coefficient of \( xy \) is zero; hence
\[ 1 + 3m + 3n = 14 + 2m + 7n \quad \text{and} \quad -9 - 7m - 22n = 0, \]
that is,
\[ m - 4n = 13 \quad \text{and} \quad 7m + 22n = -9; \]
these equations give \( m = 5, n = -2 \).
Let \( x = 7y + 8, x = 2y - 2, 3x - y + 4 = 0 \) be the equations of the sides \( BC, CA, AB \) respectively of \( \triangle ABC \); then the coordinates of \( C \) satisfy each of the equations \( x = 7y + 8 = 0, x = 2y - 2 = 0 \) and therefore also satisfy the given equation for all values of \( m \) and \( n \); similarly, the coordinates of \( A \) and of \( B \) do so. Hence when \( m = 5, n = -2 \), the equation represents the circumcircle of \( \triangle ABC \).

It may be verified that the values \( m = 5, n = -2 \) give the equation,
\[ x^2 + y^2 - 6x + 4y - 12 = 0, \]
which represents a circle through the vertices \( A(-2, -2), B(-1, 1) \) and \( C(6, 2) \).
EXERCISE 24

[Use the $L+kL'$ property in this exercise if it is applicable.]

Find the equation of the line joining the given point to the point of intersection of the given pair of lines, Nos. 1–6:

1. $(0, 0); 7x - 3y + 6 = 0, 5x + 6y - 8 = 0.$
2. $(0, 0); 2x + 5y + 1 = 0, 3x - 7y - 8 = 0.$
3. $(0, 0); x/a + y/b = 1, x/c + y/d = 1.$
4. $(0, 0); y = mx + c, y = nx + d.$
5. $(2, 3); 4x + 3y = 11, 5y = 3x + 5.$
6. $(-2, -1); 2x - 5y = 5, 3x + 2y + 6 = 0.$

7. Find the equation of the line parallel to $x + 2y = 3$ through the point of intersection of $2x - 7y = -1, 5x + 9y = 3.$
8. Find the equation of the line perpendicular to $3x + 4y = 2$ through the point of intersection of $6x - 7y = 4, 5x - 4y = 2.$
9. Find the equation of the altitude of the triangle whose sides are $2x - y - 1, 3x + 2y - 2, 5x - 8y + 7,$ prove they are concurrent.
10. Prove that, if $k$ varies, the line whose equation is $x(k+2) + y(2k-3) + 11k + 1 = 0$
passes through a fixed point and find its coordinates.

[11] Repeat No. 10 for the line, $x(2k+1) + y(k+3) = 3.$

12. Prove by inspection that if $p, q$ vary the line, whose equation is $px + qy = 2p - 2q,$ passes through a fixed point. Find this point.

[13] Prove by inspection that if $6^\circ$ varies the line, whose equation is $x \cos \theta - y \sin \theta = 3 \sin \theta,$ passes through a fixed point. Find this point.

14. Prove that the lines, $x + 3 - 4y, 2x - 4 - 3y, 2x - 5y,$ are concurrent.

[Notice that if they are concurrent the line joining the origin to the meet of the first two lines must be the third line.]

15. Prove by inspection that the lines $8x + 2y = 1, 7x + 5y = 10, 4x + 3y = 9,$ are concurrent.

[16] Prove by inspection that the lines $7x - 10y = 0, 2x - 3y = 1, x - y = 2, 4x + 3y = 9,$ are concurrent.

17. Find the value of $b$ if the lines $7x + by = 12, x + 2y = 6$ intersect on the line $5x - 7y = 0.$

18. Prove that there is a value of $k$ such that the line $(2x + 3y - 5) + k(3x - 5y + 1) = 0$
is the same as the line $13x - 9y = 7.$ What follows?

[19] Prove that there is a value of $k$ such that the line $(7x - y + 4) + k(3x - y + 2) = 0$
is the same as the line $3y = x + 2.$

20. Interpret the equation $\frac{ax + by + c}{a_{1} + b_{1} + c_{1}} = \frac{lx + my + n}{l_{1} + m_{1} + n_{1}}$ with reference to the lines $ax + by + c = 0, lx + my + n = 0.$

21. Find the equation of the line which joins the meet of $x + 2y = 3, y - 3x + 4$ to the meet of $x + 2y = 1, 5y - 2x + 1.$

22. Interpret the locus whose equation is $y^{2} - 4ax(ly + my) = 0$ with reference to the parabola $y^{2} - 4ax = 0.$ Hence find the condition that $lx + my + l = 1$ touches $y^{2} - 4ax = 0.$

23. The variable line $lx + my + l = 1$ meets the curve $2x^{2} + 3y^{2} - 6z$ at points $P, Q.$ What is the equation of the pair of lines $OP, OQ$ where $O$ is the origin. If $OP$ is perpendicular to $OQ,$ prove that $PQ$ passes through a fixed point and find its coordinates.

24. A variable line meets the given curve $xy = ax - by = 0$ at $P$ and $Q.$ If $OP$ is perpendicular to $OQ,$ where $O$ is the origin, prove that $PQ$ is fixed in direction.

25. If the line $lx + my + n = 0$ meets the curve $ax^{2} + by^{2} = 1$ at points $P, Q,$ find the equation of the pair of lines $OP, OQ$ where $O$ is the origin. Hence find the condition that $lx + my + n = 0$ is a tangent to $ax^{2} + by^{2} = 1.$

26. If $P = (q + r)x + \frac{y}{pq}, Q = (t + s)x + \frac{y}{pt}, R = (p + q)x + \frac{y}{pq}$ prove that $y(R - Q) + q(R - P) + r(P - Q) = 0.$

What can now be said about the lines $P = c, Q = c, R = c ?$

27. Prove that the locus whose equation is $k(y^{2} - 4ax) + [2x - (m + n)y + 2amz][2x - (p + q)y + 2apz] = 0$
meets the parabola, $y = ax^{2}, y = 2at,$ at the points $M, N, P, Q$ given by $t = m, t = n, t = p, t = q.$ Find the conditions for the locus to be a circle.

28. Prove that the locus whose equation is $k(2y - c^{2})[x + mny - c(m + n)] + [x + qny - c(p + q)] = 0$
meets the curve $x = a, y = c$ at the points $M, N, P, Q$ given by $t = m, t = n, t = p, t = q.$ Find the conditions for the locus to be a circle.

29. A variable line meets the circle $x^{2} + y^{2} - 4x - 6y + 3 = 0$ at $P, Q, R$ is the foot of the perpendicular from the origin $O$ to $PQ.$ If $OP$ is perpendicular to $OQ,$ prove that the locus of $R$ is the circle $2x^{2} + 2y^{2} - 4x - 6y + 3 = 0.$ (N)

30. If $L_{1} = mL_{2} + m_{1}L_{3} + n_{1}, L_{2} = mL_{3} + m_{2}L_{3} + n_{2}, L_{3} = mL_{1} + m_{3}L_{2} + n_{3},$ and if $A$ is the mean of the lines, $L_{1} = 0, mL_{1} + mL_{2} = 0,$ and if $C$ is the mean of the lines $L_{2} = 0, mL_{1} + mL_{2} = 0,$ prove that constants $p, q, r$ can be found so that both $A$ and $C$ lie on $pL_{1} + qL_{2} + rL_{3} = 0.$ Hence find the equation of $AC.$
CHAPTER 7
THE PARABOLA

7.1. The circle and the parabola are the curves which the reader meets most frequently in elementary work.

The simplest definition of a parabola is the statement that, with reference to two selected perpendicular lines as coordinate axes, it is the locus of a point \( P(x, y) \) given by the parametric equations,

\[
x = at^2, \quad y = 2at,
\]

where \( a \) is constant. The equivalent \( x, y \) equation is

\[
y^2 = 4ax.
\]

7.1.1. Corresponding to each value of \( t \), there is just one value of \( x \) and one value of \( y \) and therefore just one point \( P(x, y) \) of the locus, called 'the point \( t \)' of the locus.

If \( a \) is a positive constant, \( x \) is positive for all values of \( t \), and \( y \) is positive or negative according as \( t \) is positive or negative; also corresponding to each positive value of \( x \), there are two numerically equal values of \( y \) of opposite signs. Hence the curve is situated wholly to the right of \( y'Ox \) and is symmetrical about \( x'Ox \), see Fig. 57.

The axis of symmetry of the parabola, here taken as \( x'Ox \), is called the axis of the parabola.

The point \( O \) where the axis of the parabola meets the curve, here taken as the origin, is called the vertex of the parabola.

7.1.2. The line \( x = 0 \) meets the parabola \( y^2 = 4ax \) where \( y^2 = 0 \); therefore the \( y \)-axis, \( x = 0 \), meets the parabola at two coincident points, coincident with \( O(0, 0) \), and so \( y'Oy \) is the tangent to the parabola \( y^2 = 4ax \) at its vertex \( O(0, 0) \).

7.1.3. If \( S \) is the point \( (a, 0) \) on the axis of the parabola \( y^2 = 4ax \) and if \( PN \) is the perpendicular from any point \( P \) on the parabola to the axis, see Fig. 57, then the statement that the equation of the locus of \( P(x, y) \) is \( y^2 = 4ax \) is equivalent to the geometrical property that, if \( O \) is the vertex of the parabola,

\[
PN^2 = 4OS \quad \text{ON}.
\]

7.2] FOCUS AND DIRECTRIX

7.1.4. All parabolas have the same shape. If \( b > a > 0 \), the parabola \( y^2 = 4bx \) is a magnification of the parabola \( y^2 = 4ax \) in the ratio \( b : a \). For example, the point \( (b, k) \) lies on \( y^2 = 8x \) if \( k^2 = 8b \), and this is precisely the condition that the point \( (3b, 3k) \) lies on \( y^2 = 24x \). Hence the parabola \( y^2 = 24x \) is an enlargement of the parabola \( y^2 = 8x \) in the ratio \( 3 : 1 \).

7.2. Focus and Directrix

The equation \( y^2 = 4ax \) can be interpreted in another way so as to give what is often regarded as the fundamental property of a parabola.

The equation \( y^2 = 4ax \) may be written

\[
y^2 = (x + a)^2 - (x - a)^2
\]

that is,

\[
(x - a)^2 + y^2 = (x + a)^2.
\]

To interpret the equation in this form, take the fixed point \( S(a, 0) \) and the fixed line \( DD' \), \( x + a = 0 \), and draw the perpendicular \( PM \) from \( P(x, y) \) to \( DD' \), see Fig. 57; then

\[
(x - a)^2 + y^2 = SP^2 \quad \text{and} \quad (x + a)^2 = PM^2,
\]

and so the equation means that

\[
SP^2 = PM^2.
\]

Hence a parabola is the locus of a variable point \( P \) which moves so that its distance \( SP \) from a fixed point \( S \) is equal to its distance \( PM \) from a fixed line \( DD' \).

The fixed point \( S(a, 0) \) is called the focus and the fixed line \( DD' \), \( x + a = 0 \), is called the directrix of the parabola \( y^2 = 4ax \). The property, \( SP = PM \), is called the focus-directrix property of the parabola.

If \( SO \) meets \( DD' \) at \( X \), \( SO = OX \) and \( XS = 2a \).

The chord \( LS' \) through the focus \( S \) perpendicular to the axis \( OS \) is called the latus rectum.

\[
\text{L.S'} = SL \text{ by symmetry about } x'Ox;
\]

\( SL \) is called the semi latus rectum and its length is denoted by \( l \).

If \( LK \) is drawn perpendicular to the directrix, then \( SL = LK \); but \( LK - 2X - 2a \), opposite sides of a rectangle,

\[
\therefore l = SL = 2a.
\]

Any chord through the focus \( S \) is called a focal chord.
Example 1. Interpret the locus whose equation is \( y^2 + 12x = 0 \).

The equation can be written \( y^2 = (x - 3)^2 - (x + 3)^2 \),

that is, \( (x + 3)^2 + y^2 = (x - 3)^2 \).

Hence if \( S \) is the point \((-3, 0)\) and if \( DD' \) is the line, \( x = -3 \), a point \( P(x, y) \) of the locus moves so that \( SP = PM \), where \( PM \) is the perpendicular from \( P \) to \( DD' \).

Thus the locus of \( P \) is the parabola whose focus is \( S(-3, 0) \) and whose directrix is the line \( DD', x = -3 = 0 \); the vertex of the parabola is the origin \( O \) and the axis \( OS \) points along \( O \rightarrow x' \), see Fig. 58.

**Fig. 58**

Example 2. Interpret the locus whose equation is \( y^2 - 3x - 2y - 5 = 0 \).

Complete the square, \( y^2 - 2y = y^2 - 2y + 1 = (y - 1)^2 \);

then the equation can be written \((y - 1)^2 - 1 - 3x - 5 = 0\),

that is, \((y - 1)^2 = 3(x + 2)\).

The locus can be interpreted more easily if the equation is simplified by choosing another point as origin.

Denote by \( A \) the point \((-2, 1)\) and draw \( AX, AY \) parallel to \( Ox, Oy \), see Fig. 59.

If the coordinates of any point \( P \) are \( x, y \) referred to the original axes \( Ox, Oy \) and are \( X, Y \) referred to the new origin \( A \) and the new parallel axes \( AX, AY \), then

\[ X = x + 2 \] and \[ Y = y - 1. \]

that is, \[ x = X - 2 \] and \[ y = Y + 1. \]

As a check, notice that \( x = -2, y = 1 \) for \( A \) and so \( X = 0, Y = 0 \) for \( A \).

Hence with this change of axes, the locus equation \((y - 1)^2 = 3(x + 2)\) becomes \[ y^2 = 3x. \]

This is a parabola whose vertex is given by \( X = 0, Y = 0 \), that is, by \( x = -2, y = 1 \), and this is the point \( A \).

Further the focus \( S \) is given by \( X = 3, Y = 0 \),

that is, by \( x = \frac{3}{2}, y = 0 + 1 = 1 \), see Fig. 59.

Similarly, the directrix (not shown in Fig. 59) is the line \( X = -\frac{3}{2}, \) that is, the line \( x = -2 \).

7.3. General Equation of a Parabola. It is important to remember that the equation of a parabola assumes the simple form, \( y^2 = 4ax \) or \( x = at^2, y = 2at \), only if the axes of reference are chosen in a special way. It may be impossible to do this, as for example in a problem which concerns two parabolas having different vertices. The equation of a parabola referred to any selected pair of perpendicular lines as axes can be found by using a geometrical property such as the focus-directrix relation or the property in 7.13, p. 88, \( PM^2 = 4OS \cdot ON \).

Example 3. Find the equation of a parabola whose focus is \( S(-1, 2) \) and whose directrix is \( D'D, x = -3 + y = 7 = 0 \).

Let \( P(x, y) \) be a point on the parabola. Draw the perpendicular \( PM \) from \( P \) to \( D'D \).

By the focus-directrix relation, \( SP = PM \).

\[
SP^2 = (x + 1)^2 + (y - 2)^2 \quad \text{and} \quad PM^2 = [(2x - 3y + 7)(2x + 3y + 7)]^2
\]

\[
SP^2 = (x + 1)^2 + (y - 2)^2 = (2x - 3y + 7)^2 / (2x - 3y + 7)^2 / 13,
\]

\[
13(x^2 + y^2 + 4x - 6y + 9) = 4x^2 - 12xy + 9y^2 + 28x - 42y + 49.
\]

\[
x^2 + y^2 + 4x + 12xy - 2x - 10y + 16 = 0.
\]

The equation of the parabola may be written

\[ (3x + 2y)^2 - 2x - 10y + 16 = 0. \]

7.3.1. The terms of second degree in the general equation of a parabola are a perfect square, as illustrated by Example 3.

If the focus is \( S(h, k) \) and the directrix is \( D'D, x = \cos \alpha + y \sin \alpha = p \), the method of Example 3 shows that the equation of the parabola is

\[ (x - h)^2 + (y - k)^2 = x \cos \alpha + y \sin \alpha = p^2 \]

This equation is of the form

\[ x^2 + y^2 - (x^2 \cos^2 \alpha + 2xy \sin \alpha \cos \alpha + y^2 \sin^2 \alpha) + 2gx + 2fy + c = 0, \]

that is, \( x^2(1 - \cos^2 \alpha) - 2xy \sin \alpha \cos \alpha + y^2(1 - \sin^2 \alpha) + 2gx + 2fy + c = 0 \),

that is, since \( 1 - \cos^2 \alpha = \sin^2 \alpha \) and \( 1 - \sin^2 \alpha = \cos^2 \alpha \),

\[ x \sin \alpha - y \cos \alpha = 2gx + 2fy + c = 0. \]

**EXERCISE 25**

Sketch the parabola whose focus and directrix are given and find its equation, Nos. 1-8:

1. \( (0, 0); x + 2y = 0 \).  
2. \( (0, 0); x - 2y = 0 \).  
3. \( (0, 2a); y = 0 \).  
4. \( (1, 3); x - y = 2 \).  
5. \( (-1, -1); x + y = 1 \).  
6. \( (-2, 3); 2x = 3y \).

Sketch the parabola and find its equation, Nos. 7-8:

7. \( \text{Vertex} (0, 0); \) focus \((0, 0)\).  
8. \( \text{Vertex} (0, k); \) focus \((0, k)\).
9. A is the point (3, -2) referred to Ox, Oy. Find referred to axes $AX, AY$ the coordinates of the points $(1, 4), (-5, 2), (b, k), (0, 0)$ referred to Ox, Oy and the new equations of the line $2x - y = 8$ and the circle $(x - 3)^2 + (y + 2)^2 = 1$.

10. A is the point $(-1, -4)$ referred to Ox, Oy. Find referred to axes $AX, AY$ the coordinates of the points $(2, 0), (-2, -1), (b, k), (0, 0)$ referred to Ox, Oy and the new equations of the line $3x - 2y = 5$ and the curve $y^2 - 5x + 9y + 11 = 0$.

11. Find the coordinates of A referred to Ox, Oy if the curve whose equation is $2y^2 - 3x + 12y + 21 = 0$ becomes referred to axes $AX, AY$ the parabola $y^2 = Ax$ and find the value of k.

Interpret the locus whose equation is given, Nos. 12-18:

12. $x^2 - y^2 = (y - 3)^2$.  
13. $x^2 + y^2 = (x + 5)^2$.  
14. $x^2 = 4by$.

15. $(x - 1)^2 + (y + 2)^2 = (x + 3)^2$.  
16. $(x + 2)^2 + (y - 1)^2 = (y - 3)^2$.

17. $(x + 1)^2 + (y - 3)^2 = 4(x - 2y + 4)^2$.

18. $(x + 1)^2 + (y - 1)^2 = 4(x + y + 1)^2$.

Find the coordinates of the focus and vertex and the equation of the directrix of the parabola whose equation is given and illustrate by a sketch, Nos. 19-30:

19. $y^2 = 6x$.  
20. $y^2 + x = 0$.  
21. $x^2 + 5y = 0$.

22. $y^2 = 8x + 4$.  
23. $x^2 = 12y + 8$.  
24. $x^2 = 6 - 2y$.

25. $y^2 - 2x - 4y - 2 = 0$.  
26. $y^2 - x - 6y + 10 = 0$.

27. $x^2 + 4y + 9 + y = 0$.  
28. $x^2 - 4x + 9 = 0$.

29. $(x - h)^2 = 4c^2(y - k)$.

30. $(y - k)^2 = 4c^2(x + h)$.

31. The vertex of a parabola is $(1, 2)$ and its axis is parallel to $3x - 4y$ and the length of its latus rectum is 2. Find the equations of: (i) the axis; (ii) the tangent at the vertex; (iii) the parabola.

32. $L, M, N$ are the points $(a, a), (-a, -a), (0, 0)$ respectively; $P$ divides $MN$ in the ratio $t : 1$ and $Q$ divides $LP$ in the ratio $t : 1$. Prove that, when $t$ varies, the locus of $Q$ is the parabola $(y + a)^2 = 4ax$, and find the coordinates of the focus. (N)

Use a geometrical argument to write down the value of $k$ if the given equation represents a parabola, Nos. 33, 34:

33. $x^2 + y^2 = k(3x + 4y - 2)^2$.

34. $(x - 4)^2 + (y + 3)^2 = k(5x - 2y + 1)^2$.

35. If $P(x, y)$ is any point in a plane, find a geometrical meaning of the function $y^2 - 4ax$ by reference to the point $8(a, 0)$ and the line $D'y = B$, $x + a = 0$. Show on a sketch the region in which $y^2 - 4ax$ is negative.

7.4. Equation of a Chord. If $P_1, P_2$ are the points $t_1, t_2$ of the parabola $x = at^2, y = 2at$, the equation of the chord $P_1P_2$ is $x = \frac{1}{2}(t_1 + t_2)y + at_1t_2 = 0$.

Gradient of $P_1P_2 = \frac{2at_1 - 2at_2}{at_1^2 - at_2^2} = \frac{-2}{t_1 + t_2}$, where $t_1 + t_2$.

The equation of the chord $P_1P_2$ is $2x - y(t_1 + t_2) = 2at_1 - 2at_2(t_1 + t_2) - 2at_1t_2$.

that is, $x = \frac{1}{2}y(t_1 + t_2) + at_1t_2 = 0$.

This result was proved on p. 80 by a method which suggests how the equation obtained by writing $t_1$ for $t_2$ should be interpreted.

7.4.1. Equation of a Tangent. The equation of the tangent to the parabola $x = at^2, y = 2at$ at the point $P_1, \, t = t_1$, is $x - t_1y + at_1^2 = 0$.

The line $lx + my + na = 0$ meets $x = at^2, y = 2at$ where $lx^2 + 2mnt + na = 0$.

Hence, as explained on pp. 74, 75, $lx + my + na = 0$ is the tangent at $P_1, \, t = t_1$, if the quadratic in $t$ is equivalent to 

$(-t_1)^2 - 2lt_1 + t_1^2 = 0$.

that is, $t : m = n = 1 ; \quad t_1 : t_2$.

The equation of the tangent at $P_1, \, t = t_1$, is $x - t_1y + at_1^2 = 0$.

7.4.2. The equation of the tangent to the parabola $y^2 = 4ax$ at the point $P_1(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.

If $x_1 = at_1^2$ and $y_1 = 2at_1$, the equation of the tangent at $P_1$ is $x - t_1y + at_1^2 = x - y(y_1/2a) + x_1 = 0$.

that is, $2ax - yy_1 + 2ax_1 = 0$.

7.4.3. The equation of the tangent at $(x_1, y_1)$ can be written down by the rule given for the circle on p. 57.

In the $(x, y)$ equation of the parabola, replace $x^2$ by $xx_1$ and $y^2$ by $yy_1$, replace $2x$ by $x + x_1$ and $2y$ by $y + y_1$.

Thus with $y^2 = 4ax$ we associate $yy_1 = 2a(x + x_1)$, and, interchanging $x$ and $y$, with $x^2 = 4ay$ we associate $xx_1 = 2b(y + y_1)$.

Similarly, the equation of the tangent at $(x_1, y_1)$ to the parabola $y^2 + 2x^2 + 2fy + c = 0$ is $yy_1 + f(x + x_1) + f(y + y_1) + c = 0$.

It is left to the reader to prove this by showing that the result of eliminating $x$ and $x_1$ from (i), (ii) and the relation $y_1^2 + 2xx_1 + 2fy_1 + c = 0$ is $y^2 + 2yy_1 + y_1^2 = (y - y_1)^2 = 0$;

hence the line (ii) meets the parabola (i) at two coincident points.
7.4.4. The equation of the tangent to $y^2 = 4ax$ having its gradient equal to $m$ is

$$y = mx + a/m.$$

The equation of the tangent at $(at^2, 2at)$ is $x - ty + at^2 = 0$; put $t = 1/m$, then the equation of the tangent at $(a/m^2, 2a/m)$ is $x - y/m + a/m^2 = 0$, that is, $mx - y + a/m = 0$.

Example 4. Prove that the angle $\theta$ between the tangents from $T(h, k)$ to $y^2 = 4ax$ is given by

$$\tan \theta = \pm \left(\sqrt{k^2 - 4ah}/(h + a)\right).$$

Let $m_1, m_2$ be the gradients of the tangents from $T$, then

$$\tan \theta = (m_1 - m_2)/(1 + m_1 m_2).$$

The tangent, $y = mx + a/m$, passes through $T(h, k)$ if

$$k = mh - a/m;$$

$m_1, m_2$ are the roots of the quadratic, $m^2 - m(h + a) = 0$;

$m_1 + m_2 = h + a$ and $m_1 m_2 = a/h$;

$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2 = h^2 - 4ah; h = (k^2 - 4ah)/h^2$;

$m_1 - m_2 = \pm \sqrt{(k^2 - 4ah)/h^2};$ also $1 + m_1 m_2 - 1 + a/h = (h + a)/h$;

$$\tan \theta = \pm \left(\sqrt{k^2 - 4ah}/(h + a)\right).$$

7.5. If $PT$ is the tangent to a curve at the point $P$, the line $PG$ at right angles to $PT$ is called the normal at $P$ to the curve and $P$ is called the foot of the normal from $G$.

7.5.1. Equation of a Normal. The equation of the normal to the parabola $x = at^2, y = 2at$ at the point $P_t$, $t = t_1$, is

$$t_1 x + y = at_1^2 + 2at_1.$$

The equation of the tangent at $P_t$ is $x - t_1y + at_1^2 = 0$.

.: the equation of the normal at $P_t$ is

$$t_1 x + y = \ldots = t_1(at_1^2 + 2at_1) - at_1^2 + 2at_1.$$

7.5.2. The equation of the normal to the parabola $y^2 = 4ax$ at the point $P_2(x_1, y_1)$ is $y = x + 2ay + y_1(x_1 + 2a)$.

The equation of the tangent at $P_1(x_1, y_1)$ is $yy_1 = 2ax + 2ax_1$.

.: the equation of the normal at $P_1(x_1, y_1)$ is

$$2ay + y = \ldots = 2ay_1 + y_1 x_1.$$

Example 5. If the normal at $P$, $t = p$, to $x = at^2, y = 2at$ meets the curve again at $Q$, $t = q$, find $q$ in terms of $p$.

The normal $PQ$ at $t = p$, $y = p + ap^2 + 2ap$, meets $x = at^2, y = 2at$ where

$$pa^2 + 2at = ap^2 + 2ap;$$

the roots of this quadratic in $t$ are $t = p, t = q$;

$p + q = -2a/(pa), \quad q = -p/p - 1(p^2 + 1)/p$.

Check: The quadratic in $t$ can be solved by factors:

$$pa^2 + 2at - (ap^2 + 2ap) = a(-p)(pt + p^2 + 2) = 0,$$

.: $t = p$ or $t = -(p^2 + 2)/p$.

7.5. Tangent and Normal

EXERCISE 26

Find the values of $t$ which determine the points of intersection of the parabola $x = at^2, y = 2at$ with the given lines, Nos. 1-6:

1. $2x - y = 12a$. \hspace{1cm} (2) $x + 2y - 5a$. \hspace{1cm} (3) $2y = x + 9a$.

7. $x - 3y + 16a$. \hspace{1cm} (5) $my = mx + a$. \hspace{1cm} (6) $mx - y = am^2 + 2am$.

7. Find the equation of the chord $PQ$ of the parabola, $x = at^2, y = 2at$ if $P, Q$ are given by $t = 3, t = -2$; \hspace{1cm} (ii) $t = 3, t = -1/p$.

8. Find the equations of the tangent and normal to the parabola $x = at^2, y = 2at$ at the points $(i) t = -1$, \hspace{1cm} (ii) $t = 0$; \hspace{1cm} (iii) $t = 1$.

9. The line joining $P, t = p$, on $x = at^2, y = 2at$ to a point $C$ meets the parabola again at $Q$, $t = q$. \hspace{1cm} Find $q$ if $C$ is: $i) (c, 0)$; \hspace{1cm} (ii) $(h, k)$.

10 A parabola is given by $x = p^2, y = 4t$. \hspace{1cm} Find the equations of (i) the chord $t_1, t_2$; \hspace{1cm} (ii) the tangent and normal at the point $t_1$; \hspace{1cm} (iii) the tangent which is parallel to the normal at the point $t = 1$.

11. Find the point on $x = at^2, y = 2at$ at which the normal is parallel to the tangent at the point $t_1$.

12. If $P_1(x_1, y_1), P_2(x_2, y_2)$ are points on $y^2 = 4ax$, prove that the equation of $P_1P_2$ is $4ax - y(y + y_2) + y_2y_2 = 0$.

13. Prove that the gradient of the chord joining $(x_1, y_1), (x_2, y_2)$ on $x^2 - 4ky = \frac{1}{4}(x_1 + x_2)^2/k$.

14. Find $c$ if $y = x + c$ touches $y^2 = x$ and find the equation of the normal at the point of contact.

15. Find the equations of the tangent and of the normal to $y^2 = 12x$ which are parallel to $x + y = 0$.

16. Find the equations of the tangent and of the normal to $y^2 = 12x$ which are parallel to $2x + 3y = 0$.

17. Find the equation of the normal to $y^2 = 8x$ at the point $(2, 4)$ and the coordinates of the second point where it meets the curve.

18. The parabola $y^2 = 4x, x^2 = 4y$ meet at the origin $O$ and at $P$. \hspace{1cm} Find the coordinates of the second point where the tangent $PT$ at $P$ to $y^2 = 4x$ meets $x^2 = 4y$ and the coordinates of the second point where the normal $PG$ at $P$ to $x^2 = 4y$ meets $y = 4x$.

19. A variable normal to $y^2 = 4ax$ meets $Ox, Oy$ in $G, H$. \hspace{1cm} Prove that the equation of the locus of the mid-point of $OH$ is $xy = 2x(x^2 - a)$.

20. The angle between the tangents from a variable point $T$ to $y^2 = 4ax$ is $60^\circ$. \hspace{1cm} Prove that the equation of the locus of $T$ is $y^2 - 4ax = 3(x + a)^2$.

21. Prove that $x^2 + y^2 = a^2$ intersects $y^2 = 2ax + a^2$ at an angle $45^\circ$. \hspace{1cm} (N)

22. The normal to $y^2 = 4ax$ at $P(at^2, 2at)$ meets the curve again at $Q$. \hspace{1cm} Prove that $PQ^2 = 16a^3(t^2 + 1)/a^2$.

23. The tangent at a variable point $P$ on $y^2 = 4ax$ meets $Oy$ at $Q$; the normal at $P$ meets $Ox$ at $R$. \hspace{1cm} Prove that the equation of the locus of the mid-point of $QR$ is $2y^2 = a(x - a)$. (L)
24. $PQ$ is a chord of $y^2 = 4ax$ parallel to $x + my = 0$. $PHQK$ is a rectangle with sides parallel to $Ox, Oy$. Prove that $H$ and $K$ lie on $(y + 4am)^2 = 4ax.$ If this parabola meets $y^2 = 4ax$ at $R$, prove the tangent at $R$ to $y^2 = 4ax$ is parallel to $PQ$. (O)

[25] $y^2 = 4ax$ meets $x^2 = 4y$ at the origin $O$ and $P$; the tangent at $P$ to $y^2 = 4ax$ meets $x^2 = 4y$ at $Q$; the tangent at $P$ to $x^2 = 4y$ meets $y^2 = 4ax$ at $R$. Prove that $QR$ is a common tangent.

26. Prove that the equations of the tangents common to $y^2 = 4ax$ and $x^2 + y^2 - 14ax + 24a^2 = 0$ are $x + 2y = 8a - 3a = 0, x = 2y - 3a = 0$. (L)

27. Prove that the parabola, $x = 4at^2, y = 4at$ meets $y^2 - 4ax + 16a^2 + 48 = 0$ at the points $P, t = 3$, and at three coincident points $Q$. Prove that $PQ$ is a diameter of the circle and is the normal at $Q$ to the parabola. (OC)

28. If the normal to $y^2 = 4ax$ at $P(at^2, 2at)$ meets $OQ$ at $G$, prove that the equation of the circle $OPG$, where $O$ is the origin, is $x^2 + y^2 - ax(y^2 + 2) = 0$.

If $p^2 > 4a$, prove this circle meets $y^2 = 4ax$ again at points $Q, R$ such that $PQ$, $PR$ are the normals at $Q, R$.

7.6. Geometrical and Coordinate Methods. Many properties of the parabola can be proved easily by the methods of pure geometry. A coordinate method should be used if, and only if, it gives a simpler solution than a geometrical method. In some cases the best procedure is to combine geometry with algebra in a single proof.

7.6.1. Standard Notation. It saves time to denote some of the special points and lines associated with the parabola by particular letters, unless otherwise stated.

The notation is illustrated by Fig. 60:

$O$ is the vertex. $S$ is the focus. The directrix $DV$ meets $OS$ at $T$.

Tangent $TP$ at $P$ meets $OS, DV$ at $T, R$.

Normal $PG$ meets $OS$ at $G$.

$PN, PM$ are the perpendiculars to $OS, DV$.

$L$ is the latus rectum; $l = 2a$.

$F$ is the foot of the perpendicular from $S$ to $PT$.

Fig. 60

Tangent $TP$ at $P$ meets $OS, DV$ at $T, R$.

Normal $PG$ meets $OS$ at $G$.

$PN, PM$ are the perpendiculars to $OS, DV$.

$L$ is the latus rectum; $l = 2a$.

$F$ is the foot of the perpendicular from $S$ to $PT$.

7.6.2. Prove that with the standard notation, see Fig. 60:

(i) $TO = ON$; (ii) $ST = SP$; (iii) $SPMT$ is a rhombus.

(i) Take axes so that the parabola is $y^2 = 4ax$; let $P$ be $(x_1, y_1)$. Tangent $PT, y_1 = 2a(x - x_1)$, meets $OS, y = 0$, at $T(-x_1, 0)$; but $N$ is $(x_1, 0)$, $: : TO = ON$.

(ii) $SO = OX$, $: : ST = SO + OT = OX + NO = NX$.

Since $NXMP$ is a rectangle, $NX = PM$, $: : ST = PM$; $: :$ by the focus-directrix relation, $ST = SP$.

(iii) $ST$ is equal and parallel to $PM$, $: : SPMT$ is a parallelogram; but $ST = SP$, $: :$ the parallelogram is a rhombus.

7.6.3. Prove that with the standard notation, see Fig. 60:

(i) $PT$ bisects $\angle SPM$ and $SM$ bisects $PT$ at right angles;

(ii) $TY = YP$ and $SY$ bisects $\angle TSP$.

(i) Since $SPMT$ is a rhombus, the diagonals $SM, PT$ bisect each other at right angles and bisect $\angle TSP$ and $\angle SPM$.

(ii) Since $SM, PT$ is perpendicular to $PT, SM$ meets $PT$ at the foot $Y$ of the perpendicular from $S$ to $PT$; $: : TY = YP$ and $\angle TSY = \angle PST$.

7.6.4. Prove that with the standard notation, see Fig. 60:

(i) $Y$ lies on the tangent at $O$; (ii) $SY = SO + SP$.

(i) Since $TO = ON$ and $TY = YP$, $OY$ is parallel to $NP$; $: : OY$ is perpendicular to $XOS$; $: : OY$ is the tangent at $O$.

(ii) Since $\angle OSY = \angle YSP$ and $\angle YOS = \angle SYP$, $\triangle OSY$ is similar to $\triangle SYP$; $: : SO : SY = SY : SP$.

7.6.5. Prove that with the standard notation, see Fig. 60, $\angle RSP$ is a right angle and $RP$ bisects $\angle SRM$.

Since $SP = MP$ and $\angle SPR = \angle MPR$, $\angle RSP = \angle RMP$; $: : \angle RSP = \angle RMP = 1\text{rt.} \angle$ and $\angle RSP = \angle RMP$.

7.6.6. Prove that with the standard notation, see Fig. 60:

(i) $SO = SP$; (ii) $NG = 2OS$.

(i) $\angle TPG = 1\text{rt.} \angle$ and $ST = SP$, $: : SG = ST = SP$.

(ii) $\angle PNG$ is similar to $\triangle YOS$, since corresponding sides are parallel, $: : NG = NP$, $NT = 2$, $: : NG = 2OS$.

7.6.7. Prove that, if $PSQ$ is a focal chord, then the tangents at $P$ and $Q$ meet at right angles on the directrix.

Let the tangents at $P, Q$ meet the directrix at $R, R'$, then $SR$ and $SR'$ are each perpendicular to the chord $PSQ$; $: : R'$ is the same point as $R$.

Hence also $RP, RQ$ are the bisectors of $\angle SRD, \angle SDR'$; $: : RP$ is perpendicular to $RQ$. 

\[\text{\}7.6\]
7.6.8. As a comparison of coordinate methods with geometrical methods, we give an alternative proof of the property in 7.6.7:
The tangents at the extremities $P, Q$ of a focal chord of a parabola meet at right angles on the directrix.

Let $P, Q$ be the points $t = p, t = q$ on the parabola, $x = at^2, y = 2at$.
The chord $PQ$, $2x - y(p + q) + 2apq = 0$ passes through $S(a, 0);
\therefore 2a + 2apq = 0; \therefore pq = -1$.

Since the equations of the tangents at $P, Q$ are

$$x - py + ap^2 = 0, \quad x - qy +aq^2 = 0,$$

their gradients are equal to $1/p, 1/q$.

but $pq = -1, \therefore$ the tangents at $P, Q$ are at right angles.

The tangents at $P, Q$ meet at the point, $x = ap, y = a(p + q); but pq = -1, \therefore$ the tangents meet at a point on the line $x = -a$.

7.6.9. Prove that the circumcircle of the triangle formed by three tangents to a parabola passes through the focus $S$.

By 7.6.4, the foot of the perpendicular from $S$ to each tangent lies on the tangent at the vertex $O$, and so the feet of the perpendiculars from $S$ to the three tangents are collinear; therefore, by the converse of the Simson-line theorem, $S$ lies on the circumcircle of the triangle.

Example 6. $K$ is a fixed point and $P, Q$ are variable points given by $t = k, t = p, t = q$ on the parabola $x = at^2, y = 2at$. If $\angle PKQ$ is a right angle, prove that $PQ$ passes through a fixed point.

The gradients of $KP, KQ$ are equal to $2/(k+p), 2/(k+q)$.

Since $KP$ is perpendicular to $KQ, the product of their gradients equals $-1;
\therefore (k+p)(k+q) = -4; \therefore (k^2 + 4) + k(p + q) + pq = 0$.

Comparison of this relation with $2x - y(p + q) + 2apq = 0$, the equation of $PQ$, shows that $PQ$ passes through the point given by

$$2a(k^2 + 4) = -y = k = 2a;
\therefore PQ$ passes through the fixed point $a(k^2 + 4), -2ak$.

EXERCISE 27

[The standard notation described on p. 96 is used in this exercise.]

1. If $P$ is $(at^2, 2at)$ on $y^2 = 4ax$, prove $SP = a(1 + t^2)$.

2. Prove $LX$ is the tangent at $L$.

3. Prove that: (i) $TX = SN; (ii) PG = 2ST$.

4. If the line through $P$ perpendicular to $OP$ cuts $OS$ at $K$, prove that $NK = 4OS$.

5. Prove that the circle on $SP$ as diameter touches $OY$.

7.6] $E_1, E_2$ are given points on $OS$ equivalent from $S, E_1Z_1, E_2Z_2$ are the perpendiculars from $E_1, E_2$ to a variable tangent. Prove that $E_1Z_1^2 - E_2Z_2^2$ is constant.

7. If $SO = 3a$, prove the perpendicular from $S$ to $PG$ equals $\sqrt{6}$. (O)

8. If $PN, PO$, produced each way, cuts the tangents at $L$ and $L'$ in $K$ and $K'$, prove that $PK : PK' = SN^2$.

9. $P, Q$ are points on $y^2 = 4ax$ such that $PQ$ passes through $(4a, 0)$. Prove that $OP$ is perpendicular to $OQ$.

10. $K$ is a point on the directrix; $OK$ meets the parabola again at $P$. Prove that the tangent at $P$ is parallel to $SK$.

11. The tangents at $P, Q$ on $y^2 = 4ax$ intersect at right angles at $T$; the normals at $P, Q$ meet at $K$. Prove $TK$ is parallel to Oz. \( N \)

12. The tangents from a point $U$ to $y^2 = 4ax$ meet the tangent at $O$ in $H, K$; $UF$ is the perpendicular from $U$ to $Ox$. Prove that $OH, OK = a, OF$.

13. A chord $P_1P_2$ meets the axis of a parabola at $K$; the tangents at $P_1$ and $P_2$ meet the axis at $T_1$ and $T_2$. Prove $OT_1, OT_2 = OK^2$.

14. A chord $P, P_2$, when produced, passes through $X; P_1N_1, P_2N_2$ are the ordinates of $P$, $P_2$. Prove:

(i) $ON_1, ON_2 = OS^2$; (ii) $P_1N_1, P_2N_2 = XS^2$.

15. $Y$ is the mid-point of a chord of a parabola which makes $45^\circ$ with the axis. Prove that $VL$ or $V'L'$ is parallel to the axis.

16. The chord $PQ$ passes through $S$ and the chord $PO$ is produced to meet the directrix at $K$. Prove $QK$ is parallel to the axis.

17. $P, Q$ are the points $t = p, t = q$ on $x = at^2, y = 2at$; $QO$ produced meets $PT$ at right angles at $K$. (i) Prove that $pq + 2 = 0$. (ii) Prove that $QO : OK = SL^2$.

18. The tangents at $P$ and $Q$ to $y^2 = 4ax$ meet at $U$; $PO, PQ, UO$, when produced, meet the directrix at $P', Q', U'$. Prove that $U'$ is the mid-point of $PQ$.

19. The normal at $P$ on $x = at^2, y = 2at$ meets $Ox$ at $G$. Prove that $2a < PG^2/OG < 4a$. (L)

20. $P, Q$ are the points $t = p, t = q$ on $x = at^2, y = 2at$. The tangents at $P, Q$ meet the lines through $Q, P$ parallel to $Ox$ at $H, K$ respectively. Prove: (i) $PK = QH$; (ii) area $PQHK = 2a(p + q)^2$. (L)

21. $PC$ produced meets $y^2 = 4ax$ again at $Q$; $H$ is the point on $PQ$ such that $O$ is the mid-point of $PH$. $PN, HK, QN'$ are the perpendiculars to $Ox$. Prove $ON', KN' = 4a^2$. (C)

22. If $O, S, P, R$ lie on a circle, prove $\triangle POR = \frac{1}{2} \triangle PSR$.

23. $PSQ$ is a focal chord; the line through $Q$ perpendicular to $PQ$ meets the tangent at $P$ in $K$. Prove the directrix bisects $QK$. (N)
7.7. Chord of Contact. If $P$ and $Q$ are the points of contact of the tangents from a point $U$ to a parabola, $PQ$ is called the chord of contact of the tangents from $U$. The line $PQ$ is also called the polar of $U$ with respect to the parabola, and $U$ is called the pole of the line $PQ$ with respect to the parabola.

The terms, polar and pole, are used in the same sense for a circle or ellipse or hyperbola.

7.7.1. If $PQ$ is the chord of contact of the tangents from a point $U(x', y')$ to $y^2 = 4ax$, the equation of $PQ$ is

$$yy' = 2a(x + x').$$

First Method. The equation of the tangent at $(at^2, 2at)$ is

$$x - ty + at^2 = 0;$$

this line passes through $U(x', y')$ if $x' - ty' + at^2 = 0$.

Therefore by hypothesis the quadratic

$$at^2 - ty' + x' = 0$$

has two roots $t = p, t = q$, which determine $P, Q$, see Fig. 61, and is equivalent to

$$(t - p)(t - q) = t^2 - (p + q) + pq = 0;$$

$$: t = p, t = q; y : x ;$$

the equation of $PQ$, $2x - (p + q)y + 2apq = 0$, is equivalent to

$$2ax - yy' + 2ax' = 0.$$  

Note. The quadratic, $at^2 - ty' + x' = 0$, has no roots if $y'^2 - 4ax' < 0$; this is the condition that $U(x', y')$ lies inside the parabola $y^2 = 4ax = 0$, in which case no tangents can be drawn from $U$ to $y^2 = 4ax$.

Second Method. Let $P, Q$ be the points $(x_1, y_1), (x_2, y_2)$. $U(x', y')$ lies on the tangent at $P, y = 2a(x + x_1)$.

$$y' = 2a(x + x_1).$$

This numerical relation is precisely the condition that $P(x_1, y_1)$ lies on the line,

$$y - 2a(x + x_1).$$

In exactly the same way, it can be proved that $Q(x_2, y_2)$ also lies on the line,

$$y' = 2a(x + x_2);$$

therefore this is the equation of the line $PQ$.

7.7.2. It is easy to remember the equation of the polar of a given point because it has the same form as the equation of a tangent. Similarly, using the second method in 7.7.1 and the formula for the equation of a tangent in 7.4.3, p. 93, it follows that the polar of $(x', y')$ with respect to

$$y^2 = 22$$

is

$$yy' + 2x + 2yy' + c = 0.$$  

7.7.3. The pole of a given line $PQ, lx + my + na = 0$, with respect to the parabola $y^2 = 4ax$ is the point $U \left( \frac{na}{l}, -\frac{2ma}{l} \right)$.

Let $U$ be the point $(x', y')$; then the equation of the polar $PQ$ of $U$ is

$$yy' = 2a(x + x');$$

the line, $2ax - yy' + 2ax' = 0$, is equivalent to

$$2a - \frac{y}{m} = 2ax'; \quad \therefore \quad x' = \frac{na}{l}, y' = -\frac{2ma}{l}.$$  

Example 7. $P, Q, R$ are points on $x = at^2, y = 2at$ given by $t = p, t = q, t = r$. $PQ, PR, QR$ are the polars of $QR, RP, PQ$. Prove that the orthocentre of $\triangle PQR$ lies on the directrix.

The equation of $QR$ is

$$2x - y(q + r) + 2arq = 0;$$

as in 7.7.3, the pole $P'$ of $QR$ is

$$P'(a, a(q + r)).$$

$QR'$ is the tangent at $P$,

the equation of $QR' = x - y + 2pr = 0$; the equation of the perpendicular from $P'$ to $QR'$ is

$$y = -2p + q + a(q + r).$$

This line meets the directrix, $x = -a$, where $y = opr + a(q + r)$; the symmetry of this expression for the $y$-coordinate in terms of $p, q, r$ shows that the perpendiculars from $Q'$ to $RP'$ and from $R'$ to $FP'$ meet the directrix at the same point. Thus the orthocentre of $\triangle P'Q'R'$ is the point $(-a, opr + a(q + r))$ on the directrix.

Exercise 28

1. Find the polar of $(-4, 3)$ with respect to $y^2 = 12x$.

2. Find the pole of $2x - y - 33 = 0$ with respect to $y^2 = 12x$.

3. Find the points of contact of the tangents to $y^2 = 8x$ from $(-2, 3)$.

4. Find the equation of the tangents to $y^2 = 6x$ from $(4, 7)$.

5. Prove the polar of $(-a, k)$ with respect to $y^2 = 4ax$ passes through $S$.

6. Prove the pole of $x + my = a$ with respect to $y^2 = 4ax$ lies on $DD'$.

7. If with the data of Example 7, see Fig. 62, $R'$ is the mid-point of $P'O$, prove: (i) $2p = q + r$; (ii) $QR' = P'Q$ parallel to $QR$.

8. A variable chord $PQ$ of $y^2 = 4ax$ passes through the fixed point $(c, 0)$; $T$ is the pole of $PQ$. Prove: (i) $T$ lies on $x + c = 0$; (ii) the centroid of $\triangle PQT$ lies on $y^2 = a(3x - c)$. (L)

9. $U$ is the pole of the chord $PQ$ of $y^2 = 4ax$; $PH, QK$ are the perpendiculars to the directrix. Prove $UH^2 = UK^2 = US^2 = SP \cdot SQ$.  


10. Find the poles of \(az + by + c = 0\) with respect to each of the parabolas, \(y^2 = 4ax\), \(x^2 = 4by\). Hence prove that this line is a common tangent of the parabolas.

11. The tangents to a parabola at \(P_1, P_2\) meet at \(U\) and cut the axis at \(T_1, T_2\). Prove the line through \(U\) parallel to \(P_1P_2\) bisects \(T_1T_2\).

[12] \(P, Q\) are variable points on \(y^2 = 4ax\) such that \(\angle POQ\) is a right angle. Prove that the pole of \(PQ\) lies on a fixed line.

13. Prove that the locus of the pole with respect to \(y^2 = 4ax\) of a variable tangent to \(y^2 = 4ax\) is the parabola, \(ax^2 = 4by\).

14. The normal at the variable point \(P(a^2, 2at)\) on \(y^2 = 4ax\) meets the parabola again at \(Q\). Prove that the equation of the locus of the pole of \(PQ\) is \(xy + 2at(y^2 + 2ax) = 0\). (N)

[15] A variable chord \(PQ\) of \(y^2 = 4ax\) passes through the fixed point \((2a, b)\). Prove the pole of \(PQ\) lies on the line \(2ax - by + 4a^2 = 0\).

16. Prove that the perpendicular from \((-4a, 3a)\) to its polar with respect to \(y^2 = 4ax\) does not meet \(y^2 = 4ax\). (OC)

17. The chords \(PQ, PQ'\) are the polars of \(U, U'\) with respect to \(y^2 = 4ax\). If \(U'\) lies on \(PQ\) produced, prove \(U\) lies on \(PQ'\) produced.

[18] \(P_1P_2, P_3P_4\) are the ordinates of variable points \(P_1, P_2\) on \(y^2 = 4ax\) such that \(ON_1, ON_2\) is constant. Prove that the locus of the pole of \(P_1P_2\) is two straight lines.

19. \(P, Q\) are variable points \(t=p, t=q\) on \(x=at^2, y=2at\); \(T\) is the pole of \(PQ\). Prove area \(\Delta TPO=\frac{1}{2}a^2(p-q)^2\). If \(V\) is the mid-point of \(PQ\) and if area \(\Delta TPO\) is constant, prove that the length of \(TV\) is constant. (N)

20. The tangents at variable points \(P_1, P_2\) on a parabola meet at \(U\) and cut the tangent at the vertex at \(Y_1, Y_2\). If \(Y_1Y_2\) is of constant length, prove that the locus of \(U\) is an equal parabola.

[21] A chord \(PQ\) of \(y^2 = 4ax\) passes through \((-a, 0)\); the tangents at \(P, Q\) meet at \(U\); the normals at \(P, Q\) meet at \(K\). Prove that \(UK\) is bisected by the axis.

22. \(P_1, P_2\) are the points \(t=p_1, t=p_2\) on \(x=at^2, y=2at\); the chords \(P_1Q_1, P_2Q_2\) intersect at \((c, 0)\). Find the values of the parameters of \(Q_1, Q_2\). If the equation of \(P_1P_2\) is \(ax + by + az = 0\), prove the equation of \(Q_1Q_2\) is \(ax - cby + az = 0\). If the pole of \(P_1P_2\) is \((h, k)\), prove that the pole of \(Q_1Q_2\) is \((c^2/h, -ck)/h)\). If \(c\) is constant and \(P_1, P_2\) vary, prove that \(P_1P_2\) meets \(Q_1Q_2\) on a fixed line perpendicular to \(OX\).

23. The perpendicular from a variable point \(U\) to its polar with respect to \(y^2 = 4ax\) meets the pole at \(H\) and the axis at \(K\). If \(UK = 3UH\), prove the locus of \(U\) is a parabola and find its vertex and latus rectum.

7.8. If \(PQ\) is a variable chord of a parabola parallel to the tangent \(KZ\) at a fixed point \(K\), the locus of the mid-point \(V\) of \(PQ\) is the line through \(K\) parallel to the axis of the parabola.

Choose axes so that the equation of the parabola is \(x=at^2, y=2at\). Let the points \(K, P, Q\) be given by \(t=1, t=\alpha, t=\beta\).

Then \(PQ\).

\[
x = y(p + q) + 2apq = 0,
\]

is parallel to \(KZ\),

\[
-x - ky + ak^2 = 0;
\]

\[
\therefore \quad p + q = 2k.
\]

The mid-point \(V\) of \(PQ\) is

\[
\left\{ \frac{1}{2}(ap^2 + aq^2), \frac{1}{2}(2ap + 2aq) \right\},
\]

\[
\therefore \quad y\text{-coordinate of } V = a(p + q) = 2ak = y\text{-coordinate of } K;
\]

\[
\therefore \quad KV\text{ is the line } y = 2ak\text{ parallel to the axis.}
\]

7.8.1. The line which bisects each chord parallel to the tangent at a given point \(K\) is called a diameter of the parabola, and \(K\) is called the vertex of the diameter.

The gradient of the tangent \(KZ\) at \(K(at^2, 2at)\) is \(1/k\); therefore the equation of the diameter which bisects chords making an angle \(0^\circ\) with the axis is

\[
y = 2ak\cot t^\circ.
\]

7.8.2. If the mid-point \(V\) of a chord \(PQ\) of the parabola \(y^2 = 4ax\) is \((x, y)\), then the equation of \(PQ\) is

\[
yy_1 - 2ax = (y_1^2 - 2ax_1) = 0.
\]

If the line through \(V\) parallel to the axis meets \(y = 4ax\) at \(K\), the chord \(PQ\) is parallel to the tangent at \(K\).

Since the \(y\) coordinate of \(K = y\text{-coordinate of } V = y_1\), the tangent at \(K\) is parallel to \(yy_1 = 2ax_1\),

\[
\therefore \text{ the equation of } PQ\text{ is } yy_1 - 2ax = \ldots = y^2 - 2ax_1.
\]

7.8.3. If \(K\) is the vertex of the diameter which bisects the chord \(PQ\) at \(V\) and if \(U\) is the pole of \(PQ\), then \(V\) is the mid-point of \(UV\).

Choose axes so that the equation of the parabola is \(y^2 = 4ax\).

Let \(V\) be the point \((x, y_1)\) and let \(U\) be the point \((x, y_2)\).

Since \(KV\) is parallel to the axis, the \(y\) coordinate of \(K\) equals \(y_1\); also \(K\) is a point on \(y^2 = 4ax\). \(K\) is the point \((y_1^2/4a, y_1)\).

By 7.8.2, the equation of \(PQ\) is \(yy_1 - 2ax = (y_1^2 - 2ax_1) = 0\).

Since \(PQ\) is the polar of \(U(x_2, y_2)\), this equation is equivalent to

\[
yy_2 - 2ax = 2ax_1 = 0.
\]

\[
\therefore \quad y_2 = y_1 \quad \text{and } 2ax_1 = y_1^2 - 2ax_1;
\]

\[
U(x_2, y_2) \text{ is the point } (y_1^2/2a - x_1, y_1); \quad \text{also } V \text{ is } (x_1, y_1);
\]

\[
\therefore \text{ the mid-point of } UV \text{ is } K(y_1^2/4a, y_1).
7.8. A line drawn through the point \( V(x_1, y_1) \) in the direction making an angle \( 6^\circ \) with \( Ox \) meets the parabola \( y = 4ax \) in \( P, Q \).

(i) Prove that \( VP \cdot VQ = (y_1^2 - 4ax_1) \cot^2 6^\circ \).

(ii) If \( V \) is the mid-point of \( PQ \) and if the line through \( V \) parallel to the axis, meets the parabola, focus \( S \), at \( K \) and meets the directrix \( DD \) at \( R \), prove that \( PV^2 = 4RK \cdot KV = 4SK \cdot KY \).

(iii) By 2.9.2, p. 17, if a step of directed length \( r \) is taken from \( V(x_1, y_1) \) in direction making angle \( 6^\circ \) with \( Ox \), the point of arrival is

\[
(x_1 + r \cos 6^\circ, y_1 + r \sin 6^\circ)
\]

and this point lies on \( y = 4ax \) if

\[
(y_1 + r \sin 6^\circ)^2 = 4a(x_1 + r \cos 6^\circ),
\]

that is,

\[
r^2 \sin^2 6^\circ = 2r(y_1 \sin 6^\circ - 2a \cos 6^\circ) + y_1^2 - 4ax_1 = 0.
\]

The roots of this quadratic in \( r \) are the directed lengths \( V \to P, V \to Q \), here denoted by \( VP, VQ \);

\[
\therefore VP \cdot VQ = - (y_1^2 - 4ax_1) \cot^2 6^\circ.
\]

(ii) If \( V \) is the mid-point of \( PQ \), \( VQ = - VP \), then \( VP + VQ = 0 \);

\[
\therefore \text{the sum of the roots of the quadratic is zero;}
\]

\[
y_1 \sin 6^\circ - 2a \cos 6^\circ = 0 \Rightarrow \cot 6^\circ = y_1 / (2a)\]

also \( PV^2 = VP \cdot VQ = 4(ax_1 - y_1^2) \cot^2 6^\circ \).

\[
V, K, R \text{ are the points } (x_1, y_1), (y_1^2 / (4a), y_1), (-a, y_1);
\]

\[
KV = x_1 - y_1^2 / (4a) = 4(ax_1 - y_1^2) / (4a)
\]

and

\[
RK = y_1^2 / (4a) - (-a) = (4a \cot^2 6^\circ + 4a^2) / (4a) = a \cot^2 6^\circ;
\]

\[
PV^2 = 4RK \cdot KV = 4SK \cdot KY, \text{ by focus-directrix relation.}
\]

Example 8. If the line \( 3x - 2y + 8 = 0 \) cuts at \( P \) and \( Q \) the parabola \( y^2 - 30x + 6y - 40 = 0 \), find the coordinates of the mid-point \( V \) of \( PQ \).

The equation of the tangent at \( (x_1, y_1) \) to the parabola is

\[
y_1 - 15(x + x_1) + 3(y + y_1) - 40 = 0,
\]

that is,

\[
15x - (y + 3)y = 3y_1 - 15x_1 - 40;
\]

this is parallel to \( 3x - 2y + 8 = 0 \) if \( (y_1 + 3) / 15 = 8 / 15 \), that is, if \( y_1 = 7 \).

Since the equation of the parabola can be written \( (y + 3)^2 = 30(x + 49/30) \),

the axis of the parabola is parallel to \( x = 0 \);

\[
\therefore V \text{ lies on the line } y = y_1 = 7 \text{ which cuts } 3x - 2y + 8 = 0, \text{ where } x = 2;
\]

\[
\therefore V \text{ is the point } (2, 7).
\]

Note. The form of the question implies that the line cuts the parabola. This may be verified as follows:

The line \( x = 2 \) meets the parabola where \( y^2 + 6y - 100 = 0 \), that is, where

\[
y = -3 + \sqrt{109} > 7 \text{ and } y = -3 - \sqrt{109} < 7;
\]

\[
\therefore \text{the point } V(2, 7) \text{ lies inside the parabola.}
\]
7.9. Concurrent Normals. (i) If the normal to \( x = at^2, y = 2at \) at the point \( P_1, t = t_1 \), passes through \( F(h, k) \), then \( t_1 \) is a root of the cubic
\[
at^3 + t(2a - h) - k = 0.
\]
(ii) Conversely if \( t = t_1 \) is a root of \( at^3 + t(2a - h) - k = 0 \), the point \( P_2 \), \( t = t_1 \), is the foot of a normal from \( F(h, k) \) to \( x = at^2, y = 2at \).

(i) The equation of the normal at \( P_1 \) is \( y + t_1 x = 2at_1 + at_1^2 \).
If this normal passes through \( F(h, k) \), \( k + t_1 h = 2at_1 + at_1^2 \);.
\( t_1 = t_1 \) is a root of the cubic, \( at^3 + t(2a - h) - k = 0 \).
(ii) Reverse the argument in (i).
If \( at_1^3 + t_1(2a - h) - k = 0 \), then \( k + t_1 h = 2at_1 + at_1^2 \);
the line \( y + t_1 x = 2at_1 + at_1^2 \) passes through \( F(h, k) \);
the normal at \( P_2 \), \( t = t_2 \), passes through \( F(h, k) \).

7.9.1. If three normals can be drawn to \( x = at^2, y = 2at \) from \( F(h, k) \) and if the feet of the normals are given by \( t = t_1, t = t_2, t = t_3 \), then
\[
\Sigma t = t_1 + t_2 + t_3 = 0,
\]
and
\[
\Sigma t_1 t_2 = t_1 t_2 + t_1 t_3 + t_2 t_3 = (2a - h)/a,
\]
and
\[
t_1 t_2 t_3 = k/a.
\]

Since it is given that the normal at each of the points \( t = t_1, t = t_2, t = t_3 \) passes through \( F(h, k) \), it follows from 7.9 that there are three roots, \( t_1, t_2, t_3 \), of the cubic, \( at^3 + t(2a - h) - k = 0 \), and the values of \( \Sigma t, \Sigma t^2, \Sigma t^3 \) are as stated.

7.9.2. \( P_1, P_2, P_3 \) are the feet of the normals from \( F(h, k) \) to the parabola
\( x = at^2, y = 2at \), then

(i) the circle \( P_1 P_2 P_3 \) passes through the vertex of the parabola:
(ii) the equation of circle \( P_1 P_2 P_3 \) is \( x^2 + y^2 - (h + 2a)x - \frac{1}{2}ky = 0 \).

(i) The values of \( t \) which determine \( P_1, P_2, P_3 \) are the roots \( t_1, t_2, t_3 \) of
\[
\Delta t^3 + t(2a - h) - k = 0;
\]
\( t_1 + t_2 + t_3 = 0 \).

Denote equation of circle \( P_1 P_2 P_3 \) by \( x^2 + y^2 + 2gx + 2fy + c = 0 \); then the values of \( t \) which determine the points \( P_1, P_2, P_3 \) where the circle meets \( x = at^2, y = 2at \) are the roots \( t_1, t_2, t_3 \) of the quartic
\[
\Delta t^4 + 4a^2 t^2 + 2gat + 2f = c = 0.
\]
\( t_1 + t_2 + t_3 = 0 \); but \( t_1 + t_2 + t_3 = 0 \), \( t_4 = 0 \).
\( P_4 \) is the vertex \( (0, 0) \) determined by \( t = 0 \).
(ii) Since \( t_4 = 0, c = t_1 t_2 t_3 t_4 = 0 \);
\( t_1, t_2, t_3 \) are the roots of \( \Delta t^3 + (4a + 2g)t^2 + 4f = 0 \);
but \( t_1, t_2, t_3 \) are also the roots of \( \Delta t^3 + (2a - h)t^2 - k = 0 \);
\( 4a + 2g = -2a - h \), that is, \( 2g = -2a - h \) and \( 4f = -k \);
the equation of circle \( P_1 P_2 P_3 \) is \( x^2 + y^2 + (2a - h)x - \frac{1}{2}ky = 0 \).

7.9.3. Any cubic equation has at least one root and may have three roots, distinct or repeated. Hence at least one normal, and not more than three normals, can be drawn from a point to a parabola.

Example 9. If two of the normals from a variable point \( F(h, k) \) to \( x = at^2, y = 2at \) are coincident, find the equation of the locus of \( F \) and prove it touches each normal of the parabola.

Two normals from \( F(h, k) \) are coincident if two roots of
\[
a t^3 + (2a - h)t - k = 0
\]
are equal. Denote the roots by \( t_1, t_2, t_3 \), then
\( t_1 + t_2 + t_3 = 0 \).
Symmetric functions of the roots \( t_1, t_2, t_3 \), \( -2t_1 \) give
\( (2a - h)/a = t_1^2 - 2t_1^2 - 2t_1^2 = -2t_1^2 \) and \( k/a = -2t_1^2 \);
the locus of \( F(h, k) \) is given by the parametric equations
\( x = 2a + 3at^2, y = -2at^2 \).

Any normal, \( y + ax = 2am + an^2 \), meets the locus of \( F \) where
\( -3at^2 + m(2a + 3at^2) = 2am + an^2 \) or \( 2a^2 - 3mt^2 + n^2 = 0 \),
that is, \( (t - m)(2i + n - m) = (t - m)(2i + m) = 0 \).

Hence the normal, \( y + mx = 2am + an^2 \), touches the locus of \( F \) at the point \( t = m \), and cuts it again at the point \( t = -\frac{1}{2}m \). The reader should sketch the locus of \( F \); it has the same form as the curve, \( x = t^2, y = t^2 \) or \( y = x^2 \).
7.9.4. Since for all values of $m$ the normal $y = mx + 2am + am^2$ to $y^2 = 4ax$ touches the curve, $x = 2a + 3at^2$, $y = -2at^2$, the normal is said to envelop the curve and the curve is called the envelope of the normals.

Example 10. If two of the normals from $F$ to $x = at^2$, $y = 2at$, are coincident with the normal $FP$ at $P(at^2, 2at)$, then the centre, circle $F$, radius $FP$, meets the parabola at three coincident points, coincident with $P$, and at one other point given by $t = -3m$.

By 7.9.3, $F$ is the point $(2a + 3am^2, -2am^2)$; therefore the equation of the circle, centre $F$, passing through $P(at^2, 2at)$, is

$$x^2 + y^2 - 2ax - 3am^2 x + 4am^2 y = c,$$

where $c = a^2 m^4 + 4a^3 m^3 - 2a^2 m^2 + 2a^3 m + 8a^2 m - 2a^2 m^2 = 0$.

Hence the circle meets $x = at^2$, $y = 2at$ at the points given by the roots of

$$a^2 t^4 + 4a^2 t^2 - 2at(2 + 3m^2) + 8a^2 m - 2a^2 m^2 = 0,$$

that is,

$$t^4 - 6mt^3 + 8m^2 t^2 - 2am^2 t + 3m = 0,$$

that is,

$$(t^2 - 3m^2 + 8m^2 m^2)(t + 3m + 2m^2) = 0,$$

$\therefore$ the circle cuts the parabola at three coincident points, coincident with $t = m$, and at the point $t = -3m$.

This circle is called the circle of curvature at $P$ and its centre $F$ is called the centre of curvature at $P$. Thus the envelope of the normals is the locus of the centres of curvature.

Example 11. The normals to a parabola at variable points $P, Q$ meet at $F$. If the direction of $PQ$ is fixed, prove the locus of the focus is a line.

Choose axes so that the parabola is given by $x = at^2$, $y = 2at$. Let $R$ be the foot of the third normal from $F$ to the parabola and let $P, Q, R$ be determined by $t = p$, $t = q$, $t = r$.

Since gradient of $PQ$ equals $2(p + q)$, $p + q = constant = k$, say.

By 7.9.1, $p + q + r = 0$, $k + r = 0$, $r = -k$.

The circle at the fixed point $R, t = -k$.

EXERCISE 30

1. If the feet of the normals from $(30, -24)$ to the parabola, $x = 2t^2$, $y = 4t$, are given by $t = p$, $t = q$, $t = r$, find $p, q, r$.

[2] Prove that the normal at $F(0, 6)$ on $y^2 = 4ax$ passes through $K(12, -51)$ and find the coordinates of the feet $Q, R$ of the other normals from $K$. Verify that $QR$, $OF$ make supplementary angles with $OQ$.

3. Prove that the normals to $y^2 = 4ax$ at points $P, Q, R$ given by $y = 2a, y = 10a, y = -12a$ are concurrent. Find the point of concurrence and the equation of the circle $PQR$.

4. $PSQ$ is a variable focal chord of $y^2 = 4ax$; the normals at $P, Q$ meet at $F$. Prove the equation of the locus of $F$ is $y^2 = a(x - 3a)$.

5. $P, Q$ are variable points on $x = at^2$, $y = 2at$, given by $t = p$, $t = q$, $PQ$ is parallel to $y = a$. Prove the locus of the meeting of the normals at $P$ and $Q$ is the line $2x - y = 12a$. (L)

6. The normals to $P = at^2$, $y = 2at$ at the points given by $t = p$, $t = q$, $t = r$ meet at $K$. Find $h$ and $k$ in terms of $p, q, r$.

7. If the normals to $x = at^2$, $y = 2at$ at the points $t = p$, $t = q$ meet at the point $t = r$ on the parabola, prove that $pq = 2$.

8. The normals at $P, Q, R$ on a parabola, vertex $O$, focus $S$, meet at $F$; $FN$ is the perpendicular to $OS$. Prove $2P + S + 2R + 2S + 2O = 20N$.

9. The normals at $P, Q, R$ on $y^2 = 4ax$ meet at $(h, k)$; $(x, y)$ is the pole of $PQ$. Prove $h = 2x - x_1 + x_1 y_1 / a$ and $k = -x_1 y_1 / a$.

10. Prove that the equation of the circle which passes through the three points on $x = at^2$, $y = 2at$, given by the roots of $t^2 - 2at + 2at - 2at = 0$, is $x^2 - y^2 + x + 2at - 2at - 2at + 2at = 0$.

11. The normals to $y^2 = 4ax$ at $P, Q, R$ meet at $(h, k)$; $QR, RP, PQ$ meet $y = 0$ at $G, H, K$. Prove that the perpendiculars to $QR, RP, PQ$ at $G, H, K$ meet at $(0, 4h)$.

12. $P, Q, R$ are the feet of the normals from $(h, k)$ to $y^2 = 4ax$. Prove the coordinates of the centroid, circumcentre and orthocentre of $\triangle PQR$ are respectively $(2h - 4a, 0)$, $(a / 2, h / 4)$, $(4a, -6a - h / 4)$.

13. The normals to $y^2 = 4ax$ at $P, Q, R$ concur at $F(h, k)$. If $\angle PFE$ is a right angle, prove:

(i) $PQ = a + h_k$ (ii) $R$ is the point $(k^3 / a, -2a)$.

14. $P, Q, R$ are the feet of the normals from a variable point on the given line $y = c$. Prove the sides of $\triangle PQR$ touch $x^2 = 2cy$.

15. A circle, centre $K$, passes through the origin and cuts $y^2 = 4ax$ again at $P, Q, R$.

(i) Prove the normals at $P, Q, R$ are concurrent.

(ii) If $F$ is the point of concurrence and if $K$ lies on $lx + my + n = 0$, prove that $F$ lies on $2lx + my + 4(a + n) = 0$. (L)

16. $F$ is a variable point on $y^2 = a(x - a)$; the circle on $OF$ as diameter meets $y^2 = 4ax$ again at $P, Q, R$. Prove the normals at $P, Q, R$ to $y^2 = 4ax$ concur at a point on $y^2 = a(x + a)$.

17. $P$ and $Q$ are the feet of the normals to a parabola from a variable point on the parabola. Prove that $PQ$ passes through a fixed point.

18. The normal at a variable point $P$ on a parabola meets the axis at $G$; $P$ is the mid-point of $PG$; $Q$ and $R$ are the feet of the other normals from $P$ to the parabola. Prove that $QR$ cuts the axis at a fixed point.
CHAPTER 8

THE RECTANGULAR HYPERBOLA

8.1. The simplest definition of a rectangular hyperbola is the statement that, with reference to two selected perpendicular lines as coordinate axes, it is the locus of a point \( P(x, y) \) determined by the parametric equations

\[
    x = ct, \quad y = \frac{c}{t}.
\]

where \( c \) is a positive constant.

The equivalent \( x, y \) equation is

\[
    xy = c^2 \quad \text{or} \quad y = \frac{c^2}{x}.
\]

8.1.1. The curve \( xy = c^2 \) has the same shape as the curve \( xy = 1 \), that is, the same shape as the graph of \( 1/x \), whose form is indicated by the following tables of values:

<table>
<thead>
<tr>
<th>( xy = 1 )</th>
<th>( x )</th>
<th>0-001</th>
<th>0-01</th>
<th>0-1</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &gt; 0 )</td>
<td>( y )</td>
<td>1000</td>
<td>100</td>
<td>10</td>
<td>1</td>
<td>0-1</td>
<td>0-01</td>
<td>0-001</td>
</tr>
<tr>
<td>( xy = 1 )</td>
<td>( x )</td>
<td>-0-001</td>
<td>-0-01</td>
<td>-0-1</td>
<td>-1</td>
<td>-10</td>
<td>-100</td>
<td>-1000</td>
</tr>
<tr>
<td>( x &lt; 0 )</td>
<td>( y )</td>
<td>-1000</td>
<td>-100</td>
<td>-10</td>
<td>-1</td>
<td>-0-1</td>
<td>-0-01</td>
<td>-0-001</td>
</tr>
</tbody>
</table>

These tables illustrate three peculiarities of the form of the curve whose equation is \( xy = c^2 \), see Fig. 65.

(i) The curve consists of two disconnected branches. For one branch, the coordinates \( x, y \) are both positive; for the other branch they are both negative. Accordingly, one branch lies in the first quadrant and the other branch lies in the third quadrant.

(ii) The curve approaches indefinitely close to the \( x \)-axis from above at one end and from below at the other end of \( x'ox \) and approaches indefinitely close to the \( y \)-axis from the right at one end and from the left at the other end of \( y'oy \).

(iii) The curve does not meet either \( x'ox \) or \( y'oy \). The curve whose equation is \( xy = c^2, \ (c + 0) \), cannot pass through any point whose \( x \)-coordinate is zero because \( xy = 0 \) if \( x = 0 \); similarly, it cannot pass through any point whose \( y \)-coordinate is zero because \( xy = 0 \) if \( y = 0 \). The fact that there is no value of \( y \) when \( x = 0 \) is represented by a break in the curve, which jumps up abruptly from large negative values of \( y \) to large positive values of \( y \) when the values of \( x \) increase through zero.

8.1.2. If \( P(x, y) \) is a variable point on the curve \( xy = c^2 \), the value of \( y \) decreases steadily but remains positive when \( x \) increases through positive values and can be made as near zero as we please by choosing a sufficiently large value of \( x \). This is described by the phrase that when \( x \) tends to plus infinity, \( y \) tends downwards to zero and is written,

\[
    \text{when } x \to + \infty, \ y \to 0+.
\]

where the symbol \( 0+ \) means an approach to zero through positive values. Geometrically, we say that when \( P(x, y) \) moves along the curve \( xy = c^2 \) so that \( x \to + \infty \), the shortest distance of \( P \) from the line \( y = 0 \) tends to zero, and we call the line \( y = 0 \) an asymptote of the curve.

Similarly, if \( P(x, y) \) is a variable point on the curve \( xy = c^2 \), the value of \( x \) decreases steadily but remains positive when \( y \) increases through positive values and can be made as near zero as we please by choosing a sufficiently large value of \( y \). This is described by the phrase that when \( y \) tends to plus infinity, \( x \) tends downwards to zero and is written,

\[
    \text{when } y \to + \infty, \ x \to 0+.
\]

Geometrically we say that when the point \( P(x, y) \) moves along the curve \( xy = c^2 \) so that \( y \to + \infty \), the shortest distance of \( P \) from the line \( x = 0 \) tends to zero, and we call the line \( x = 0 \) an asymptote of the curve.

8.1.3. If \( x, y = c^2 \), then \((-x), (-y) = c^2 \).

(i) If \( P_1(x_1, y_1) \) lies on \( xy = c^2 \), then \( P_1(-x_1, -y_1) \) also lies on \( xy = c^2 \).

Therefore any chord \( P_1P_1' \) which passes through the origin \( O \) is bisected at \( O \).

For this reason, \( O \) is called the centre of the curve and any chord through \( O \) is called a diameter. Thus the curve \( xy = c^2 \) is symmetrical about its centre, which is also the point of intersection of the asymptotes. Since the curve is symmetrical about the origin, \( y \to 0 \) when \( x \to + \infty \) and \( x \to 0 \) when \( y \to - \infty \), where the symbol \( 0- \) means an approach to zero through negative values.

Hence the curve is also asymptotic to the line \( y = 0 \) when \( x \to - \infty \), approaching \( y = 0 \) from below and is also asymptotic to the line \( x = 0 \) when \( y \to - \infty \), approaching \( x = 0 \) from the left.

Fig. 65
8.1.4. Corresponding to each value of the parameter \( t \), except \( t = 0 \), there is just one value of \( x \) and one value of \( y \) given by the parametric equations of the rectangular hyperbola

\[ x = ct, \quad y = \frac{c}{t}, \quad (c > 0) \]

and therefore just one point \( P(x, y) \) which may therefore be called without ambiguity 'the point of the locus; the coordinates of \( P \) then satisfy the equation, \( xy = c^2 \).

Conversely if the coordinates of \( P(x, y) \) satisfy the equation \( xy = c^2 \), a unique value of \( t \), \( t \rightarrow 0 \), is determined by \( x = ct, \quad y = \frac{c}{t} \), (c > 0) and for this value of \( t \), \( y = \frac{c^2}{x} \), (c > 0).

8.1.5. Axes of Symmetry. If \( P \) is the point \((h, k)\) given by \( t = t_1 \), then the point \( Q, t = 1/t_1 \), is \((c/t_1, ct_1)\), that is, \((k, h)\), see Fig. 66; hence if \( P', Q' \) are the points given by \( t = -t_1 \), \( t = -1/t_1 \), \( P' \) is the point \((-h, -k)\) and \( Q' \) is the point \((-k, -h)\). Therefore \( P, Q, P', Q' \) are points on the rectangular hyperbola such that \( PP', QQ' \) bisect each other at \( O \), and the gradients of \( PQ, PQ' \) are \(-1, +1\), respectively. Thus \( PP'Q' \) is a rectangle whose sides are bisected at right angles by the lines \( y = x, \quad y = -x \). Hence if \( P \) is any point on the rectangular hyperbola \( xy = c^2 \), the images \( Q, Q' \) of \( P \) in the lines \( x - y = 0, \quad x + y = 0 \) also lie on \( xy = c^2 \) and therefore each of the pair of perpendicular lines \( x - y = 0, \quad x + y = 0 \) is an axis of symmetry of the curve; they are called the axes of the rectangular hyperbola \( xy = c^2 \) and make \( 45^\circ \) with each asymptote.

\[ \text{Fig. 66} \]

The line \( x - y = 0 \) cuts \( xy = c^2 \) at \( A(c, c) \) and \( A'(-c, -c) \), see Fig. 67;

\[ OA^2 = OA'^2 = 2c^2 \]

\( A \) and \( A' \) are called the vertices and \( AA' \) is called the transverse axis of the rectangular hyperbola. The length of \( AA' \) is denoted by \( 2a \);

\[ a^2 = OA^2 = 2c^2 \]

The line \( x + y = 0 \) does not cut \( xy = c^2 \); it is called the conjugate axis. The hyperbola is called 'rectangular' because its asymptotes are at right angles.

\[ \text{Fig. 67} \]

8.1.6. If \( PH, PK \) are the perpendiculars from any point \( P \) of a rectangular hyperbola to its asymptotes, taken as coordinate axes \( x'Ox, y'Oy \), and if \( A \) is a vertex, then the statement that the equation of the locus of \( P(x, y) \) is \( xy = c^2 \) is equivalent to the geometrical properties, see Fig. 67,

(i) \( \angle PH \cdot PK = \frac{c}{4} \angle OA^2 \).

(ii) Area of rectangle \( OHPK = c^2 = \frac{1}{4} \angle OA^2 \).

8.1.7. The equation \( xy = c^2 \) can also be interpreted so as to show the relation between the distances of any point of the curve from its transverse axis \( A'OA \), \( x - y = 0 \), and from its conjugate axis \( B'OB \), \( x + y = 0 \), see Fig. 67.

Let \( Pn \) and \( Pn' \) be the perpendiculars from a point \( P(x, y) \) on \( xy = c^2 \) to \( A'OA \), \( x - y = 0 \), and to \( B'OB \), \( x + y = 0 \), respectively; then

\[ PN = \pm (x-y)/\sqrt{2} \quad \text{and} \quad Pn = \pm (x+y)/\sqrt{2}; \]

\[ Pn^2 - PN^2 = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2 = 2xy = 2c^2 = a^2; \]

\[ Pn^2 - PN^2 = OA^2. \]

8.1.8. If the transverse axis \( A'OA \) and the conjugate axis \( B'OB \) of a rectangular hyperbola are taken as \( x \)-axis and \( y \)-axis, the equation of the rectangular hyperbola is

\[ x^2 - y^2 = a^2. \]

where \((a, 0), \*(-a, 0)\) denote the vertices \( A, A' \).

If \( P(x, y) \) is any point on the hyperbola, referred to \( A'OA \) and \( B'OB \) as \( x \)-axis and \( y \)-axis, see Fig. 68, then, with the notation of 8.1.7, \( nP = ON = x \) and \( NP = On = y \); therefore the algebraic equivalent of the property, \( Pn^2 - PN^2 = OA^2 \), is \( x^2 - y^2 = a^2 \).

The dotted curve in Fig. 68 represents the rectangular hyperbola

\[ x^2 - y^2 = a^2. \]

The coordinate axes \( x'Ox, y'Oy \) are axes of symmetry and are therefore the axes of

\[ x^2 - y^2 = -a^2. \]

\[ y'Oy, x = 0, \text{meets } x^2 - y^2 = -a^2 \text{ at } B(0, a) \text{ and } B'(0, -a); \]

\[ B, B' \] are the vertices and \( B'OB \) is the transverse axis.

\( A'OA, y - 0 \), does not meet \( x^2 - y^2 = -a^2 \), it is the conjugate axis.

The loci, \( x^2 - y^2 = a^2, x^2 - y^2 = -a^2 \), are called conjugate hyperbolas; the transverse axis of either is the conjugate axis of the other, and they have the same asymptotes, \( x - y = 0, \quad x + y = 0 \).

With the asymptotes as axes \( Ox, Oy \), the equations of these conjugate rectangular hyperbolas are of the form, \( xy = c^2, \quad xy = -c^2 \).
Example 1. Find the equation of the rectangular hyperbola having the perpendicular lines, \(3x + 2y - 1 = 0\), \(2x - 3y + 8 = 0\) as asymptotes and passing through \((-2, 1)\). Find (i) the coordinates of its center C, (ii) the equations of its axes and the coordinates of its vertices \(A, A'\).

If \(P(x, y)\) is a point on the rectangular hyperbola, the product of its distances, \(\pm (3x + 2y - 1)/\sqrt{9 + 4}, \pm (2x - 3y + 8)/\sqrt{4 + 9}\), from the asymptotes is constant;

\[
\therefore \text{the equation of the rectangular hyperbola is}
\]
\[
(3x + 2y - 1)(2x - 3y + 8) = k
\]

where \(k\) is a constant whose value is determined by the fact that the point \((-2, 1)\) lies on the curve;

\[
\therefore k = (-6 + 2 - 1)(-4 - 3 + 8) = -5.
\]

\[
\therefore \text{the equation of the rectangular hyperbola is}
\]
\[
(3x + 2y - 1)(2x - 3y + 8) = -5.
\]

(i) The center \(C\) is the point of intersection of the asymptotes, and so its coordinates are given by the equations
\[
3x + 2y - 1 = 0, \quad 2x - 3y + 8 = 0;
\]

hence \(C\) is the point \((-1, 2)\).

(ii) A point \((x, y)\) lies on an axis if it is equidistant from the asymptotes;

\[
\therefore \text{the equations of the axes are}
\]
\[
(3x + 2y - 1)/\sqrt{13} = \pm (2x - 3y + 8)/\sqrt{13}.
\]

We can distinguish between these two equations by using the conditions that the transverse axis meets and the conjugate axis does not meet the hyperbola.

If a point \((x, y)\) lies on the hyperbola, the values of \(3x + 2y - 1\) and \(2x - 3y + 8\) are opposite in sign because their product equals \(-5\) and this is negative;

\[
\therefore \text{the equation of the transverse axis is}
\]
\[
(3x + 2y - 1) = -(2x - 3y + 8)
\]

and the equation of the conjugate axis is
\[
(3x + 2y - 1) = +(2x - 3y + 8).
\]

Hence the equation of the transverse axis is \(5x - y + 7 = 0\), and the equation of the conjugate axis is \(x + 3y - 9 = 0\).

The vertices \(A, A'\) are given by the equations
\[
(3x + 2y - 1)(2x - 3y + 8) = -5 \quad \text{and} \quad (3x + 2y - 1) = -(2x - 3y + 8),
\]

\[
\therefore (2x - 3y + 8)^2 = 5 \quad \text{and} \quad 8x - y + 7 = 0,
\]

\[
\therefore 2x - 3y = -8 \pm \sqrt{5} \quad \text{and} \quad 5x - y = -7;
\]

Hence the vertices are
\[
A\left\{ + \sqrt{5}/13, 2 + (\sqrt{5}/13) \right\}, A'\left\{ - \sqrt{5}/13, 2 - (\sqrt{5}/13) \right\}.
\]

8.1.9. Since by definition the asymptotes of a rectangular hyperbola are at right angles, their equations are of the form,

\[
px + qy + r = 0, \quad qx - py + s = 0,
\]

and so the equation of a rectangular hyperbola is of the form,

\[
(px + qy + r)(qx - py + s) = k;
\]

when expanded, this equation is of the form,

\[
a(x^2 - y^2) + 2hxy + 2gx + 2fy + c = 0.
\]

Thus if the general equation of the second degree represents a rectangular hyperbola, the sum of the coefficients of \(x^2\) and \(y^2\) is zero.

Example 2. Interpret the locus whose equation is

\[
xy + 5x - 3y - 7 = 0.
\]

Since \((x - 3)(y + 5) = xy + 5x - 3y - 15\), the equation of the locus can be written

\[
(x - 3)(y + 5) - 7 - 15 = -8.
\]

The distances of a point \(P(x, y)\) from the perpendicular lines, \(x - 3 = 0\), \(y + 5 = 0\), are \(\pm (x - 3), \pm (y + 5)\); therefore if \(P\) lies on the given locus, the product of its distances from these lines is constant.

\[
\therefore \text{the locus is a rectangular hyperbola whose asymptotes are} \ x - 3 = 0 \quad \text{and} \ y + 5 = 0.
\]

Any point \(P(x, y)\) on the hyperbola is situated so that the values of \(x - 3\) and \(y + 5\) are opposite in sign because their product equals \(-8\) and this is negative;

\[
\therefore \text{the equation of the transverse axis is}
\]
\[
(x - 3) = -(y + 5), \quad \text{that is}, \ x + y + 2 = 0.
\]

Hence also the equation of the conjugate axis is
\[
(x - 3) = + (y + 5), \quad \text{that is}, \ x - y - 8 = 0.
\]

It is left to the reader to show that the vertices are the points \((3 + \sqrt{8}, -5 - \sqrt{8}), (3 - \sqrt{8}, -5 + \sqrt{8})\) and then sketch the locus.

Alternatively, the locus can be interpreted more easily, as on p. 90, by a change of origin.

Choose the point \((3, -5)\) as the new origin and draw \(X'CX, Y'CY\) parallel to \(x'Ox, y'Oy\).

If the coordinates of a point \(P\) are \(x, y\) referred to axes \(x'Ox, y'Oy\) and are \(X, Y\) referred to axes \(X'CX, Y'CY\), then

\[
X = x - 3, \quad Y = y + 5 \quad \text{or} \quad x = X + 3, \ y = Y + 5.
\]

Hence referred to axes \(X'CX, Y'CY\), the equation of the locus is

\[
XY = -8.
\]

This is a rectangular hyperbola whose asymptotes are \(X = 0, Y = 0\), and the two branches lie in the second and fourth quadrants into which the plane is divided by the perpendicular lines \(X'CX, Y'CY\).
EXERCISE 31

Find the equation of the rectangular hyperbola having the given lines as asymptotes and passing through the given point, Nos. 1–9.
Find also the coordinates of the centre and the equation of the transverse axis and illustrate by a sketch.

1. \[ x = 0, y = 0; (-3, 5). \]
2. \[ x = 0, y + 1 = 0; (2, 3). \]
3. \[ x - 3 = 0, y + 1 = 0; (1, 2). \]
4. \[ x + 4 = 0, y - 5 = 0; (0, 0). \]
5. \[ x = f, y = g, (f \neq 0); (0, 0). \]
6. \[ x + y = 0, x - y = 0; (0, 2). \]
7. \[ 2x - y = 0, x + 2y + 4 = 0; (1, 2). \]
8. \[ 3x + 4y - 4 = 0, 4x - 3y + 3 = 0; (2, 1). \]
9. \[ ax + by + c = 0, bx - ay + d = 0; (a, 0). \]

Find the equations of the asymptotes and the coordinates of the centre and vertices of the loci, Nos. 10–21. Illustrate by a sketch.

10. \[ xy = -16. \]
11. \[ (x-2)(y-3) = 1. \]
12. \[ (x+4)(y-1) = 4. \]
13. \[ (x-3)(y+2) = -9. \]
14. \[ xy - 2x - 5y + 1 = 0. \]
15. \[ xy - x - 2y - 6 = 0. \]
16. \[ xy + 5x - 3y - 11 = 0. \]
17. \[ 3xy + 2y + 1 = 0. \]
18. \[ x^2 + y^2 = 4. \]
19. \[ 2x^2 - 3y = 0. \]
20. \[ 2x^2 - 3xy - 2y^2 = 4. \]
21. \[ 12x^2 - 7xy - 12y^2 = 25. \]

22. Find the equation of the rectangular hyperbola which passes through the four points (0, 2), (3, 0), (1, 1), (1, 1).

23. The transverse axis of a rectangular hyperbola is \( x - 3y = 4 \), the conjugate axis is \( 3x + y = 2 \); the point (0, 2) lies on the curve. Find the equations of the curve and its asymptotes and the coordinates of its vertices.

24. Find the equation of the rectangular hyperbola whose vertices are \( -2, -3 \) and \( 1, 3 \). Find also the equations of its asymptotes.

25. Express the equation, \( (3x + 4y)(4x - 3y) + 2x + 11y - 5 = 0 \), in the form \( 2x + 3y + p)(4x - 3y + q) = r \). Hence find the coordinates of the centre and vertices of the rectangular hyperbola.

26. \( PM \) is the perpendicular from a variable point \( P(x, y) \) to the fixed line \( DD', x + b = 0 \). If \( OP = e \), \( PM \), where \( O \) is the origin and \( e \) is a positive constant, find the equation of the locus of \( P \). Find the value of \( e \) if the locus is a rectangular hyperbola and then find the coordinates of its vertices and centre. \( O \) is called the focus and \( DD' \) is called the diameter of the hyperbola.

27. Express \( x^2 - y^2 = a^2 \) in the form \( (x - p)^2 + y^2 = 2(x - q)^2 \); find \( p \) and \( q \) in terms of \( a \); then interpret the locus.

8.2 CHORD AND TANGENT

8.2.1. Equation of a Chord. If \( P_1, P_2 \) are the points \( t = t_1, t = t_2 \) of the hyperbola, \( x = c t, y = c t \), the equation of the chord \( P_1 P_2 \) is

\[ x + t_1 y = c(t_1 + t_2). \]

Gradient of \( P_1 P_2 = \left( \frac{c}{t_2} - \frac{c}{t_1} \right) = \frac{1}{t_1 t_2}. \)

\[ \therefore \text{ the equation of the chord } P_1 P_2 \text{ is } \]

\[ \frac{x}{t_1} + \frac{y}{t_2} = \frac{c}{t_1} \pm \frac{c}{t_2}. \]

Alternatively, the line \( bx + my + nc = 0 \) meets \( x = ct, y = ct \) at the points \( t \pm t_1, t \pm t_2 \) if \( t_1, t_2 \) are the roots of the equation

\[ l(t + m) + nc = 0, \]

that is, \( l^2 + n^2 + m = 0. \)

This equation is equivalent to

\[ (t - t_1)(t - t_2) = t^2 - 2t(t_1 + t_2) + (t_1 t_2) = 0. \]

\[ \therefore \text{ the equation of } P_1 P_2 \text{ is } x(t_1 y - c(t_1 + t_2)) = 0. \]

It can be proved as on p. 93 that the equation obtained by writing \( t_1 \) for \( t_2 \) is the equation of the tangent at \( P_1, t = t_1 \).

8.2.2. Equation of a Tangent. The equation of the tangent to the hyperbola, \( x = ct, y = ct \), at the point \( P_1, t = t_1 \), is

\[ x + t_1 y - 2ct = 0. \]

As in 8.2.1, the line \( bx + my + nc = 0 \) meets \( x = ct, y = ct \) where

\[ l^2 + n^2 + m = 0. \]

Hence \( bx + my + nc = 0 \) is the tangent at \( P_1, t = t_1 \), if this quadratic equation in \( t \) is equivalent to

\[ (t - t_1)^2 = t^2 - 2t_1 t + t_1^2 = 0, \]

that is, if

\[ l : m = -1 : -2t_1 : t_1^2. \]

\[ \therefore \text{ the equation of the tangent at } P_1, t = t_1, \text{ is } x + t_1 y - 2ct = 0. \]

8.2.3. The equation of the tangent to the hyperbola \( xy = c^2 \) at the point \( P_1(x_1, y_1) \) is

\[ xy_1 + yx_1 = 2c^2. \]

The equation of the tangent at \( P_1(c t_1, c t_1) \) is \( x + t_1 y - 2t_1 c = 0 \); this equation can be written \( x t_1 y + y t_1 c - 2c^2 = 0 \); if \( x_1 = c t_1 \) and \( y_1 = c t_1 \), the equation of the tangent at \( P_1(x_1, y_1) \) is \( xy_1 + yx_1 - 2c^2 = 0. \)
8.2.4. The easiest way of remembering the equation of the tangent at \((x_1, y_1)\) is to associate it with the \((x, y)\) equation of the curve and make the rule:

In the \((x, y)\) equation of the curve, replace \(xy\) by \(\frac{1}{2}(xy_1 + yx_1)\).

Thus with \(xy = c^2\) we associate \(\frac{1}{2}(xy_1 + yx_1) = c^2\).

The reader will probably find it easier to remember this form of the equation of the tangent than the form, \(x + t_1^2 y = 2ct_1 = 0\), for the tangent at \((c_{t_1}, c_{t_1})\); if so, he can then obtain from the equation \(x_1 y_1 + y_1 x_1 = 2c^2\), by writing \(ct_1, c_{t_1}\) in place of \(x_1, y_1\), the equation \(x_1 t_1 + y_1 t_1 = 2c^2\), that is, \(x + y = 2c^2\).

Combining this rule with the rule given on p. 93 for the equation of the tangent to a parabola, the equation of the tangent at \((x_1, y_1)\) to the rectangular hyperbola whose general equation, see 8.1.9, p. 115, is

\[a(x^2 - y^2) + 2ax + 2by + c = 0\]

can be written down in the form

\[a(xx_1 - yy_1) + b(xy_1 + yx_1) + c(y + y_1) + c = 0\]

We shall not, however, at this stage prove the general formula because there will be no occasion for using it as yet; we give it here merely to illustrate the application of the rule to particular cases which occur, see 8.3.2, 8.3.3, p. 119.

8.2.5. Equation of a Normal. The equation of the normal to the hyperbola, \(x = ct, y = -ct\), at the point \(P_1, t = t_1\), is

\[t_1^2 x - t_1 y = c(t_1^2 - 1)\]

The equation of the tangent at \(P_1\) is \(x + y = 2ct_1\).

:. the equation of the normal at \(P_1(c_{t_1}, c_{t_1})\) is

\[t_1^2 x - y = t_1^2 (c_{t_1} - c_{t_1})\]

that is,

\[t_1^2 x - t_1 y = c_{t_1} - c\]

8.2.6. The equation of the normal to the hyperbola \(xy = c^2\) at the point \(P_1(x_1, y_1)\) is \(xx_1 - yy_1 = x_1^2 - y_1^2\).

The equation of the tangent at \(P_1\) is \(x_1 y + y_1 x = 2c^2\).

:. the equation of the normal at \(P_1(x_1, y_1)\) is

\[xx_1 - yy_1 = x_1^2 - y_1^2\]

Example 3. If the normal at \(P, t = p\), to \(x = ct, y = -ct\), meets the curve again at \(Q, t = q\), find \(q\) in terms of \(p\).

The normal at \(P, t = p\), is \(p^2 x - py = c(p^2 - 1)\); it meets \(x = ct, y = -ct\) where \(p^2 - p^2 - t = c(p^2 - 1)\), that is, \(p^2 - p^2 + p = 0\), either way: sum of roots, \(p + q = (p^2 - 1)/t\), \(q = -1/p\); or say: product of roots, \(pq = -p/p^2 = -1/p^2\), \(q = -1/p^2\).

8.3. Points of the rectangular hyperbola \(x^2 - y^2 = a^2\) are given by \(x : y = -(t^2 + 1):(t^2 - 1):2t, t \neq 0\). \((a > 0)\)

Let \((x, y)\) be any given point of the rectangular hyperbola, see 8.1.8, p. 113, whose equation is \(x^2 - y^2 = (x + y)(x - y) = a^2\); then there is just one value of \(t\) such that \(x + y = at, t \neq 0\); and for this value of \(t, x - y = a^2 \frac{x+y}{y}\), \(a/t\).

\[\therefore 2x = at + a/t\]

\[2y = at - a/t\]

\[\therefore \frac{x}{y} = a = (t^2 + 1):(t^2 - 1):2t, t \neq 0\]

Conversely, if \(t \neq 0, x^2 - y^2 = \frac{1}{2}a^2(t^2 + 1)/2 - \frac{1}{2}a^2(1-t^2)/2 - a^2\).

8.3.1. The equation of the tangent at \(t = t_1\) to the hyperbola \(x : y = (t^2 + 1):(t^2 - 1):2t\) is

\[(1 + t_1^2)x + (1 - t_1^2)y - 2t_1 a = 0\]

The line \(lx + my + na = 0\) touches the curve at \(t = t_1\) if

\[l(t^2 + 1) + m(t^2 - 1) + 2nt = l^2 + m^2 + 2nt - (m - n) = 0\]

is equivalent to

\[(l - t_1)^2 t^2 - 2t_1 t + t_1^2 = 0\]

\[\therefore lx + my + na = 0\] is the tangent at \(t = t_1\) if \(\frac{l^2 + m^2}{1} = 2l \frac{1}{1} = -2t_1^2 = \frac{1}{1}\)

From these equal ratios, it is easy to find the values of \(l : m : n\).

The best way of doing so here, and in similar examples which occur frequently, is to use the ratio theorem:

If \(\frac{a}{b} = \frac{c}{d}\), then each ratio \(\frac{a+c}{b+d} = \frac{a-c}{b-d} = \frac{pa+qc}{pd+qd}\).

Thus

\[2t = \frac{(l + m) + (l - m)}{1 + t_1^2} = \frac{2t_1}{1 + t_1^2} = \frac{2t_1}{1 + t_1^2} = \frac{1}{1 - t_1^2}\]

\[\therefore \text{the tangent at } t = t_1 \text{ is } (1 + t_1^2)x + (1 - t_1^2)y - 2t_1 a = 0\]

As soon as this ratio-method is understood, most of the working should be done mentally; we merely write

\[\frac{l+m}{1-2t_1} = \frac{l-m}{t_1^2} = \frac{2t}{1 + t_1^2} = \frac{1}{1 - t_1^2}\]

8.3.2. The equation of the tangent at \(P_1(x_1, y_1)\) to \(x^2 - y^2 = a^2\) is \(xx_1 - yy_1 = a^2\).

By 8.3.1, the equation of the tangent at \(t = t_1\) is \(\frac{1}{2}a(t^2 + 1) + 1/t = (\frac{1}{2}a(t^2 - 1))/t = a = 0\).

:. the equation of the tangent at \((x_1, y_1)\) is \(xx_1 - yy_1 = a^2 = 0\).

8.3.3. The equation of the normal at \(P_1(x_1, y_1)\) to \(x^2 - y^2 = a^2\) is \(xy_1 + yx_1 = 2x_1 y_1\).

Since the equation of the tangent at \((x_1, y_1)\) is \(xx_1 - yy_1 = a^2\), the equation of the normal at \((x_1, y_1)\) is \(xy_1 + yx_1 = x_1 y_1 + y_1 x_1 = 2x_1 y_1\).
8.3.4. The parametric representation in 8.3 of points on \( x^2 - y^2 = a^2 \) was found by using the fact that a variable line \( x + y = at \), parallel to an asymptote, meets the curve at just one point; hence each point of the curve is determined by just one value of \( t \).

Another convenient form of the parametric equations for the hyperbola \( x^2 - y^2 = a^2 \) can be found by using the fact that a variable line \( y = (x + a)t \) through the vertex \( A'(-a, 0) \) meets the curve again at just one point.

The coordinates of the second point of intersection of the line \( y = (x + a)t \) with the curve \( y^2 - x^2 - a^2 = (x + a)(x - a) \) satisfy the equation, \( ty = x - a \).

\[ y = tx + at, \quad ty = x - a; \]

hence, if \( t^2 + 1 \),

\[ x = \frac{a(1 + t^2)}{1 - t^2}, \quad y = \frac{2ta}{1 - t^2}. \]

These equations represent all points of the curve \( x^2 - y^2 = a^2 \) except the vertex \( A'(-a, 0) \). It is easy to see that the point \( (x, y) \) moves along the curve to positions indefinitely near to \( A'(-a, 0) \) when \( t \to \infty \) and when \( t \to -\infty \).

It is often convenient to write these parametric equations in the form \( x : y : a = (1 + t^2) : 2t : (1 - t^2), \ t^2 + 1 \).

8.4. Asymptote and Tangent. The equation of the tangent at \( (x_1, y_1) \) to \( xy = c^2 \) is \( xy_1 + yx_1 = 2c^2 \).

Since \( x_1y_1 = c^2 \), the equation of the tangent can be written

\[ \frac{c^2x}{x_1^3} + \frac{y}{x_1} = \frac{2c^2}{x_1}. \]

If the point \( (x_1, y_1) \) moves along the curve so that \( x_1 \to \infty \) or \( x_1 \to -\infty \), the limiting form of this equation of the tangent is the equation of the asymptote \( y = 0 \).

Similarly, the equation of the tangent at \( (x_1, y_1) \) can be written

\[ x + \frac{c^2y}{y_1^3} = \frac{2c^2}{y_1}. \]

Therefore if the point \( (x_1, y_1) \) moves along the curve so that \( y_1 \to \infty \) or \( y_1 \to -\infty \), the limiting form of this equation of the tangent is the equation of the asymptote \( x = 0 \).

This characteristic of an asymptote is sometimes described by the phrase: ‘An asymptote is a tangent at infinity to the curve.’ This statement is permissible if it is regarded merely as an abbreviation for the statement about limiting forms and is convenient because theorems about tangents (see 8.5.3) can be stated with greater generality if asymptotes are regarded as tangents.

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EXERCISE 32

1. Find the coordinates of the points of intersection of the hyperbola \( xy = 6 \) with the lines: (i) \( x - y + 1 = 0 \); (ii) \( 3x + 2y = 12 \).

2. Find the values of \( t \) which give the points of intersection of \( x = ct, \ y = ct^2 \) with: (i) \( 2x - 3y + 6c = 0 \); (ii) \( x + ky + 2ck = 0 \).

Find the equations of the tangent and normal to the given curve at the given point, Nos. 3-10:

3. \( xy = -24 \); (3, -8).

4. \( 3xy = 4 \); (-2, -2).

5. \( x = ct, \ y = ct^2; \ t = 1 \).

6. \( x = ct, \ y = -ct^2; \ t = -\frac{1}{2} \).

7. \( xy - 3x = y = 1 \); (3, 5).

8. \( xy = 2y = 8 \); (-1, 2).

9. \( x^2 - y^2 = 9 \); (5, 4).

10. \( x^2 - y^2 = 3 \); (1, -2).

11. A variable line passes through \( (4, 2) \) and meets \( x'Ox, \ y'Oy \) at \( P, Q \). Prove the equation of the locus of the mid-point of \( PQ \) is the curve \( xy = x + 2y \). Verify that the curve passes through \( (0, 0) \) and \( (4, 2) \) and that the tangents to the curve at these points are parallel. (L)

12. Find \( m \) in terms of \( b, c \) if \( y = mx + b \) touches the curve \( x = ct, \ y = ct^2 \), and find in terms of \( m \) the value of \( t \) which gives the point of contact.

13. Find the condition that the line \( x/a + y/b = 1 \) touches the curve \( xy = c^2 \) and the coordinates of the point of contact.

14. The line joining \( (a, 0) \) to the point \( t_1 \) on \( x = ct, \ y = ct^2 \) meets the curve again at the point \( t_2 \). Find \( t_2 \) in terms of \( a, \ c, t_1 \). [Start by writing down the equation of the chord \( t_1t_2 \).]

15. The line joining the point \( (b, 0) \) to the point \( t_1 \) on \( x = ct, \ y = ct^2 \) meets the curve again at the point \( t_2 \). Find \( t_2 \) in terms of \( b, c, t_1 \).

16. Prove that the locus of the centroid of the triangle formed by a variable tangent to \( xy = c^2 \) and its asymptotes is \( bx = my = 4c^2 \). (L)

17. The tangent at \( P \) to \( xy = c^2 \) meets the asymptotes at \( T, T' \); the normal at \( P \) meets the axes of the hyperbola at \( G, g \). Prove that \( TT'g \) is a rhombus.

18. The tangent at \( P \) to \( xy = c^2 \) meets \( Oy \) at \( K \); \( POQ \) is a diameter. \( KOQ \) meets \( Ox \) at \( H \). Prove \( \triangle QOH = \frac{1}{2} \triangle AQK \).

19. The chord joining the points \( t_1 \) and \( t_4 \) on \( x = ct, \ y = ct^2 \) is perpendicular to the chord joining the points \( t_2 \) and \( t_3 \). Prove that the chords \( t_2t_4 \) and \( t_1t_3 \) are at right angles, as also the chords \( t_1t_4, \ t_2t_3 \). Interpret these results geometrically.

20. Two parallel tangents to a rectangular hyperbola, centre \( O \), are met by a third tangent in \( P \) and \( Q \). Prove \( OP, \ OQ \) are equally and oppositely inclined to each asymptote. (L)
21. If \( lx - my + nz = 0 \) meets the points \( t = p, t = q \), the hyperbola, \( x: y: a = (t^2 + 1): (t^2 - 1) : 2t \), find in terms of \( p, q \): (i) the ratio \( l : m : n \); (ii) the equation of the chord \( pq \).

22. If \( lx - my + nz = 0 \) is the tangent at the point \( t = t_1 \) to the hyperbola, \( x: y: a = (1 + t^2): 2t: (1 - t^2) \), find in terms of \( t_1 \): (i) the ratio \( l : m : n \); (ii) the equations of the tangent and normal at the point \( t = t_1 \).

23. The foot of the perpendicular from \( O \) to the tangent at a variable point \( P \) on \( xy = c^2 \) is \( Q \). (i) Prove \( OP \cdot OQ \) is constant. (ii) Find the equation of the locus of \( Q \) and sketch it. (OC)

(24) If the rectangular hyperbola, \( xy = c^2 \) and \( x^2 - y^2 = a^2 \), cut at \( P \), prove the tangents at \( P \) to the hyperbolas are at right angles.

25. If the values \( t_1, t_2, t_3 \) of \( t \) which determine the points \( P_1, P_2, P_3 \) on the hyperbola, \( x = \alpha, y = \gamma t \) are the roots of \( s + pt^q + rt^r + t = 0 \), find \( t_1 \) in terms of \( r \) if: (i) the tangent at \( P_1 \) is parallel to \( P_2P_3 \); (ii) \( \angle P_1P_2P_3 \) is a right angle.

26. Find the \((x, y)\) equation of the curve given by the parametric equations, \( x = (1 + t^2)/(1 - t^2), y = 2t/(1 - t^2) \). Sketch the curve and show how the values of \( t \) change along the curve.

(27) Repeat No. 26 for the curve, \( x = t^2 + 1/t, y = t - 1/t \).

28. The normal to \( xy = c^2 \) at \( P \) meets the curve again at \( Q \). Prove \( PQ = OP \cdot OQ \).

29. The normal to \( xy = c^2 \) at a variable point \( P \) meets the curve again at \( Q \). Prove that the equation of the locus of the mid-point of \( PQ \) is \( 4x^2y^2 + c^2(x^2 - y^2)^2 = 0 \). (L)

8.5. CHORD OF CONTACT. If \( P \) and \( Q \) are the points of contact of the tangents from a point \( U \) \((x', y')\) to the hyperbola \( xy = c^2 \), then the equation of \( PQ \) is

\[ xy' + yx' = 2c^2. \]

Let \( P, Q \) be the points \((x_1, y_1), (x_2, y_2)\).

The tangent at \( P \), \( xy_1 + yx_1 = 2c^2 \), passes through \( U \) \((x', y')\),

\[ x'y_1 + y'x_1 = 2c^2. \]

This numerical relation is precisely the condition that \( P(x_1, y_1) \) lies on the line whose equation is \( x'y + y'x = 2c^2 \).

It can be proved in exactly the same way that \( Q(x_2, y_2) \) also lies on the line whose equation is \( x'y + y'x = 2c^2 \); therefore the line, \( x'y + y'x = 2c^2 \), passes through \( P \) and through \( Q \).

8.5.1. If \( P \) and \( Q \) are the points of contact of the tangents from a point \( U \) to a hyperbola, the line \( PQ \) is called the polar of \( U \) with respect to the hyperbola and \( U \) is called the pole of \( PQ \).

It is easy to remember the equation of the polar of a given point because its form, as in the case of the parabola, see 7.7.2, p. 100, is the same as the form of the equation of the tangent; this is also true if the equation of the hyperbola is taken in its general form and can be proved by the same method as was used in 8.5.

8.5.2. The pole of a given line \( PQ \), \( lx + my + nz = 0 \), with respect to the hyperbola \( xy = c^2 \) is the point \( U \left( -\frac{2mc}{n}, -\frac{2lc}{n} \right) \).

Let \( U \) be the point \((x', y')\), then the equation of the polar \( PQ \) of \( U \) is

\[ xy' + yx' = 2c^2; \]

\[ \therefore \text{the line } xy' + yx' - 2c^2 = 0 \text{ is equivalent to the line } lx + my + nz = 0, \]

\[ \therefore \frac{y'}{x'} = -\frac{2c}{n} \quad \therefore \frac{x'}{y'} = -\frac{2mc}{n}, \quad \frac{y'}{y} = -\frac{2lc}{n}. \]

8.5.3. Two distinct tangents can be drawn from \( U \) \((x', y')\) to the hyperbola \( xy - c^2 = 0 \) if \( x'y' - c^2 < 0 \), and no tangent can be drawn from \( U \) to the hyperbola if \( x'y' - c^2 > 0 \).

The tangent to \( xy - c^2 = 0 \) at \((x, y)\) is \( x + ty = 2ct = 0 \), and this passes through \( U \) \((x', y')\) if \( x' + ty' = 2ct = 0 \); two distinct tangents can be drawn from \( U \) if the quadratic equation, \( t^2y' - 2ct + x' = 0 \), has two unequal roots, and no tangent can be drawn if the quadratic has no root; two tangents can be drawn from \( U \) if \( c^2 > x'y' \); and no tangent can be drawn if \( c^2 < x'y' \).

The conditions, \( c^2 > x'y', c^2 < x'y' \), can be interpreted geometrically:

It is natural to say that \( U \) is 'inside' the hyperbola if \( U \) and the centre \( O \) are on opposite sides of the curve and is 'outside' the hyperbola if \( U \) and \( O \) are on the same side of the curve, see Fig. 60.

Hence \( U \) is inside the curve if, and only if, there is just one point \( P \) on the curve such that \( P \) divides \( UO \) in the ratio \( k : 1 \) where \( k > 0 \); then \( P \) is the point \((x'/1+k, y'/1+k)\).

Thus \( U \) is inside the hyperbola if, and only if, \( x'y' - c^2 > 0 \).
8.6. The mid-points of the chords of the rectangular hyperbola \( xy = c^2 \) which are parallel to the diameter \( OK \), \( y + mx = 0 \), lie on the diameter \( OH \), \( y - mx = 0 \).

Let \( P_1 \), \( P_2 \) be the points \((ct_1, c/t_1), (ct_2, c/t_2)\), then the equation of \( P_1P_2 \) is \( x + t_2y = c(t_1 + t_2) \);

\[ \therefore P_1P_2 \text{ is parallel to } OK, y + mx = 0, \text{ if } t_2 = 1/m. \]

Let \((h, k)\) be the mid-point \( V \) of \( P_1P_2 \); then the equation of \( OV \) is

\[ \frac{x}{h} = \frac{y}{k} \text{ that is, } y = \frac{k}{h}x; \]

but \(2h = c(t_1 + t_2) \) and \(2k = c\left(\frac{1}{t_1} + \frac{1}{t_2}\right) = c\left(t_1 + t_2\right)/t_1t_2\),

\[ \therefore \text{the equation of } OV \text{ is } y = x/(t_1t_2), \text{ that is, } y = mx. \]

\[ \therefore \text{if } P_1P_2 \text{ is parallel to the diameter } y + mx = 0, \text{ its mid-point } V \text{ lies on the diameter } y - mx = 0. \]

8.6.1. If \( OH \) is the diameter which bisects chords parallel to the diameter \( OK \), then the diameter which bisects chords parallel to \( OH \) is the diameter \( OK \).

With the notation of 8.6, the diameter which bisects any chord \( P_1P_2 \) parallel to the diameter \( OK \), \( y + mx = 0 \), is the diameter \( OH \), \( y - mx = 0 \), for all values of \( m \); therefore the diameter which bisects any chord \( Q_1Q_2 \) parallel to the diameter \( OH \), \( y + (-m)x = 0 \), is the diameter \( y - (-m)x = y + mx = 0 \), that is, the diameter \( OK \).

Two diameters \( OH, OK \) of a rectangular hyperbola such that either bisects chords parallel to the other are called conjugate diameters.

8.6.2. The asymptotes of a rectangular hyperbola are the bisectors of the angle between two conjugate diameters.

With the asymptotes as axes \( xo, yo \), the equation of the rectangular hyperbola is \( xy = c^2 \) and the equations of any two conjugate diameters can be taken as \( y + mx = 0 \), \( y - mx = 0 \), and these lines make equal angles with \( xo \) and with \( yo \).

Example 4. Find the equation of the chord \( P_1P_2 \) of the rectangular hyperbola \( xy = c^2 \) whose mid-point is \( V(h, k) \).

The gradient of \( OV \) is \( k/h \); \( \therefore \) the gradient of \( P_1P_2 \) is \(-k/h\); \( \therefore \) the equation of \( P_1P_2 \) in \( kx + hy = \ldots = kh + hk = 2hk \).
**Example 5.** Find the equation of the chord \( P_1P_2 \) of the rectangular hyperbola \( x^2 - y^2 = a^2 \) whose mid-point is \( V(h, k) \).

\( OV \) either meets \( x^2 - y^2 = a^2 \) or meets the conjugate hyperbola \( x^2 - y^2 = -a^2 \) at a point \( P \) such that the tangent at \( P \) is parallel to \( P_1P_2 \).

Since \( P \) lies on \( OV \), its coordinates are of the form \( (nh, nk) \);

\( \therefore \) the tangent at \( P \) either to \( x^2 - y^2 = a^2 \) or to \( x^2 - y^2 = -a^2 \) is parallel to \( nhx - nk = 0 \);

\( \therefore P_1P_2 \) is parallel to \( hx - ky = 0 \) and passes through \( V(h, k) \);

\( \therefore \) the equation of \( P_1P_2 \) is \( hx - ky = h^2 - k^2 \).

**EXERCISE 33**

1. Find the equation of the polar of \((-3, -5)\) with respect to the hyperbola \( xy + 30 = 0 \).

2. Find the poles of the line \( 4x - 3y = 5 \) with respect to the hyperbola:
   (i) \( xy = 6 \); (ii) \( x^2 - y^2 = 3 \).

3. Find the point of contact of the tangents from \((24, -48)\) to the hyperbola \( xy = 24 \).

4. Find the equations of the tangents from \((8, -12)\) to the hyperbola \( xy = 12 \).

5. Find the point of intersection of the tangents at the points \( t = p \), \( t = q \) to the hyperbola, \( x = at, y = ct \).

6. Prove that the pole of any tangent to \( xy = c^2 \) with respect to \( xy = -c^2 \) lies on \( xy = c^2 \).

7. Find the condition that the pole of \( lx + my + nc = 0 \) with respect to \( xy = c^2 \) lies on \( xy = -c^2 \).

8. Find the coordinates of the mid-point of the chord intercepted by \( xy = c^2 \) on \( 3x - 4y = 1 \).

9. Find the locus of mid-points of chords parallel to \( 2x + 3y = 0 \) intercepted by:
   (i) \( xy = c^2 \); (ii) \( x^2 - y^2 = a^2 \).

10. Prove that \( y = m_1x, y = m_2x \) are conjugate diameters of \( x^2 - y^2 = a^2 \) if \( m_1m_2 = 1 \).

11. If \( U \) is the pole of a chord \( PQ \) of a hyperbola, centre \( O \), and if \( V \) is the mid-point of \( PQ \), prove \( UV \) lies on \( OV \).

12. The mid-point of a variable chord \( PQ \) of \( xy = c^2 \) lies on a fixed line parallel to \( OY \); prove the locus of the pole of \( PQ \) is a line parallel to \( Ox \).

13. \( P \) and \( Q \) are points outside a rectangular hyperbola. If \( Q \) lies on the polar of \( P \), prove that \( P \) lies on the polar of \( Q \).

14. Find the condition that the pole of the line \( lx + my + n = 0 \) with respect to \( xy = c^2 \) lies on the line \( 'lx + my' + n' = 0 \). If this condition is satisfied, prove that the pole of \( lx + my + n = 0 \) lies on \( lx + my + n = 0 \).

15. \( PQ \) is the normal at a variable point of \( xy = c^2 \). Prove that the equation of the locus of the pole of \( PQ \) with respect to \( x^2 - y^2 = a^2 \) is \( c^2(x^2 - y^2) = a^2xy \).

8.7. **POLARS AND DIAMETERS**

16. Find the condition that:
   (i) the point \((h + r \cos \theta, k + r \sin \theta)\) lies on \( x^2 - y^2 = a^2 \);
   (ii) the sum of the roots of the quadratic in \( r \) obtained in (i) is zero. Interpret the result.

17. If \( P, Q \) are the points \( t = t_1^2, t = 1/t_1 \) on \( x = ct, y = c/t \), prove that the circle on \( OP \) as diameter passes through \( Q \).

18. A variable tangent to \( y^2 = 4ax \) cuts the hyperbola \( xy = c^2 \) at \( P, Q \). Prove that the locus of the mid-point of \( PQ \) is the parabola \( 2y^2 + ax = 0 \).

8.7. (i) If \( P_1, P_2 \) are points on the same branch or different branches of a rectangular hyperbola, centre \( O \), and if \( P_1P_2 \) meets the asymptotes at \( R_1, R_2 \), then \( R_1P_1 = P_2R_2 \).

(ii) If the tangent at \( P \) meets the asymptotes at \( T_1, T_2 \), then \( T_1P = PT_2 = OP \).

Take the asymptotes as axes \( Ox, Oy \) and let the equation of the hyperbola be \( xy = c^2 \).

\[ \text{Fig. 71} \]

(i) If mid-point of \( P_1P_2 \) is \( V(h, k) \), by Example 4, p. 124, the equation of \( P_1P_2 \) is

\[ kx + hy = 2hk; \]

\( \therefore P_1P_2 \) meets \( y = 0 \) at \( R_2(2h, 0) \) and meets \( x = 0 \) at \( R_4(0, 2k) \);

\( \therefore \) mid-point of \( R_2R_4 \) is \((h, k)\);

\( \therefore \) \( R_4, R_2, P_1, P_2 \) have the same mid-point.

(ii) **First Method.** If \( P \) is the point \((x', y')\), the equation of tangent \( TP_1P_2 \) is

\[ xy' + yx' = 2c^2 = 2x'y'. \]

Hence as in (i), of which it is the limiting form when \( P_1, P_2 \) coincide, \( T_1, T_2 \) are the points \((2x', 0), (0, 2y')\).

\( \therefore \) mid-point of \( T_1T_2 \) is \((x', y')\);

\( \therefore T_1P = PT_2 \).

Since \( P \) is mid-point of hypotenuse \( T_1T_2 \) of \( \triangle OT_1T_2 \), \( OP = T_1P \).

**Second Method.** By 8.6.4, the tangent \( T_1P_1T_2 \) at \( P \) is parallel to the diameter conjugate to \( OP \).

By 8.6.2, conjugate diameters make equal angles with the asymptotes.

\( \therefore \) \( \angle PT_1O = \angle POT_1 \) and \( \angle PT_2O = \angle POT_2 \);

\( \therefore \) \( PO = PT_1 \) and \( PO = PT_2 \).
8.7.1. The property in 8.7 (i) gives a simple construction for any number of points of a rectangular hyperbola when the asymptotes and one point \( P_1 \) are given: Draw a line through the given point \( P_1 \) and let it cut the asymptotes at \( R_1, R_2 \); construct \( P_2 \) on \( R_1R_2 \) such that \( R_2P_2 = P_1R_1 \); then \( P_2 \) lies on the curve.

8.7.2. (i) The orthocentre of a triangle \( PQR \) inscribed in a rectangular hyperbola lies on the curve.

(ii) If \( PQ, PR \) are perpendicular chords of a rectangular hyperbola, the tangent \( PT \) at \( P \) is perpendicular to \( QR \).

Let \( P, Q, R \) be the points \( t = p, t = q, t = r \) on the rectangular hyperbola, \( x = ct, y = ct \).

(i) Let the perpendicular from \( P \) to \( QR \) cut the curve again at \( H, t = h \).

By 8.2.1, p. 117, the gradients of chords \( QR, PH \) are \(-1/pr, -1/ph\); \( \therefore -1/pr(-1/ph) = -1 \), \( h = -1/(pgq) \).

(ii) The perpendicular from \( P \) to \( QR \) meets the curve again at the point \( H, t = -1/(pgq) \). The symmetry of this result shows that the perpendicular from \( Q \) to \( RP \) and from \( R \) to \( PQ \) also meet the curve again at the point \( H, t = -1/(pgq) \).

\( : \) the orthocentre of \( \triangle PQR \) is the point \( H, t = -1/(pgq) \).

(iii) \( PQ \) is perpendicular to \( PR \), \( : -1/pr(-1/pr) = -1 \), \( p^2pr = -1 \).

But the gradients of the tangent \( PT \) at \( P \) and of the chord \( QR \) are \(-1/pr^2 \) and \(-1/pr\);

\( : \) PT is perpendicular to \( QR \).

Note. The property (ii) is the limit of the property (i) when the orthocentre \( H \) of \( \triangle PQR \) moves along the curve towards \( P \).

8.7.3. A variable chord \( PQ \) of a rectangular hyperbola is fixed in direction. If the circle on \( PQ \) as diameter meets the hyperbola again at \( H, H' \), then \( H \) and \( H' \) are fixed points at the extremities of a diameter of the hyperbola.

Since \( PQ \) is a diameter of the circle \( PQQH'H \), \( \angle PHQ \) and \( \angle PHQ' \) are right angles;

\( \therefore \) by 8.7.2 (ii), the tangents at \( H \) and \( H' \) to the hyperbola are perpendicular to \( PQ \) and so are parallel to each other and are fixed in direction.

\( : \) \( H \) and \( H' \) are fixed points at the extremities of a diameter of the hyperbola.

A coordinate method of proof is indicated in Exercise 34, No. 25. For a converse of this property, see 8.7.4 (ii).

8.7.4. A circle meets the rectangular hyperbola, \( x = ct, y = ct \), at points \( P, Q, R, S \), given by \( t = p, t = q, t = r, t = s \). Prove:

(i) \( pqr = 1 \);

(ii) if \( PQ \) is a diameter of the hyperbola, then \( RS \) is a diameter of the circle;

(iii) if \( H \) is the orthocentre of the triangle \( PQR \), then \( SH \) is a diameter of the hyperbola.

(i) \( x^2 + y^2 + 2px + 2qy + k = 0 \) be the equation of circle \( PQR \);

\( \therefore \) \( p, q, r, s \) are the roots of \( c^2t^2 + 2gct + 2ct + c^2 = 0 \);

\( \therefore \) product of roots, \( pqr \), equals \( c^2c^2 \), that is, 1.

(ii) If \( PQ \) is a diameter of the hyperbola, \( q = -p \), \( : p^2rs = -1 \).

But the gradients of \( PR, PS \) are \(-1/pr, -1/ps \);

\( \therefore \triangle RPS \) is a right-angled triangle, \( : RS \) is a diameter of circle \( RPS \).

(iii) By 8.7.2, \( H \) is the point \( t = h \) on the hyperbola such that \( pqrh = -1 \);

\( \therefore pqrh = -pqs; \therefore h = -s; \therefore SH \) is a diameter of the hyperbola.

EXERCISE 34

[The following examples refer to a rectangular hyperbola, centre \( O \).]

1. If a variable tangent meets the asymptotes at \( T_1, T_2 \), prove that \( \triangle OT_1T_2 \) is of constant area.

2. The tangent at \( P \) meets an asymptote \( Ox \) at \( R \); the line through \( P \) parallel to \( Ox \) meets \( Ox \) at \( Q, R \). Prove that the mid-point of \( RR \) lies on the hyperbola.

3. The tangents at \( P_1, P_2 \) meet an asymptote at \( R_1, R_2 \). Prove that \( P_1P_2 \) bisects \( R_1R_2 \).

4. The tangent at a point \( P \) of \( 2xy = c^2 \) meets \( 2xy = b^2 \) at \( Q, Q' \). Prove that \( P \) is the mid-point of \( QQ' \).

5. If the chord \( P_1P_2 \) and the tangent at \( P_1 \) meet on one asymptote, prove that the line through \( P_2 \) parallel to the other asymptote bisects \( P_1P_2 \).

6. \( R_1, R_2 \) are the feet of the perpendiculars from points \( P_1, P_2 \) on a hyperbola to the asymptotes \( Ox, Oy \) respectively. If \( P_1R_1 \) is the tangent at \( P_1 \), prove \( P_1R_1 \) is the tangent at \( P_2 \).

7. \( H, K \) are fixed points on a rectangular hyperbola and \( P \) is a variable point on the curve. Prove that \( PH, PK \) meet an asymptote at \( Q, R \), prove that the length of \( QR \) is constant.
8.7 THE RECTANGULAR HYPERBOLA

[8.7] The tangent at \( P \) meets the asymptote \( Oxy \) at \( R \); the line \( RQ \) parallel to the asymptote \( Ox \) meets the curve at \( Q \). \( PQ \) meets \( Ox, Oy \) at \( H, K \). Prove \( HP = PQ = QK \).

9. The tangent at \( P \) meets the asymptotes at \( T_1, T_2 \); the normal at \( P \) meets \( OT_1, OT_2 \) at \( G_1, G_2 \). Prove \( T_1G_1 = OT_1 \cdot G_1Q \).

[10] The tangent at a variable point \( P \) meets the asymptotes at \( R, R' \); the line through \( P \) parallel to \( OR' \) cuts the curve at \( Q \). Prove the locus of the mid-point of \( PQ \) is a rectangular hyperbola.

11. Two chords \( PQ, P'Q' \) of a rectangular hyperbola meet at right angles at \( K \); prove that \( KP \cdot KQ = P'K \cdot KQ' \).

[12] If \( x \cos \theta + y \sin \theta = p \) is a normal to \( xy = c^2 \), prove that \( \frac{p^2}{\cos^2 \theta} + \frac{1}{c^2} = 0 \).

13. Describe the region in which the point \( P(x, y) \) lies if \( xy + a^2 < b^2y - c^2 \) where \( a, b, c \) are constants.

[14] If \( PQ \) is a chord normal to \( xy = c^2 \) at \( P \) and if \( PP' \) is a diameter, prove that \( \triangle OPP' \) is a right angle.

15. If a chord \( P_1P_2 \) of a rectangular hyperbola is parallel to the tangent \( PT \) at \( P \), and if \( PP_1, PP_2 \) meet an asymptote at \( R, T \) respectively, prove that \( RP_1, RP_2 = TP \).

[16] \( OT \) is the perpendicular to a tangent to \( xy = c^2 \) and meets the curve at \( Q \). Prove that \( OT \cdot OQ = 2c^2 \).

17. The normal at a point \( P \) meets the transverse axis \( OA \) at \( G \). Prove that \( PG = OP \).

18. A variable chord \( PQ \) of \( xy = c^2 \) passes through the fixed point \( E(h, k) \). Prove that the locus of the mid-point of \( PQ \) is a hyperbola with asymptotes parallel to \( Ox, Oy \) and with its centre at the mid-point of \( OE \).

[19] The normal at a point \( P \) meets the hyperbola again at \( Q \); \( V \) is the mid-point of \( PQ \). Prove \( \angle POV \) is a right angle.

20. The normal at a point \( P \) meets the hyperbola again at \( Q \). Prove that \( PQ^2 = SO^2 + OQ^2 \).

21. Find the condition that the circle through the points \( P, Q, R \) of the hyperbola \( x = ct, y = ct \), given by \( t = p, t = q, t = r \), touches the hyperbola at \( P \).

22. \( P, Q, R \) are the points of the hyperbola \( x = ct, y = ct \), given by \( t = p, t = q, t = r \); find the coordinates of the centre of the circle \( PQR \) in terms of \( c, p, q, r \).

23. A circle cuts a rectangular hyperbola at \( P, Q, R, S \). Prove that the diameter of the hyperbola perpendicular to \( PQ \) bisects \( KS \).

24. \( P \) and \( Q \) are given points on a rectangular hyperbola. A circle is drawn through \( P \) and \( Q \) to touch the hyperbola at \( R \). Prove that there are two positions of \( R \) which are at the extremities of a diameter.

8.7 GEOMETRICAL PROPERTIES

25. The extremities of a variable chord of \( xy = c^2 \) are \( P(cp, c/p), Q(cq, c/q) \). What can be said about \( PQ \) if \( pq \) is constant? Prove that the circles whose equations are \( (x - cp)(x - c/p) + (y - c/p)(y - c/q) = 0 \) form a coaxal system if \( pq \) is constant. Identify these circles.

[26] \( P_1, P_2 \) are variable points on \( x = ct, y = ct \), given by \( t = t_1, t = t_2 \), such that \( at_1 + bt_2 + c(t_1 + t_2) + d = 0 \) where \( a, b \) are constants and \( b^2 + ad \). Prove that \( P_1P_2 \) passes through a fixed point and find its coordinates. Prove that, if \( a + d = 0 \), the fixed point lies on the conjugate axis.

27. If the circle \( x^2 + y^2 + 2gx + 2fy + k = 0 \) meets \( x^2 = c^2 \) at \( (x_1, y_1), (x_2, y_2) \), \( (x_3, y_3), (x_4, y_4) \), prove \( x_1x_2x_3x_4 = y_1y_2y_3y_4 = c^4 \).

28. If the points of contact of the tangents from \( T(b, h) \) to the hyperbola, \( x = ct, y = ct \), are \( t = p, t = q \), find the values of \( p + q \) and \( pq \) in terms of \( c, h, k \). If the tangents meet one asymptote at \( E_1, E_2 \) and the other at \( P_1, P_2 \), prove that \( \triangle E_1P_1E_2 = \triangle TP_1F_1 = 2e \sqrt{(c^2 - b^2)} \).

29. \( P, Q, R, H \) are points on \( x = ct, y = c/t \) given by \( t = g, t = q, t = r, t = h \); \( PH \) meets \( QR \) at \( D \). If \( H \) is the orthocentre of \( \triangle PQR \), prove that \( HP \cdot HD = p(h - t)q(h - t) \).

30. \( HH' \) is a diameter of a rectangular hyperbola; \( P \) is any point on the curve. Prove that the chords \( PH, PH' \) make equal angles with each asymptote. If \( Q \) is another point on the curve on the same branch as \( P \) and on the same side of \( HH' \) as \( P \), prove that \( \angle PQH = \angle PH'Q \).

31. \( P \) is a point on a rectangular hyperbola whose vertices are \( A, A' \); \( OE, OF \) are the asymptotes; \( H \) is the orthocentre of the triangle \( PAA' \). Prove \( \angle POE = \angle HOF \). (N)

[32] The diameter which bisects a chord \( P_1P_2 \) of a rectangular hyperbola at \( V \) meets the curve at \( P \), \( P' \); prove \( P, P' \) are collinear with \( P \), \( P' \).

33. \( PP' \) is a diameter of a rectangular hyperbola; \( Q \) is a point on the same branch as \( P \). Prove that \( \angle PP'Q \) is equal to the angle which the tangent \( PP' \) makes with \( PQ \).

[34] The chords \( QR, RP, PQ \) of a rectangular hyperbola meet an asymptote at \( P', Q', R' \) respectively. Prove that the lines through \( P', Q', R' \) perpendicular to \( QR, RP, PQ \) are concurrent.

35. Find the condition that the normal to \( x = ct, y = c/t \) at the point \( t = p \) passes through the point \( (h, k) \). If the normals at the points \( t = t_1, t_2 \) meet at \( (h, k) \), prove that: (i) \( t_1t_2, t_1t_2 = -1 \); (ii) \( \Sigma t_1t_2 = 0 \). Prove also that \( h = \Sigma t_1 \) and \( k = \Sigma \frac{1}{t_1} \).

36. If the normal at the point \( (x_1, y_1) \) on \( xy = c^2 \) passes through the given point \( (h, k) \), prove that \( (x_1, y_1) \) lies on the rectangular hyperbola \( x^2 - y^2 - kx + ky = 0 \). Show that this equation can be written in the form \( (x - y - p)(x - y + q) = pq \) and find the coordinates of the centre of this hyperbola in terms of \( h, k \).
CHAPTER 9
THE ELLIPSE

9.1. Orthogonal Projection. Fig. 72 represents portions of two planes $OAH$, $OAh$, intersecting at an angle $\alpha$. The foot $P$ of the perpendicular from a point $P$ in the plane $OAH$ to the plane $OAh$ is called the orthogonal projection or, for short, the projection of $P$ on the plane $OAh$. The line of intersection $A'OAh$ of the two planes is called the axis of projection and may be taken as the coordinate axis $X'OY$ for points in the plane $OAH$ and as the coordinate axis $x'oy$ for points in the plane $OAh$; the origin for each plane is denoted by $O$, and the lines $OY$, $Oy$ perpendicular to $OA$ in the planes $OAH$, $OAh$ are the $y$-axes in the two planes. For the sake of brevity, it will be assumed in Section 9.1 that $p$, $q$, $p_1$, $q_1$, etc., denote the projections of points $P$, $Q$, $P_1$, $Q_1$, etc., in the plane $OAH$ on the plane $OAh$, and that the coordinates of the projections of the points $(X, Y)$, $(X_1, Y_1)$, etc., are denoted by $(x, y)$, $(x_1, y_1)$, etc. Hence, with the notation of Fig. 72,

$$z = ON = X$$ and $$y = NP = NP \cos \alpha = Y \cos \alpha;$$

$$\therefore X = x \quad \text{and} \quad Y = y \sec \alpha.$$

![Fig. 72](image)

9.1.1. If a variable point $P(X, Y)$ describes a circle, centre $O$, radius $a$, in the plane $OAH$, the equation of the locus of the orthogonal projection $P(x, y)$ of $P$ on the plane $OAh$ is

$$x^2 + y^2 \sec^2 \alpha = a^2.$$  

The coordinates of $P(X, Y)$ satisfy the equation $X^2 + Y^2 = a^2$, but $X = x$ and $Y = y \sec \alpha$,  

$$\therefore \quad \text{the coordinates of } P(x, y) \text{ satisfy the equation } x^2 + y^2 \sec^2 \alpha = a^2.$$
9.2. If \( P_1(x_1, y_1) \) is a point on the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), then \( x_1^2/a^2 + y_1^2/b^2 = 1 \), and so the point \( P_2(x_1, -y_1) \) also lies on the ellipse. Therefore the chord \( P_1P_2 \) is bisected at right angles by \( x'Ox \), and so \( x'Ox \) is an axis of symmetry of the ellipse. The chord \( P_1P_2 \) perpendicular to \( x'Ox \) is called a double ordinate. Similarly, the points \((-x_1, y_1)\) and \((-x_1, -y_1)\) also lie on the ellipse and so \( y'Oy \) is an axis of symmetry and any chord \( P'OP \) through \( O \) is bisected at \( O \).

The axis of symmetry \( x'Ox \) meets the ellipse at \( A(a, 0) \), \( A'(-a, 0) \) and the axis of symmetry \( y'Oy \) meets the ellipse at \( B(0, b) \), \( B'(0, -b) \); therefore \( AA' = 2a \) and \( BB' = 2b \), where \( a > b > 0 \), see Fig. 74 (ii).

The two axes of symmetry \( A'OA \), \( B'O \) are called the principal axes of the ellipse, their point of intersection \( O \) is called the centre and any chord \( P'OP \) through the centre \( O \) is called a diameter.

\( A \) and \( A' \) are called the vertices; the segment \( AA' \), length \( 2a \), is called the major axis; the segment \( BB' \), length \( 2b \), is called the minor axis.

9.2.3. If the ordinate \( NP \) of a point \( P \) on the ellipse meets the auxiliary circle at the point \( Q \) corresponding to \( P \), the angle \( \theta \) which \( OQ \) makes with \( Ox \), see Fig. 75, is called the eccentric angle of \( P \). With this notation,

\[
NQ = a \sin \theta, \quad NP = (b/a)NQ = b \sin \theta.
\]

\( Q \) is the point \((a \cos \theta, a \sin \theta)\) and \( P \) is the point \((a \cos \theta, b \sin \theta)\). Hence any point \( P(x, y) \) on the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is given by

\[
x = a \cos \theta, \quad y = b \sin \theta, \quad (a > b > 0),
\]

and then \( P \) may be called 'the point \( Q \)' on the curve and the relations serve as parametric equations of the ellipse.

With \( Ox \) as initial line and \( O \) as pole, if the vectorial angle of \( Q \) is \( \theta \), see 2.10, p. 18, then the eccentric angle of the point \( P \) of the ellipse which corresponds to \( Q \) is equal to \( \theta \).

9.2.4. The tangent at a point \( Q \) on a circle meets it at just one point \( Q \) (regarded as two coincident points) and therefore the projection of the tangent meets the ellipse at just one point \( P \), (also regarded as two coincident points) and is called the tangent at \( P \) to the ellipse.

The tangent at \( Q(X_1, Y_1) \) to \( x^2 + y^2 = a^2 \) is \( XX_1 + YY_1 = a^2 \).

\[
\text{the tangent at } P(x, y) \text{ to } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}
\]

\[
x x_1 + (ay/b)(ay_1/b) = a^2,
\]

that is,

\[
x x_1/a^2 + y y_1/b^2 = 1.
\]

Therefore the tangent at \((a \cos \theta, b \sin \theta)\) is

\[
(x/a) \cos \theta + (y/b) \sin \theta = 1.
\]
9.2.5. By 9.2, a circle can be constructed whose orthogonal projection is a given ellipse. Hence by 9.1.3 properties of the ellipse which can be expressed in terms of ratios of segments of the same line or of parallel lines can be deduced from corresponding properties of the circle, and by 9.2.3 eccentric angles of points on the ellipse can be replaced by vectorial angles of the corresponding points on the circle.

9.2.6. If the eccentric angles of the points $H$, $K$ on an ellipse differ by a right angle, then the mid-points of chords parallel to the diameter $HOH'$ lie on the diameter $KOK'$.

Conversely if the mid-point of a chord parallel to $HOH'$ lies on $KOK'$, then the eccentric angles of $H$, $K$ or of $H'$, $K'$ differ by a right angle.

\[ \text{Fig. 76} \]

Reverse the notation in 9.1; then $h$, $k$, etc., denote the points on the circle whose projections are the points $H$, $K$, etc., on the ellipse. Since the eccentric angles of $H$, $K$ differ by a right angle, the radii $Oh$, $Ok$ of the circle are at right angles; therefore the diameter $kOk'$, being perpendicular to a chord $pq$ parallel to $hOk'$, bisects it.

\[ \therefore \text{by 9.1.3 (i), } KOK' \text{ bisects the corresponding chord } PQ \text{ parallel to } HOH'. \]

Conversely, it follows by reversing the argument, that the diameters $hOk'$, $kOk'$ of the circle are at right angles and so the vectorial angles of $h$, $k$ or of $h'$, $k'$ differ by a right angle; therefore the eccentric angles of $H$, $K$ or of $H'$, $K'$ do so.

It follows from this property and its converse that if a diameter $KOK'$ bisects a chord $PQ$ parallel to $HOH'$, then the diameter $HOH'$ bisects a chord $PQ'$ parallel to $KOK'$. The pair of diameters $HOH'$, $KOK'$ are then called conjugate diameters; they are the projections of a pair of perpendicular diameters of the corresponding circle.

Note. It is easy to deduce that if $OH$, $OK$ are conjugate semi-diameters, the tangents at $H$, $K$ are parallel to $OK$, $OH$ respectively.

9.2.7. (i) The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$.

(ii) If $\theta_1, \theta_2$, where $0^\circ < \theta_1 < \theta_2 < 360^\circ$, are the eccentric angles of $P_1, P_2$, area of sector $P_1OP_2$ is $\pi ab(\theta_2 - \theta_1)/360$.

(i) The area of the circle, diameter $AA'$, is $\pi a^2$;

\[ \therefore \text{by 9.1.5 (ii), p. 134, area of ellipse } = \pi a^2 \cos a \text{ where } \cos a = b/a, \]

\[ \therefore \text{area of ellipse } = \pi ab. \]

(ii) The vectorial angles of points $P_1$, $P_2$ on the circle corresponding to $P_1$, $P_2$ are equal to $\theta_1$, $\theta_2$,

\[ \therefore \text{by 9.1.5 (ii), area of sector } P_1OP_2 = \pi a^2(\theta_2 - \theta_1)/360; \]

\[ \therefore \text{by 9.1.5 (ii), area of sector } P_1OP_2 = \pi ab \frac{b}{360} \frac{\theta_2 - \theta_1}{a}. \]

Example 1. $OH$, $OK$ and $OP$, $OQ$ are pairs of conjugate semi-diameters of an ellipse. If the tangent at $H$ cuts $OP$, $OQ$ when produced at $P'$, $Q'$ prove that $P'H$. $HQ' = OK^2$.

\[ \text{Fig. 77} \]

Express the required property in terms of ratios of segments of parallel lines:

\[ P'H \quad HQ' \quad OK \quad OK = 1. \]

In Fig. 77, $Oh$, $Ok$ and $Op$, $Oq$ are pairs of perpendicular radii of a circle whose projection is the given ellipse.

Since $\angle p'q' = 1 \text{ rt. } \angle = \angle Ohp'$,

\[ \therefore \text{by 9.1.3 (ii), p. 133, } \frac{P'H}{OK} = \frac{HQ'}{OK} = 1. \]

Example 2. If two chords $p_1q_1$, $p_2q_2$ of a circle intersect at $t$, then $tp_1$, $tq_1 = tp_2$, $tq_2$. Obtain a corresponding property for an ellipse.

Draw the radii $Oh$, $Ok$ parallel to $p_1q_1$, $p_2q_2$.

Consider the orthogonal projection of the figure and use 9.1.3 (iii), p. 133.

\[ \text{where } Oh, Ok \text{ are semi-diameters of the ellipse parallel to } P_1Q_1, P_2Q_2, \]

\[ \therefore TP_1 . TQ_1 = TP_2 . TQ_2 = \text{OH}^2 . \text{OK}. \]

The reader should illustrate this property by a sketch.
EXERCISE 35

1. The tangents at points P, Q on an ellipse meet at T; OH, OK are semi-diameters parallel to TP, TQ. Prove that:
   (i) \( TP : TQ = OH : OK \); (ii) OT bisects PQ.

2. \( PP' \) is a diameter of an ellipse. Prove that the diameters parallel to the chords \( PQ, QP' \) are conjugate.

3. A tangent to an ellipse meets two conjugate diameters at H, K. Prove the other tangents from H, K are parallel.

4. The tangents at P and Q to an ellipse, centre O, meet at T; OT cuts \( PQ \) at V and the ellipse at H. Prove \( OV \cdot OT = OH^2 \).

5. A tangent to \( x^2/a^2 + y^2/b^2 = 1 \) cuts \( x^2/a^2 + y^2/b^2 = k^2 \) \((k > 1)\) at Q, R. Prove that \( OQ \perp PR \).

6. An ellipse touches the sides BC, CA, AB of \( \triangle ABC \) at P, Q, R. Prove that \( BP \cdot CQ \cdot AR = PC \cdot QA \cdot RB \).

7. OH, OK are conjugate semi-diameters of \( x^2/a^2 + y^2/b^2 = 1 \). Prove that area of \( \triangle HOK \) equals \( \frac{1}{4}ab \).

8. If \( OH, OK \) are conjugate semi-diameters of \( x^2/a^2 + y^2/b^2 = 1 \) and if the eccentric angle of \( H \) is \( \theta \), prove that
   (i) \( OH^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta \);
   (ii) \( OH^2 = OK^2 = a^2 + b^2 \).

9. OH, OK are conjugate semi-diameters of an ellipse; a chord \( PQ \) parallel to \( OQ \) meets \( OH \) at \( V \). Prove \( O V^2 \cdot P V^2 = O H^2 \cdot O K^2 \).

10. If \( PQ \) is a diameter of circle \( PQR \), then \( RP^2 + RQ^2 = PQ^2 \). Obtain a corresponding property for an ellipse.

11. If a line cuts a circle at \( P, Q \) and a concentric circle at \( H, K \), then \( PH = KQ \). Obtain a corresponding property for two ellipses.

12. \( POP' \) is a diameter of an ellipse; the tangent at \( P \) meets a chord \( P'Q \) at \( R \). Prove that the tangent at \( Q \) bisects \( PR \).

13. OH, OK are conjugate semi-diameters of an ellipse; \( HRQ \) is drawn parallel to a semi-diameter \( OP \) to meet \( OK \) at \( R \) and the ellipse again at \( Q \). Prove \( HR \cdot HQ = 2OP^2 \).

14. The triangle of greatest area which can be inscribed in a circle is equilateral. Obtain a corresponding property for the ellipse, \( x^2/a^2 + y^2/b^2 = 1 \), and find the greatest area of an inscribed triangle.

15. T is a variable point on the tangent \( CT \) to an ellipse at a fixed point \( C \); \( P \) is the point on the ellipse such that the tangent at \( P \) bisects \( CT \). Prove that \( TP \) passes through a fixed point.

9.3. The equation of an ellipse assumes the simple form
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0,
\]
only if the x-axis and y-axis are taken as the major and minor axes of the ellipse respectively. This standard equation is equivalent to the following geometrical property:

If \( PM, PN \) are the perpendiculars from a point \( P \) on an ellipse to its principal axes \( BC, BA \) respectively, then, see Fig. 78,

\[
\left( \frac{MP}{CA} \right)^2 + \left( \frac{NP}{CB} \right)^2 = 1.
\]

Fig. 78

Example 3. The equations of the major axis \( A'CA \) and minor axis \( B'C'B \) are \( x + 2y = 1 \), \( 2x - y = 3 \), respectively. \( A'A = 3 \), \( B'B = 4 \). Find the equation of the ellipse.

With the notation of Fig. 78, \( CA = 2 \), \( CB = 2 \).

\[
\therefore \text{if } P(x, y) \text{ lies on the ellipse, } \frac{PM^2}{3^2} + \frac{PN^2}{2^2} = 1.
\]

But \( PM = \pm (2x_1 - y_1 - 3)/\sqrt{(2^2 + 1^2)} \) and \( PN = \pm (x_1 + 2y_1 - 4)/\sqrt{(1^2 + 2^2)} \),

\[
\frac{(2x_1 - y_1 - 3)^2}{5 \times 9} + \frac{(x_1 + 2y_1 - 4)^2}{5 \times 4} = 1;
\]

\( \therefore \) the equation of the ellipse is

\[
4(2x - y - 3)^2 + 9(2x + 2y - 4)^2 = 180.
\]

This becomes after expansion and reduction

\[
5x^2 + 4xy + 8y^2 - 24x - 24y = 0.
\]

Example 4. Interpret the locus whose equation is

\[
x^2 + 2y^2 = 150.
\]

The equation can be written in the form

\[
\frac{x^2}{50} + \frac{y^2}{75} = 1.
\]

This represents an ellipse whose centre is at the origin \( O \) and whose principal axes are along \( x'Ox \) and \( y'Oy \).

If \( y = 0, x^2 = 50 \), \( \therefore x = \pm \sqrt{50} = \pm 5\sqrt{2} \);

\( \therefore \) the ellipse cuts \( x'Ox \) at \( A(5\sqrt{2}, 0), A'(-5\sqrt{2}, 0) \); \( \therefore A'A = 10\sqrt{2} \).

If \( x = 0, y^2 = 75 \), \( \therefore y = \pm \sqrt{75} = \pm 5\sqrt{3} \);

\( \therefore \) the ellipse cuts \( y'Oy \) at \( B(0, 5\sqrt{3}), B'(0, -5\sqrt{3}) \); \( \therefore BB' = 10\sqrt{3} \).

Hence, since \( BB' > A'A \), the major axis is along \( y'Oy \) and the minor axis is along \( x'Ox \).
Example 5. Interpret the locus whose equation is

\[ 3x^2 + 2y^2 + 12x - 12y - 120 = 0. \]

The equation can be made more intelligible by completing squares as on p. 14; the equation can be written

\[ 3(x^2 + 4x) + 2(y^2 - 6y) = 120, \]

that is,

\[ 3(x + 2)^2 - 12 + 2(y - 3)^2 - 9 = 120, \]

that is,

\[ 3(x + 2)^2 + 2(y - 3)^2 = 150. \]

This equation can be interpreted more easily if we choose the point \( C(-2, 3) \) as a new origin and draw axes \( CX, CY \) parallel to \( Ox, Oy \), as on p. 90. If the coordinates of any point \( P \) are \( x, y \) referred to axes \( Ox, Oy \) and are \( X, Y \) referred to axes \( CX, CY \), then in Fig. 70

\[ CN' = X - x + 2, \quad NP = Y - y - 3. \]

Hence the equation of the locus referred to \( CX, CY \) as axes is

\[ 3X^2 + 2Y^2 = 150. \]

Therefore as in Example 4, p. 139, the locus is the ellipse whose centre is \( C(-2, 3) \) and whose major axis is of length \( 10 \sqrt{3} \) along \( CY \) and whose minor axis is of length \( 10 \sqrt{2} \) along \( CX \).

EXERCISE 36

Find the coordinates of the centre, the equations of the major and minor axes, and the lengths of the semi-axes of the given ellipse, Nos. 1-11. Illustrate by a sketch.

1. \( 9x^2 + 25y^2 = 36. \)
2. \( 3x^2 + 2y^2 = 30. \)
3. \( 5x^2 + 8y^2 = 10. \)
4. \( 4(x - 1)^2 + 9(y + 4)^2 = 144. \)
5. \( 25(x + 2)^2 + 4(y + 1)^2 = 900. \)
6. \( 10(x + 2)^2 + 5(y - 2)^2 = 12. \)
7. \( 2(x - 1)^2 + 5(y + 4)^2 = 30. \)
8. \( x^2 + 5y^2 + 6x + 10y = 6. \)
9. \( 3x^2 + y^2 - 12x - 6y + 9 = 0. \)
10. \( 2x^2 + 6y^2 + 6x - 6y + 18 = 0. \)
11. \( 5x^2 + y^2 - 32x - 6y + 60 = 0. \)

12. Interpret the locus, \( ax^2 + by^2 + 2gx + 2fy + c = 0, a > b > 0, \) if:

(i) \( d > 0, \) (ii) \( d = 0, \) (iii) \( d < 0, \) where \( d = 4g^2/a + 4f^2/b - c. \)

13. \( PN \) is the perpendicular from a point \( P \) on an ellipse to the major axis \( A'O'A; \) the minor axis is \( B'O'B. \) Prove that

\[ PN^2 : A'N : NA = OB^2 : OA^4 \]

14. Write down the lengths of the perpendiculars from \( P(x_1, y_1) \) to the lines \( x + y - 1 = 0, \) \( x - y - 3 = 0. \) Hence interpret the locus,

\[ (x + y - 1)^2/p^2 + (x - y - 3)^2/q^2 = 1, q > g > 0. \]

9.4. Parametric equations of a point on the ellipse \( x^2/a^2 + y^2/b^2 = 1 \) is suggested by writing the equation in the form

\[ y^2/b^2 = 1 - x^2/a^2 = \left(1 + x/a\right)\left(1 - x/a\right). \]

The equation of any chord \( A'P \) through the vertex \( A'(-a, 0) \) can be written

\[ y/b = t\left(1 + x/a\right); \]

\( A'P \) meets the ellipse where

\[ y/b = t\left(1 + x/a\right) \]

that is, \( y/b = 1 - x/a \) if \( x + a. \)

\[ \therefore \text{the second point of intersection } P \text{ of } A'P \text{ with the ellipse is given by} \]

\[ y/b = t\left(1 + x/a\right), \quad y/b = 1 - x/a, \]

\[ 1 + x/a = y/b = \frac{1}{t} \quad \text{... (i)} \]

This form is often convenient; alternatively, \( x \) and \( y \) can be expressed explicitly in terms of \( t, \) using the ratio method, see S.3.1, p. 119.

Each ratio in (i) \( = \frac{1 + x/a}{1 - x/a} = 1 + \frac{2x/a}{1 - f^2} = 2t/(1 + t^2), \)

\[ \therefore x/a : y/b : (1 - t^2) : 2t : (1 + t^2). \]

This parametric representation can be derived from \( x = a \cos \theta, \) \( y = b \sin \theta. \)

Write \( t = \tan \frac{\theta}{2}; \) then \( \cos \theta = (1 - t^2)/(1 + t^2) \) and \( \sin \theta = 2t/(1 + t^2); \)

\[ \therefore x = a(1 - t^2)/(1 + t^2), \quad y = 2bt/(1 + t^2). \]

To each value of \( t \) corresponds just one point on the ellipse; conversely, each point on the ellipse except \( A'(-a, 0) \) is determined by just one value of \( t. \)

There is no value of \( t \) if \( \theta = 180^\circ, \) that is, if \( x = -a, \) \( y = 0; \)

but when \( t \rightarrow +\infty \) or \( t \rightarrow -\infty, \) \( \theta \rightarrow 180^\circ, \therefore x \rightarrow -a \) and \( y \rightarrow 0. \)
9.4.1. Chord and Tangent. If \( P_1, P_2 \) are the points \( t = t_1, \ t = t_2 \) of the ellipse, 
\[ x(a + (t_1 + t_2)y/b - (1 + t_2) = 0, \]
then the equation of the chord \( P_1P_2 \) is 
\[ (1 - t_1t_2)x/a + (t_1 + t_2)y/b - (1 + t_1t_2) = 0. \]
and the equation of the tangent at \( P_1 \) is 
\[ (1 - t_1^2)x/a + 2t_1y/b - (1 - t_1^2) = 0. \]
The parametric equations of the ellipse can be written, see p. 141, 
\[ (x/a) + (y/b)(1 + t_1t_2) = t_1 + t_2 = 0, \]
that is, the line, \( (1 - x/a - y/b)(t_1 + t_2) + (1 + x/a)t_2 = 0, \) meets the ellipse where 
\[ t_1^2 - 2t_1(t_1 + t_2) + t_2^2 = (t_1 - t_2) = 0, \]
that is, at the points \( P_1, t = t_1; \)
\[ \text{the equation of the tangent at } P_1 \] 
\[ (t_1^2 - 1)x/a - 2t_1y/b + (t_1^2 + 1) = 0. \]
Further, if \( P_1(x_1, y_1) \) is the point \( t = t_1, \)
\[ x_1/a = (1 - t_1^2)/(1 + t_1^2) \] \[ y_1/b = 2t_1/(1 + t_1^2); \]
\[ \text{the equation of the tangent at } P_1(x_1, y_1) \]
to \( x^2/a^2 + y^2/b^2 = 1 \) is 
\[ xx_1/a^2 + yy_1/b^2 = 1. \]
In particular, the equation of the tangent at the point \( (a \cos \theta, b \sin \theta) \) is 
\[ x(a \cos \theta, b \sin \theta) = x^2/a^2 + y^2/b^2 = 1. \]
\[ \text{Note. This result was obtained in 9.2.4, p. 135, by orthogonal projection.} \]
\[ \text{It is easy to prove the equation of the tangent at } (x_1, y_1) \]
because the rule on p. 57 for a circle applies to an ellipse. The equation of the tangent at the point \( t_1 \) can then be written down by expressing \( x_1 \) and \( y_1 \) in terms of \( t_1. \)

9.4.2. If the line \( lx + my + n = 0 \) touches \( x^2/a^2 + y^2/b^2 = 1, \)
\[ a^2l^2 + b^2m^2 = n^2 \]
and conversely.
Let the point of contact be \( (x_1, y_1); \)
then \( lx + my + n = 0 \) is equivalent to \( xx_1/a^2 + yy_1/b^2 = 1 = 0; \)
\[ x_1/2 = x/a^2 + y_1/b^2 = 1; \]
\[ x = -a^2/n, \ y_1 = -b^2m/n; \]
but \( (x_1, y_1) \) lies on \( xx_1/a^2 + yy_1/b^2 = 1, \)
\[ \text{Conversely, if } \]
\[ a^2l^2 + b^2m^2 = n^2; \]
the point \( P(-a^2/n, -b^2m/n) \) \( \text{lies on } xx_1/a^2 + yy_1/b^2 = 1, \)
\[ \text{the tangent at } P \]
is \( x(-a^2/n)(a^2x) + y(-b^2m)(y/b^2) = 1, \)
that is, 
\[ lx + my + n = 0. \]

9.4.3. Equations of Tangents in specified Directions.
(1) By 9.4.2, the line \( x \cos \alpha + y \sin \alpha = p \) touches \( x^2/a^2 + y^2/b^2 = 1 \) if 
\[ p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha; \]
\[ x \cos \alpha + y \sin \alpha = \pm \sqrt{(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} \]
touches \( x^2/a^2 + y^2/b^2 = 1 \) for all values of \( \alpha. \)
(2) By 9.4.2, the line \( y = mx + c \) touches \( x^2/a^2 + y^2/b^2 = 1 \) if 
\[ c^2 = a^2m^2 + b^2; \]
\[ y = mx + \sqrt{(a^2m^2 + b^2)} \]
touches \( x^2/a^2 + y^2/b^2 = 1 \) for all values of \( m. \)

9.4.4. Normal. If \( P \) is a point on an ellipse, the line \( PQ \) through \( P \)
perpendicular to the tangent at \( P \) is called the normal \( \text{at } P \) to the ellipse, and \( P \) is called the foot of a normal from \( Q \) to the ellipse.
The equation of the normal to \( x^2/a^2 + y^2/b^2 = 1 \) at \( P(x_1, y_1) \) is 
\[ \frac{ax}{x_1} - \frac{by}{y_1} = a^2 - b^2. \]
The normal at \( P(x_1, y_1) \) is perpendicular to \( xx_1/a^2 + yy_1/b^2 = 1; \)
\[ : \text{the equation of the normal at } P(x_1, y_1) \] 
\[ a^2x + b^2y = \ldots \]
\[ x_1 = \ldots = \frac{a^2x_1}{x_1} - \frac{b^2y_1}{y_1} = a^2 - b^2. \]
Hence the equation of the normal at the point \( (a \cos \theta, b \sin \theta) \) is 
\[ \frac{ax}{x_1} = \frac{by}{y_1} = a^2 - b^2. \]
Also the equation of the normal at the point \( t \) given by 
\[ \frac{ax}{x_1} = \frac{by}{y_1} = a^2 - b^2. \]
is 
\[ \frac{1 - t^2}{2t} = 1 + t^2. \]

Example 6. Find the equations of the tangents from the point \( (3, -1) \) to the ellipse \( x^2/a^2 + y^2/b^2 = 1. \)
The equation of the ellipse can be written \( x^2/a^2 + y^2/b^2 = 1. \)
\[ \because \text{by 9.4.3, (2), } y = mx \pm \sqrt{(a^2m^2 + b^2)} \]
touches the ellipse and passes through \( (3, -1) \) if \( 1 - 3m = \pm \sqrt{(a^2m^2 + b^2)} \).
\[ \because \text{the gradients of the tangents from } (3, -1) \text{ are given by} \]
\[ (1 + m^2)^2 = (3m^2 + 1)/(1 + m^2)^2; \]
\[ : \text{is } 4m^2 + 36m + 4 = (3m + 2)(15m + 2); \]
\[ : \text{is } m = -\frac{8}{15} \text{ or } -\frac{2}{15}. \]
Hence the equations of the tangents from \( (3, -1) \) are 
\[ 2x + 3y = \ldots = 6 - 3 = 3 \] \[ \text{and } 2x + 15y = \ldots = 6 - 15 = -9. \]
that is, 
\[ 2x + 3y = 3 = 0 \] \[ \text{and } 2x + 15y + 9 = 0. \]
THE ELLIPSE

9.4.5. If \( x + \beta, \ a - \beta \) are the eccentric angles of the points \( P, Q \) on \( x^2/a^2 + y^2/b^2 = 1 \), then the equation of \( PQ \) is

\[
\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = \cos \beta.
\]

First Method. \( P, Q \) are

\[
\{a \cos (x + \beta), b \sin (x + \beta)\}, \{a \cos (x - \beta), b \sin (x - \beta)\},
\]

\( \therefore \) gradient of \( PQ \)

\[
\frac{b \sin (x + \beta) - \sin (x - \beta)}{a \cos (x + \beta) - \cos (x - \beta)} = \frac{2b \cos x \sin \beta}{-2a \sin \alpha \sin \beta} = \frac{b \cos \alpha}{a \sin \alpha}.
\]

\( \therefore \) equation of \( PQ \) is

\[
(x/a) \cos \alpha + (y/b) \sin \alpha = \cos \alpha \cos \beta,
\]

that is,

\[
(x/a) \cos \alpha + (y/b) \sin \alpha = \cos \alpha \cos \beta.
\]

Second Method. Let \( x^2/a^2 + y^2/b^2 = 1 \) be the orthogonal projection of the circle \( X^2 + Y^2 = a^2 \) and let \( P, Q \) be the projections of points \( p, q \) on the circle; then \( Op, Oq \) make angles \( x + \beta, \ a - \beta \) with \( OX \); hence, if \( \text{Oh} \) is the perpendicular to \( pq \), \( \angle hOX = x, \angle pOh = \beta, Oh = a \cos \beta \);

\( \therefore \) the perpendicular form of the equation of \( PQ \) is

\[
X \cos x + Y \sin x = a \cos \beta;
\]

that is,

\[
(x/a) \cos \alpha + (y/b) \sin \alpha = \cos \alpha \cos \beta.
\]

If the eccentric angles of \( P_1, P_2 \) are denoted by \( \theta_1, \ \theta_2 \),

then \( x + \beta = \theta_1, \ x - \beta = \theta_2 \),

and

\[
\frac{x}{a} \cos \theta_1 + \frac{y}{b} \sin \theta_1 = \cos \theta_1 \cos \beta.
\]

The equation of \( P_1P_2 \) is

\[
\frac{x}{a} \cos \theta_1 \cos \beta + \frac{y}{b} \sin \theta_1 \cos \beta = \cos \beta.
\]

It is left to the reader to show that this equation is equivalent to

\[
(1 - t_1t_2)x/(a + t_1t_2)y/b = 1 - t_1t_2,
\]

see 9.4.1, p. 142, if \( \tan \theta_1 = t_1 \) and \( \tan \theta_2 = t_2 \), see 9.4, p. 141.

Example 7. If \( 0 + \beta, \ 0 - \beta \) are the eccentric angles of points \( P, Q \) on \( x^2/a^2 + y^2/b^2 = 1 \), where \( \theta \) varies and \( \beta \) is constant, prove that \( PQ \) touches a fixed ellipse and find its equation.

First Method. By 9.4.5, the equation of \( PQ \) can be written

\[
x \cos \beta/a \cos \beta + y \sin \beta,b \cos \beta = 1.
\]

By 9.4.1, p. 142, \( x \cos \theta/A + y \sin \theta/B = 1 \) touches \( x^2/A^2 + y^2/B^2 = 1 \) at \( (A \cos \theta, B \sin \theta) \).

\( \therefore \) \( PQ \) touches \( x^2/(a \cos \beta)^2 + y^2/(b \cos \beta)^2 = 1 \),

that is, the similar ellipse \( x^2/a^2 + y^2/b^2 = \cos^2 \beta \).

Second Method. With the notation of the second method of 9.4.5,

\( \text{Oh} = a \cos \beta = \text{constant} \).

\( \therefore \) \( PQ \) touches the circle \( X^2 + Y^2 = (a \cos \beta)^2 \).

\( \therefore \) \( PQ \) touches the ellipse \( x^2/a^2 + y^2/b^2 = \cos^2 \beta \).

Example 8. Find the coordinates of the foot of the normals to \( 5x^2 + 12y^2 = 120 \),

\( \parallel \) to \( 2x - y = 0 \).

If \( (x, y) \) is the foot of a normal \( \parallel \) to \( 2x - y = 0 \), the tangent at \( (x_1, y_1) \), \( 6ax_1 + 12by_1 = 120 \), is parallel to \( x + 2y = 0 \).

\( \therefore \) \( 5x_1 = 6y_1 \), where \( 5x_1^2 + 12y_1^2 = 120 \).

\( \therefore \) \( 5x_1^2 + 60y_1^2 = 600 \), \( y_1^2 = 900/60 = 15/4 \), \( y_1 = \pm 3/4 \).

The foot of the normals are \( (3, 2/3) \) and \( (-3, -2/3) \).

EXERCISE 37

[Assume in this exercise that \( a > b > 0 \) in the equation \( x^2/a^2 + y^2/b^2 = 1 \); \( A'A' \) denotes the major axis.]

Find the equations of the tangent and normal at the given point:

\[
1. \ \ 5x^2 + 4y^2 = 24; \quad (2, -1). \\
2. \ \ 9x^2 + 2y^2 = 36; \quad (-2, 3).
\]

3. \ Prove that there is a point \( (x_1, y_1) \) on \( 4x^2 + 3y^2 = 4 \) at which the tangent is \( 2x - 3y = 4 = 0 \). Find \( x_1, y_1 \).

4. \ Prove that \( 3x - y = 7 \) is a tangent to \( 3x^2 + 2y^2 = 14 \) and find the coordinates of the point of contact.

5. \ Find the equations of the tangents to \( 9x^2 + 4y^2 = 100 \) \( \parallel \) to \( 2x - 8y = 0 \).

6. \ Find the equations of tangents to \( 6x^2 + 4y^2 = 15 \) with gradient \( -1 \).

7. \ Find the equations of the normals to \( 9x^2 + 4y^2 = 36 \) \( \parallel \) to \( 3x - 9y = 0 \).

8. \ Find the equations of normals to \( 4x^2 + 9y^2 = 1 \) with gradient 2.

9. \ Prove that \( 3x + 2y + 3 = 0 \) is a normal to \( 2x^2 + 3y^2 = 45 \) and find the coordinates of the foot of the normal.

10. \ Find the equations of the tangents from \( (1/2, 2) \) to \( 9x^2 + 2y^2 = 6 \).

11. \ Find the equations of the tangents from \( (3/4, 1/2) \) to \( x^2 + 2y^2 = 25 \).

12. \ Prove the foot of the perpendiculars from \( (2, 0), \ (-2, 0) \) to the tangent at \( (2, 1) \) to \( 5x^2 + 9y^2 = 45 \) lie on the circle \( x^2 + y^2 = 9 \).

13. \ \( P, Q \) are corresponding points on an ellipse and its auxiliary circle; prove the tangents at \( P, Q \) meet on \( AA' \%).

14. \ \( K \) is the point \( (k, k) \); \( P \) is the point \( t = p \) on the ellipse,

\( x/a : y/b : 1 = (1 - \epsilon) : 2 : (1 + \epsilon) \),

\( PK \) meets the ellipse again at \( t = q \). Find \( q \) in terms of \( a, k, p \).

15. \ \( P, Q, R \) are points on \( x/a : y/b : 1 = (1 - \epsilon) : 2 : (1 + \epsilon) \), given by \( t = p, \ t = q, \ t = r \); \( PQ, PR \) pass through the points \( (f, 0), (g, 0) \); find \( r \) in terms of \( a, q, f, g \).
16. If \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) are points on \( px^2 + qy^2 = 1 \), verify that the equation of \( P_1P_2 \) is \( p(x_1 + x_2)x + q(y_1 + y_2)y = p^2x_1x_2 + q^2y_1y_2 + 1 \).

[17] Find the coordinates of the points of contact of tangents to \( x^2/a^2 + y^2/b^2 = 1 \) parallel to \( x + y = 0 \).

18. Find the condition that \( y = mx \) touches the ellipse
\[ a^2x^2 + b^2y^2 + 2px + 2qy = 0. \]

[19] Find the equations of the normals to \( x^2/a^2 + y^2/b^2 = 1 \) which are parallel to \( ax - by = 0 \).

20. If \( lx + my + n = 0 \) is a normal to \( x^2/a^2 + y^2/b^2 = 1 \), prove that \( a^2l^2 + b^2m^2 = (a^2 - b^2)n^2 \).

21. If \( P, Q \) are eccentric angles of \( P, Q \) on \( x^2/a^2 + y^2/b^2 = 1 \); \( OQ \) is parallel to tangent at \( P \); find \( \phi \) in terms of \( \theta \).

22. If \( \pi \cos \theta + \pi \sin \theta = P \) meets \( x^2/a^2 + y^2/b^2 = 1 \) at \((x_1, y_1) \), \((x_2, y_2) \), prove \( x_1x_2 + y_1y_2 = 1 - a^2(l^2 - b^2 \sin^2 \theta) - b^2(\pi^2 - a^2 \cos^2 \theta) = a^2 \cos^2 \theta + \pi^2 \sin^2 \theta \).

23. A variable tangent to \( x^2/a^2 + y^2/b^2 = 1 \) meets \( Ox, Oy \) at \( Q, R \); \( QORP \) is a rectangle. Prove locus of \( P \) is \( x^2y^2 = b^2x^2 + a^2y^2 \).

24. A tangent to \( x^2/a^2 + y^2/b^2 = 1 \) with positive gradient \( m \) meets \( OX, OY \) at \( Q, R \). Prove area \( \triangle OQR = \frac{1}{2} [a(\sqrt{m} - b) + b(\sqrt{m})] \). Find the point of contact if area \( \triangle OQR \) is a minimum. (OC)

25. If \( P \) is tangent and normal at \( Q \) on \( x^2/a^2 + y^2/b^2 = 1 \) meet the major axis \( AA' \) at \( T \), \( O \). Prove \( OT \cdot OQ = a^2 - b^2 \).

26. The tangent at \( A, A' \) in \( H, H' \).

27. The normal to \( x^2/a^2 + y^2/b^2 = 1 \) at \( P \) meets the major axis \( AA' \) at \( G \); \( PN \) is the perpendicular to \( AA' \). Prove \( OQ \parallel ON \sin \alpha \).

28. The normal to \( x^2/a^2 + y^2/b^2 = 1 \) at \( P \) meets passing through \( B \), \( O, b \). Prove \( OQ = a^2 - a^2b^2 \).

29. The tangent to \( x^2/a^2 + y^2/b^2 = 1 \) at \( P \) also touches the circle \( x^2 + y^2 = c^2 \), where \( a^2 > c^2 > b^2 \), prove \( OP^2 = a^2b^2 - a^2b^2/c^2 \).

30. If the tangent to \( x^2/a^2 + y^2/b^2 = 1 \) at \( P \) also touches the circle \( x^2 + y^2 = c^2 \), where \( a^2 > c^2 > b^2 \), prove \( OP^2 = a^2b^2 - a^2b^2/c^2 \).

31. \( P, Q \) are variable points on \( x^2/a^2 + y^2/b^2 = 1 \); \( A \) is a vertex and \( AP, AQ \) meet \( OY \) at \( H, K \). If \( OH \cdot OK \) is constant, prove \( PQ \) passes through a fixed point. (N)

32. If \( y = mx + c, y = mx + c \) touch \( x^2/a^2 + y^2/b^2 = 1 \) and meet at \( (h, k) \), prove that \( m_1, m_2 \) are the roots of \( (mk - h)^2 = a^2m^2 + b^2 \). Find the relation between \( h, k, a, b \) if \( m_1m_2 = -1 \), and interpret the result geometrically.

9.5. Chord of Contact. If \( PQ \) is the chord of contact of the tangents from a point \( U \) to an ellipse, \( PQ \) is called the polar of \( U \) with respect to the ellipse and \( U \) is called the pole of \( PQ \) with respect to the ellipse.

9.5.1. If \( PQ \) is the chord of contact of the tangents from \( U(x', y') \) to the ellipse \( x^2/a^2 + y^2/b^2 = 1 \), the equation of \( PQ \) is \( \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \).

Let \( P \) and \( Q \) be the points \( (x_1, y_1), (x_2, y_2) \).

The tangent at \( P \), \( xx_1/a^2 + yy_1/b^2 = 1 \), passes through \( U(x', y') \), \( xx'/a^2 + yy'/b^2 = 1 \).

This relation is precisely the condition that \( P(x_1, y_1) \) lies on the line whose equation is \( xx'/a^2 + yy'/b^2 = 1 \).

In exactly the same way it can be proved that \( Q(x_2, y_2) \) lies on this line, and so this is the equation of \( PQ \).

This result can also be stated as follows:

The equation of the polar of \( U(x', y') \) with respect to the ellipse,

\[ \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1, \]

is \( \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \).

It is easy to remember this result because the equation is of the same form as the equation of the tangent, see p. 123.

9.5.2. If the line \( lx + my + n = 0 \) meets \( x^2/a^2 + y^2/b^2 = 1 \) at \( P, Q \), the tangents at \( P \) and \( Q \) meet at the point \( U(-a^2/m, -b^2/m) \).

Let the tangents at \( P \) and \( Q \) meet at \( U(x', y') \), then

\[ xx'/a^2 + yy'/b^2 = 1 \]

is equivalent to \( lx + my + n = 0 \);

\[ \frac{x'}{x} = \frac{y'}{y} = \frac{1}{1} \]

\[ \frac{a^2}{a^2} = \frac{b^2}{b^2} = \frac{1}{n} \]

\[ x' = -\frac{a^2}{n}, y' = -\frac{b^2}{n} \]

This result can also be stated as follows:

The pole of the line \( lx + my + n = 0 \) with respect to the ellipse,

\[ x^2/a^2 + y^2/b^2 = 1, \]

is \( U(-a^2/m, -b^2/m) \).

9.5.3. If the chord of contact \( PQ \) of the tangents from \( U_1 \) to the ellipse \( px^2 + qy^2 = 1 \) produces to \( U_2 \), then the chord of contact of the tangents from \( U_1 \) to \( px^2 + qy^2 = 1 \) passes through \( U_2 \).

Let \( U_1, U_2 \) be the points \( (x_1, y_1), (x_2, y_2) \), then the equation of \( PQ \) is \( px_1 + qy_1 = 1 \); but \( U_2(x_2, y_2) \) lies on \( PQ \).

The equation of the chord of contact of the tangents from \( U_1(x_1, y_1) \) to the ellipse \( px_2 + qy_2 = 1 \) is \( px_2 + qy_2 = 1 \); this chord of contact passes through \( U_1(x_1, y_1) \).

Thus \( U_1, U_2 \) are points such that the polar of either passes through the other; \( U_1 \) and \( U_2 \) are then called conjugate points.
THE ELLIPSE

9.5.4. No tangent can be drawn from \( U(x_1, y_1) \) to \( x^2/a^2 + y^2/b^2 - 1 = 0 \)
if
\[ x_1^2/a^2 + y_1^2/b^2 - 1 < 0. \]
No tangent can be drawn from \( U(x_1, y_1) \) if the line \( xx_1/a^2 + yy_1/b^2 - 1 = 0 \)
does not meet the ellipse \( x^2/a^2 + y^2/b^2 - 1 = 0 \) whose points are given by
\[ x/a : y/b : 1 = (1 - t^2) : 2t : (1 + t^2). \]
\[ : \text{ no tangent can be drawn if there is no value of } t \text{ such that} \]
\[ (1 - t^2)x_1/a^2 + 2ty_1/b^2 - (1 + t^2) = 0, \]
that is, if
\[ y_1/b^2 + (x_1/a + 1)(x_1/a - 1) = y_1/b^2 + x_1/a - 1 = 0. \]
This is the condition that \( U(x_1, y_1) \) lies 'inside' the ellipse, where
the word 'inside' is used in its natural sense.
The coordinates of points on \( x^2/a^2 + y^2/b^2 = 1, \ a > b > 0 \), satisfy the
relation,
\[ y/b = \pm \sqrt{(1 - x^2/a^2)}; \]
\[ : \text{ } U(x_1, y_1) \text{ lies 'inside' } x^2/a^2 + y^2/b^2 = 1 \text{ if} \]
either \( - \sqrt{(1 - x_1^2/a^2)} < y_1/b < 0 \) or \( 0 < y_1/b < \sqrt{(1 - x_1^2/a^2)} \),
that is, if \( x_1^2/a^2 + y_1^2/b^2 - 1 < 0. \)
Alternative methods are indicated in Exercise 38, Nos. 7, 8.

EXERCISE 38

1. Find the polar of \((1, 3)\) with respect to \(4x^2 + 9y^2 = 36.\)
2. Find the polar of \((-2, 5)\) with respect to \(3x^2 + 2y^2 = 1.\)
3. The equation of the chord of contact of the tangents from \( T \)
   to \( x^2 + 2y^2 = 9 \) is \( x - 2y + 9 = 0. \) Find: (i) the coordinates of \( T; \) (ii) the
   equations of the tangents from \( T.\)
4. Find the points \( P, Q \) where \( x + 3y = 13 \) cuts \( x^2 + 4y^2 = 65 \)
   and the point of intersection of the tangents at \( P \) and \( Q.\)
5. \( PQ \) is a chord of the ellipse \( x^2/a^2 + y^2/b^2 = 1 \) parallel to the line \( bx + ay = 0. \) Prove that the tangents at \( P \) and \( Q \) meet on the line \( bx - ay = 0. \)
6. The normal to \( x^2/a^2 + y^2/b^2 = 1 \) at \( (a \cos \theta, b \sin \theta) \) meets the
   ellipse again at \( Q. \) Find the coordinates of the pole of \( PQ.\)
7. Find the condition that there is no value of \( m \) for which
   \( y_1 = mx_1 + \sqrt{(a^2m^2 + b^2)}. \)
8. Find the condition that each of the ratios in which the line joining
   \( O \) to \( U(x_1, y_1) \) is divided by its points of intersection with
   \( x^2/a^2 + y^2/b^2 = 1 \)
is negative. Interpret the condition geometrically.
9. If the pole of \( t, x + my + n = 0 \) with respect to \( px^2 + qy^2 = 1 \)
lies on \( lx + my + n = 0, \)
prove \( l^2x^2/p + m^2y^2/q - n^2n = 0, \)

Polar and Pole

10. The eccentric angles of variable points \( P, Q \) on \( x^2/a^2 + y^2/b^2 = 1 \)
are \( \theta + \phi, \theta - \phi, \) where \( \theta \) is constant. Prove that the pole \( T \) of \( PQ \)
is \((a \cos \beta \cos \phi, b \sin \beta \sin \phi); \) and that the equation of the locus of \( T \)
is \( x^2/a^2 + y^2/b^2 = \sec^2 \beta. \) Deduce these results also by the method of
    orthogonal projection.
11. Prove that the pole of any tangent to \( x^2/a^2 + y^2/b^2 = 1 \) with respect
    to the ellipse \( x^2/a^2 + y^2/b^2 = a + b \) lies on the circle \( O, \) radius \( a + b.\)
12. A variable line \( lx + my = 1 \) passes through the fixed point \( (b, k) \)
and meets the given ellipse \( px^2 + qy^2 = 1 \) at \( P, Q. \) Prove that the tangents
    at \( P, Q \) meet on a fixed line and find its equation.
13. \( P, Q \) is the chord of contact of the tangents from a variable point\( \]
    \( T(x_1, y_1) \) on the fixed line \( bx + my = 1 \) to the given ellipse \( px^2 + qy^2 = 1. \)
    Prove that \( PQ \) passes through a fixed point and find its coordinates.
14. \( P, Q \) are variable points on \( x = a, \ x = -a; \) such that \( P, Q \) are
congruent points with respect to \( x^2/a^2 + y^2/b^2 = 1. \) Prove that the envelope
    of \( PQ \) is \( x^2/a^2 + 2y^2/b^2 = 1. \)
15. \( P_1, P_2 \) are the points \( t - t_1, t - t_2 \) on \( x/a = y/b = 1 - (t - t_1) \cdot 2t : (1 + t^2; \)
    \( x', y' \) is the pole of \( P_1P_2, \) prove \( x'/a : y'/b : 1 = (1 - t_1t_2) : (t_1 + t_2) : (1 + t_1t_2). \)
    Explain the exceptional case, \( 1 + t_1t_2 = 0. \)
16. The eccentric angles of variable points \( P, Q \) on \( x^2/a^2 + y^2/b^2 = 1 \)
are \( \theta, \ 90^\circ. \) Find the equation of the locus of the pole of \( PQ.\)
17. Prove that the locus of the pole of a variable tangent of the parabola 
    \( ay^2 = 2bx \) with respect to \( x^2/a^2 + y^2/b^2 = 1 \) is an equal parabola.
18. \( C, D \) are fixed points \((c, 0), \ (d, 0); \) \( PCQ, \ PDQ \) are chords of 
    the ellipse, \( x/a : y/b : 1 - (t - t^2) : 2t : (1 + t^2). \) \( P \) varies, prove that
    the pole of \( PQ \) lies on a fixed line parallel to \( Oy.\)
19. The line joining \( P_1(x_1, y_1) \) to \( P_2(x_2, y_2) \) cuts \( x^2/a^2 + y^2/b^2 = 1 \)
    at \( Q_1, Q_2. \) If the ratios \( P_1Q_1 : Q_1P_2 \) and \( P_2Q_2 : Q_2P_2 \) are equal in magnitude
    but opposite in sign, prove \( x_1x_2/a^2 + y_1y_2/b^2 = 1 \) and interpret this result
    geometrically.
20. \( AA' \) is the major axis of \( x^2/a^2 + y^2/b^2 = 1, \ a > b > 0; \) \( P \) is the point
    \((a \cos \theta, b \sin \theta); \) prove that the orthocentre of \( \triangle PAA' \)
    lies \( H(a \cos \theta, a \sin \theta/b) \)
and that the polar of \( H \) with respect to the auxiliary circle touches the
    ellipse.
21. \( P_1P_2 \) are the points \( t = t_1, t = t_2 \) on \( x/a : y/b : 1 - (t - t_1) : 2t : (1 + t^2; \)
    \( A, B \) are the points \((a, 0), \ (0, b); \) the poles of \( AP_1, BP_2 \) are \((ap, b); \ (ap, b); \)
    prove: (i) \( t_1t_2 + t_1 + t_2 = 1; \) (ii) the pole of \( P_1P_2 \) lies on \( x/a = y/b. \)
9.6. Starting with the definition of an ellipse as the orthogonal projection of a circle, it is natural to proceed to deduce properties of the ellipse from those of the circle; in particular, some properties of conjugate diameters and of eccentric angles were discussed on pp. 136 and 144.

Alternative methods based directly on the equation \(x^2/a^2 + y^2/b^2 = 1\) or the equation \(px^2 + qy^2 = 1\), where \(p > 0\) and \(q > 0\), are useful.

9.6.1. If \(V(h, k)\) is the mid-point of a chord \(P_1P_2\) of \(px^2 + qy^2 = 1\), the gradient of \(P_1P_2\) is equal to \(-\frac{ph}{qk}\).

First Method. Let \(P_1, P_2\) be the points \((x_1, y_1), (x_2, y_2)\), then \(x_1 + x_2 = 2h\) and \(y_1 + y_2 = 2k\).

Also \(px_1^2 + qy_1^2 = 1 \Rightarrow px_2^2 + qy_2^2 = 1\), \(q(y_2^2 - y_1^2) = -p(x_2^2 - x_1^2)\), \(q(y_2 - y_1)(y_2 + y_1) = -p(x_2 - x_1)(x_2 + x_1)\).

First gradient of \(P_1P_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{p(2h)}{q(2k)} = \frac{ph}{qk}\).

Second Method. The point reached by starting from \(V(h, k)\) and moving a directed distance \(r\) in the direction making an angle \(\theta\) with \(OX\) is \((h + r \cos \theta, k + r \sin \theta)\).

If this point lies on \(px^2 + qy^2 = 1\), then \(p(h + r \cos \theta)^2 + q(k + r \sin \theta)^2 = 1\), that is,

\[r^2(p \cos^2 \theta + q \sin^2 \theta) + 2r(\frac{ph}{qk} \cos \theta + \frac{pk}{qk} \sin \theta) + (\frac{ph^2}{qk^2} - \frac{p}{q}) = 0\]

Hence if \(V\) is the mid-point of \(P_1P_2\) when \(P_1P_2\) makes an angle \(\theta\) with \(OX\), the roots \(r_1, r_2\) of the quadratic in \(r\) are equal in magnitude and opposite in sign:

\[r_1 + r_2 = 0, \quad \frac{ph}{qk} \cos \theta + \frac{pk}{qk} \sin \theta = 0;\]

\[\text{gradient of } P_1P_2 = \frac{\sin \theta}{\cos \theta} = -\frac{ph}{qk}.

Note. It follows that equation of \(P_1P_2\) is \(phx + qky = ph^2 + qk^2\).

9.6.2. The locus of the mid-points of chords of \(px^2 + qy^2 = 1\) which are parallel to the diameter \(y = mx\), \(m > 0\), is the diameter \(y = m'x\), where \(mm' = -\frac{p}{q}\).

If \(V(h, k)\) is the mid-point of a chord of gradient \(m\), then, by 9.6.1, \(m = -\frac{ph}{qk}\); \(mk = -ph\);

\[V(h, k)\] lies on the line \(my = px;\)

\[\text{the locus of } V \text{ is the line } y = m'x \text{ where } m' = -\frac{p}{mg};\]

\[mm' = -\frac{p}{q}.

This relation is symmetrical in \(m\) and \(m'\); therefore if \(y = m'x\) bisects chords parallel to \(y = mx\), then \(y = mx\) bisects chords parallel to \(y = m'x\). Thus \(y = mx, y = m'x\) are conjugate diameters, see p. 138, if \(mm' = -\frac{p}{q}\).

9.6.3. (i) \(y = mx, y = m'x\) are conjugate diameters of \(x^2/a^2 + y^2/b^2 = 1\) if \(mm' = -\frac{b^2}{a^2}\).

(ii) If \(\theta\) and \(\theta'\) are the eccentric angles of the extremities \(H, K\) of two conjugate diameters, then \(\theta - \theta' = 0^\circ\) or \(270^\circ\) and the tangent \(HT\) at \(H\) is parallel to \(OK\).

(i) In the equation \(px^2 + qy^2 = 1\), write \(1/a^2, 1/b^2\) for \(y, q\), then the condition \(mm' = -\frac{b^2}{a^2}\) becomes \(mm' = -\frac{b^2}{a^2}\).

(ii) If \(m = \frac{b}{a} \cos \theta\) and \(m' = \frac{b}{a} \cos \theta'\) then \(\frac{a^2 \cos \theta \cos \theta'}{a^2 \cos \theta' \cos \theta} = \frac{b^2}{a^2}\).

\[\tan \theta' = -\cot \theta = \tan (90^\circ + \theta), \quad \theta' = 0 = 90^\circ \text{ or } 270^\circ.

Hence \(H, K\) may be taken as \(H(a \cos \theta, b \sin \theta), K(-a \sin \theta, b \cos \theta)\), see Fig. 80, where \(\theta\) is given a suitable value between \(0^\circ\) and \(360^\circ\).

Since the tangent \(HT\) at \(H\) is \((x/a) \cos \theta + (y/b) \sin \theta = 1\), \(HT\) is parallel to \(OK, x/(-a \sin \theta) = y/(b \cos \theta)\).

[Fig. 80]

9.6.4. \(HOH', KOK'\) are conjugate diameters of \(x^2/a^2 + y^2/b^2 = 1\) and \(QQ'\) is a chord parallel to \(KOK'\).

(i) If \(\theta\) is the eccentric angle of \(H, \) the eccentric angles of \(Q, Q'\) may be taken as \(\theta + \phi, \theta - \phi\).

(ii) The pole \(U\) of \(QQ'\) lies on \(OH\).

(i) Denote the eccentric angle of \(Q\) by \(\theta + \phi\), then by 9.4.5, p. 144, the equation of the chord joining \(Q\) to the point whose eccentric angle is \(\theta - \phi\) is \((x/a) \cos \theta + (y/b) \sin \theta = \cos \phi;\) this is the equation of a line parallel to the tangent at \(H\) \((a \cos \theta, b \sin \theta)\) and so parallel to \(KOK'\); therefore it is the equation of the chord \(QQ'\).

(ii) Let \(U\) be the point \((x_1, y_1)\), then \(xx_1/a^2 + yy_1/b^2 = 1\) is equivalent to \((x/a) \cos \theta + (y/b) \sin \theta = \cos \phi;\)

\[x_1/(a \cos \theta) = y_1/(b \sin \theta);\]

but the equation of \(OH\) is \(x/(a \cos \theta) = y/(b \sin \theta),\)

\[U(x_1, y_1)\] lies on \(OH\).

The diameter \(OH\) which bisects the chord \(QQ'\) is said to be conjugate to \(QQ'\) because it passes through the pole of \(QQ'\).
9.6.5. The coordinates of a point are not changed when its eccentric angle is increased or decreased by a multiple of 360°. It follows from 9.6.3 and 9.5.4 that if HOH', KOK' are conjugate diameters and if QQ' is a chord parallel to KK', the eccentric angles of H, H', K, K', Q, Q' can be taken as shown in Fig. 80, p. 151.

9.6.6. If OH, OK are conjugate semi-diameters of \(x^2/a^2 + y^2/b^2 = 1\) and if HOKT is a parallelogram, then

\[(1) \text{ OH}^2 + \text{ OK}^2 = a^2 + b^2; \quad (2) \text{ Area HOKT} = ab\]

(i) H and K may be taken as \((a \cos \theta, b \sin \theta), \ (-a \sin \theta, b \cos \theta)\);
\[\therefore \text{ OH}^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta \quad \text{and} \quad \text{ OK}^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta;\]
\[\therefore \text{ OH}^2 + \text{ OK}^2 = a^2 (\cos^2 \theta + \sin^2 \theta) + b^2 \cos^2 \theta + b^2 \sin^2 \theta = a^2 + b^2.\]

(ii) Let OF be the perpendicular from O to HT, see Fig. 81.
Since HT is parallel to OK, HT is the tangent at H;
\[\therefore \text{ the equation of HT is } (x/a) \cos \theta + (y/b) \sin \theta = 1;\]
\[\therefore \text{ OF} = 1/\sqrt{(\cos^2 \theta/a^2 + \sin^2 \theta/b^2)} = ab/\sqrt{(b^2 \cos^2 \theta + a^2 \sin^2 \theta)};\]
\[\therefore \text{ OF} = ab/\text{OK}; \quad \therefore \text{ area HOKT} = \text{OF} \cdot \text{OK} = ab.\]

If the lengths of the conjugate semi-diameters OH, OK are equal, the diameters are called **equi-conjugate**.
It is evident that \(\text{OH}^2 = \text{OK}^2\) if \(\theta = 45°;\)
the equations of \(\text{OH, OK}\) are then \(x/a = \pm y/b.\)

9.6.7. If Q is a point on an ellipse and if POP' is a diameter, the diameters parallel to QP, QP' are conjugate.

**First Method.** Let the sum of the eccentric angles of P and Q be 2α, then the sum of the eccentric angles of P' and Q is 2α + 180°, see Fig. 82.
\[\therefore \text{ by 9.4.5, p. 144, the gradients of } \text{ PQ, P'Q} \text{ equal } -(b/a) \cot\alpha, -(b/a) \cot(\alpha + 90°); \text{ but cot } (\alpha + 90°) = -\tan\alpha,\]
\[\therefore \text{ product of gradients of PQ, P'Q} \text{ is } -b^2/a^2;\]
\[\therefore \text{ the diameters parallel to PQ, P'Q are conjugate.}\]

9.6.8. If OH, OK are conjugate semi-diameters of \(x^2/a^2 + y^2/b^2 = 1,\)
a > b > 0, and if the normal at H meets the major axis AA' at G, then
\[\text{HG} : \text{OK} = b : a.\]
Let \(H, K\) be \((a \cos \theta, b \sin \theta), \ (-a \sin \theta, b \cos \theta)\);
draw \(HN\) perpendicular to \(AA'\) and let \(GH\) make an angle \(\phi\) with \(A' A\), see Fig. 81, p. 152.
The equation of \(HG\) is \(ax - by = a^2 - b^2, \quad \therefore \tan \phi = a \sin \theta/b \cos \theta;\)
also \(\text{OK}^2 = a^2 \sin^2 \theta + b^2 \cos^2 \theta; \quad \text{hence, working with equal ratios,}\)
\[a \sin \phi/b \cos \phi = \sqrt{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)} = \text{OK};\]
\[\therefore \text{ GH} = \text{NH} = b \sin \phi = b \sin \phi/b \cos \phi = \text{OK}.\]

**EXERCISE 39**

[In this exercise, A'O'A along Ox is the major axis and B'O'B along Oy is the minor axis of \(x^2/a^2 + y^2/b^2 = 1;\) HOH', KOK' denote a pair of conjugate diameters.]

1. Find the gradient of the tangent to \(4x^2 + 5y^2 = 100\) at the point \((-2, 3);\) hence find the equation of the chord whose mid-point is \((-2, 3).\)
2. Find the coordinates of the mid-point of the chord of \(3x^2 + 5y^2 = 120\) whose equation is \(4x - 5y = 31.\)
3. Explain why the chord of \(px^2 + qy^2 = 1\) whose mid-point is \((h, k)\) is parallel to \(phx + qky = 0\) and find its equation, (see No. 1).
4. A variable chord \(PQ\) of \(x^2 + 4y^2 = 1\) whose mid-point is \((h, k)\) is parallel to \(phx + qky = 0\) and find its equation, (see No. 1).
Illustrate the result by a sketch.

5. \(HN, KR\) are the perpendiculars from \(H, K\) to \(AA'\). Prove that
\[(i) \angle OHN = \angle ORK, \quad (ii) \angle HN: KR : ON = OR = b^2 : a^2.\]
6. The tangents at \(H, K\) meet \(OA, OB\) at \(Q, R\) respectively. Prove \(QR\) is parallel to an equi-conjugate diameter.
[7] HP is a chord parallel to an axis; prove KP is parallel to an equi-conjugate diameter.

8. P, Q are points on an ellipse, centre C, such that OQ bisects the normal chord at P. Prove that OP bisects the normal chord at Q.

[9] The normal at the point H on \(x^2/a^2 + y^2/b^2 = 1\) meets AA' at G and meets OK at F. Prove HF. HG = b^2.

10. If \(lx + my + n = 0\) meets \(px^2 + qy^2 = 1\) at P, Q, prove that the pole of PQ lies on \(pmx = qny\); deduce that the mid-point of PQ is given by the ratios, \(x : y : -m = q : p : ql^2 + pm^2\).

11. M, N, P, Q are variable points on an ellipse; the chords MN, NP, PQ are fixed in direction; prove MQ is fixed in direction.

12. The eccentric angles of points P, P', P'' on an ellipse are \(\theta_1, \theta_2, \theta_3\). If \(P_1P_2P_3\) is parallel to the tangent at \(P_1\) and if \(P_2P_3\) is parallel to the tangent at \(P_2\), prove that \(P_1P_3\) is parallel to the tangent at \(P_3\) and find \(\theta_3, \theta_2\), in terms of \(\theta_1\). (OC)

13. The normals at H and K meet AA' at G and R. Prove that \(HG + HR = b^2(a^2 + b^2)/a^2\).

14. HP is a chord perpendicular to AA'. Prove that the normals at K and P meet on one of the lines \(ax + by = 0\).

15. A variable chord PQ of \(x^2/a^2 + y^2/b^2 = 1\) passes through the fixed point \((h, k)\). Prove that the equation of the locus of the centroid of \(\Delta OPQ\) is \(x(2x - 2h)a^2 + y(3y - 2k)b^2 = 0\), and find the coordinates of the centre of this ellipse. [Use No. 3.]

16. If the normals at H and K meet at R, prove that OR is perpendicular to HK.

17. PN is the perpendicular to AA' from a point P on \(x^2/a^2 + y^2/b^2 = 1\); QQ' is a diameter; PQ, PQ' meet AA' at R, R' and make angles \(\theta, \theta'\) with AA'. Prove that:

(i) \(\tan \theta \tan \theta' = -b^2/a^2\) (ii) \(R'N. NR = PN^2 = a^2 : b^2\).

18. The perpendiculars from O to the tangents from P to \(x^2/a^2 + y^2/b^2 = 1\) lie along conjugate diameters. Prove that the equation of the locus of P is \(a^2x + b^2y = a^2 + b^2\). (OC)

19. Find the equation of the diameter which bisects the chord PQ of \(px^2 + qy^2 = 1\) formed by the line \(lx + my + n = 0\); hence find the equation of the perpendicular bisector of PQ.

20. The mid-point of a variable chord PQ of \(x^2/a^2 + y^2/b^2 = 1\) lies on \(x^2/a^2 + y^2/b^2 = 1/2\).

21. Prove that two of the lines AH, AK, BH, BK are parallel and that the equation of the locus of the points of intersection of the other two when H and K vary is \(x^2/a^2 + y^2/b^2 = x/y + y/b\).

22. The line \(lx + my = 1\) meets OH, OK at Q, R. Prove that the perpendiculars to OH, OK, at Q, R meet on the line \(a^2x + b^2y = a^2 + b^2\).

Miscellaneous Examples

EXERCISE 40

[The notation is the same as that given in the heading to Exercise 39. Use geometrical methods whenever it is simpler to do so.]

1. The tangent at P to \(x^2/a^2 + y^2/b^2 = 1\) meets AA', OB at T, I; PN, Pt are the perpendiculars to OA, OB. Prove: (i) \(ON. OT = a^2;\) (ii) \(ON. OT = b^2\).

2. The normal at H meets OA, OB, OK at G, g, F; prove that:

(i) \(HF. HG = a^2;\) (ii) \(HG. HG = b^2;\) (iii) \(HG. HG = a^2 + b^2;\)

3. The normal at P meets OA, OB at O, g; PN, Pt are the perpendiculars to OA, OB. Prove:

(i) \(GN. ON = b^2 + a^2;\) (ii) \(GN. ON = a^2 + b^2;\)

4. The tangents to an ellipse at P, Q meet at U. Prove that the triangles OPU, OQU are equal in area.

5. The tangent at A meets OH, OK at Q, R. Prove \(QA. AR = b^2;\)

6. PP' is a chord perpendicular to AA'; the chords QP, PP' meet AA' at R, R'. Prove OR. OR' = a^2.

7. Q is the point on the auxiliary circle corresponding to P on the ellipse; the normal at P meets OQ at R. Prove OR = a + b.

8. The normal at P meets AA' at G; PN is the perpendicular to AA'; PK is the tangent from P to the curve on BB' as diameter. Prove that \(PK^2 = OG. ON.\)

9. Prove that the product of the lengths of the perpendiculars from \((a \sin \alpha, 0)\) to any tangent of \(x^2 + y^2 \sec^2 \alpha = a^2 \cos^2 \alpha\) is \(a^2 \cos^2 \alpha\).

10. \(r_1\) and \(r_2\) are the lengths of two variable perpendicular semi-diameters of an ellipse. Prove \(1/r_1^2 + 1/r_2^2\) is constant.

11. If the tangent at F meets the normal at Q on BB', prove that the tangent at Q meets the normal at P on BB'. Prove that this is not possible unless \(a^2 > 2b^2\).

12. The tangent at P meets the diameter parallel to AP at Q. Prove that Q lies on the tangent at A.

13. P and Q are the eccentric angles of variable points \(P, Q\) on \(x^2/a^2 + y^2/b^2 = 1\). If \(\angle PAQ = 1\) rt. \(\angle\), prove: (i) \(\tan \frac{\theta}{2} \tan \frac{\phi}{2} = -b^2/a^2\); (ii) \(PQ \perp AA'\) at a fixed point. (L)

14. K, Q, L, R are marks in order on a straight thin rod; K and L move along given perpendicular lines. Prove that Q and R trace out ellipses with semi-axes equal to KQ, QL and KR, LR.

15. The normal at P meets AA' at G; R is a point inside the ellipse such that PR is perpendicular to AA'. Prove that the chord through R perpendicular to BG is bisected at R.

16. Two tangents cut the tangent at A in Q, R and cut the tangent at A' in Q', R'. Prove Q'R', Q'R meet on AA'.
CHAPTER 10  
FOCUS AND DIRECTRIX  

10.1. There is a property of the ellipse \( x^2/a^2 + y^2/b^2 = 1 \) which corresponds to the focus-directrix property of the parabola, see p. 89, and can be found by interpreting geometrically its equation.

For this purpose it is convenient to take the equation of the ellipse in the form given by orthogonal projection, 9.2, p. 134,

\[ x^2 + y^2 \sin^2 \alpha = a^2, \]

that is,

\[ x^2 \cos^2 \alpha + y^2 = a^2 \cos^2 \alpha, \ldots \quad (i) \]

where \( \alpha = \beta/a, a > b > 0 \).

The equation can be written

\[ x^2(1 - \sin^2 \alpha) + y^2 = a^2(1 - \sin^2 \alpha), \]

that is,

\[ x^2 + a^2 \sin^2 \alpha + y^2 = a^2 + x^2 \sin^2 \alpha. \]

To interpret this equation, use the method of 'completing squares' as follows:

\[ (x - a \sin \alpha)^2 + 2ax \sin \alpha + y^2 = (a - x \sin \alpha)^2 + 2ax \sin \alpha \]

that is,

\[ (x - a \sin \alpha)^2 + y^2 = \sin^2 \alpha (a \cos \alpha - x)^2. \quad \ldots \quad (ii) \]

Let \( S, X \) be the points \((\alpha \sin \alpha, 0), (\alpha \cos \alpha, 0)\), and draw \( XZ \) perpendicular to \( Ox \), see Fig. 83.

Let \( P N, PM \) be the perpendiculars from \( P(x, y) \) to \( Ox, XZ \), then \( SP^2 = (x - a \sin \alpha)^2 + y^2 \) and \( PM = ONX = ON \cdot ON = a \cos \alpha - x \).

Hence, if \( P \) is any point on the ellipse \( x^2 \cos^2 \alpha + y^2 = a^2 \cos^2 \alpha \), the transformed equation (ii) is equivalent to the statement,

\[ SP^2 = \sin^2 \alpha \cdot PM^2, \]

\[ SP = \sin \alpha \cdot PM. \]

Put \( e = \sin \alpha \), then \( SP = e \cdot PM \), where \( 0 < e < 1 \), and equation (i) of the ellipse becomes

\[ x^2(1 - e^2) + y^2 = a^2(1 - e^2). \quad \ldots \quad (iii) \]

10.1.1. The locus, whose equation is \( x^2(1 - e^2) + y^2 = a^2(1 - e^2) \), is symmetrical about the minor axis \( Oy \). Therefore if \( S' \) is the point \((-ae, 0)\) and if \( S'X' \) is the line, \( x = -ae \), and if \( PM' \) is the perpendicular from a point \( P \) of the ellipse to \( S'X' \), see Fig. 83,

\[ \frac{SP}{PM} = e. \quad (0 < e < 1) \]

Equation (iii) can be written

\[ \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \quad \ldots \quad (iv) \]

Since \( b = a \cos \alpha \), \( b^2 = a^2 \cos^2 \alpha = a^2(1 - \sin^2 \alpha) \),

\[ b^2 = a^2(1 - e^2) \quad \text{and} \quad a^2 - b^2 = a^2. \]

Equation (iv) then takes the standard form, \( x^2/a^2 + y^2/b^2 = 1 \).

If \( e = 0 \), equation (iii) is the equation of the circle, \( x^2 + y^2 = a^2 \).

While the value of \( e \) is increasing from 0 up towards 1, the oval shape of the curve is becoming more pronounced and for this reason \( e \) is called the eccentricity of the ellipse.

10.1.2. Focal Distances. With the notation of Fig. 83,

\[ SP = e \cdot PM = e(a/x - a) = a - ex, \]

and

\[ S'P = e \cdot PM' = e(a/x + a) = a + ex; \]

\[ : \; SP + S'P = 2a. \]

10.1.3. Focal Chord. A chord of the ellipse which passes through either focus is called a focal chord.

The focal chord \( LSL' \) perpendicular to the major axis \( AA' \) is called the latus rectum and its length is denoted by \( 2l \).

Hence from the expression for \( SP \) in 10.1.2, taking \( P \) at \( L \),

\[ l - SL = a - e(ax) = a(1 - e^2), \quad : \; l = bu/a. \]

Fig. 83
10.1.4. Notation. Time is saved by adopting the notation of Fig. 88, p. 156, as the standard notation for an ellipse, except that, if the origin is not at the centre of the ellipse, the centre is denoted by $C$; for example, $CS = ae$, $CX = a/e$, etc.

**Example 1.** Find the equation of an ellipse of eccentricity $e$ having $S(h, k)$ as one focus and $XZ$, $Ix + my + n = 0$, as the corresponding directrix.

If $P(x, y)$ is a point on the ellipse and if $PM$ is the perpendicular from $P$ to $Ix + my + n = 0$, then $SP = e \cdot PM$.

But $SP^2 = (x - h)^2 + (y - k)^2$ and $PM = \pm [(x + my + n)/\sqrt{(1 + e^2)}]$, then:

$$(x - h)^2 + (y - k)^2 = e^2(Ix + my + n)^2/((1 + e^2)),$$

the equation of the ellipse is

$$(1 + e^2)[(x-h)^2 + (y-k)^2] = e^2(IX + MY + N)^2.$$

**Example 2.** $S(-1, -3)$ is a focus and $XZ$, $x + 2y = 3$, is the corresponding directrix of an ellipse of eccentricity $3/5$. Find (i) the coordinates of the centre $C$, (ii) the lengths of the semi-axes, (iii) the equation of the other directrix $X'Z'$.

(i) Using the standard notation, $SX$ is perpendicular to $XZ$.

$2x - y = 2(-1) - (-3) = 1$.

(ii) $X$ is given by $2x - y = 1$, $x + 2y = 3$.

(iii) $X = (1, 1)$.

Since $CS = ae$ and $CX = a/e$, $CS : CX = e^2 : 1$;

$C$ divides (externally) the line joining $S(-1, -3)$ to $X(1, 1)$ in the ratio $-3 : 1$, that is, $-3 : 4 : 9$.

$C$ is the point $3(9 - 1 - 4(1)) = 3(9 - 4 - 1 - 1 - 1) = 3(9 - 4 - 1) = -3/4$.

(iv) $SX$ = length of perpendicular from $S(-1, -3)$ to $XZ$, $x + 2y = 3$.

$SX = \pm [(1 - 6 - 3)/\sqrt{(1 + 4)}] = \pm (10)/\sqrt{5}$.

but $SX = a/e - ae = 3a - 2a = 1a = 60/\sqrt{5}$.

$ae = 12/\sqrt{5}$.

Hence $b^2 = a^2(1 - e^2) = 14/4(1 - 1/2) = 16$.

(iii) Since $C$ is the mid-point of $XX'$, the coordinates $(x, y)$ of $X'$ are given by $x + 1 = 2(-1 - 1/2)$, $y + 1 = 2(-3 - 3/2)$.

$x' = (x, y) = (3/2, -3/2)$.

$x'Z'$, parallel to $XZ$, is $x + 2y = 9/2 - 3/2 = 3/2$.

that is, $x + 2y + 3 = 0$.

**Exercise 41**

Find: (i) the eccentricity; (ii) the coordinates of the foci; (iii) the length of the latus rectum; (iv) the equations of the directrices of the ellipse whose equation is given, Nos. 1-4. Illustrate by a sketch.

1. $x^2 + 16y^2 = 28$.
2. $3x^2 + 4y^2 = 48$.
3. $5(x + 1)^2 + 9(y - 2)^2 = 180$.
4. $4(x - 2)^2 + 9(y + 3)^2 = 36$.

Find: (i) the coordinates of the centre; (ii) the lengths of the semi-axes; (iii) the equation of the given ellipse, Nos. 5-10. Make a sketch.

5. $Foci (-2, 0)$, corresponding directrix $x = 1$; eccentricity $1/2$.
6. $Foci (0, 3)$, corresponding directrix $y = 2$; eccentricity $1/2$.
7. $Foci (0, -1)$ and $(0, 5)$; eccentricity $1/2$.
8. Vertices $(-3, 0)$ and $(7, 0)$; focus $(5, 0)$.
9. $Foci (1, 6)$, corresponding directrix $x + 3y + 11 = 0$; eccentricity $1/2$.
10. $Foci (-2, -1)$, corresponding directrix $x + y = 1$; eccentricity $1/2$.

Find: (i) the coordinates of the centre and foci; (ii) the eccentricity of the ellipse whose equation is given, Nos. 11, 12.

11. $5x^2 + 3y^2 + 15x - 3y = 0$.
12. $2x^2 + 5y^2 + 12x - 2y + 13 = 0$.

13. Fix two drawing-pins to a drawing board at points $S, S'$, 3 in. apart. The sides of $\triangle SSS'$ are formed by a loop of thread of total length 7 in. enclosing $S, S'$. A pencil point at $P$ moves on the board so that the loop remains triangular. Use this method to trace an ellipse. Calculate the eccentricity and the lengths of the semi-axes.

14. In $\triangle ABC$, $BC = 6$ in., $CA = 5$ in., $AB = 4$ in. Calculate the eccentricity and lengths of the semi-axes of an ellipse if: (i) its foci are $B, C$ and it passes through $A$; (ii) its foci are $A, B$ and it passes through $C$; (iii) its foci are $A, B$ and it passes through $C$.

In Nos. 15-24, the notation is that of Fig. 88, p. 156.

15. Prove $A'S, SA = b^2$.
17. Prove $SX : OX = b^2 : a^2$.
18. Prove $SX = 1/e$.
19. Prove that $OP^2 - e^2 \cdot ON^2$ is constant.
20. If $SL$ meets the auxiliary circle at $Q$, prove $SQ = b$.
21. If the line through $L$ parallel to $A'A$ cuts $SB$ at $K$, prove $SK = b$.
22. If $LA$ meets the directrix $XZ$ at $K$, prove $XX' = SX$.
23. If the normal at $P$ meets $OA$ at $G$, prove $OG = e \cdot ON$.
24. If the tangent and normal at $P$ meet $OA$ at $T$ and $G$, prove $OG \cdot OT = OS^2$.
10.2. If \( R \) is a point on the directrix \( XZ \) which corresponds to the focus \( S \), then (i) the chord of contact \( PQ \) of the tangents from \( R \) passes through \( S \), (ii) \( SR \) is perpendicular to \( PQ \).

Conversely, if \( PSQ \) is a focal chord, the tangents at \( P, Q \) meet on the directrix which corresponds to \( S \).

(i) With the standard notation of Fig. 83, let the equation of the ellipse be \( x^2/a^2 + y^2/b^2 = 1 \) and let \( R \) be the point \((ae, k)\), see Fig. 85. Then the equation of the polar \( PQ \) of \( R(ae, k) \) is

\[
\frac{x}{ae} + \frac{ky}{b^2} = 1;
\]

\( \therefore \) \( PQ \) passes through the point \( S(ae, 0) \).

(ii) The equation of the line through \( S(ae, 0) \) perpendicular to \( PSQ \) is

\[
ax - bz = 0;
\]

this line passes through \( R(ae, k) \) because \( a^2 - b^2 = a^2e^2 \); therefore the line through \( S \) perpendicular to \( PSQ \) is \( SR \).

Conversely, let the tangent at \( P \) meet the directrix at \( R' \), then by (i) the chord of contact of the tangents from \( R' \) is the line \( PS \); but \( PS \) meets the ellipse at \( Q \), therefore \( PQ \) is the chord of contact of the tangents from \( R' \).

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10.2.1. If a non-focal chord \( PQ \) meets the directrix \( XZ \) at \( R \), then \( SR \) is one of the bisectors of \( \angle PSQ \), where \( S \) is the focus which corresponds to \( XZ \).

Draw \( PM, QK \) perpendicular to \( XZ \), see Fig. 86.

By 10.1, p. 157, \( SP = e \cdot PM \) and \( SQ = e \cdot QK \);

\[
\frac{SP}{PM} = \frac{SQ}{QK} \text{ similar triangles};
\]

therefore \( SR \) is the external bisector of \( \angle PSQ \).

Note. If, in Fig. 86, \( Q \) approaches \( P \) along the curve, the limiting position of \( PQR \) is the tangent at \( P \) and the limiting position of the external bisector of \( \angle PSQ \) is perpendicular to \( SP \). This gives 10.2 (ii).

---

10.2.2. If the normal at a point \( P \) of an ellipse, foci \( S, S' \), meets \( SS' \) at \( G \), then (i) \( SG = e \cdot SP \); (ii) \( PG \) is a bisector of \( \angle SP S' \); (iii) \( PS, PS' \) make equal angles with the tangent \( TPT' \) at \( P \).

(i) The equation of the normal \( PG \) at \( P(x_1, y_1) \) to \( x^2/a^2 + y^2/b^2 = 1 \) is

\[
\frac{x}{x_1} - \frac{y}{y_1} = \frac{a^2}{b^2};
\]

\( \therefore \) \( PG \) meets \( SS' \), \( y = 0 \), where \( x = e/x_1 \), see Fig. 87;

\( \therefore \) \( GS = OS - OG = ae - e^2 x_1 = e(a - ex_1) \).

By 10.1.2, \( SP = a - ex_1 \), \( \therefore \) \( GS = e \cdot SP \).

(ii) By (i), since \( S' \) is a focus, \( GS' = e \cdot SP' \);

\( \therefore \) \( GS : GS' = e \cdot SP : e \cdot SP' = SP : SP' \);

\( \therefore \) \( PG \) is one bisector of \( \angle SP S' \).

(iii) The tangent \( TPT' \) is perpendicular to the normal \( PG \), \( \therefore \) \( TPT' \) is the other bisector of \( \angle SP S' \).

\( \therefore \) \( \angle SPT = \angle SPT' \).

---

10.2.3. If \( Y, Y' \) are the feet of the perpendiculars to the tangent \( TPT' \) at \( P \) from the foci \( S, S' \), then (i) \( Y \) and \( Y' \) lie on the auxiliary circle, (ii) \( SY \cdot SY' = b^2 \).

(i) Produce \( SY \) to meet \( SP \) produced at \( K \); join \( OY \), see Fig. 87,

\[ \angle SYP = \angle S'PY' = \angle KPY \text{ and } \angle SYP = 1 \text{rt. } \angle = \angle KPY; \]

\[ \angle SYP = \angle KPY; \]

\( \therefore \) \( SY = YK \); also \( SO = OS' \); \( \therefore \) \( OY = \frac{1}{2} S'K \).

But \( S'K = SP + PK = SP' + PS = 2a; \)

\( \therefore \) \( OY = a \).

\( \therefore \) \( Y \) lies on the auxiliary circle, centre \( O \), radius \( a \), and similarly for \( Y' \).

(ii) By 9.4.3, p. 143, the equation of any tangent to \( x^2/a^2 + y^2/b^2 = 1 \) can be written \( x \cos \alpha + y \sin \alpha = p \), where \( p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha \).

The lengths of the perpendiculars to \( x \cos \alpha + y \sin \alpha = p \) from \( S(ae, 0) \), \( S'(-ae, 0) \) are given by \( SY = p - ae \cos \alpha, \)

\( \therefore \) \( S'Y = p + ae \cos \alpha; \)

\( \therefore \) \( SY \cdot S'Y = p^2 - a^2 \cos^2 \alpha = (a^2 \cos^2 \alpha + b^2 \sin^2 \alpha) - (a^2 - b^2) \cos^2 \alpha; \)

\( \therefore \) \( SY \cdot S'Y = b^2 (\sin^2 \alpha + \cos^2 \alpha) = b^2. \)
10.2.4. If \( Y, Y' \) are the feet of the perpendiculars to the tangent at \( P \) from the foci \( S, S' \), and if \( OP, OD \) are conjugate diameters,

\[
SP = S'P = OD.
\]

(i) \( SY = SY' = \frac{a}{b} \);

(ii) \( SP \cdot S'P = OD^2 \).

(i) Draw \( OF \) perpendicular to the tangent at \( P \).

\[OF = \frac{1}{2}(SY + SY') \cdot \]

Also by 9.6.6 (ii), p. 152, since \( OD \) is conjugate to \( OP \),

\[OF \cdot OD = ab.\]

By 10.2.2, \( \angle SYP = \angle S'YP \), also \( SYP = 1 \) rt, \( \angle = \angle S'YP \),

\[
SP = S'P = \frac{1}{2}(SP + S'P) = \frac{a}{b} = OD.
\]

\[SY = SY' = \frac{1}{2}(SY + SY') \cdot \]

(ii) Hence also from these equal ratios, since \( SY, S'Y' = b^2 \),

\[SP \cdot S'P = OD^2.\]

10.2.5. If the normal at \( P \) meets the major axis \( AA' \) at \( G \), and if \( GK \) is the perpendicular from \( G \) to \( SP \), then \( PK \) is equal to the semi latus rectum \( SL \).

With the notation of Fig. 88, \( \angle GPY = 1 \) rt, \( \angle \),

\[
\frac{PK}{PG} = \cos KPG = \sin SPY = \frac{SY}{SP} = \frac{b}{OD}, \text{ by 10.2.4 (i)}.\]

By 9.6.8, p. 153, \( PG = \frac{b}{a} \); \( PK = b \), \( PG = \frac{b^2}{a} = SL \), by 10.1.3, p. 157.

**EXERCISE 42**

The meaning of the notation used in this exercise is shown in Fig. 88 and in Fig. 83, p. 136.

1. The tangent at \( P(x_1, y_1) \) to \( x^2/a^2 + y^2/b^2 = 1 \) meets \( AA' \) at \( T \). Prove that \( SP = ST = x_1 \cdot a \).

2. \( PM \) is the perpendicular from \( P \) to \( XZ \); prove that \( PG \) meets \( MS \) on the minor axis.

3. Prove the circle through \( S, S', L \) touches the auxiliary circle.

4. If \( OA, OB \) are taken as \( Ox, Oy \), prove the coordinates of \( Y \) can be taken as \( (a \cos \theta, a \sin \theta) \) that \( SY = e, YX \).

5. If \( P \) is a point of intersection of the circle on \( SS' \) as diameter with the ellipse, prove that \( PN = SX \).

6. Prove that \( OG^2 + PG^2 = 1 \).

7. \( PH, PK \) are the perpendiculars from a variable point \( P \) to fixed lines \( OA, OB \), where \( \angle AOB = 60^\circ \). If \( PH^2 + PK^2 \) is constant, prove the locus of \( P \) is an ellipse of eccentricity \( \frac{1}{2} \sqrt{6} \).

8. The tangents to an ellipse at \( P \), \( Q \) meet at \( U \); the perpendicular from \( S \) to \( PQ \) meets \( OU \) at \( H \) prove that \( H \) lies on \( XZ \).

9. The tangent at \( L \) meets \( NP \) produced at \( H \). Prove \( NH = SP \).

10. The tangent at \( P \) meets the line through \( O \) parallel to \( SP \) at \( H \). Prove that \( OH = OA \).

11. The tangent at \( P \) meets the minor axis \( BB' \) at \( t \); the line through \( S \) parallel to \( PT \) meets \( BB' \) at \( H \). Prove \( OH^2 = t^2 \cdot OB \).

12. \( PG \) meets \( OB \) at \( g \). Prove \( Sg = e \cdot Pg \).

13. \( PG \) meets \( OB \) at \( g \). Prove \( Sg^2 : OD^2 \) is constant.

14. The point \( Q \) on the auxiliary circle corresponds to \( P \); \( SH \) is the perpendicular from \( S \) to the tangent at \( Q \). Prove \( SH = SP \).

15. \( SL \) is a semi latus rectum; the tangent and normal at \( L \) meet \( OB \) at \( t, g \). Prove \( S'L = gt \).

16. \( OP, OQ \) are conjugate semi-diameters such that \( PQ \) passes through \( S \); \( P \) is \((a \cos \phi, b \sin \phi) \). Prove: (i) \( \cos \phi = \sin \phi = 1/\sqrt{2} \); (ii) \( PQ = a \).

17. \( K \) is a variable point on a fixed circle, centre \( S', \text{radius} \; c \); \( S \) is a fixed point inside the circle. Prove the perpendicular bisector of \( SK \) touches an ellipse, foci \( S, S', \text{major axis of length} \; c \). (OC)

18. The lengths of the perpendiculars from \((0, ae), (0, -ae)\) to a tangent of \( x^2/a^2 + y^2/b^2 = 1 \) are \( p, q \); prove \( p^2 + q^2 = 2a^2 \).

19. Prove that \( PD^2 = (SP - SD)^2 = 2h^2 \).

20. \( PH \) is the tangent from \( P \) to the circle whose diameter is the minor axis \( BB' \). Prove that \( SP\cdot SP = 2PH \).

21. The perpendicular bisector of a chord \( PQ \) meets \( AA' \) at \( H \). Prove that \( SH = \frac{1}{2}(SP + SQ) \).

22. Prove that: (i) \( OY \) is parallel to \( SP \); (ii) the circle on \( SP \) as diameter touches the auxiliary circle.

23. \( PG \) meets the minor axis at \( g \). Prove that the length of the tangent from \( P \) to the circle, centre \( g \), radius \( gS \), is equal to \( OD \).

24. If the tangent at \( P \) meets at \( H \) and \( H' \) the tangents at \( A \) and \( A' \) prove that the circle on \( HH' \) as diameter passes through \( S, S' \).

25. The lines through \( A, A' \), parallel to \( SP, S'P \) respectively meet at \( H, H' \) the tangent at \( P \). Prove that \( AH + A'H' = 2a \).
10.3. Director Circle. If the tangents from a variable point $U$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ are at right angles, the locus of $U$ is the circle $x^2 + y^2 = a^2 + b^2$, called the director circle or orthoptic circle.

First Method. By 9.4.3, p. 143, $y = mx ± \sqrt{(a^2m^2 + b^2)}$ is the equation of the tangent for any value of $m$, therefore the gradients $m_1, m_2$ of the tangents through $U(h, k)$ are the roots of the quadratic in $m$:

$$(k - mh)^2 = a^2m^2 + b^2$$

that is, $m_1(k^2 - a^2) - 2hk + k^2 - b^2 = 0$.

Since $UP$ is perpendicular to $UQ$, $m_1m_2 = -1$;

$$(k^2 - a^2) + 2hk + k^2 - b^2 = 0$$

$U(h, k)$ lies on the circle $x^2 + y^2 = a^2 + b^2$.

Second Method. By 10.2.3 (i), the feet $Y, H$ and $Y', H'$ of the perpendiculars from $S$ and $S'$ to $UP$, $UQ$ lie on the auxiliary circle, see Fig. 80.

Since $PUUQ = 1$ right, $\angle SUY$ and $S'Y'U'H'$ are rectangles,

$UH = SY \cdot S'Y' = b^2$.

Draw the tangent $UE$ from $U$ to the auxiliary circle, then $UE^2 = UH \cdot UH' = b^2$; $OEU = OE^2 + UE^2 = a^2 + b^2$.

$U$ lies on the circle, centre $O$, radius $\sqrt{(a^2 + b^2)}$.

For a third method, see Exercise 43, No. 13.

10.4. If, with the standard notation of Fig. 83, p. 156, $SP$ makes with $SX$ an angle $\theta$ measured counterclockwise from $0^\circ$ to $360^\circ$, and if $SP = r$, then $r(1 + \cos \theta) = l$.

Draw $LH$ perpendicular to $Z$, see Fig. 90,

then $L = SL - \epsilon \cdot LH = \epsilon \cdot SX$, $SX = l \epsilon$.

Take the $z$-axis and $y$-axis along $SX, SL$, then the equation of $XZ$ is $x - l \epsilon = 0$.

Let $P$ be the point $(x, y)$, then the positive value of the length of the perpendicular $PM$ from $P$ to $XZ$ equals $l\epsilon - x$;

$r - SP = \epsilon \cdot PM = \epsilon(l\epsilon - x) = l - \epsilon x$.

By 2.10, p. 18, $x = r \cos \theta$ for all values of $\theta$,

$r = l - \epsilon \cos \theta$; $r(1 + \epsilon \cos \theta) = l$.

10.5. Ellipses which have the same foci are called confocal. If two ellipses have the same focus $S$ and $S'$, the centre of each ellipse is the mid-point $O$ of $SS'$, the major axis of each ellipse lies along $SS'$ and the minor axis lies along the perpendicular bisector of $SS'$.

10.5.1. The equation of any ellipse confocal to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0,$$

can be expressed in the form

$$\frac{x^2}{a^2 + t} + \frac{y^2}{b^2 + t} = 1, \text{ where } b^2 + t > 0.$$

If $S, S'$ are the foci of $x^2/a^2 + y^2/b^2 = 1$, $SS' = 2ae = 2\sqrt{(a^2 - b^2)}$.

Similarly if $x^2/A^2 + y^2/B^2 = 1$ is a confocal ellipse, $SS' = 2\sqrt{(A^2 - B^2)}$.

$$A^2 - B^2 = a^2 - b^2; A^2 = a^2 - t, B^2 = b^2 + t$$

where, since $a > b > 0$, $t$ can take any value for which $b^2 + t > 0$.

10.5.2. The poles of a given line $HK$ with respect to the ellipses of a given confocal system lie on a line perpendicular to $HK$.

Denote the equation of $HK$ by $lx + my + n = 0$.

Let $x^2/a^2 + y^2/b^2 = 1$ be a given ellipse of the system, then the equation of any other ellipse of the system can be taken as

$x^2/(a^2 + t) + y^2/(b^2 + t) = 1$.

If $P(x, y)$ is the pole of $HK$ with respect to this ellipse,

$xx_1/(a^2 + t) + yy_1/(b^2 + t) - 1 = 0$ is equivalent to $lx + my + n = 0$;

$x^2_1/\{a^2 + t\} + y^2_1/\{b^2 + t\} = 1, \text{ after p. 119}$;

$P(x, y)$ lies on the line, $x/l - y/m + (a^2 - b^2)n = 0$; this line is perpendicular to $HK$, $lx + my + n = 0$.

10.6. $SY$ is the perpendicular from the focus $S$ to the tangent to

$x^2/a^2 + y^2/b^2 = 1$

at $P$. If $SP = r$ and $SY = p$, prove

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

With the notation of Fig. 88, p. 162, by 10.2.4, $S'Y' = SY$ $S'P = SP$ $S'P = Sp$,

by 10.2.3, $S'Y' = \frac{b^2}{p}$ $S'P = \frac{b^2}{p}$; by 10.1.2, $S'P = 2a - SP = 2a - r$;

$$\frac{b^2}{p(2a - r)} = \frac{S'Y'}{S'P} = \frac{SY}{SP} = \frac{p}{r}$$

This is called the $(p, r)$ equation of an ellipse, with a focus as pole.
EXERCISE 43

The meaning of the notation used in this exercise is shown in Fig. 88, p. 162, and in Fig. 83, p. 158.

1. \( TH \) is the tangent from a point \( T \) on the directrix \( XZ \) to the director circle. Prove that \( TH = TS \).

2. A tangent to the auxiliary circle meets the director circle at \( T, U \). Prove that \( TU = 2OB \).

3. Two ellipses whose eccentricities are equal are called similar. Prove that the ellipses, \( x^2/a^2 + y^2/b^2 = 1 \), \( x^2/a^2 + y^2/b^2 = k^2 \), are similar.

4. The perpendiculars from \( S, S' \) to a variable pair of conjugate diameters meet at \( Q \). Prove that the locus of \( Q \) is a similar ellipse. [See No. 3.]

5. \( PSQ \) is a focal chord. Prove \( \frac{1}{SP} + \frac{1}{SQ} = \frac{2}{SL} \).

6. If the circles, whose diameters are the parallel chords \( x = p, x = q \) of \( x^2/a^2 + y^2/b^2 = 1 \), pass through \( (a, 0) \), prove \( p + q = 2a/(\sqrt{a^2 - \epsilon^2}) \). If one chord is \( x = 0 \) and the other is \( QQ' \), prove \( \epsilon = 1/\sqrt{2} \) and that the tangent at \( Q \) makes an intercept \( 3a/\sqrt{2} \) on \( Oy \). [OC]

7. The normal at a variable point \( P \) of an ellipse meets \( OA, OB \) at \( C, g \). Prove that the locus of the mid-point of \( OG \) is an ellipse similar to the point-ellipse.

8. Prove \( \frac{1}{S^2} + \frac{1}{T^2} = \frac{1}{FG} \) is constant. [Use 10.6.]

9. The tangent at \( T \) meets \( AA' \) at \( T \); \( PN \) is the perpendicular from \( P \) to \( AA' \). Prove that the circle on \( NT \) as diameter cuts the auxiliary circle orthogonally.

10. The tangent at \( P \) meets \( AA' \) at \( T \); \( AP, AT \) meet the line through \( T \) perpendicular to \( AA' \) at \( H, H' \). Prove \( HT = TH' \).

11. If \( \epsilon^2 > \frac{1}{2} \), prove that there are two points \( P, Q \) on \( x^2/(a^2 + y^2/b^2) = 1 \), other than \((0, b)\), the normals at which pass through \( B(0, b) \) and that \( FB = QB = a/\epsilon \). [C]

12. Prove that
\[
\begin{align*}
x \cos a + y \sin a &= \sqrt{(a^2 \cos^2 a + b^2 \sin^2 a)}, \\
x \sin a - y \cos a &= \sqrt{(a^2 \sin^2 a + b^2 \cos^2 a)}
\end{align*}
\]
are the equations of two perpendicular tangents to \( x^2/(a^2 + y^2/b^2) = 1 \). What result is obtained by squaring and adding, for the two equations? Interpret the answer geometrically.

13. The tangent at \( (x', y') \) of \( x^2/(a^2 + y^2/b^2) = 1 \) meets the auxiliary circle at \( Y, Y' \). Prove that the equations of \( OY, OY' \) are \( bx \sin \phi = ay(\cos \phi \pm \epsilon) \) and that \( OY, OY' \) cannot lie along conjugate diameters.

14. If \( 0, \phi \) are the eccentric angles of \( P, Q \) on \( x^2/(a^2 + y^2/b^2) = 1 \), prove the pole of \( PQ \) is \( (a \cos \phi/(\cos \phi + \sin \phi), b \sin \phi/(\cos \phi + \sin \phi)) \). Deduce that if \( UV \) is a parallelogram which circumescibe the ellipse and if \( U, V \) are the points \((x_1, y_1), (x_2, y_2)\), then
\[
\begin{align*}
(x_1^2/(a^2 + y_1^2/b^2) - 1)(x_2^2/(a^2 + y_2^2/b^2) - 1) &= 1.
\end{align*}
\]
Prove also that, if \( UV, UU' \) are parallel to conjugate diameters, \( U, U', V \) lie on the similar ellipse \( x^2/(a^2 + y^2/b^2) = 2 \).

15. If the tangents at \( P, Q \) to \( x^2/(a^2 + y^2/b^2) = 1 \) meet at \( U(x_1, y_1) \), prove that \( SU^2 = SP \cdot SQ = (x_1^2/(a^2 + y_1^2/b^2) - 1) \).
CHAPTER 11

THE HYPERBOLA

11.1. The ellipse, \( x^2/a^2 + y^2/b^2 = 1 \), \( a > b > 0 \), was introduced as the orthogonal projection of the circle, \( x^2 + y^2 = a^2 \), on to a plane making an angle \( \alpha \) with the plane of the circle, where \( \cos \alpha = b/a \).

Similarly, the locus, \( x^2/a^2 - y^2/b^2 = 1 \), \( a > b > 0 \), is the orthogonal projection of the rectangular hyperbola, \( x^2 - y^2 = a^2 \) (p. 113), on to a plane making with the plane \( XOY \) an angle \( \alpha \), where \( \cos \alpha = b/a \), taking the line of intersection of the planes as axes \( OX, Oy \).

Further the locus, \( x^2/a^2 - y^2/b^2 = 1 \), \( b > a > 0 \), is the orthogonal projection of the rectangular hyperbola \( x^2 - y^2 = b^2 \) on to a plane making with the plane \( XOY \) an angle \( \beta \), where \( \cos \beta = a/b \), taking the line of intersection of the planes as axes \( OX, Oy \).

The locus whose equation is

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (a > b > 0 \text{ or } b > a > 0)
\]

is called a general hyperbola or, for short, a hyperbola.

Thus a hyperbola can be regarded as the orthogonal projection of a rectangular hyperbola and is therefore of the same form and possesses those properties of the rectangular hyperbola which survive orthogonal projection, for example, the property that the mid-points of parallel chords are collinear.

To avoid needless repetition, it will be assumed in future that in any reference to the hyperbola, \( x^2/a^2 - y^2/b^2 = 1 \), \( a \) and \( b \) denote positive numbers.

11.1.1. Many processes used for the ellipse can be applied to the hyperbola and, when this happens, taking the principal axes as \( Ox, Oy \), it is merely necessary to write \(-b^2\) for \(b^2\) to obtain the corresponding result, for example, the locus of points from which tangents to \( x^2/a^2 - y^2/b^2 = 1 \) are at right angles is the circle (the director circle), \( x^2 + y^2 = a^2 - b^2 \), provided that \( a^2 > b^2 \). There are, however, some fundamental differences between the ellipse and the hyperbola, corresponding to the differences between the circle, \( x^2 + y^2 = a^2 \), and the rectangular hyperbola, \( x^2 - y^2 = a^2 \).

11.1.2. As far as possible, the same names are given to elements of the hyperbola as to corresponding elements of the ellipse and the same notation is used. \( Ox, Oy \) are axes of symmetry of the hyperbola \( x^2/a^2 - y^2/b^2 = 1 \) and are called its principal axes; \( O \) is called the centre and any chord \( POP' \) through \( O \) is called a diameter.

11.1.3. For any point \((x, y)\) on \( x^2/a^2 - y^2/b^2 = 1 \), \( x^2/a^2 = y^2/b^2 + 1 \Rightarrow \)

\[
x^2 > a^2; \quad x < -a \text{ or } x > a.
\]

Thus there are no points of the locus between the lines \( x = -a \) and \( x = a \); but for any selected value of \( x \) less than \(-a \) or greater than \(a \), there are two values of \( y \) equal in magnitude but opposite in sign. Further, when \( x^2 \) is large, so also is \( y^2 \).

Again, \( y^2/b^2 < x^2/a^2 \), \( \therefore y^2 < b^2x^2/a^2 \);

therefore the line \( y = mx \) does not meet the hyperbola \( x^2/a^2 - y^2/b^2 = 1 \) at any point if \( m > b/a \) or if \( m < -b/a \).

If \(-b/a < m < b/a \), the line \( y = mx \) cuts the curve at two points \( P, P' \), where \( PP' \) is a diameter. In particular, \( x'Ox \) cuts the curve at \( A(a, 0), A'(-a, 0) \); \( A \) and \( A' \) are called the vertices and \( AA' \) is called the transverse axis, its length is \( 2a \); but \( y'Oy \) does not meet the curve, and so there is no 'minor axis' of a hyperbola; \( y'Oy \) is called the conjugate axis. The circle whose diameter is \( AA' \) is called the auxiliary circle, see Fig. 91.

To sketch the hyperbola \( x^2/a^2 - y^2/b^2 = 1 \), first draw the lines \( E'Ox, \frac{x}{a} = 0 \), and \( F'Oy, \frac{y}{b} = 0 \), whose gradients are given by

\[
m = b/a \quad \text{and} \quad m = -b/a.
\]

These lines divide the plane into four regions, and all points of the curve lie in those two of the regions which contain \( x'Ox \), see Fig. 91.

![Fig. 91](image)

11.1.4. If \( PH, PK \) are the perpendiculars from a point \( P(x_1, y_1) \) on the hyperbola \( x^2/a^2 - y^2/b^2 = 1 \) to the lines \( E'Ox, \frac{x}{a} = 0 \), \( F'Oy, \frac{y}{b} = 0 \),

then \( PH \cdot PK = a^2b^2/(a^2 + b^2) \text{ is constant} \).

\( PH, PK \) equal the numerical values of

\[
\frac{x_1-a-y_1/b}{\sqrt{(1/a^2+1/b^2)}} \quad \frac{x_1+a+y_1/b}{\sqrt{(1/a^2+1/b^2)}}
\]

but \( x_1^2/a^2 - y_1^2/b^2 = 1 \), \( \therefore (x_1-a-y_1/b)(x_1+a+y_1/b) = 1 \);

\( PH \cdot PK = 1/(1/a^2+1/b^2) = a^2b^2/(a^2+b^2) \).
11.1.5. If \( PH \) and \( PK \) are the perpendiculars from a point \( P(x, y) \) of the hyperbola \( x^2/a^2 - y^2/b^2 = 1 \) to the lines \( E'OE, x/a - y/b = 0 \), and \( F'OF, x/a + y/b = 0 \), the length of \( PK \) can be made as large as we please by taking points on the curve for which \( x \) and \( y \) assume sufficiently large positive values and so, since \( PH < PK \), the length of \( PH \) can be made as small as we please but is never zero. Thus the curve approaches indefinitely closely the line \( E'OE, x/a - y/b = 0 \), from below on the right of \( O \) but never meets it, see Fig. 93. Also since there is symmetry about the centre \( O \), the curve approaches indefinitely closely the line \( E'OE \) from above on the left of \( O \) but never meets it. Hence as in S.1.2, p. 111, \( E'OE \) is called an asymptote of the curve, and the curve is asymptotic to \( E'OE \) both on the right and on the left of \( O \).

Similarly, since the curve is symmetrical about \( x'Ox \), the line \( F'OF, x/a + y/b = 0 \), is an asymptote and the curve approaches \( F'OF \) from above on the right of \( O \) and from below on the left of \( O \).

Fig. 93

11.2. The locus whose equation is

\[
x^2(1-e^2) + y^2 = a^2(1-e^2).
\]

is an ellipse if \( 0 < e < 1 \), see p. 156, and the focus-directrix property can then be established by re-grouping the terms of this equation in each of the forms

\[
(x - ae)^2 + y^2 = e^2(ae - x)^2 \quad \text{and} \quad (x + ae)^2 + y^2 = e^2(ae + x)^2.
\]

Exactly the same re-arrangements can be made if \( e > 1 \) and can be interpreted in the same way.

If \( e > 1 \), it is natural to write equation (i) in the form

\[
x^2(a^2 - 1) - y^2 = a^2(e^2 - 1), \quad e > 1.
\]

Since \( e^2 - 1 \) is positive, \( a^2(e^2 - 1) \) can be replaced by \( b^2 \), giving

\[
x^2/a^2 - y^2/b^2 = 1, \quad e > 1.
\]

where \( c > 1 \) and \( b^2 = a^2(e^2 - 1) \).

Also, with this notation, equations (ii) are re-arrangements of (iv).

11.2.1. Let \( S, S' \) be the points \( (ae, 0), (-ae, 0) \) and \( X, X' \) the points \( (ae, 0), (-ae, 0) \) and let \( XZ, XZ' \) be the lines \( x = ae, x = -ae, \) see Fig. 93, then the re-arrangements in 11.2 (ii) of the equation,

\[
x^2 - y^2 = b^2 = a^2(e^2 - 1),
\]

show that, if \( P(x, y) \) is a point of this hyperbola, then

\[
SP = e \cdot PM \quad \text{and} \quad S'P = e \cdot PM', \quad e > 1,
\]

where \( PM, PM' \) are the perpendiculars from \( P \) to \( XZ, XZ' \).

The points \( S(ax, 0), S'(ae, 0) \) are called the foci; the lines \( XZ, x = ae \), and \( XZ', x = -ae \), are called the directrices corresponding to the foci \( S \) and \( S' \); \( e \) is called the eccentricity.

Since \( e > 1 \) for a hyperbola, \( ae > ae > 0 \) and \( ae < ae < 0 \), therefore \( X \) lies between \( O \) and \( S \), also \( X' \) lies between \( O \) and \( S' \).

11.2.2. Let the perpendicular at the vertex \( A(a, 0) \) of \( x^2/a^2 - y^2/b^2 = 1 \) to its transverse axis \( OA \) meet the asymptote \( OE, x/a - y/b = 0 \), at \( E \), then \( E \) is the point \( (a, b) \), see Fig. 94; \( AE = b \).

Let \( \angle EOA = \alpha \), then \( \tan \alpha = b/a \).

Further, \( OF^2 = OA^2 + AE^2 = a^2 + b^2 = a^2 + a^2(e^2 - 1) = a^2 \).

\[
OE = a \cdot OS \quad \text{and} \quad e = OE/|OA| = \sec \alpha.
\]

Fig. 94

11.2.3. The foot \( Y \) of the perpendicular from the focus \( S \) to an asymptote \( OE \) lies on the auxiliary circle and on the directrix.

With the data of Fig. 93, \( OY = OS \cos \alpha = ae(1/e) = a = OA \); therefore \( Y \) lies on the auxiliary circle, centre \( O \), radius \( OA \).

Draw the perpendicular \( YX \) from \( Y \) to \( OS \),

\[
OX = OY \cos \alpha = a(1/e) = a/e,
\]

therefore \( XY \) is the directrix.

Note. Since \( \angle SYO = 1 \) rt. \( \angle \), \( SY \) is the tangent from \( S \) to the auxiliary circle and \( SY = a \tan z = a(b/a) = b \).

Hence if \( O \) and \( A \) and the asymptote \( OE \) are given, the positions of \( S \) and the directrix can be constructed.
11.2.4. Focal Distances. With the notation of Fig. 93, p. 170, by 11.2, equations (ii), if \( P(x, y) \) is a point on \( x^2/a^2 - y^2/b^2 = 1 \),

\[
SP^2 = (x - ae)^2 + y^2 = e^2(a/e - x)^2 = (a - ex)^2
\]

and

\[
SP^2 = (x + ae)^2 + y^2 = e^2(a/e + x)^2 = (a + ex)^2;
\]

\[
\therefore SP = a - ex \text{ if } x > a \quad \text{and} \quad SP = a - ex \text{ if } x < a.
\]

also

\[
SP = a - ex \text{ if } x > a \quad \text{and} \quad SP = a - ex \text{ if } x < a; \]

\[
\therefore SP = 2a \text{ if } P \text{ lies on the branch which encloses } S
\]

and

\[
SP = 2a \text{ if } P \text{ lies on the branch which encloses } S'.
\]

11.2.5. Focal Chord. A chord of the hyperbola which passes through either focus is called a focal chord. In particular, the chord \( LSL' \) perpendicular to \( OS \) is called the latus rectum and its length is denoted by \( 2l \).

Hence, since \( SP = ex - a \) when \( x > a \),

\[
l = SL = e(ae - a) = a(e^2 - 1) = b^2/a.
\]

Example 1. Find the equation of the hyperbola having as asymptotes \( CE, \ 2x - y - 4 = 0 \), and \( CF, \ 2x - 11y + 14 = 0 \), and passing through \((3, 4)\).

Find also (i) the equations of the transverse axis \( ACA' \) and the conjugate axis \( BCB' \), (ii) the coordinates of \( A, A', C \), and (iii) the length of \( CA \) and the eccentricity.

The product of the distances of a point \( P(x, y) \) of the hyperbola from the asymptotes \( CE, \ CF \) is constant,

\[
\{(2x + y - 4)/\sqrt{5}\}(2x - 11y + 14)/\sqrt{125}\] is constant;

\[
(2x + y - 4)(2x - 11y + 14) = k, \text{ where } k \text{ is constant}.
\]

Hence the hyperbola passes through \((3, 4)\) if

\[
k = (3 + 4 - 4)(6 - 34 + 14) = -144;
\]

\[
\therefore \text{ the equation of the hyperbola is}
\]

\[
(2x + y - 4)(2x - 11y + 14) = -144.
\]

(i) The point \( P(x, y) \) lies on an axis if it is equidistant from the asymptotes, therefore the equations of the axes are

\[
(2x + y - 4)/\sqrt{5} = \pm (2x - 11y + 14)/\sqrt{125},
\]

that is,

\[
5(2x + y - 4) = \pm (2x - 11y + 14);
\]

If \( P(x, y) \) lies on the hyperbola, \( 2x + y - 4 \) and \( 2x - 11y + 14 \) are of opposite signs because their product equals \(-144\);

\[
\therefore \text{ the equation of } ACA' \text{ is } 5(2x + y - 4) = -(2x - 11y + 14),
\]

and the equation of \( BCB' \) is \( 5(2x + y - 4) = +(2x - 11y + 14) \).

Hence \( AA' = 2x - y - 1 = 0 \) and \( BB' = 4x + 8y - 17 = 0 \).

(ii) \( AA', 5(2x + y - 4) = -(2x - 11y + 14), \) meets the hyperbola, where

\[
5(2x + y - 4)^2 = -144;
\]

\[\therefore A, A' \text{ are given by } 2x + y - 4 = \pm 12/\sqrt{5}, \text{ and } 2x - y - 1 = 0;
\]

hence \( A, A' \) are \((1\frac{1}{2} + 3/\sqrt{5}, 1\frac{1}{2} + 6/\sqrt{5}) \), \((1\frac{1}{2} - 3/\sqrt{5}, 1\frac{1}{2} - 6/\sqrt{5}) \).

Also \( C \) is the mid-point of \( AA' \), \( \therefore C \) is the point \((1\frac{1}{2}, 1\frac{1}{2}) \).

11.2.6. Asymptotes and Axes

(iii) Hence from (ii), \( CA^2 = (\frac{12}{\sqrt{5}})^2 + (\frac{12}{\sqrt{5}}^2) = 9 + \frac{36}{5} = \frac{54}{5} \), \( \therefore CA = 3 \).

The gradients of \( CE, CA \) are \(-2, 2 \); therefore the acute angle \( \alpha \) which \( CE \) makes with \( CA \) is given by \( \tan \alpha = \frac{2 - 2}{1 - 4} = \frac{4}{3} \).

\[
\therefore \cos \alpha = \sqrt{(1 + \tan^2 \alpha)} = \sqrt{(1 + \frac{16}{9})} = \frac{5}{3}.
\]

Exercise 44

Illustrate the examples in this exercise by sketches.

Write down the equations of the asymptotes and find the eccentricity of the given hyperbola, Nos. 1-3.

1. \( x^2 - 4y^2 = 1 \) \( 2 \) \( 3 \) \( 16x^2 - 9y^2 = 36 \).

Find the equations of the asymptotes and transverse axis, Nos. 4, 5.

4. \( x^2 - 3xy + 2y^2 = 1 \) \( 5 \) \( x^2 - xy - 2y^2 = 1 \).

Find the eccentricity, coordinates of focal, equations of directrices and asymptotes, and length of latus rectum, Nos. 6-10.

6. (i) \( 5x^2 - 4y^2 = 20 \); (ii) \( 5x^2 - 4y^2 = -20 \).

7. (i) \( 9x^2 - 16y^2 = 144 \); (ii) \( 9x^2 - 16y^2 = -144 \).

8. \( x^2 - y^2 = 18 \). \( 9 \) \( 3y^2 - x^2 = 12 \).

9. \( 9y^2 - 7x^2 = 25 \).

Find the equation of the hyperbola, coordinates of centre, equations of transverse axis and conjugate axis, and eccentricity, Nos. 11-15.

11. Asymptotes, \( 5x - 2y = 0, 5x + 2y = 0 \), passing through \((0, 0) \).

12. Asymptotes, \( y = cx, \ y = -cx \), passing through \((0, 0) \).

13. Asymptotes, \( x + 2y = 1, 4x + 3y = 7 \), passing through \((1, 1) \).

14. Asymptotes, \( 3x + 4y = 5, 4x + 3y = 2 \), passing through \((0, 0) \).

15. Asymptotes, \( x - y = 1, x + y = 3 \), passing through \((0, 0) \).

Find equation of hyperbola and length of transverse axis, Nos. 16-20.

16. Focus \((0, -1) \), corresponding directrix \( y = -5 \); eccentricity 3.

17. Focus \((4, 0) \), corresponding directrix \( x = 1 \); eccentricity 2.

18. Focus \((-1, -2) \), corresponding directrix \( 3x + 4y = 4 \); eccentricity \( \frac{\sqrt{5}}{2} \).

19. Focus \((6, 0) \), asymptotes \( 4x + 3y = 0, 4x - 3y = 0 \).

20. Directrix \( x = 3 \), asymptotes \( 2x + y = 0, 2x - y = 0 \).

21. Express the equation \( (x + 3y)(2x - y) - 3x - 16y = 18 \) in the form \((x + 3y + p)(2x - y + q) = r \).

Interpret the locus.

22. If, with the data of Fig. 94, p. 171, \( EK \) is drawn perpendicular to \( EO \) to meet \( OA \) produced at \( K \), prove \( AK \) semi-latus rectum.

23. If the centre \( O \), the asymptote \( OE \) and the focus \( S \) are given, show how to construct the corresponding directrix and vertex.

24. If the centre \( O \), the vertex \( A \) and the focus \( S \) are given, show how to construct the asymptotes.
11.3. A parametric representation of points on \( x^2/a^2 - y^2/b^2 = 1 \) is suggested, as on p. 141, by writing the equation in the form
\[
y^2 = \frac{x^2}{a^2} - 1 = \left( \frac{x}{a} + 1 \right) \left( \frac{x}{a} - 1 \right).
\]
The equation of any chord \( A'P \) through \( A'(-a, 0) \) can be written
\[
y = \left( \frac{x}{a} + 1 \right).
\]
then \( A'P \) meets the curve again where \( \frac{y}{b} = \frac{x}{a} - 1 \) if \( x + a \).

\[
\therefore \text{the second point of intersection } P \text{ of } A'P \text{ with the hyperbola is given by}
\]
\[
x/a - 1 = y/b = x/a + 1
\]
\[
t^2 = t = \frac{2x/a}{t^2 + 1} = \frac{2}{1 - t^2}.
\]

\[
\therefore \text{x/a : y/b : 1 = (1 + t') : 2t : (1 - t^2)}.
\]

Further, each ratio in (i) \( \frac{2x/a}{t^2 + 1} = \frac{2}{1 - t^2} \).

\[\therefore x/a : y/b : 1 = (1 + t') : 2t : (1 - t^2) \quad \ldots \quad (ii)\]

11.3.1. Chord and Tangent. If \( P_1, P_2 \) are the points \( t = t_1, t = t_2 \) of the hyperbola,
\[
x/a : y/b : 1 = (1 + t^2) : 2t : (1 - t^2),
\]
then the equation of the chord \( P_1P_2 \) is
\[
(t_1t_2 + 1)x/a - (t_1 + t_2)y/b + (t_1t_2 - 1) = 0.
\]
and the equation of the tangent at \( P_1 \) is
\[
(t_1^2 + 1)x/a - 2t_1y/b + (t_1^2 - 1) = 0.
\]
The parametric equations of the hyperbola can be written, see 11.3 (i),
\[
(x/a - 1) : y/b : (x/a + 1) = t^2 : t = 1.
\]
\[
\therefore \text{the line, } (x/a - 1) - (y/b)((t_1 + t_2) + (x/a + 1)t_1t_2 = 0, \text{ meets the hyperbola}
\]
\[
t^2 - t(t_1 + t_2) + t_1t_2 = (t - t_1)(t - t_2) = 0,
\]
that is, at the points \( P_1, t = t_1 \), and \( P_2, t = t_2 \).

\[
\therefore \text{the equation of } P_1P_2 \text{ is } (t_1t_2 + 1)x/a - (t_1 + t_2)y/b + (t_1t_2 - 1) = 0.
\]
Similarly the line, \( (x/a - 1) - (y/b)(2t_1) + (x/a + 1)t_1^2 - 0 \) meets the hyperbola where \( t^2 - 2t_1 + t_1^2 = (t - t_1)^2 = 0 \),
that is, at two coincident points, coincident with \( P_1, t = t_1 \).

\[
\therefore \text{the equation of the tangent at } P_1 \text{ is}
\]
\[
(t_1^2 + 1)x/a - 2t_1y/b + (t_1^2 - 1) = 0.
\]
Further if \( P_1(x_1, y_1) \) is the point \( t = t_1 \),
\[
x/a = (1 + t_1^2)/(1 - t_1^2), \quad y_1/b = 2t_1/(1 - t_1^2);
\]
\[
\therefore \text{the equation of the tangent at } P_1(x_1, y_1) \text{ to } x^2/a^2 - y^2/b^2 = 1 \text{ is}
\]
\[
\frac{x}{a^2} - \frac{y}{b^2} = 1.
\]

11.3.2. The identification of points of the hyperbola \( x^2/a^2 - y^2/b^2 = 1 \) with the values given to \( t \) in the parametric equations
\[
x/a : y/b : 1 = (1 + t^2) : 2t : (1 - t^2)
\]
is indicated in Fig. 96. The vertex \( A(a, 0) \) is the point \( t = 0 \).

When the values of \( t \) increase from 0 to 1, \( t \) remaining less than 1, the values of \( x \) and \( y \) are positive and become large when \( t \) approaches 1 from below; we then say
\[
x \to +\infty \text{ and } y \to +\infty \text{ when } t \to +1-,
\]
where the symbol \(-1-\) means an approach to \(+1\) through values less than 1. This range of values of \( t \) determines points of the curve which approach the asymptote \( E'OE \) in the first quadrant from below, see Fig. 96. There is no value of \( x \) or \( y \) when \( t = 1 \), but there is an abrupt change when \( t \) increases beyond the value 1.

When \( t > 1 \), the values of \( x \) and \( y \) are negative and become numerically large when \( t \) approaches 1 from above; we then say
\[
x \to -\infty \text{ and } y \to -\infty \text{ when } t \to +1+,
\]
where the symbol \(+1+\) means an approach to \(+1\) through values greater than 1.

Thus the range of values of \( t \) in which \( t \) approaches 1 from above determines points of the curve which approach the asymptote \( E'OE \) in the third quadrant from above.

The values of \( x \) and \( y \) corresponding to large values of \( t \) are best examined by writing the parametric equations in the form
\[
x/a : y/b : 1 = (1/t^2 + 1) : 2/t : (1/t^2 - 1).
\]
When \( t \) is large and positive, \( 1/t \) is small and positive; therefore when \( t \to +\infty \), \( x/a \to -1 \) and \( y/b \to 0 \);
hence, as \( t \) increases, the corresponding points of the curve approach the vertex \( A'(-a, 0) \), and it is customary to say that \( A'(-a, 0) \) is the point \( t = +\infty \), but this must be regarded merely as an abbreviation for the statement about the limiting values of the coordinates of a point on the curve when \( t \to +\infty \).

The arrows along the curve in Fig. 96 show the direction in which the curve is traced when \( t \) increases from 0 to \(+1-\) and from \(+1+\) to \(+\infty\).

If \( t = t_1 \) gives \((x_1, y_1)\), that is, the image of \((x_1, y_1)\) in \( x'Ox \). Hence, as shown in Fig. 96, the range of values when \( t \) decreases from 0 to \(-\infty\) determines the image of \((x_1, y_1)\) in \( x'Ox \). In particular, the vertex \( A'(-a, 0) \) may also be called the point \( t = -\infty \).
11.3.3. Asymptote and Tangent. By 11.3.1, p. 174, the equation of the tangent at the point \( t \) to \( x/a : y/b = 1 : (1 + t^2) : 2t : (1 - t^2) \) is

\[
(t^2 + 1)x/a - 2ty/b + (t^2 - 1) = 0;
\]

the limiting forms of this equation when \( t \to 1 \) from above or below and when \( t \to -1 \) from above or below are

\[
2x/a - 2y/b = 0 \quad \text{and} \quad 2x/a + 2y/b = 0;
\]

these are the equations of the asymptotes \( E'O'F', F'O'F' \), see Fig. 96.

Hence an asymptote may be regarded as the limit of a tangent when its point of contact tends to infinity along the curve, and its equation may be found by using this property.

Example 2. \( P, Q \) are the points \( t-p, t-q \) on the curve \( x : y : a = t : (t-2) : (t-1)(t-3) \).

(i) Find the equations of the chord \( PQ \), the tangent at \( P, Q \), and the asymptotes.

(ii) Explain why the origin is said to lie on the curve and find the other points \( H, K \) where the curve meets \( Ox, Oy \).

(i) The line \( lx + my + na = 0 \) meets the curve at \( P, Q \) if \( t = p, t = q \) are the roots of

\[
U + m(t-2) + n(t-3) = 0;
\]

therefore, the equation is equivalent to

\[
(t^2 - (p+q)t + pq) = 0;
\]

\[
n = \frac{m(l+4m-4n)t + 3n - 2m}{l} = \frac{p+q}{p+q};
\]

\[
each ratio \frac{2m}{3-pq}, \text{ and each ratio} \frac{2}{3-pq} \text{ (see p. 119);}
\]

\[
t : m = \frac{(pq - 2p - 2q + 5)x + (3 - pq)y + 2a}{0};
\]

\[
the equation of \( PQ \) is \((pq - 2p - 2q + 5)x + (3 - pq)y + 2a = 0\).
\]

Hence equation of tangent at \( P \) is

\[
(p^2 - 4p + 5)x + (3-p)py + 2a = 0.
\]

If \( t \to 1 \), the corresponding point \((x, y)\) tends to infinity along the curve and the limiting form of the equation of the tangent is \( 2x + 2y = 0 \); the equation of one asymptote is \( x + y + a = 0 \).

Similarly, taking the limit when \( t \to 3 \), it follows that the equation of the other asymptote is \( 2x - 6y + 2a = 0 \), that is, \( x - 3y + a = 0 \).

(ii) The parametric equations may be written

\[
x : y : a = t : (t-2) : (1-t)/4; \quad (1 - t)/(1 - 3t),
\]

\[
x \to 0 \text{ and } y \to 0 \text{ when } t \to \infty \text{ and when } t \to -\infty;
\]

\[
the origin may be regarded as the point of the curve determined by \( t = \infty \) or by \( t = -\infty \).
\]

The curve meets \( y = 0 \) again where \( t = 2 \); \( H \) is \((-2a, 0)\); the curve meets \( x = 0 \) again where \( t = 0 \); \( K \) is \((0, -3a)\).

11.4. If the equation of a locus is written in the form

\[
px^2 + qy^2 = 1,
\]

the locus is an ellipse if \( p = 1/a^2 \) and \( q = 1/b^2 \); it is a hyperbola if \( p = 1/a^2 \) and \( q = 1/b^2 \) or if \( p = -1/a^2 \) and \( q = 1/b^2 \).

Hence when it is possible to work with the equation \( px^2 + qy^2 = 1 \) without specifying the signs of \( p \) and \( q \), the results which are obtained hold both for an ellipse and for a hyperbola, and it is then convenient to call the locus a central conic. It is not always possible to do so because there are properties of the ellipse which are not properties of the hyperbola and vice-versa, for example, the ellipse is the orthogonal projection of a circle and does not possess asymptotes; but each curve, unlike the parabola, has a centre and two foci.

11.4.1. The equation of the tangent to \( px^2 + qy^2 = 1 \) at \((x_1, y_1)\) is

\[
pqx_1 + qyy_1 = 1.
\]

This was proved for the ellipse on p. 142 and by a similar method for the hyperbola on p. 174; it is therefore true for a central conic.

Some other results proved in Chapter 9 for the ellipse can be proved by precisely the same methods for the locus \( px^2 + qy^2 = 1 \) and therefore hold also for the hyperbola.

We now state these results but leave it to the reader to prove them for himself as a useful form of revision. Should any difficulty be experienced, it is merely necessary to re-write in the new notation the proofs given for the ellipse, see pp. 142, 143, 147.

11.4.2. The line \( lx + my + n = 0 \) touches \( px^2 + qy^2 = 1 \) if, and only if,

\[
I^2/p + m^2/q = n^2, \quad \text{where} \quad I^2/p + m^2/q > 0.
\]

11.4.3. The lines \( x \cos \alpha + y \sin \alpha = \pm \sqrt{(1/p \cos^2 \alpha + (1/q \sin^2 \alpha)} \) touch \( px^2 + qy^2 = 1 \) for all values of \( \alpha \), such that

\[
(1/p \cos^2 \alpha + (1/q \sin^2 \alpha > 0)
\]

11.4.4. The lines \( y = \pm \sqrt{m^2/p} \pm 1/q \) touch \( px^2 + qy^2 = 1 \), for all values of \( m \) such that \( m^2/p + 1/q > 0 \).

11.4.5. The equation of the normal to \( px^2 + qy^2 = 1 \) at \((x_1, y_1)\) is

\[
(x_1/px_1 - y_1/qy_1) = 1/p - 1/q.
\]

11.4.6. The equation of the chord of contact \( PQ \) of the tangents from \( U(x', y') \) to \( px^2 + qy^2 = 1 \), called the polar of \( U \), is

\[
p(x'x - 2y'y) = 1 - p/(lx_1 + my_1).
\]

11.4.7. If \( lx + my + n = 0 \) meets \( px^2 + qy^2 = 1 \) at \( P, Q \), the tangents at \( P \) and \( Q \) meet at the point \( U\{U/(lx_1 + my_1), -m/(qx_1)\} \), called the pole of \( PQ \).
EXERCISE 45

Find the equations of the tangent and normal to the given hyperbola at the given points, Nos. 1, 2.

1. \( 5x^2 - 4y^2 = 0; (3, 3) \).

2. \( 2x^2 - 9y^2 = -1; (-2, 1) \).

3. Prove that there is a point \((x_1, y_1)\) on \(x^2 - 6y^2 = 1\) such that the tangent at \((x_1, y_1)\) is \(5x - 12y = 1\). Find \(x_1, y_1\).

4. Prove that \(3x - 4y + 10 = 0\) is a tangent to \(3x^2 - 2y^2 + 20 = 0\) and find the coordinates of the point of contact.

5. Find the equations of the tangents to \(4x^2 - 9y^2 = 27\) which are parallel to \(4x + 3y = 0\) and find their points of contact.

6. Find the equations of the normals to \(2x^2 - 5y^2 - 2 = 0\) which are parallel to \(5x = 3y\) and find the feet of these normals.

7. Prove that \(4x + 3y = 32\) is a normal to \(3x^2 - 5y^2 + 5 = 0\) and find the foot of the normal.

8. Find the points of contact of the tangents from \((-\frac{1}{2}, -\frac{5}{2})\) to \(x^2 - 4y^2 = 1\) and the equations of the tangents.

9. Find the values of \(m\) and \(c\) such that \(y = mx + c\) passes through \((-3, -3)\) and touches \(2x^2 - 9y^2 = 9\).

10. If \(P(x_1, y_1)\), \(P_x(x_1, y_1)\) are points on \(x^2/a^2 - y^2/b^2 = 1\), prove that the gradient of \(P_1P_x\) equals \(b^2(x_1 + x_1)/a^2(y_1 + y_1)\).

Deduce that the equation of \(P_1P_x\) is

\[
\frac{2(x_1 + x_1)}{a^2} \cdot \frac{y(y_1 + y_1) - 1 + \frac{2}{a^2}x_1y_1}{b^2}.
\]

If \((h, k)\) is the mid-point of \(P_1P_x\), deduce that \(P_1P_x\) is parallel to \(2(h^2 - yk)/b^2 = 0\) and then write down the equation of \(P_1P_x\) in terms of \(h, k\).

11. Use the results obtained in Example 2, p. 176, to sketch the hyperbola, \(x : y : a = t : (t-2) : (t-3)\), and show on the sketch, as in Fig. 96, the ranges of values of \(t\) which determine corresponding portions of the curve. Express the ratios \(t : t - 1\) in terms of \(x, y, a\) and then write down the \((x, y)\) equation of the hyperbola. Verify that the hyperbola passes through the origin.

12. Find the equation of the tangent at the point \(t = 1\), to the hyperbola, \(x : y : a = 1 : (t-1) : (t-2)\). Find the equations of the asymptotes and the coordinates of the centre. Explain why the curve passes through the origin \(O\) and find the equation of the tangent at \(O\). Sketch the curve.

11.4 TANGENT AND NORMAL

In Nos. 13-16, \(P, Q\) are the points \(t = p, \ t = q\) on the hyperbola

\(x/a : y/b : 1 = (t^2 + 1) : (t^2 - 1) : 2t\).

13. Prove that the equation of the chord \(PQ\) is

\(x(1+pq)/a + y(1-pq)/b = p + q\),

and deduce the equations of the tangent and normal at \(P\). Find also the equations of the asymptotes.

14. If the tangents at \(P\) and \(Q\) meet at \((h, k)\), prove \(t = p, \ t = q\) are the roots of the equation \(P(h/a - k/b) - 2t + (h/a + k/b) = 0\) and deduce that \(h/a - k/b = \frac{1}{p-q} + \frac{1}{p+q}\).

15. If the circle \(x^2 + y^2 + 2px + 2qy + c = 0\) meets the hyperbola at \(P, Q, R, S\) given by \(t = p, q, r, s\), prove \(pqrs = 1\). Deduce that \(PQ, RS\) make supplementary angles with the transverse axis.

16. Find the relation between \(p, q, (i)\) if \(PQ\) is a diameter of the hyperbola, \((ii)\) if \(PQ\) passes through \((a, b)\).

In Nos. 17-19, \(P, Q\) are the points \(t = p, \ t = q\) on the hyperbola

\(x/a : y/b : 1 = (1 + t^2) : 2 - (1 - t^2)\).

17. If \(pq + r(p + q) + d = 0\), where \(c, d\) are constants, \(c^2 + d^2\), prove that the variable chord \(PQ\) passes through a fixed point.

18. The line joining \(P\) to the given point \(O(c, 0)\) meets the hyperbola again at \(Q\), find \(q\) in terms of \(p\).

19. If the variable chord \(PQ\) passes through the given point \((h, k)\), prove the tangents at \(P\) and \(Q\) meet on the line \(ka^2/x - ky/b^2 = 1\).

20. A variable point \(P\) moves on a fixed line parallel to an asymptote \(EO\) of \(x^2/a^2 - y^2/b^2 = 1\); prove that the polar of \(P\) passes through a fixed point on \(EO\).

21. A point moves so that the difference of its distances from the fixed points \((e, 0), (-e, 0)\) is constant and equals \(2b\), where \(e > b > 0\). Find the equation and the eccentricity of the locus.

22. The normal at a variable point \(P(x_1, y_1)\) on \(x^2/a^2 - y^2/b^2 = 1\) meets the transverse axis at \(G\) and the conjugate axis at \(g\). Prove that the locus of the mid-point of \(GP\) is the hyperbola

\(a^2x^2 - b^2y^2 = \frac{1}{2}(a^2 + b^2)^2\).

23. The normal at \(P(x_1, y_1)\) to \(x^2/a^2 - y^2/b^2 = 1\) meets the transverse axis at \(G\); \(OF\) is the perpendicular from the centre \(O\) to the tangent at \(P\). Prove that \(OF = b\).

24. The perpendicular from a variable point \(R\) to the chord of contact of the tangents from \(R\) to \(x^2/a^2 - y^2/b^2 = 1\) passes through the fixed point \((h, k)\). Prove that \(R\) lies on a fixed rectangular hyperbola whose centre is \(\{a^2h/(a^2 + b^2), b^2k/(a^2 + b^2)\}\).
11.5. In the proofs given for the following properties of the ellipse, the equation of the locus was taken in the form \( px^2 + qy^2 = 1 \), see pp. 147, 150, and so these are also properties of the hyperbola.

11.5.1. If the polar of \( U_1 \) passes through \( U_2 \), then the polar of \( U_2 \), if it exists, passes through \( U_1 \).

11.5.2. If \( V(b, k) \) is the mid-point of a chord of \( px^2 + qy^2 = 1 \), the gradient of the chord equals \( -(ph)/(qk) \).

11.5.3. The locus of the mid-points of chords of \( px^2 + qy^2 = 1 \) which are parallel to the diameter \( y = mx, m \neq 0 \), is the diameter \( y = m'x \), where \( mm' = -p/q \).

11.5.4. If the locus of the mid-points of chords of \( px^2 + qy^2 = 1 \) which are parallel to the diameter \( y = mx, m \neq 0 \), is the diameter \( y = m'x \), then the locus of the mid-points of chords parallel to the diameter \( y = m'x \) is the diameter \( y = mx \).

This property follows from the fact that the relation \( mm' = -p/q \) is symmetrical in \( m \) and \( m' \). The diameters \( y = mx, y = m'x \) where \( mm' = -p/q \) are called conjugate diameters.

Hence, if \( y = mx, y = m'x \) are conjugate diameters of \( x^2/a^2 - y^2/b^2 = 1 \), then

\[
mm' = \frac{b^2}{a^2}
\]

11.6. There are important relations between the hyperbolas,

\[
x^2/a^2 - y^2/b^2 = 1 \quad \text{and} \quad x^2/a^2 - y^2/b^2 = -1,
\]

represented by the continuous and dotted curves in Fig. 97, p. 181.

(i) These hyperbolas have the same asymptotes.

The lines \( x/a - y/b = 0, x/a + y/b = 0 \) are asymptotes of each hyperbola.

(ii) The tangents at the vertices \( A(a, 0), A'(-a, 0) \) of the first hyperbola meet the tangents at the vertices \( B(0, b), B'(0, -b) \) of the second hyperbola on the common asymptotes.

The tangents at \( A, A' \) are \( x = a, x = -a \); the tangents at \( B, B' \) are \( y = b, y = -b \); therefore the points of intersection lie on \( x/a + y/b = 0 \).

(iii) These hyperbolas have the same pairs of conjugate diameters.

\( y = mx, y = m'x \) are conjugate diameters of each hyperbola if

\[
mm' = b^2/a^2.
\]

(iv) Each line through the origin, except for the asymptotes, meets one, but only one, of the two hyperbolas. Further, if \( y = mx, y = m'x \) are conjugate diameters and if \( y = mx \) meets \( x^2/a^2 - y^2/b^2 = 1 \), then \( y = m'x \) meets \( x^2/a^2 - y^2/b^2 = -1 \).

By 11.1.3, p. 169, \( y = mx \) meets \( x^2/a^2 - y^2/b^2 = 1 \) only if \(-b/a < m < b/a\); similarly, \( y = mx \) meets \( x^2/a^2 - y^2/b^2 = -1 \) only if \( m < -b/a \) or \( m > b/a \); therefore, if \( m = \pm b/a \), \( y = mx \) meets one, and only one, of the two hyperbolas. Further if \( mm' = b^2/a^2 \) and if \( -b/a < m < b/a \), then \( m' = -b/a \) or \( m' > b/a \).

11.6.1. The hyperbolas, \( x^2/a^2 - y^2/b^2 = 1, x^2/a^2 - y^2/b^2 = -1 \) are called conjugate hyperbolas.

From the relation \( \sec^2 \theta - \tan^2 \theta = 1 \), it follows that the coordinates of any point \( P \) on \( x^2/a^2 - y^2/b^2 = 1 \) can be expressed in the form

\[
(a \sec \phi, b \tan \phi)
\]

where \( \theta \) may have any value from \( 0^\circ \) to \( 360^\circ \), except \( 90^\circ, 270^\circ \).

Similarly, since \( \tan^2 \theta - \sec^2 \theta = -1 \), the coordinates of any point \( Q \) on the conjugate hyperbola can be taken as \( (a \tan \theta, b \sec \theta) \).

For a geometrical interpretation of \( \theta \), see Exercise 46, No. 7.

11.6.2. If \( OH, y = mx \), meets \( x^2/a^2 - y^2/b^2 = 1 \) at \( P, P' \), then the conjugate diameter \( OK, y = m'x \), meets the conjugate hyperbola at two points \( Q, Q' \). It is customary to call \( P, P' \) and \( Q, Q' \) the extremities of the conjugate diameters \( OH, OK \) of \( x^2/a^2 - y^2/b^2 = 1 \) and to call the lengths of \( FOP, QQ' \) the lengths of those conjugate diameters of either hyperbola, although \( Q, Q' \) do not lie on \( x^2/a^2 - y^2/b^2 = 1 \) and \( P, P' \) do not lie on \( x^2/a^2 - y^2/b^2 = -1 \).

![Fig. 97](image-url)

11.6.3. If \( P \) is the point \((a \sec \theta, b \tan \theta)\) on \( x^2/a^2 - y^2/b^2 = 1 \) and if \( Q, Q' \) are the extremities of the diameter conjugate to \( OP \), then \( Q, Q' \) can be taken as \((a \tan \theta, b \sec \theta), (-a \tan \theta, -b \sec \theta)\).

Since \( OQ \) is conjugate to \( OP \), the point \( O \) lies on the conjugate hyperbola and may be taken as \((a \tan \phi, b \sec \phi)\).

Gradient of \( OP = (b \sec \phi)/(a \sec \theta) = (b \sin \phi)/a \), and gradient of \( OQ = (b \sec \phi)/(a \tan \phi) = b/(a \sin \phi)\).

By 11.5.4, \( \frac{b \sin \theta}{a} \cdot \frac{b}{a \sin \phi} = \frac{b^2}{a^2} \cdot \sin \phi = \phi \).

\( \phi = 0 \) or \( \phi = 180^\circ - \theta \).
11.6.4. If \( P, P' \) are points on \( x^2a^2 - y^2b^2 = 1 \) and \( Q, Q' \) are points on \( x^2a^2 - y^2b^2 = -1 \) such that \( POP', QQ' \) are conjugate diameters, then

(i) the tangents at \( P, P' \) and the tangents at \( Q, Q' \) form a parallelogram whose vertices lie on the asymptotes and whose sides are parallel to \( QQ', POQ' \);

(ii) \( OP^2 - OQ^2 = a^2 - b^2 \);

(iii) if the tangents at \( P \) and \( Q \) meet at \( R \), the area of the parallelogram \( PQOR \) is equal to \( ab \);

(iv) if the tangent at \( P \) meets the asymptotes at \( R \) and \( r \), \( P \) is the mid-point of \( Rr \) and the area of the triangle \( ORr \) is equal to \( ab \).

\[ \text{Fig. 98} \]

(i) By 11.6.3, \( P, Q \) can be taken as \( (a \sec \theta, b \tan \theta), (a \tan \theta, b \sec \theta) \).

The tangent at \( P \) is \( x/a \sec \theta - y/b \tan \theta = 1 \), and the tangent at \( Q \) is \( x/a \tan \theta - y/b \sec \theta = -1 \);

\[ \therefore \text{each tangent meets } x/a = y/b \text{ at the point } R \text{, see Fig. 98 given by} \]

\[ x/a = y/b = 1/(a \sec \theta - b \tan \theta). \]

Further \( P', Q' \) are the points \( (-a \sec \theta, -b \tan \theta), (-a \tan \theta, -b \sec \theta) \), therefore the tangents at \( P' \) and \( Q' \) meet at the point \( R' \), given by

\[ x/a = y/b = 1/(a \sec \theta + b \tan \theta). \]

and are respectively parallel to the tangents at \( P \) and \( Q \).

Similarly, the tangents at \( P \) and \( Q' \) meet \( x/a = -y/b \) at the same point \( r \), and the tangents at \( P' \) and \( Q \) meet \( x/a = -y/b \) at the same point \( r' \).

Also the equations of \( QQ' \) and \( PO' \) are

\[ x/(a \tan \theta) = y/(b \sec \theta) \quad \text{and} \quad x/(a \sec \theta) = y/(b \tan \theta); \]

\[ \therefore \text{QQ' is parallel to the tangent at } P \text{ and } POP' \text{ is parallel to the tangent at } Q. \]

11.6.5. The segment of a line intersected by a hyperbola and the segment intersected by its asymptotes have the same mid-point.

First suppose the points of intersection \( P_1, P_2 \) of the line with the hyperbola lie on the same branch, see Fig. 98.

Let \( P_1, P_2 \) meet the asymptotes at \( H_1 \) and \( H_2 \).

If \( V \) is the mid-point of \( P_1 P_2 \), \( OV \) meets the branch on which \( P_1, P_2 \) lie at a point \( P \); then the tangent at \( P \) is parallel to the diameter conjugate to \( OPV \) and is therefore parallel to \( P_1 P_2 \).

By 11.6.4 (iv), if the tangent at \( P \) meets the asymptotes \( OH_1, OH_2 \) at \( R, r \), then \( RP = PR \); therefore \( H_1 V = FH_2 \);

therefore \( V \) is the mid-point of \( H_1 H_2 \) and of \( P_1 P_2 \).

Next suppose the points of intersection \( Q_1, Q_2 \) of the line with the hyperbola lie on different branches, see Fig. 98.

If \( W \) is the mid-point of \( Q_1 Q_2 \), \( OW \) meets the conjugate hyperbola at a point \( Q \); then the tangent at \( Q \) to the conjugate hyperbola is parallel to the diameter conjugate to \( OWQ \) and is therefore parallel to \( Q_1 Q_2 \).

Let \( Q_1, Q_2 \) meet the asymptotes at \( K_1, K_2 \).

By 11.6.4 (iv), if the tangent at \( Q \) meets the asymptotes \( OK_1, OK_2 \) at \( R, r \), then \( RQ = QR \); therefore \( K_1 W = WK_2 \);

therefore \( W \) is the mid-point of \( K_1 K_2 \) and of \( Q_1 Q_2 \).

11.6.6. The property in 11.6.5 may also be stated as follows:

If \( P_1, P_2 \) are points on the same or different branches of a hyperbola and if \( P_1 P_2 \) meets the asymptotes at \( H_1, H_2 \), then the directed length \( H_1 P_1 = \text{directed length } P_2 H_2 \).

This property gives a simple construction for any number of points of a hyperbola when the asymptotes and one point \( P_1 \) are given; the method is the same as that given in 8.7.1, p. 128.
11.6.7. If a variable chord $P_1P_2$ of a hyperbola is fixed in direction and meets an asymptote at $H_1$, then $H_1P_1 \cdot H_1P_2$ is constant.

The coordinates of any point $H_1$ on the asymptote $x/a = y/b$ of $x^2/a^2 - y^2/b^2 = 1$ may be taken as $(at, bt)$.

Denote by $\theta$ the constant angle which $P_1P_2$ makes with $Ox$; then the point on $H_1P_1P_2$, whose directed distance from $H_1$ is $r$, is the point $(at + r \cos \theta, bt + r \sin \theta)$ and this lies on $x^2/a^2 - y^2/b^2 = 1$ if

$$(at + r \cos \theta)^2/a^2 - (bt + r \sin \theta)^2/b^2 = 1.$$

The directed lengths $H_1P_1$, $H_1P_2$ are the roots of the r-quadratic

$$r^2(\cos^2 \theta/a^2 - \sin^2 \theta/b^2) + 2r(\cos \theta/a - \sin \theta/b) - 1 = 0;$$

$$H_1P_1 \cdot H_1P_2 = \left(\cos^2 \theta/a^2 - \sin^2 \theta/b^2\right) = \text{constant},$$

since $\theta$ is constant.

Values of the constant are given by taking special positions for $H_1$.

With the notation of Fig. 98, p. 182,

$$H_1P_1 \cdot H_1P_2 = \alpha P_1 \cdot \alpha P_2 = \alpha P \cdot \alpha P = \alpha P^2 = \alpha O^2$$

and

$$K_1Q_1 \cdot K_2Q_2 = OP \cdot OP = OP(\alpha O) = -\alpha O^2.$$

11.6.8. If the lines through a variable point $P$ of a hyperbola parallel to the asymptotes $OE, OF$ meet $OF, OE$ at $N, M$, then

$$ON \cdot OM = \text{constant}.$$

Let the tangent at $P$ meet $OE, OP$ at $R, r$.

By 11.6.4 (iv), $\alpha P = \alpha R$;

$$\alpha O = \alpha OR \quad \text{and} \quad \alpha OM = \alpha OR.$$

Also by 11.6.4 (iv), area $\Delta ORr$ is constant;

but $\angle ORr$ is fixed,

$$\alpha OR = \text{constant},$$

$$\alpha ON = \frac{1}{2} \alpha OR \quad \text{or} \quad \alpha OR = \frac{1}{2} \alpha ON = \frac{1}{4} \alpha O.$$

The value of the constant can be found by taking $P$ at the vertex $A$. With the notation of Fig. 97, p. 181,

$$\frac{1}{2} \alpha O = \text{constant} = OF \cdot OE = \alpha O^2 = \frac{1}{2} \alpha a^2 + \frac{1}{2} \alpha b^2.$$

by 11.2.2, p. 171, \(\alpha ON = \frac{1}{4} \alpha OR \quad \text{or} \quad \alpha OR = \frac{1}{4} \alpha O = \frac{1}{4} \alpha O = \frac{1}{4} \alpha O^2 = \frac{1}{4} \alpha O^2.$$

EXERCISE 46

1. Find the equation of the chord of $3x^2 - 2y^2 = 3$ whose mid-point is $(9, 12)$ and find the equations of the tangents parallel to this chord.

2. Find the equation of the chord of $2x^2 - 5y^2 = 20$ whose mid-point is $(8, 3)$ and find the pole of this chord.

3. Find the equation of the diameter of $3x^2 - 2y^2 = 1$ conjugate to the diameter $3x - 2y = 0$ and find the coordinates of the mid-point of the chord whose equation is $3x - 2y = 19$. 

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(4) Find the coordinates of the mid-point of the chord of $7x^2 - 2y^2 = 10$ whose equation is $7x - 3y = 20$.

5. Find the equation of the locus of the mid-points of chords of $8x^2 - 9y^2 = 20$ which are parallel to $2x - 3y = 0$.

6. If $P$ is a point on $x^2/a^2 - y^2/b^2 = 1$, prove that the polar of $P$ with respect to $x^2/a^2 - y^2/b^2 = 1$ touches $x^2/a^2 - y^2/b^2 = 1$.

7. If $PN$ is the perpendicular from $P$ (a sec $\theta$, b tan $\theta$) to the transverse axis of $x^2/a^2 - y^2/b^2 = 1$ and if $NP_1$ is the tangent from $N$ to the auxiliary circle, prove that $P$ is the angle $OP_1$ makes with Ox.

8. $P, Q$ are the points $(a \cos \phi, b \sin \phi), (a \sec \phi, b \tan \phi)$ on $x^2/a^2 - y^2/b^2 = 1$. If $\phi$ varies, prove: (i) $PQ$ passes through a fixed point; (ii) the tangents at $P, Q$ meet on a fixed line.

9. If $U$ is the pole of a chord $P_1P_2$ of $x^2/a^2 - y^2/b^2 = 1$, prove that $OU$ produced bisects $P_1P_2$.

10. A line cuts $x^2/a^2 - y^2/b^2 = k_1$ at $P_1, Q_1$ and cuts $x^2/a^2 - y^2/b^2 = k_2$ at $P_2, Q_2$. Prove $P_1Q_1$ and $P_2Q_2$ have the same mid-point.

11. If the polar of $(a \sec \theta, b \tan \theta)$ with respect to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ passes through $(a \sec \theta - k, b \tan \theta - k)$, prove that $k/\alpha = \alpha /k$. Interpret the result geometrically.

12. A circle, centre $O$, cuts a hyperbola at $P, Q$; $PQ$ meets the asymptotes at $H, K$. Prove that $OH = CK$.

13. (i) Find the equation of the chord of $x^2/a^2 - y^2/b^2 = 1$ whose mid-point is $(a, b)$; (ii) If a variable chord passes through $(2a, 2b)$, prove its mid-point lies on $x/a + y/b = 2$.

14. $P$ and $Q$ are variable points on $x^2/a^2 - y^2/b^2 = 1$, $x^2/a^2 - y^2/b^2 = 1$ respectively. If $OP, OQ$ are conjugate semi-diameters and if $R$ is the point such that $P$ is the mid-point of $QR$, prove that $R$ lies on $x^2/a^2 - y^2/b^2 = 3$.

15. $PP', QQ'$ are variable conjugate diameters of a hyperbola; $R$ is a variable point on the curve. Prove $RP^2 + RP'^2 - RO^2 - RO'^2$ is constant.

16. With the data of No. 14, prove that the normals at $P$ and $Q$ to the two hyperbolas meet on one of the lines $ax = \pm by$.

17. The tangent at a point $P$ of a hyperbola, centre $O$, meets the asymptotes at $R, r$; the normal at $P$ meets the circle $ORr$ at $H, K$. Prove $HK$ is a diameter of the circle and deduce that $H$ and $K$ lie on the axes of the hyperbola.

18. Verify that the point, $x = a(t + t'/(1 - t^2)), y = 2bt/(1 - t')$ is the point $(a \sec \theta, b \tan \theta)$ if $t = \tan \phi$. Deduce from 11.3.1, p. 174, that the equation of the chord joining $(a \sec \theta, b \tan \theta)$ to $(a \sec \phi, b \tan \phi)$ is $(x/a) \cos \left(\frac{1}{2}(\theta - \phi)\right) - (y/b) \sin \left(\frac{1}{2}(\theta + \phi)\right) = \cos \left(\frac{1}{2}(\theta + \phi)\right)$. 
11.7. Many focal properties of the hyperbola are the same as, or similar to, focal properties of the ellipse and can be proved by the same or similar methods. We now state some of these results but leave it to the reader to prove them as a useful form of revision.

11.7.1. If \( U \) is a point on the directrix \( XZ \), which corresponds to the focus \( S \), then (i) the chord of contact \( PQ \) of the tangents from \( U \) passes through \( S \), (ii) \( SU \) is perpendicular to \( PQ \); and conversely.

The proofs are similar to the proofs for the ellipse in 10.2, p. 160.

11.7.2. If a non-focal chord \( PQ \) meets the directrix \( XZ \) at \( U \), then \( SU \) is one of the bisectors of \( \angle PSQ \) where \( S \) is the focus corresponding to \( XZ \).

The proof is the same as for the ellipse in 10.2.1, p. 160.

11.7.3. If \( S, S' \) are the foci of a hyperbola, whose eccentricity is \( e \), and if the normal at \( P \) meets \( SS' \) at \( G \), then, see Fig. 100,

(i) \( SG = e \cdot SP \) and \( S'G = e \cdot S'P \);
(ii) \( PG \) is a bisector of \( \angle SPS' \);
(iii) \( PS, PS' \) make equal angles with the tangent at \( P \).

The proof of (i) is similar to the proof for the ellipse in 10.2.2, p. 161; the proof of (ii), (iii) are the same as those in 10.2.2, p. 161.

\[ \begin{align*}
\text{Fig. 100}
\end{align*} \]

11.7.4. If \( Y, Y' \) are the feet of the perpendiculars from the foci \( S, S' \) of \( x^2/a^2 - y^2/b^2 = 1 \) to the tangent \( PT \) at any point \( P \), then, see Fig. 100.

(i) \( Y \) and \( Y' \) lie on the auxiliary circle;
(ii) \( SY, SY' = bj \);
(iii) If \( OP, OQ \) are conjugate semi-diameters,

\[ SP : SY = SP : SY' = OQ : bj \]

and

\[ SP : S'P = OQ : bj^2. \]

The proofs are similar to those for the ellipse in 10.2.3, p. 161, and 10.2.4, p. 162.

\[ \begin{align*}
11.7.5. \text{If the tangents from a variable point } U \text{ to the hyperbola, } x^2/a^2 - y^2/b^2 = 1, a^2 > b^2, \text{ are at right angles, the locus of } U \text{ is the circle } x^2 + y^2 = a^2 - b^2, \text{ called the director circle.}
\end{align*} \]

The proof is similar to the proof for an ellipse in 10.3, p. 164.

If \( a^2 < b^2 \), the hyperbola is a rectangular hyperbola and the director circle is the point-circle, \( x^2 + y^2 = 0 \), at the centre \( O \).

If \( a^2 > b^2 \), there are no points from which the tangents to the hyperbola are at right angles.

11.7.6. The hyperbola whose equation is

\[ \frac{x^2}{a^2 + t} + \frac{y^2}{b^2 + t} = 1, \ a^2 + t > 0 > b^2 + t, \]

has the same foci as the ellipse, \( x^2/a^2 + y^2/b^2 = 1, a > b > 0. \)

The equation of the hyperbola can be written in the form

\[ x^2/A^2 - y^2/B^2 = 1, \ A^2 = a^2 + t \text{ and } B^2 = -(b^2 + t). \]

By 11.2.1, p. 161, the foci of this hyperbola are \((c, 0), (-c, 0)\), where \( c^2 = A^2 + B^2 = (a^2 + t) - (b^2 + t) = a^2 - b^2 = ae^2 \), where \( e \) is the eccentricity of \( x^2/a^2 + y^2/b^2 = 1 \) whose foci are \((ae, 0), (-ae, 0)\).

It follows from 10.5.1, p. 165, that the locus whose equation is

\[ x^2/(a^2 + t) + y^2/(b^2 + t) = 1 \]

has the same foci as \( x^2/a^2 + y^2/b^2 = 1 \), whether the former locus is an ellipse or a hyperbola, and the two foci are called confocal.

Conversely, it follows as on p. 165 that any hyperbola confocal with \( x^2/a^2 + y^2/b^2 = 1, a > b > 0 \), can be denoted by

\[ x^2/(a^2 + t) + y^2/(b^2 + t) = 1, a^2 + t > 0 > b^2 + t. \]

11.7.7. If a hyperbola and an ellipse have the same foci \( S, S' \), and meet at \( P \), then the tangents \( PT, PT' \) are at right angles.

First Method. By 11.7.3 (iii), the tangent \( PT \) at \( P \) to the hyperbola is the internal bisector of \( \angle SPS' \), see Fig. 100, and by 10.2.2 (iii), p. 161, the tangent \( PT' \) at \( P \) to the ellipse is the external bisector of \( \angle SPS' \), see Fig. 87.

\[ \therefore \ PH \text{ and } PK \text{ are perpendicular lines.} \]

Second Method. Let \( P(x_1, y_1) \) be a point of intersection of the hyperbola \( \frac{x^2}{a^2 + t} + \frac{y^2}{b^2 + t} = 1, a^2 + t > 0 > b^2 + t \) with the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

The equations of the tangents \( PT, PT' \) are

\[ \frac{x_1}{a^2 + t} + \frac{y_1}{b^2 + t} = 1, \frac{x_1}{a^2} + \frac{y_1}{b^2} = 1. \]

But

\[ \frac{x_1}{a^2 + t} + \frac{y_1}{b^2 + t} = 1 \quad \text{and} \quad \frac{x_1}{a^2} + \frac{y_1}{b^2} = 1, \]

\[ \therefore \ x_1 \left( \frac{1}{a^2} - \frac{1}{a^2 + t} \right) + y_1 \left( \frac{1}{b^2} - \frac{1}{b^2 + t} \right) = 0; \quad \therefore \ x_1 \left( \frac{1}{a^2 + t} - \frac{1}{a^2} \right) + y_1 \left( \frac{1}{b^2} - \frac{1}{b^2 + t} \right) = 0; \]

this is the condition for the lines \( PT, PT' \) to be at right angles.
11.8. Standard Notation. The following notation is used in
Exercise 47 in connection with the hyperbola \( x^2/a^2 - y^2/b^2 = 1 \), where
\( a^2 + b^2 = c^2 \).

Vertices, \( A(a, 0), A'(-a, 0) \). Foci, \( S(ae, 0), S'(-ae, 0) \).

\( B, B' \) denote \((0, \pm b)\).

AA' meets the directrices \( XZ, X'Z' \) at \( X(\pm c, 0), X'(\pm c, 0) \).

Asymptotes, \( OE, ax - yb = 0 \) and \( OP, ax + yb = 0 \).

Latus rectum, \( LSL' = 2a \). \( FN \) is the ordinate of \( P \).

Tangent \( PT \) and normal \( PG \) at \( P \) meet \( AA' \) at \( T \) and \( G \).

EXERCISE 47

1. Prove: (i) \( ON \cdot OT = a^2 \); (ii) \( OG = c \). ON; (iii) \( OG \cdot OT = a^2 + b^2 \).

2. If \( P \) is a variable chord of \( x^2/a^2 - y^2/b^2 = 1 \) perpendicular to \( AA' \).

Prove that \( AP_1 \) meets \( AP_2 \) on \( x^2/a^2 + y^2/b^2 = 1 \).

3. If the line through a point \( P \) of a hyperbola parallel to an asymptote
meets the directrix at \( Z \), prove \( SP = PZ \).

4. If the normal at \( P \) is parallel to one asymptote and meets the
axes at \( G, g \), prove that the other asymptote bisects \( GP \).

5. \( P, Q \) are variable points \((a \sec \theta, b \tan \theta), (a \sec \phi, b \tan \phi)\) on
\( x^2/a^2 - y^2/b^2 = 1 \). If \( \theta + \phi = 90^\circ \), prove that the pole of \( PQ \) lies on \( y = b \) and
that the normals at \( P, Q \) meet on a fixed line \( g = k \).

6. The tangent at \( P \) to \( x^2/a^2 - y^2/b^2 = 1 \) meets \( x^2/a^2 - y^2/b^2 = k \) at \( P_1 \) and \( P_2 \).

Prove that \( P_1 P = P_2 P \).

7. \( FN \) meets \( OE \) at \( F \). Prove that the perpendicular at \( P_1 \) to \( OE \)
passes through \( G \).

8. If the tangent at \( P \) meets \( SL \) on one asymptote, prove that \( SP \) is parallel
to the other asymptote.

[9] If \( PT \) meets \( OE \) at \( R \), prove that \( \angle PRG \) is constant.

10. If \( P \) is a variable point \((a \sec \theta, b \tan \theta)\) on \( x^2/a^2 - y^2/b^2 = 1 \); \( AP, AP' \)
meet \( x=a=y=b \) at \( R, R' \). Prove that: (i) \( R \) is the point \((at, bt) \), where
\( t = \tan (1/2) \); (ii) the length of \( EE' \) is constant.

11. If \( OQ \) is the semi-diameter conjugate to \( OP \), prove that
\( PG : OQ = b : a \).

[12] Prove that the pole with respect to \( x^2/a^2 - y^2/b^2 = 1 \) of a tangent
at \( x^2/a^2 + y^2/b^2 = 1 \) lies on \( x^2/a^2 + y^2/b^2 = 1 \).

13. The normal at a point \( H \) on \( xy = c^2 \) meets \( x^2 - y^2 = a^2 \) at \( P_1 \) and \( P_2 \).

Prove \( PH = HP_2 \).

14. If the normal at \( P \) to \( x^2/a^2 - y^2/b^2 = 1 \) meets the conjugate axis at \( g \),
prove that \( SP^2 : SP : SP' = a^2 : b^2 \).

[15] If the normal at \( P \) meets the axes at \( G, g \), and if \( OP, OQ \) are
conjugate semi-diameters, prove \( PG : gP = OQ^2 \).

11.8] GEOMETRICAL PROPERTIES

16. If \( P \) and \( Q \) are points on \( x^2/a^2 - y^2/b^2 = 0 \); \( PH, QK \) are parallel to
\( OE, OF \) and meet \( OE, OF \) at \( H, K \). Prove \( HK, PQ \) are parallel.

17. If \( PT \) and the line through \( S \) parallel to \( PT \) meet \( BB' \) at \( t, K \).

Prove that \( SK = e \cdot ST \).

18. Prove that \( AG, AK = c \cdot ON = a \).

19. If \( x \cos \theta + y \sin \theta = 0 \) meets \( x^2/a^2 - y^2/b^2 = 1 \) at \((x_1, y_1), (x_2, y_2)\),
prove \( x_1 \cos \theta + y_1 \sin \theta = 0 = x_2 \cos \theta + y_2 \sin \theta \).

20. If \( PQ, Q'P' \) are chords of a hyperbola. If \( PP' \) is a diameter, prove that \( PQ, Q'P' \) are parallel to conjugate diameters.

21. The line through \( S \) parallel to \( OE \) meets the hyperbola at \( Q \) and
meets \( OP \) at \( H \). Prove: (i) \( SQ = S \cdot SL \); (ii) \( QH = SQ \).

22. The normal at \( P \) meets \( AA', BB' \) at \( G, g \); \( GH, gH \) are the
perpendiculars from \( G, g \) to \( SP \). Prove: (i) \( PH : PG = b : QK \), where
\( OP, OQ \) are conjugate semi-diameters; (ii) \( PH = SL \) and \( PH = OA \).

[Draw \( NY \) perpendicular to \( PT \).]

23. If \( PG, PT \) meet \( BB' \) at \( g, t \), prove that: (i) the circle \( SPS' \) passes
through \( g \) and \( t \); (ii) \( gS \cdot e = g \).

24. \( Y \) is the foot of the perpendicular from \( S \) to \( PT \). Prove that \( Y \) can
be taken as \((a \cos \phi, a \sin \phi) \) and deduce \( SY = e \cdot YX \).

25. If the length of the perpendicular from \( S \) to \( PT \) is \( p \) and if \( SP = r \)
and \( P \) lies on the branch enclosing \( S \), prove \( \sqrt{p^2 + \sqrt{a^2 - 2pr}} \).

Find the corresponding relation if \( P \) lies on the other branch.

26. \( PK \) is the tangent from a point \( P \) on a hyperbola to its auxiliary
circle. Prove \( PK \) equals the semi-minor axis of the ellipse through \( P \)
confocal with the hyperbola.

27. If \( p \) and \( q \) are the lengths of the semi-conjugates of the centros
to a pair of parallel tangents, prove that \( p^2 - q^2 \) is constant.

28. A circle touches \( AA' \) at \( O \) and touches \( x^2/a^2 - y^2/b^2 = 1 \) at \( P \).

If \( OP, OQ \) are conjugate semi-diameters, prove \( OQ = OS \).

29. Prove that the locus of the pole with respect to the hyperbola
\( x^2/a^2 - y^2/b^2 = 0 \) of a tangent to the circle, centre \( S \), radius \( \frac{1}{2} \),
is a hyperbola whose centre divides \( OX \) externally in the ratio \( 1 : 1 \).

[30] \( O \) is the point \( (a \sec \theta, b \tan \theta) \) on \( x^2/a^2 - y^2/b^2 = 1 \); the perpendicular
at \( P \) to \( AP \) meets \( OA \) at \( K \). Prove: (i) \( OK = \frac{1}{2} \sqrt{1 + b^2} \) \((a \cos \theta) \);
(ii) \( OK = SL \).

31. If two central conics intersect at a given point \( P \) and are confocal,
prove one is an ellipse and the other is a hyperbola.

32. If \( P \) is the focal chord perpendicular to \( PG \); \( OP, OQ \) are conjugate
semi-diameters. Prove:

(i) \( P_1 S \cdot SP_2 : LS \cdot SL = OQ^2 : OB^2 \); (ii) \( P_1 S \cdot SP_2 = PG^2 \).
QUICK REVISION PAPERS

Q.R. 1-8 (Ch. 1-3)

Q.R. 1
1. Find \( h, k \) if \( (h, 0) \) and \( (-2, k) \) lie on \( 2x - y = 6 \).
2. Sketch the graph of \( y = x(3-x) \).
3. Find the distance between \( (5, -3) \) and \( (9, -6) \).
4. Interpret the locus, \((x+1)^2+y^2=1 \).
5. Find the gradient of the line joining \( (3, 7), (1, 3) \).
6. Find the equations of the lines passing through \(-4, 5\) and:
   (i) parallel to; (ii) perpendicular to \( 2x + 3y = 1 \).

Q.R. 2
1. Find \( m \) if \( y = mx - 3 \) passes through \( (4, 5) \).
2. Sketch the line whose equation is \( \frac{x}{7} - \frac{y}{5} = 1 \).
3. Find the length of the tangent from \( (2, 0) \) to \( 4x^2 + 4y^2 = 9 \).
4. Find the equation of the circle, centre \((1, -4)\), radius 3.
5. Find the gradient of the line \( 6x - 8y - 5 = 0 \).
6. Find the length of the perpendicular from \((1, -1)\) to \( x - 2y = 1 \).

Q.R. 3
1. Find \( c \) if \((2c, -3c)\) lies on \( 5x + 3y = 4 \).
2. Sketch the line \( y = x \tan 90^\circ - 3 \) if:
   (i) \( 0^\circ = 40^\circ \); (ii) \( 0^\circ = 130^\circ \).
3. Find the distance between \((-7, -3)\) and \((-2, 2)\).
4. The step from \( P(-4, +3) \) to \( Q \) is \((+10)\) units in direction making \(120^\circ\) with \( x'Ox \), measured counterclockwise; find coordinates of \( Q \).
5. Find the equation of the line joining \((-1, 7), (5, -2)\).
6. Find the equations of the lines through \((-3, -7)\) parallel and perpendicular to:
   (i) the line \( x = 1 \); (ii) the line \( 2x - 5y = 1 \).

Q.R. 4
1. Find the intercepts on the axes made by \( 5x - 4y = 8 \).
2. Find the equation of the circle, centre \((2, -3)\), through the origin.
3. Find the equations of lines through \((4, 0)\) with gradients \( \frac{2}{3}, -\frac{1}{2} \).
4. Find the distance of \((1, -2)\) from \( 8x + 15y = 12 \).
5. Find the tangent of the angle between \( x + y = 2 \) and \( 3y = 2x + 1 \).
6. Is \((3, 6)\) inside or outside the circle \( x^2 + y^2 = 64 \)?

Q.R. 5
1. Sketch the line, \( y = \frac{1}{3}x - 1 \) and \( y = 6 - 2x \).
2. Find \( k \) if \( y^2 = 4x \) passes through \((-4, 6)\).
3. Interpret the locus, \((x+3)^2 + (y-2)^2 = 25 \).
4. The step from \( P(3, -4) \) to \( Q \) makes \(150^\circ\) with \( x'Ox \), counterclockwise and is \((-6)\) units long; find the coordinates of \( Q \).
5. Find the angle between the lines \( 3x - 4y + 1 = 0, x + 7y - 8 = 0 \).
6. Find the condition that \((h, k)\) is at unit distance from \( 3x - 4y + 10 = 0 \) on the side of the line opposite to the origin.

Q.R. 6
1. Sketch the line, \( x \cos 20^\circ + y \sin 20^\circ + 2 = 0 \).
2. Find \( c \) if \( 2x - 5y = c \) passes through \((-1, -2)\).
3. Interpret and simplify the equation, \((x - 3)^2 + (y + 1)^2 = x^2 + y^2 \).
4. Find the equation of the circle, centre \((0, a)\), radius \(a\).
5. Find \( k \) if \( 2x + ky - 1 = 0 \) is perpendicular to \( 3x - 5y = 2 \).
6. Find whether the points \((0, 0), (6, 3)\) are on the same side or on opposite sides of the line \( 3x - 5y = 2 \).

Q.R. 7
1. Find \( a, b, c \) if \((a, 0), (b, 3), (c, -5)\) lie on \( y = (3+x)(1-x) \).
2. Find equation of line joining \((b, k), (b + r \cos 90^\circ, k + r \sin 60^\circ)\).
3. Find the equations of the lines through the origin parallel to and perpendicular to the line \( kx + my + n = 0 \).
4. Find the equations of the angle-bisectors of \( x + y = 4, 7x - 17y = 4 \).
5. Find the angle between the pair of lines \( 2x^2 - 7xy + 2y^2 = 0 \).
6. Find area of triangle with vertices \((0, 0), (1, 2), (3, -4)\).

Q.R. 8
1. Sketch the line, \( x \cos 130^\circ + y \sin 130^\circ + 3 = 0 \).
2. The sides of the rectangle \( ABCD \) are parallel to \( Ox, Oy \); \( A, C \) are \((-3, 5), (4, -2)\). Find the equations of \( OB, OD, BD \).
3. Interpret the locus of \( P(x, y) \) if the gradients given by \( (y-1)/(x-5) \) and \( (y+3)/(x-2) \) are equal. Find its equation.
4. Find the line through \((am^2, 2am)\) perpendicular to \( my = x + am^2 \).
5. Find the condition that \((-2 + r \cos 90^\circ, 3 + r \sin 90^\circ)\) lies on \( x^2 + y^2 = 1 \). Find \( r \) if \( \tan \theta = -\frac{1}{1} \). Interpret the result.
6. Find area of triangle with vertices \((2, -2), (4, 1), (9, -1)\).
Q.R. 9

1. Find $a, b$ if $y = ax^2 + bx$ passes through $(2, 0), (1, 1)$.
2. Find the relation between $h, k$ if the distance of $P(h, k)$ from $(1, 0)$ equals the distance of $P$ from $y = -2x$.
3. Find length of perpendicular from $(c \cos \alpha, c \sin \alpha)$ to $x \cos \alpha + y \sin \alpha = p$.
4. $A, B$ are $(1, 5), (6, 2)$; find the points which divide $AB$ in the ratios: (i) $2:3$; (ii) $-5:2$; (iii) $-1:4$.
5. Find centre and radius of circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$.
6. Find $c$ if $7x - y - c$ touches $x^2 + (y - 3)^2 = 2$.

Q.R. 10

1. $A, B$ are $(3, 5), (7, -2)$; in what ratio does $x - y = 1$ divide $AB$?
2. Interpret and simplify $(x - 1)^2 + (y - 2)^2 = (x + 3)^2 + (y - 1)^2$.
3. Find equation of line joining $(0, c/p), (c, c/q)$.
4. Find mid-point of join of $(-2, 5), (3, -1)$.
5. State equation of tangent to $x^2 + y^2 + 2x + 6y + 15 = 0$ at $(2, 1)$.
6. Find the equation of the circle which passes through the points of intersection of $2x - y = 1$ with $x^2 + y^2 = 9$ and through the origin.

Q.R. 11

1. Find foot of perpendicular from $(a, 0)$ to $my = x + am^2$.
2. Find the pair of lines through $(0, 0)$ making $30^\circ$ with $x + y = 1$.
3. Prove $P(25, -15)$ is collinear with $A(-10, 5), B(11, -7)$; find $AP : PB$.
4. Find length of tangent from $(3, -5)$ to $4x^2 + 4y^2 = 15$.
5. Find chord of $x^2 + y^2 - 4x + 2y + 20 = 0$ whose mid-point is $(3, -4)$.
6. Find equation of circle through $(9, 0)$ touching $Oy$ at $(9, 0)$.

Q.R. 12

1. Find the common points of $3x + y = -2$ and $y = x(x - 6)$.
2. Find the condition for $(r \cos 9\theta, r \sin 9\theta)$ to lie on the circle $x^2 + y^2 = 2x$; interpret the result.
3. If $ax - 5y = 11, 10x + by = 14$ meet at $(2, -1)$, prove they are perpendicular.
4. Prove $(5, 4), (-1, 2), (-3, -7), (3, -5)$ are corners of a parallelogram.
5. Interpret the equation $x^2 + y^2 + 2x - 4y + 5 = 0$.
6. Prove the circles $x^2 + y^2 = 16x - 15, x^2 + y^2 = 10x - 9$ touch each other.
Q.R. 17

1. A line cuts Ox, Oy at A, B; (3, -2) is the mid-point of AB. Find the equation of AB and the point where it cuts 2x + 3y = 3 = 0.

2. For what value of k is (x - 5y - 1) + k(4x + y + 5) = 0 perpendicular to 3x - 5y = 8 = 0. Interpret the result.

3. If the circles \( x^2 + y^2 = a^2 \), \( (x - 5)^2 + (y - 12)^2 = 64 \) touch each other externally, find a and the coordinates of the point of contact.

4. Explain why \( r^2 - 4y + 1 = 0 \) touches the curve \( y^2 = 4ax \) and find the equation of the normal at its point of contact.

5. Find the chord of \( y^2 = 4ax \) whose mid-point is \( (h, 2a) \).

6. Interpret the locus \( (3x - y - 1)(x + 3y - 17) + 100 = 0 \) and find the coordinates of its vertices.

Q.R. 18

1. Find, h, k if the origin is the centroid of the triangle whose vertices are \((5, -3), (-7, 8), (h, k)\).

2. If the line joining \( A(-2, 1) \) to \( B(8, 7) \) meets \( x^2 + y^2 = 25 \) at \( P, Q \), find the values of \( AP : PB \) and \( AQ : QB \).

3. \( A, B \) are the points \((-2, 2), (3, 1)\). Find the equation of the circle on \( AB \) as diameter. If \( ACBD \) is a square, find the coordinates of \( C, D \).

4. Find where \( y = 2cx \) cuts the curve \( x^2 + y^2 = axy \).

5. Find the equations of the tangent and normal to \( y^2 = 4ax \) which are parallel to the line \( x = 2y \).

6. \( T \) is a variable point on a fixed diameter of \( x^2 - y^2 = a^2 \), prove that the chord of contact of the tangents from \( T \) is fixed in direction.

Q.R. 19

1. If the points \( A(-4, 5), B(8, 12), C(14, k) \) are collinear, find the values of \( AC : CB \) and the value of \( k \).

2. Find the equation of the line joining the origin to the point of intersection of the lines \( 2x - y + 3 = 0, x + 3y - 2 = 0 \).

3. Interpret the equation \( (x^2 + y^2 - 2x + 3y - 3) + k(2x + 3y - 5) = 0 \) and find the locus of the centre of the circle if \( k \) varies.

4. Find the values of \( t \) which determine the points of the curve \( x = at^2, y = at \) which lie on \( 2x + 13y - 32 = 0 \).

5. Find the pole of \( 2x - y(p + q) + 2pq = 0 \) with respect to \( y^2 = 4ax \).

6. If the tangents at points \( P, Q \) on \( xy = a^2 \) meet at \( T \), prove that \( OT \) bisects \( PQ \).

Q.R. 20

1. If \( (x_1 + d \cos \alpha, y_1 + d \sin \alpha) \) lies on \( x \cos \alpha + y \sin \alpha = p = 0 \), find \( d \) in terms of \( x_1, y_1, \alpha, p \) and interpret the result.

2. Find the equation of the tangent at \( (h + r \cos \theta, k + r \sin \theta) \) to the circle \( (x - h)^2 + (y - k)^2 = r^2 \) and the equation of the parallel tangent.

3. Find the equations of the two circles concentric with the circle \( x^2 + y^2 = 10x - 16y + 40 = 0 \) and touching \( x^2 + y^2 = 6x - 4y = 3 = 0 \).

4. Find \( l : m : n \) in terms of \( t_1, t_2 \) if \( lx + my + nz = 0 \) meets the curve \( x = at_1/(t^2 - 1), y = at_2/(t - 1) \) at points given by \( t = t_1, t = t_2 \).

5. \( PQ \) is a focal chord of \( y^2 = 4ax \); \( P \) is \( (at^2, 2at) \); prove that the length of \( PQ \) is \( at^2 + 1/t \).

6. \( OH, OK \) are conjugate semi-diameters of \( xy = c^2 \); \( OP, OQ \) are perpendicular to \( OH, OK \); prove \( OP, OQ \) are conjugate semi-diameters.

Q.R. 21

1. If the line \( y - 2x - 1 \) bisects the angle between \( 9y = 13x - 4 \) and \( y = mx + c \), find \( m \) and \( c \).

2. \( A, B, C \) are \((3, 1), (9, 5), (5, 0)\). Find the coordinates of the mid-points \( P, Q, R \) of \( BC, CA, AB \) and the areas of the triangles \( ABC, PQR \). Find also the coordinates of their centroids.

3. Find the equation of the circle which passes through \((0, 0), (2, 0), (0, 4)\) and prove the circle touches \( x + 2y = 10 \).

4. Find \( l : m : n \) if \( lx + my + nz = 0 \) meets \( x = ct, y = ct^2 \) at \( t = p \) (repeated), and find the tangent at \( P \).

5. Prove that \( x = y = 3a \) is a normal to \( y^2 = 4ax \).

6. Find the condition that the pole of \( px + qy + r = 0 \) with respect to \( xy = c^2 \) lies on the line \( lx + my + n = 0 \).

Q.R. 22

1. Find the line joining the point of intersection of \( 3x + y + 1 = 0, 2x - 2y + 1 = 0 \) to the point of intersection of \( 3x - y - 1 = 0, 2x + 2y = 1 \).

2. Determine by calculation whether the point \((2, 1)\) is inside or outside the triangle whose vertices are \((-2, -1), (4, 3), (5, -4)\).

3. Prove that the lines \( 2x - y = 4, 5x + 6y = 16 \) are related so that the pole of either with respect to \( x^2 + y^2 = 16 \) lies on the other.

4. If the circle \( x^2 + y^2 + 2px + 2my + t = 0 \) meets \( x = ct, y = ct^2 \) at \( t = p, q, r, s \), find the coordinates of the centre in terms of \( p, q, r, s \).

5. \( PN, PM \) are the perpendiculars from a point \( P \) on \( y^2 = 4ax \) to \( Ox, Oy \); prove that \( MN \) touches \( y^2 + 16ax = 0 \).

6. Find the condition that the sum of the roots of the \( r \)-quadratic \( (h + r \cos \theta)^2 - (k + r \sin \theta)^2 = a^2 \) is zero. Interpret the result.
Q.R. 23

1. The lines \(x - y = 3\), \(3x + 2y = 4\) are sides of a parallelogram; \((-2, 3)\) is one vertex. Find the equations of the diagonals.

2. Find the equation of the circle, centre \((\frac{1}{2}, -1\frac{1}{2})\), radius \(2\frac{1}{2}\), and the equation of the tangent at \((2, \frac{3}{2})\).

3. \(PQ\) is a normal chord to \(x^2 + y^2 = 9\), at the point \(P, t = -2\), find the value of \(t\) which gives \(Q\).

4. Find the equation of chord of \(xy = c^2\), having \((h, k)\) as mid-point.

5. Find the eccentricity and the coordinates of the foci of the ellipse; (i) \(x^2 + 3y^2 = 36\); (ii) \(3x^2 + 2y^2 = 24\).

6. Find the equation of the transverse axis of the hyperbola passing through \((1, -1)\), with asymptotes \(2x - 3y = 1\), \(x + 4y = 6\).

Q.R. 24

1. Find the coordinates of the circumcentre of the triangle \(ABC\) whose vertices are \((1, 5)\), \((-5, 3)\), \((-9, -7)\).

2. The tangents to \(x^2 + y^2 - 6x + 3y + 5 = 0\) at \(A(3, 1)\), \(B(-5, -3)\) meet at \(P\); prove \(AP\) equals the diameter of the circle.

3. If \(x + 2y = 1\) meets \(y^2 = 4x\) at \(P, Q\), find mid-point of \(PQ\).

4. Prove \(x + y + 2x = 0\) touches \(xy = c^2\) and \(y^2 = 8x\).

5. Find the equation of common chord of the ellipse \(2x^2 + 4y^2 + 18x - 16y - 6 = 0\).

6. Prove the normals to \(x^2/a^2 - y^2/b^2 = 1\), \(a^2/y^2 - b^2/x^2 = 1\) at \((a \tan 0, b \sec 9)\), \((b \sec 9, -a \tan 0)\) respectively are at right angles.

Q.R. 25

1. Find the equation of the line joining \((-3, 4)\) to the point of intersection of \(2x + 4y + 2 = 0\), \(5x - 3y + 25 = 0\).

2. \(A, B, C\) are \((1, 6)\), \((-3, -2)\), \((7, -3)\); find the equation of the common chord of the circles whose diameters are \(AB, BC\).

3. Find the condition that the pole of the chord of \(x = at^2, y = 2at\) whose ends are the points \(t = p, t = q\), lies on the directrix.

4. Find the equation of the transverse axis, the coordinates of the vertices, and the eccentricity of \(xy + 5x - 3y + 1 = 0\).

5. Find the equations of two tangents, gradients \(\frac{1}{2}\) and \(-2\), to \(x^2 + 6y^2 = 15\). Explain why they meet on \(x^2 + y^2 = 15 + 2\).

6. Find the angle between the asymptotes of the hyperbola \(3x^2 - y^2 + 1 = 0\) and the coordinates of its foci.

Q.R. 26

1. Describe the locus of \(P(x, y)\) determined by the gradient relations: (i) \((y + 3)/(x - 1) = (y - 5)/(x + 2)\); (ii) \((y + 3)/(x - 1) = -(x + 2)/(y - 5)\).

2. Find the locus of \(P\) if the length of the tangent to \(x^2 + y^2 = 4\) from \(P\) is double the length of the tangent to \(x^2 + y^2 - 8y = 16\) from \(P\).

3. Find the angle of intersection of \(y^2 = 12x, y^2 = 16x - 12\).

4. \(P, Q\) are points \(t = p, t = q\) on \(x = kt, y = k/t\), where \(p, q\) are the roots of \(ax^2 + bx + c = 0\); find the pole of \(PQ\) in terms of \(a, b, c, k\).

5. Find length of chord, whose mid-point is \((1, 1)\); of \(3x^2 + 4y^2 = 25\).

6. Find the asymptotes of \(x^2 - 6xy - 7y^2 - 2x + 4y + 1 = 0\) and the equation of the conjugate axis.

Q.R. 27

1. Find the condition that \((h, k)\) is at distance \(2\) units from \(12x - 5y - 3 = 0\) on the origin side of the line.

2. Find \(k\) if \(x^2 + y^2 - 6x + 10y + k(x^2 - 8y + 5) = 0\) is a circle of radius zero. Interpret the results in detail.

3. Find the equation of the parabola whose focus is \((-1, -2)\) and directrix \(3x + 2y = 0\). Find the coordinates of its vertex.

4. The tangent at a point \(P\) on \(x^2 - y^2 = a^2\) meets an asymptote at \(Q\); prove that the mid-point of \(PQ\) lies on \(x^2 - y^2 = \frac{1}{2}a^2\).

5. Find the equations of the tangents to \(16x^2 + 25y^2 = 400\) from \((-4, -3)\) and the tangent of the angle between them.

6. Find \(l, m, n\) if \(2x + my + n = 0\) meets the hyperbola \(xy = 1\); \((t^2 + 1) : (t^2 - 1) : 2t\) at: (i) \(t = 2, t = 3\); (ii) \(t = 2\) repeated.

Q.R. 28

1. Find the coordinates of the ends of the diameter of the circle \(x^2 + y^2 + 2x - 4y - 20 = 0\) parallel to \(4x - 2y\).

2. If the point which divides the line joining \(P_1(x_1, y_1)\) to \(P_2(x_2, y_2)\) in the ratio \(k : 1\) lies on \(x^2 + y^2 = a^2\), find the quadratic which gives the values of \(k\). Find the condition that these values are of the form \(\pm c\) and then prove \(P_1\) lies on the polar of \(P_2\).

3. A chord of \(y^2 = 8ax\) touches \(y^2 = 4ax\), prove that its mid-point lies on \(y^2 = 16ax\).

4. \(P, t = p, P, t = q\) lie on \(x/a : y/b = 1 : (1 - t^2) : 2t : (1 + t^2)\), find the condition that \(PQ\) is: (i) a diameter; (ii) a focal chord.

5. If \(a, b, c\) are constants, prove that the locus of \((x \tan (0 + z), b \tan 0)\) is a hyperbola and find its asymptotes.

6. If the tangents to \(3x^2 - 6y^2 = 12\) from \((h, k)\) are at right angles, prove that \(5k^2 + 5h^2 = 8\).
TEST PAPERS

1-8 (Ch. 1-5)

Test Paper 1

1. If $P$, $Q$ are the points $(a \cos \theta, b \sin \theta)$, $(-a \sin \theta, b \cos \theta)$, and if $O$ is the origin, prove that the value of $OP^2 + OQ^2$ does not depend on $\theta$, and find the value of $OP^2 - (OQ^2 - OQ^2)$. 

2. $A$, $B$, $C$ are the points $(0, 3)$, $(-1, 1)$, $(1, 1)$ is the orthocentre of the triangle $ABC$. Find: (i) the coordinates of $C$ (ii) the area of the triangle $ABC$ (iii) the value of the tangent $ABC$. (L) 

3. Find the coordinates of the centre of the inscribed circle of the triangle whose sides are $y = 0$, $4x - 3y = 0$, $12x + 5y = 28$, and prove that the equation of the inscribed circle is $2x^2 + 9y^2 - 24x - 12y + 16 = 0$. 

4. $A$, $B$, $C$ are the points $(6, 2)$, $(-2, -3)$, $(8, -8)$ respectively. $BC$ is divided at $P$ in the ratio $1:2$; $AP$ is divided at $Q$ in the ratio $5:3$. Find the coordinates of $P$ and of $Q$. 

5. $A$ is the point $(3, -4)$ and $O$ is the origin. Find the equation of the circle on $OA$ as diameter and the equations of the tangents to this circle which are parallel to $Oz$. 

6. Prove that the circles whose equations are $x^2 + y^2 + ax + by = 0$ and $x^2 + y^2 - cx = 0$ touch each other if $a^2 + b^2 = c^2$ and in this case find the coordinates of the point of contact. 

Test Paper 2

1. The equations of the sides $BC$, $CA$, $BA$ of $\triangle ABC$ are $x - y = 1$, $x + 2y + 1 = 0$, $x - 2y = 3$ respectively. Find: (i) the equation of the line through $B$ perpendicular to $AC$ (ii) the length of the perpendicular from $B$ to $AC$. 

2. Find the value of $k$ if the line joining $A(-2, -3)$ to $B(5, -7)$ is cut by the line $3x - 2y + 32 = 0$ in the ratio $k:1$. Find for this value of $k$ the point $C$ on $AB$ for which $AC:CB = k:1$. 

3. $A$, $B$, $C$ are the points $(2, 2)$, $(-2, -4)$, $(8, -2)$ respectively. Prove that the triangle $ABC$ is right angled and isosceles. Find the points at which $AB$ and $BC$ cut the circle $x^2 + y^2 - 5x + y = 0$. (OC) 

4. Prove that $x + 2y = 0$ is one of the tangents from the origin $O$ to the circle $x^2 + y^2 + 2x - 6y + 5 = 0$ and find the equation of the other tangent from $O$. Find the angle between the two tangents. 

5. The circle, centre $C$, whose equation is $x^2 + y^2 - 4x + 2y + 3 = 0$, cuts $Ox$ at $A$ and $B$. Find the values of $a$ and the equation of the circle $ABC$ if the circles are equal. 

Test Paper 3

1. Lines are drawn through $A(6, 1)$ and $C(3, -5)$ parallel and perpendicular to $2x = 3y$ to form the rectangle $ABCD$. Find: (i) the length of $AD$; (ii) the area of $ABCD$. 

2. The lines $OA$, $OB$ make angles $\tan^{-1} \frac{1}{2}$, $\tan^{-1} \frac{1}{3}$ with $Ox$ and on the positive side of it; $OA \parallel OB$ and the length of the perpendicular from $O$ to $AB$ is $\sqrt{3}$. Find the equation of $AB$. (N) 

3. The vertices $A$, $B$, $C$ of a triangle are $A(a, 0)$, $B(0, b)$, $C(0, c)$. If $P(x, y)$ is a variable point such that the areas of the triangles $PCA$, $PBC$ are equal, prove that the locus of $P$ is two lines and find their equations. 

4. Prove that the points $(-17, 7)$, $(-12, 8)$, $(1, -5)$, $(0, 0)$ lie on a circle and find the coordinates of its centre and its radius. 

5. Prove that one of the segments into which the line $x + y + 2 = 0$ divides the circle $x^2 + y^2 - 4x + 2y - 4 = 0$ contains an angle of $45^\circ$. 

6. Prove that the circles $x^2 + y^2 = 1$, $x^2 + y^2 - 8x + 7 = 0$, $x^2 + y^2 - 6y + 5 = 0$ touch each other. Find the coordinates of the point from which the tangents to the circles are equal. 

Test Paper 4

1. $ABCD$ is a parallelogram. The equations of $AB$, $BC$, $CA$ are $4x + 5y = 18$, $7x + 2y = 0$, $11x + 7y = 0$ respectively. Find the equations of $AD$, $CD$, $BD$. 

2. Prove that the points $A(-2, -1)$, $B(4, 3)$, $C(6, 0)$, $D(0, -4)$ are the vertices of a rectangle. If the line $x = 3$ cuts $AB$, $DC$ in $P$, $Q$, find the area of the trapezium $PBCQ$. (C) 

3. $A$, $B$ are the points $(4, -5)$, $(-6, 2)$ respectively. Find the coordinates of points $P$, $Q$, $R$ on $AB$ such that $AP:PB = 2:3$, $AQ:QB = 3:5$, $AR:RB = 7:4$. Find also the ratio $QP:PR$. 

4. $A$, $B$ are the points $(2, -1)$, $(-3, 2)$. Find the equation of the circle on $AB$ as diameter. Prove that $5x + 3y + 18 = 0$ is the equation of a tangent to the circle and find the equation of the parallel tangent. 

5. The length of the tangent from the origin to a circle is 6 units; the equation of the chord of contact of the tangents from the origin is $3x + 2y = 18$. Find the equation of the circle. 

6. $ABCD$ is a square circumscribing the circle $x^2 + y^2 = a^2$ with its sides parallel to $Ox$, $Oy$; a tangent to the circle meets $AB$, $AD$ at $P$, $Q$ respectively. Prove that $BP: DQ = 2a^2$. 

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Test Paper 5

1. The equations of the lines $AB$, $BC$, $CD$, $DA$ are $x = y$, $2y = 5x + 3$, $7y = 3x + 2$, $2x + y = 0$ respectively. Find the equations of $AC$, $BD$.

2. $A$, $B$, $C$ are the feet of the perpendiculars from $(0, -4)$ to $x + y = 0$, $y + 2x = 1$, $3y - 9x + 32 = 0$. Find the equation of $AB$ and prove that it passes through $C$. (OC)

3. The vertices of a triangle are $A(3, 5)$, $B(-2, 1)$, $C(4, -7)$; $BC$, $OA$, $AB$ meet at $P$, $Q$, $R$. Find the values of the ratios $BP : PC$, $CQ : QA$, $AE : EB$ and the value of their product.

4. $A$, $B$, $C$ are the points $(5, 3)$, $(7, -13)$, $(4, 7)$. Prove the circle on $AB$ as diameter touches the circle, centre $C$, radius 3 units.

5. $A$, $B$ are the points $(7, -2)$, $(3, -6)$; $P$ is a variable point such that $PA : PB = 3 : 2$. Prove the locus of $P$ is a circle and find the radius.

6. Describe the system of circles given by $x^2 + y^2 + 2gx = 16$, where $g$ varies. Prove: (i) only one circle of the system touches $x + 2y = 8$, (ii) the radical axis of the circle $x^2 + y^2 + 2gx + 8y + 24 = 0$ and the variable circle of the system passes through a fixed point on $Oy$. (N)

Test Paper 6

1. The vertices of a triangle are $A(3, 6)$, $B(-5, 0)$, $O(3, -2)$. Find the equations of the perpendicular bisector of $BC$ and the internal bisector of the angle $BAC$ and their point of intersection.

2. The vertices of a triangle are $P(0, 1)$, $Q(3, 2)$, $R(-2, -1)$; $PQRS$ is a parallelogram and $PQHK$ is a square with $HK$, $RS$ on the same sides of $PQ$. (i) Find the coordinates of $S$. (ii) Prove $PQRS$ is a rhombus and find its area. (iii) Find the coordinates of $H$, $K$. (O)

3. $G$ is the centroid of the triangle $ABC$ whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$; $D$ is the point $(x_4, y_4)$; $P$, $Q$ are the mid-points of $AB$, $CD$. Prove that the point which divides $DG$ in the ratio $3 : 1$ is the mid-point of $PQ$.

4. Find the equation of the circle passing through $(1, 0)$, $(0, 1)$ and having its centre on the line $x + 2y - 6 = 0$. Prove that this circle cuts the circle $x^2 + y^2 = 3$ orthogonally.

5. Prove that $x^2 + 4xy + 4y^2 = 25$ represents two parallel lines and find the distance between them. Find the equations of the two circles which touch both lines and pass through the origin. (L)

6. A variable line, distinct from $Ox$ or $Oy$, touches the fixed circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ and cuts $Ox$, $Oy$ at $P$, $Q$. Prove that the equation of the locus of the mid-point of $PQ$ is $(x - a)(y - a) = \frac{1}{2}a^2$.

Test Paper 7

1. Prove that the points $(2, 1)$, $(4, 2)$, $(5, 6)$, $(3, 4)$ are the vertices of a parallelogram and that the points $(1, 1)$, $(2, -1)$, $(6, -2)$, $(4, 0)$ are the vertices of a parallelogram congruent to the first parallelogram.

2. The equations of the altitudes $AD$, $BE$, $CF$ of the triangle $ABC$ are $x + y = 0$, $x - 4y$, $2x - y$ respectively. If $A$ is the point $(t, -t)$, find the coordinates of $B$ and $C$. If $t$ varies, prove that the locus of the centroid of $\triangle ABC$ is $x + 5y = 0$. (OC)

3. A line through the origin $O$ cuts the lines $y = ax + b$, $y = cx + d$, where $a = c$, at $P$, $Q$. If $OP : OQ = k : 1$, find the equation of $OP$.

4. Find the equation of the circle which passes through the origin and through the points of intersection of the circles $x^2 + y^2 + 4x - 5 = 0$, $4x^2 + y^2 - 8x + 2y - 1 = 0$.

5. The length of the tangent from the origin to a circle is $k$ units and the equation of the chord of contact of the tangents from the origin to the circle is $ax + by = 1$. Find the equation of the circle and prove that $k^2(a^2 + b^2) > 1$.

6. $P$ is a point outside the circle $S$ and $PQ$ is a diameter of a circle $S'$ orthogonal to $S$; Prove that $Q$ lies on the chord of contact of the tangents from $P$ to $S$.

Test Paper 8

1. A line passes through the point $(8, 6)$ and cuts $Ox$, $Oy$ at $A$, $B$ such that $OA : OB = 2 : 3$. Find the equation of $AB$ and the distance of the mid-point of $AB$ from $O$.

2. Prove that the lines $x + 3y = 1$, $13x + 9y = 1$, $x + 3y = 7$, $13x + 9y = 31$ are the sides of a rhombus. Find the area of the rhombus and the tangent of the acute angle of the rhombus.

3. The vertices of a triangle are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$; $O$ is the origin; $AO$, $BO$, $CO$ meet $BC$, $CA$, $AB$ at $P$, $Q$, $R$. Find the values of the ratios $BP : PC$, $CQ : QA$, $AR : EB$ and the value of their product.

4. Prove that the common chords of the circles $x^2 + y^2 - 6y - 1 = 0$, $x^2 + y^2 - 2x - 2y + 1 = 0$, $x^2 + y^2 + 2x - 4y - 21 = 0$, taken in pairs, are concurrent, and find their point of intersection.

5. $P$ is a variable point on the given line $x \cos \alpha + y \sin \alpha = p = 0$, where $p = 0$; $O$ is the origin; $OP$ is produced to $Q$ so that $OP : OQ = k^2$, where $k$ is constant. Prove the locus of $Q$ is a circle and find the coordinates of its centre.

6. If the circle $x^2 + y^2 + 2px + 2y + c = 0$, prove $2(p + q) - r = 2q + 2r - c$. (L)
Test Paper 9

1. Find the equations of the angle bisectors of the lines whose equations are \( x - y = 4 \) and \( 3x + 7y = 2 \).

2. \((-1, 3), (2, 7)\) are two vertices of the rectangle \( ABCD \); \( BD \) is parallel to \( OX \). Find: (i) the coordinates of \( B \) and \( D \); (ii) the tangent of the angle \( AC \) makes with \( BD \).

3. Prove that the points \((8, 2), (-8, -4), (-1, 3)\) lie on a circle and find its centre and radius.

4. The normal at \( P \) to \( y^2 = 4ax \) meets \( OX \) at \( G \); \( PN \) is the perpendicular from \( P \) to \( OX \). Prove that the length of a tangent from the origin \( O \) to the circle, centre \( G \), radius \( GP \), is equal to \( ON \).

5. The normal at \( P \) to \( y^2 = 4ax \) meets the curve again at \( Q \); \( O \) is the origin. If \( OQ \) is at right angles to \( OP \), prove that \( OP \) is bisected by the latus rectum.

6. \( P \) is a point on the rectangular hyperbola \( x^2 - y^2 = a^2 \), centre \( O \), foci \( S, S' \). Prove that \( SP, S'T = OP^2 \).

Test Paper 10

1. \( A, B \) are the points \((25, 2), (10, -10)\), and the centroid of the triangle \( ABC \) is the point \((7, 4)\). Find the coordinates of \( C \) and prove that the triangle \( ABC \) is right angled. (L)

2. Find the equations of the two circles which touch \( OX \) and pass through the points \((1, 2)\) and \((3, 4)\). If the circles cut one another at an angle \( \theta \), prove that \( \cos \theta = 0.6 \).

3. \( P \) is a variable point on the circle \( x^2 + y^2 - 3x - 5y + 2 = 0 \) outside each of the circles \( x^2 + y^2 - 2x - 4y + 1 = 0 \), \( x^2 + y^2 + x - y - 2 = 0 \). Prove the ratio of the lengths of the tangents from \( P \) to these circles is constant and find its value.

4. A variable chord \( PQ \) of \( y^2 = 4ax \) passes through the point \((2a, 0)\). Prove that the locus of the mid-point of \( PQ \) is \( y^2 = 2a(5 - 2x) \).

5. The normal at \( P \) to \( y^2 = 4ax \) meets the curve again at \( Q \). If the tangents at \( P, Q \) meet at \( T \), prove that the mid-point of \( TP \) lies on the directrix.

6. One asymptote of a rectangular hyperbola is \( 3x - 5y - 7 = 0 \), and the curve passes through the points \((2, -3)\) and \((4, 1)\). Find: (i) the equation of the hyperbola; (ii) the equation of the transverse axis.

Test Paper 11

1. The vertices of a triangle are the points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\). If the origin is the orthocentre of the triangle, prove that \( x_1x_2y_3 + y_1y_2x_3 + x_1y_2y_3 = 0 \).

2. Find the equation of the locus of the centre of a variable circle which passes through the origin and touches the line \( 3x + 4y = 5 \). If the locus cuts \( OX \) at \( A \) and \( B \) and cuts \( OY \) at \( C \), prove that \( OA \cdot OB = OC \cdot OD = 9 : 10 \).

3. Prove that the limiting points of the system of coaxal circles determined by the circles \( 3x^2 - 5y^2 - 6x + 2 = 0 \), \( 3x^2 + 5y^2 - 6y + 1 = 0 \) are the points \((4, 3), (4, -3)\). Verify that the perpendicular bisector of \( AB \) is the radical axis of the coaxal system.

4. \( K \) is a fixed point on the axis of \( y^2 = 4ax \); \( KR \) is the perpendicular from \( K \) to the tangent at a variable point \( P \); the line through \( P \) parallel to \( OX \) meets \( KR \) at \( Q \). Prove that the locus of \( Q \) is a line parallel to \( OY \).

5. \( SL \) is the semi latus rectum of the parabola \( y^2 = 4ax \), focus \( S \). If the normal at \( L \) meets the curve again at \( Q \), prove \( LQ = 8a \sqrt{2} \).

6. The tangent at \( P(x_1, y_1) \) to \( x^2 - y^2 - a^2 \), centre \( O \), makes angle \( \theta \) with \( OP \). Prove: (i) \( a^2 \cot \theta = \pm 2xy \); (ii) \( OP^2 = a^2 \cos \theta \). (C)

Test Paper 12

1. One vertex of a rhombus is \((0, 4)\), one diagonal is \( 3y - x - 2 \), and two sides are parallel to \( y = x \). Find the coordinates of the other vertices and the area of the portion of the rhombus which lies in the first quadrant. (L)

2. Find the ratio in which the line joining \((0, \frac{1}{2})\) to \((1, 1)\) is divided by the points in which \( AB \) meets the circle \( x^2 + y^2 = 1 \).

3. A circle, centre in the first quadrant, touches \( OY \) and touches externally \( x^2 + y^2 - 4x = 0 \). Prove its centre lies on \( y^2 = 10x + 5 \). If the circle also touches \( OX \), prove the abscissa of the point of contact is \( 5 + \sqrt{30} \). (OC)

4. \( U \) is the pole of a chord \( PQ \) of \( y^2 = 4ax \); \( S \) is the focus; \( PQ \) meets \( OX \) at \( K \). Prove that \( SK \) is equal to the distance of \( U \) from the directrix.

5. Prove that \( 4x + 2y + a = 0 \) is the equation of the common tangent of the parabolas \( y^2 = 4ax \), \( 2x^2 = ay \), and that the distance between the points of contact is \( \frac{7}{2} \) times the distance of the origin from the common tangent. (L)

6. Prove that the locus of a variable tangent to \( xy = c^2 \) with respect to \( x^2 + y^2 = a^2 \) is \( 4c^2 x + ay = a^2 \).
Test Paper 13

1. A and B are the points (−2, 0), (1, 0); P and Q are the points (5, 12), (0, k). Find k and k if the angles PAQ and PBQ are right angles.

2. If the circle $x^2+y^2-a^2=2pg(x\cos a+y\sin a)$ is equal to the circle $x^2+y^2-a^2=2qg(x\cos a+y\sin a)$, find the relation between $p$ and $q$.

3. P and Q are the points on the parabola, $x=at^2$, $y=2at$, given by $t=p$, $t=q$. If the circle on PQ as diameter touches the tangent at the vertex, prove that $p^2-q^2=\pm 2p$.

4. Find the coordinates of the vertex and focus of the parabola $3x^2-4x-2y+50=0$. Find also the equations of the tangents from the point $(4, -5)$ to the parabola.

5. The normals at variable points $P, Q$ on $y^2=4ax$ meet at right angles at R. Prove that the locus of R is the parabola $y^2=a(x-3a)$. State the coordinates of its vertex and focus.

6. Show that there are two values of $b$ such that $xy-c^2$ can be expressed in the form, $(x-b)^2+(y-b)^2=2(x+y-b)/\sqrt{2}$. Interpret the result geometrically.

Test Paper 14

1. Prove that the lines through the origin which make 45° with the line $y=mx$ are given by $(1-m^2)(x^2-y^2)+4mxy=0$.

2. The centres of two circles, radii $r_1$, $r_2$, where $r_1>r_2$, are $(-a, 0), (a, 0)$. The equations of their external common tangents are $x\cos \alpha \pm y \sin \alpha = a \cos \beta$. Express $r_1, r_2$ in terms of $a, \alpha, \beta$ and prove that the internal common tangents are $x \cos \beta \mp y \sin \beta = a \cos \alpha$.

3. S is a given circle; $S'$ is a variable circle which passes through a fixed point O. If the radical axis of $S$ and $S'$ passes through a fixed point A, prove that the centre of $S'$ lies on a fixed line perpendicular to OA. (OC)

4. The line PQ meets the parabola given by $y=ax^2+y=2at$ at the points $t=p, t=q$, where $p, q$ are the roots of $t^2+2bt+c=0$. Find the equation of PQ in terms of $a, b, c$.

5. P and Q are the points of contact of the tangents to a given parabola from a variable point $T$ on a given diameter. Prove that $TP^2-TQ^2$ varies as $PQ^2$.

6. A circle, centre $O$, cuts the rectangular hyperbola, $x=at, y=ct$ at M, N, P, Q; H, K are the mid-points of MN, PQ and O is the origin. Prove that $OHOK$ is a parallelogram.

Test Paper 15

1. Prove that the product of the lengths of the perpendiculars from A(4, 0) and B(−4, 0) to the line $3x\cos \theta +5y\sin \theta =16$ does not depend on the value of $\theta$.

2. $C_1, C_2, C_3$ are three coaxial circles; $\lambda_{12}$ is the ratio of the powers of a point on $C_1$ with respect to $C_2$ and $C_3$; $\lambda_{12}$ and $\lambda_{13}$ have similar meanings. Prove that $\lambda_{12} \cdot \lambda_{13} \cdot \lambda_{13} = 1$.

3. Two fixed circles cut at $O$; a variable line OPQ cuts the circles again at $P$ and $Q$. $R$ is the mid-point of $PQ$. Prove that the locus of $R$ is a circle.

4. A variable tangent to a parabola cuts two fixed tangents $TH, TK$ at $P, Q$. Prove that the length of the projection of $PQ$ on the directrix is constant.

5. $P$ is a variable point on the parabola $y^2=x$; $PQ, PR, PQ$ are normals to $y^2=x, Q, R$. Prove: (i) $ QR$ passes through $(-\frac{1}{2}, 0)$; (ii) the circle $PQR$ passes through the origin; (iii) the orthocentre of the triangle $PQR$ lies on the parabola $y^2=2x+3$.

6. $P, Q, R$ are points on the rectangular hyperbola $xy=c^2$; $QR, RP, PQ$ meet one of the asymptotes at $L, M, N$. Prove that the perpendiculars at $L, M, N$ to $QR, RP, PQ$ are concurrent.

Test Paper 16

1. Prove that the lines whose equations are $x+(1+y)=1+p, x+(1-q)y=1-q$ meet at a point on the line $x+y=1$, without finding the coordinates of the point where the lines meet.

2. $P, T_1, P, T_2$ are the tangents from $P(x_1, y_1)$, $P_2(x_2, y_2)$ to the circle $x^2+y^2=a^2$. Find the condition for $P_1T_1^2+P_2T_2^2$ to be equal to $P_1P_2^2$ and interpret this condition geometrically.

3. $PQ$ is a diameter of a variable circle which passes through the fixed points $(a, 0), (-a, 0)$. If $P$ moves on the fixed line $y=c$, prove that the equation of the locus of $Q$ is $x^2-cy+a^2$.

4. $PQ$ is a focal chord of the parabola, $x=at^2, y=2at$; $P, Q$ are given by $t=p, t=q$. Find: (i) $q$ in terms of $p$; (ii) the lengths of $SP, SQ$ in terms of $p$. Prove $SP \cdot SQ = a \cdot PQ$.

5. The polars of the points $P, Q$ with respect to $y^2=4ax$ make angles $\theta, \phi$ with OX; PH, QK are the perpendiculars from $P, Q$ to the polars of $Q, P$. Prove $PH \cdot QK = \sin \theta \cdot \sin \phi$.

6. M, N, P, Q are points on the rectangular hyperbola, $x=at, y=ct$, such that $MN$ is perpendicular to $PQ$; $H, K$ are the mid-points of $MN$ and $PQ$. Prove: (i) the circle on $HK$ as diameter passes through the origin $O$; (ii) the line joining the mid-points of $MP, NQ$ is a diameter of this circle.
Test Paper 17

1. \(PH, PK\) are the perpendiculars from a variable point \(P\) to the fixed lines \(y = -x \tan \alpha,\ y = x \tan \alpha\), where \(0 < \alpha < 90^\circ\). If \(PH^2 = PK^2 = c^2\) where \(c\) is constant, prove the locus of \(P\) is a rectangular hyperbola whose transverse axis equals \(2c/\sqrt{\sin 2\alpha}\).

2. A variable chord \(PQ\) of \(y^2 = 4ax\) passes through \((-a, 0)\); prove that the normals at \(P\) and \(Q\) meet on \(y^2 = a(x + a)\).

3. If the polar \(PQ\) of \(U\) with respect to \(y^2 = 4ax\) makes an angle \(45^\circ\) with \(OU\), prove that the foot of the perpendicular from \(U\) to \(PQ\) lies on the latus rectum.

4. If \(a^2 > 25\), prove there is a value of \(\theta\) between \(0^\circ\) and \(90^\circ\) such that the tangent to \(x^2/a^2 + y^2/b^2 = 1\) at \((a \cos \theta, b \sin \theta)\) meets the normal at \((a \cos \theta, -b \sin \theta)\) on the minor axis. (C)

5. Find the equations of the asymptotes and of the transverse axis of the hyperbola \(x^2 - 6xy - 7y^2 - 2x + 4y + 1 = 0\).

6. A variable line passes through the given point \((h, k)\) and meets \(x^2/\alpha^2 - y^2/\beta^2 = 1\) at \(P, Q\). Prove that the equation of the locus of the midpoint of \(PQ\) is \(b^2x^2 - a^2y^2 = b^2hx - a^2ky\).

Test Paper 18

1. Find the equation of the reflection of the line \(2y = x + 3\) in the line \(3y = 2x + 2\).

2. The line \(x + h = 0, y + 0\) cuts \(x^2 + y^2 = \alpha^2\) at \(C, D\). The distance of a variable point \(P\) from \(CD\) equals the tangent from \(P\) to \(x^2 + y^2 = \alpha^2\). Prove the locus of \(P\) is a parabola. Find the locus if \(h = 0\). (OC)

3. The normal to \(y^2 = 4ax\) at \(P(q, p)\) meets the curve again at \(Q(lp, pq);\) \(O\) is the origin. Prove: (i) \(p^2 + q^2 + 2 = 0\); (ii) the angle between \(QP\) and normal at \(Q\) equals angle \(OP\) makes with \(OQ\).

4. If the ordinates \(PN, QM\) at \(P, Q\) to the hyperbola \(x^2/a^2 - y^2/b^2 = 1\) at \(Q\), prove the normal at \(P, Q\) to these hyperbolas meet on \(OQ\).

5. If the pole of a chord \(PQ\) of \(x^2/a^2 + y^2/b^2 = 1\) is \((x_1/a^2 + y_1/b^2) = 2\), prove that \(OP, OQ\), where \(O\) is the origin, are conjugate semi-diameters of \(x^2/a^2 + y^2/b^2 = 1\).

6. Prove that the equation of the tangent to \(x = h/(t^2 - 1), y = k/d/(t^2 - 1)\) at the point \(t = t_1\) is \(kx(1 + t^2_1) - 2kt_1y + kh = 0\). Find the equations of the asymptotes and the coordinates of the vertices and foci.

Test Paper 19

1. A variable line passes through a fixed point \(C\) and cuts two given perpendicular lines \(OA, OB\) at \(P, Q\). Prove the locus of the mid-point of \(PQ\) is a rectangular hyperbola.

2. The circles, \((x-a)^2 + y^2 = a^2, (x-b)^2 + y^2 = b^2\), cut at \(P\) and \(Q\); prove that the equation of the circle on \(PQ\) as diameter is \((a^2 + b^2)(x^2 + y^2) - 2(a^2x + b^2y) + (a^2 - b^2)^2 = 0\).

3. \(PQ\) is a chord of \(y^2 = 4ax\) perpendicular to \(OZ\); the tangent at \(P\) meets \(OZ\) at \(T\); the perpendicular from \(Q\) to \(PT\) meets \(OZ\) at \(H\). Prove that the mid-point of \(TH\) lies on the directrix.

4. Prove that the foot of the normals to \(x^2/a^2 + y^2/b^2 = 1\) from \((h, k)\) lie on the curve given by \(x = a h/(a^2 + t), y = b h/(b^2 + t)\).

5. \(P, \ Q, \ R\) are points on \(x / \alpha = 1 + t^2, y / \beta = 1 + t^2\) given by \(t = n, t = q, t = r\). (i) If \(U\) is the pole of \(PQ\), find the condition for \(y = mx\) and the diameter \(OU\) to be conjugate. (ii) If \(r = 1\), prove \(QR\) is a diameter. (ii) If the tangent at \(P\) meets two conjugate diameters at \(U, V\), prove the other tangents from \(U, V\) are parallel.

6. \(O\) is the mid-point of the side \(BC\) of a given triangle \(ABC\). \(P\) is a variable point on \(OA\). If the perpendiculars to \(PB, PC\) at \(B, C\) meet at \(O\), prove the locus of \(Q\) is a hyperbola, centre \(O\), with asymptotes perpendicular to \(OA\) and \(BC\).

Test Paper 20

1. A variable chord \(PQ\) of the circle \(x^2 + y^2 = \alpha^2\) subtends a right angle at the given point \((f, g)\). Prove that the locus of the mid-point of \(PQ\) is the circle \(x^2 + y^2 = f^2 - f + 1/4(f^2 + g^2 - \alpha^2) = 0\).

2. U is the pole of a chord \(PQ\) of a parabola; \(PH, QK, UF\) are the perpendiculars from \(P, Q, U\) to a tangent. Prove \(PH, QK = UF\).

3. Prove that the parabolas \(y^2 = 4(a + f), y^2 = 4(b + h)\) have the same focus and axis. \(P, Q\) are variable points on these parabolas such that the tangents at \(P, Q\) meet at angles at \(R\). Prove \(R\) lies on \(x + a = 0\) and the line through \(R\) parallel to \(Oz\) bisects \(PQ\).

4. \(PQ\) is a chord of \(xy = c^2\) normal at \(P; \ HK\) is a chord parallel to \(PQ\). Prove \(PH, QK\) meet at \(R, \angle ROP = 90^\circ\).

5. Prove that the gradients of the tangents to \(x^2/a^2 + y^2/b^2 = 1\) from \(P(h, k)\) are the values of \(m\) given by \(m^2(h^2 - a^2) - 2mhk + (k^2 - b^2) = 0\). If the angle between the tangents to \(P\) is \(45^\circ\), prove \(P\) lies on the curve \((x^2 - a^2)^2 + (y^2 - b^2)^2 = (x^2 + y^2)^2 = 4xy^2\).

6. The normal to \(x^2/a^2 - y^2/b^2 = 1\) at \(P\) meets \(Oz, Oy\) at \(G, g\); if \(P\) is produced to \(Q\) so that \(GP = PQ\). Prove that \(Q\) lies on the hyperbola \(x^2/a^2 - 4b^2y^2/(a^2 - b^2)^2 = 4\) and that the eccentricity equals \(1/(a/b + b/a)\).
Test Paper 21

1. The vertices of a triangle are \( A(-1, 3), B(-1, 1), C(2, 4) \). Find the equation of the circumcircle of \( \triangle ABC \) and the coordinates of the circumcentre and centroid and orthocentre.

2. \( P, Q \) are variable points on \( y^2 = 4ax \) such that the normals at \( P, Q \) meet on the given line \( y = b \); prove that the tangents at \( P \) and \( Q \) meet on \( xy + ab = 0 \).

3. \( QSR \) is a focal chord of \( x = at^2, y = 2at \); \( PQ \) is the normal at \( P \); \( P, Q \) are the points \( t = p, t = r \). Prove \( r \left( p^2 + 2 \right) = p \) and deduce there are two positions \( P, Q \) of \( P \) which give the same point \( R \). If \( R \) varies, prove \( P_1 P_2 \) passes through a fixed point.

4. Prove that the points of intersection of \( x^2(a^2 - k^2) + y^2(b^2 - k^2) = 1 \) and \( x^2(a^2 + y^2) = 1 \) lie on the circle \( x^2 + y^2 = a^2 + b^2 - k^2 \). (Li)

5. \( PP' \) is a diameter of \( x^2/a^2 + y^2/b^2 = 1, a > b > 0 \); eccentricity \( \cos \alpha \). \( SY, SY' \) and \( SZ, SZ' \) are the perpendiculars from the focus \( S, S' \) to the tangents at \( P, P' \). Prove \( YZ = PZ = Y'Z = \cos \alpha \). If \( YZ' Z' \) is a square and if \( P \) is \( (a \cos \theta, b \sin \theta) \), prove \( \sin \theta = \pm \tan \alpha \).

6. \( H, P \) and \( Q \) are the points on \( x/a : y/b : 1 = (1 + t^2) : 2t : (1 - t^2) \) given by \( t = p, t = q \); \( H \) is fixed and \( P, Q \) vary so that \( HP \) is perpendicular to \( HQ \). Prove \( PQ \) passes through a fixed point.

Test Paper 22

1. The equations of the sides of a parallelogram are \( u = ax + by + c = 0, \ u' = ax + by + c' = 0, \ v = bx + my + n = 0, \ v' = bx + my + n' = 0 \). Prove the equation of one diagonal is \( w = u - v \) and find the equation of the other.

2. If two vertices of a variable triangle of constant area are given, prove the locus of the orthocentre is two parabolas.

3. The normals to \( y^2 = 4ax \) at \( P, Q \) meet at a point \( R \) on the curve; \( S \) is the focus. Prove \( SP \cdot SQ = a \cdot SR \).

4. \( P \) is a point of intersection of \( x^2 - y^2 = a^2, xy = c^2 \); the tangent at \( P \) to \( x^2 - y^2 = a^2 \) meets its asymptotes at \( H, K \); the tangent at \( P \) to \( xy = c^2 \) meets its asymptotes at \( R, T \). Prove \( HEKT \) is a square.

5. The normal to \( x^2/a^2 + y^2/b^2 = 1 \) at \( P \left( a \cos \theta, b \sin \theta \right) \) meets \( Ox \) at \( Q \). Prove \( \sin \phi = \cos \phi = b \). \( P'Q' \) is taken on \( PG \) so that \( PQ = k \cdot P'Q' \), where \( k \) is a constant and \( \phi \) varies, prove the locus of \( Q \) is an ellipse or, if \( k = \pm \sin \phi \), a circle.

6. The tangent to \( x^2/a^2 + y^2/b^2 = 1 \) at \( (x_1, y_1) \) meets the asymptotes at \( R(a_1, b_1), R'(a_2, b_2) \). Prove: (i) \( 2a_1 x + b_1 y = a_1 b_1 + b_2 y = 2 \left( a_1 b_1 + b_2 y \right) \); (ii) circumcentre of \( \triangle ORR' \) lies on \( a_2 x^2 + b_2 y^2 = 4 \left( a_1 b_1 + b_2 y \right) \); (iii) orthocentre of \( \triangle ORR' \) lies on \( a_2 x^2 - b_2 y^2 = (a_1 b_1) \).

CHAPTER 12

PAIRS OF LINES

12.1. If the equations of two intersecting lines are
\[ ax + by + c_1 = 0, \]
\[ ax + by + c_2 = 0, \]
where \( a_1b_2 + a_2b_1 \), the coordinates of their point of intersection are found by solving for \( x \) and \( y \); the solution may be written
\[ \frac{x}{b_2c_3 - b_3c_2} - \frac{y}{a_2c_3 - a_3c_2} = \frac{1}{a_1b_2 - a_2b_1}. \]

At first sight, this result may appear complicated but in fact it can be remembered easily by introducing a 'pattern-notation' as a visual aid.

The cross-product \( b_2c_3 - c_2b_3 \), is denoted by the pattern \( |b_2c_3| \).

This pattern is called a determinant and with this notation
\[ \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \frac{1}{a_1b_2 - a_2b_1}. \]

The determinants in the denominators are derived from a 'skeleton' of the two equations,
\[ a_2 b_2 c_2 \\ a_3 b_3 c_3 \]
from which are omitted in turn the first, second, and third columns, bearing in mind that a MINUS sign is attached to the determinant in the second column.

\[ \text{The point of intersection of these lines lies on a third line} \]
\[ ax + by + c_3 = 0. \]

If
\[ a_2 b_2 c_2 - b_1 a_2 c_2 + c_1 a_2 b_2 = 0. \]

The left side of this relation can be remembered easily by introducing a similar 'pattern-notation' of 3 rows and 3 columns,
\[ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \]

This is the condition for the three lines to be concurrent; it represents the relation,
\[ a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3) = 0. \]
12.1.1. The meaning of a determinant formed by 3 rows and 3 columns, as defined in 12.1, can be stated as follows:

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} = a_1 A_1 + b_1 B_1 + c_1 C_1
\]

where \( A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \) and \( B_1 = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \) and \( C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \).

These determinants of 2 rows and 2 columns are carved out of the given determinant by the following operations:

- For \( A_1 \), omit the row and column through \( a_1 \);
- For \( B_1 \), omit the row and column through \( b_1 \) and attach a minus sign; and
- For \( C_1 \), omit the row and column through \( c_1 \).

Determinants are not only convenient aids for remembering formulas of a special type, but it will be found that their use simplifies many algebraic processes; this is due to properties which arise from the form of their construction.

12.1.2. The value of a determinant is unaltered if columns and rows are interchanged. In symbols

\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}
\]

Left side: \( -a_1 b_2 c_3 - a_2 a_1 b_3 - a_3 b_2 c_1 + a_3 a_2 b_1 + a_1 b_3 c_2 + a_2 c_1 b_3 \)

\[= a_1(b_2 c_3 - a_2 c_1) - a_2(b_3 c_1 - a_3 c_2) + a_3(b_1 c_2 - a_1 c_3)\]

\[= a_1(b_2 c_3 - a_2 c_1) - a_1(a_2 b_3 - a_3 b_2) + a_3(a_1 b_2 - a_2 b_3)\]

\[= \text{right side.}\]

12.1.3. If two rows of a determinant are interchanged, the numerical value is unaltered but the sign is changed.

For example,

\[
\begin{vmatrix}
  a_1 & b_2 & c_3 \\
  a_2 & b_3 & c_1 \\
  a_3 & b_1 & c_2 \\
\end{vmatrix} = -\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}
\]

Left side:

\[= a_1(b_2 c_3 - a_2 c_1) - a_2(b_3 c_1 - a_3 c_2) + a_3(b_1 c_2 - a_1 c_3)\]

\[= -a_1(b_2 c_3 - a_2 c_1) + a_2(b_3 c_1 - a_3 c_2) - a_3(b_1 c_2 - a_1 c_3)\]

\[= \text{right side.}\]

Other interchanges can be treated in the same way. The change of sign is due to the fact that the interchange of two rows is equivalent to the interchange of two suffixes in the expanded form of the determinant, and this causes a change of sign of each term.

12.1.4. If two rows of a determinant are identical, the value of the determinant is zero.

Denote the value of the determinant by \( \Delta \).

If two rows are identical, the interchange of these rows makes no difference; therefore by 12.1.3, \( \Delta = -\Delta; \therefore \Delta = 0. \)

12.1.5. If all elements of one row or column are multiplied by a constant \( k \), the value of the determinant is multiplied by \( (\text{or divided by}) \ k. \)

This is true because each term in the expanded form of the determinant contains one, and only one, element from each row.

12.1.6. The value of a determinant is not altered by adding to the elements of a row or column of the corresponding elements of the other rows. For example,

\[
\begin{vmatrix}
  a_1 & b_2 & c_3 \\
  a_2 & b_2 & c_3 \\
  a_3 & b_2 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  a_1 + a_2 + a_3 & b_2 + b_2 + b_2 & c_3 + c_3 + c_3 \\
  a_2 & b_2 & c_3 \\
  a_3 & b_2 & c_3 \\
\end{vmatrix}
\]

With the notation of 12.1.1.

\[
\text{right side} = (a_1 + a_2 + a_3)A_1 + (b_2 + b_2 + b_2)B_1 + (c_3 + c_3 + c_3)C_1
\]

\[= (a_1 A_1 + b_2 B_1 + c_3 C_1) + p(a_2 A_1 + b_2 B_1 + c_3 C_1) + q(a_3 A_1 + b_2 B_1 + c_3 C_1).\]

But \( a_2 A_1 + b_2 B_1 + c_3 C_1 \)

\[= a_2 b_2 c_3 - b_2 a_2 c_3 - b_2 a_3 c_3 + a_3 a_2 b_3 + a_3 b_2 c_3 - a_3 c_3 b_3\]

\[= \text{right side.}\]

\[\therefore \text{by 12.1.4, } a_2 A_1 + b_2 B_1 + c_3 C_1 = 0; \text{ similarly } a_3 A_1 + b_2 B_1 + c_3 C_1 = 0; \]

\[\therefore \text{right side} = (a_1 A_1 + b_2 B_1 + c_3 C_1) + 0 + 0 = \begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}\]

Similarly, it can be proved that multiples of the elements of the first and third rows can be added to corresponding elements of the second row without change of value; similarly also for additions to the third row.

12.1.7. The general properties just proved for rows hold also for columns because by 12.1.2 the value is unaltered if rows and columns are interchanged. For example,

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} = \begin{vmatrix}
  a_1 + p b_1 + q c_1 & b_1 & c_1 \\
  a_2 + p b_2 + q c_2 & b_2 & c_2 \\
  a_3 + p b_3 + q c_3 & b_3 & c_3 \\
\end{vmatrix}
\]
12.2. Use of Determinants.

12.2.1. The equation of the line through \((x_1, y_1), (x_2, y_2)\) is

\[
\begin{vmatrix}
x & y & 1 \\
x_1 & y_1 & 1 \\
x_2 & y_2 & 1
\end{vmatrix} = 0.
\]

The stated equation represents a line because, when the determinant is expanded, it has the form \(ax + by + c = 0\). Further, the equation is satisfied by \(x = x_1, y = y_1\) and by \(x = x_2, y = y_2\) because by 12.1.4

\[
\begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1
\end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1
\end{vmatrix} = 0.
\]

12.2.2. The points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) are collinear if

\[
\begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1
\end{vmatrix} = 0.
\]

By the given condition, \((x_1, y_1)\) lies on \(x \quad y \quad 1 = 0\).

12.2.3. The area of the triangle, vertices \(A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)\) is the numerical value of \(\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\).

The equation of \(BC\) is \(x \quad y \quad 1 = 0\); therefore the equation of \(BC\) is

\[
\begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1
\end{vmatrix} = 0.
\]

The equation of \(BC\) is

\[
\begin{vmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1
\end{vmatrix}.
\]

Hence, since \(BC = \sqrt{(x_3-x_2)^2+(y_3-y_2)^2}\), area \(\triangle ABC = \frac{1}{2}AD, BC = \text{numerical value of} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\).

12.2.4. Elimination. If values of \(x, y, z\), not all zero, satisfy simultaneously

\[
\begin{align*}
a_1x + b_1y + c_1z &= 0, \\
a_2x + b_2y + c_2z &= 0, \\
a_3x + b_3y + c_3z &= 0,
\end{align*}
\]

then

\[
\begin{vmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{vmatrix} = 0.
\]

The proof is the same as is used in 12.1, p. 209, to obtain the condition that three lines, whose equations are given, are concurrent.

---

12.2. USE OF DETERMINANTS

Example 1. Solve: \(7x + 3y = 4, \quad 5y - 2x = 3\).

Arrange the equations in the form, \(ax + by + c = 0\);

\[
\begin{vmatrix}
x & y & 1 \\
7 & 3 & -1 \\
-2 & 5 & -3
\end{vmatrix} = 0.
\]

\[
\therefore \quad \begin{vmatrix}
x & y & 1 \\
7 & 3 & -1 \\
-2 & 5 & -3
\end{vmatrix} = 0.
\]

\[
\therefore \quad x = 11 \quad y = 29.
\]

Example 2. Find the value of \(c\) if the lines

\[12x - 7y + c = 0, \quad 2x - 3y - 1 = 0, \quad 3x + y - 4 = 0\]

are concurrent.

The lines are concurrent if

\[
\begin{vmatrix}
x & y & 1 \\
2 & -3 & -1 \\
-1 & 3 & -1
\end{vmatrix} = 0.
\]

\[
\therefore \quad 12(12 + 1) - \begin{vmatrix}
-7 & -8 & 9 \\
5 & 4 & 6
\end{vmatrix} = 0;
\]

\[
\therefore \quad 156 - 35 + 11c = 0; \quad \therefore \quad 11c = 121; \quad \therefore \quad c = -11.
\]

Example 3. Evaluate \(\triangle = \begin{vmatrix} 5 & 4 & 0 \\ 9 & 7 & 11 \end{vmatrix}\).

From the elements of row 1, subtract 3 times the elements of row 2,

\[
\therefore \quad \triangle = \begin{vmatrix} 16 & 12 & 17 \\ 5 & 4 & 0 \\ 9 & 7 & 11 \end{vmatrix} = 0.
\]

To the elements of column 3, add the elements of column 1,

\[
\therefore \quad \triangle = \begin{vmatrix} 5 & 5 & 5 \quad 11 \\ 11 & 11 & 11 \\ 9 & 7 & 20 \end{vmatrix}.
\]

Example 4. Find the line joining the points \((ap^3, 2ap), (aq^3, 2aq), p \neq q\).

The equation of the line is

\[
\begin{vmatrix} x & y & 1 \\ ap^3 & 2ap & 0 \\ aq^3 & 2aq & 1 \end{vmatrix} = 0.
\]

From the elements of row 2, subtract the elements of row 3,

\[
\begin{vmatrix} x & y & 1 \\ ap^3 & 2ap & 0 \\ aq^3 & 2aq & 1 \end{vmatrix} = 0.
\]

Divide the elements of row 2, by the common factor, \(a(p - q)\),

\[
\begin{vmatrix} x & y & 1 \\ p + q & 2 & 0 \\ aq^3 & 2aq & 1 \end{vmatrix} = 0.
\]

that is,

\[
2x - y(p + q) + 2apq = 0.
\]
PAIRS OF LINES

12.2.5. When applying the rules for simplification of determinants, it is advisable to modify only one row or only one column at a time. Methods of simplification may be stated in an abbreviated form; for instance in Example 3, p. 213, the two processes may be indicated by the phrases, row 1 - 3 x row 2 and col. 3 + col. 1. The simplest way of evaluating a determinant is to reduce it so that there are two zeros in any one row or in any one column. If at any stage there is a common factor of the elements of a row (or column), divide the elements of that row or column by the common factor and write the factor outside the determinant.

EXERCISE 48

Write down in ratio-form the solutions of the simultaneous equations and then simplify the solutions, Nos. 1-6:

1. \( x + 2y + 3 = 0 \)
   \( 3x + y + 2 = 0 \)
   \( 2x - y - 1 = 0 \)
   \( 3y = 2x + 7 \)
   \( 3x + y + 2 = 0 \)
   \( x + 3y = 1 \)

Find \( c \) if the following lines are concurrent, Nos. 7, 8:

7. \( x + y = 1 = 0 \)
   \( 2x + 3y + 4 = 0 \)
   \( 5x + 6y + c = 0 \)

8. \( 4x + 7 = y = 0 \)
   \( x - 2y = 3 = 0 \)
   \( 3x - 7y = 11 = 0 \)

Evaluate the following determinants, giving short reasons for each step, Nos. 9-17:

9. \[
\begin{vmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{vmatrix}
\]

10. \[
\begin{vmatrix}
1 & -1 & -1 \\
-1 & 1 & 1 \\
1 & -1 & 1
\end{vmatrix}
\]

11. \[
\begin{vmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{vmatrix}
\]

12. \[
\begin{vmatrix}
1 & 4 & 5 \\
20 & -5 & 30 \\
1 & 6 & 3
\end{vmatrix}
\]

13. \[
\begin{vmatrix}
1 & 6 & 16 \\
-18 & 45 & 27 \\
15 & 7 & 13
\end{vmatrix}
\]

14. \[
\begin{vmatrix}
5 & 8 & 7 \\
5 & 5 & 8 \\
7 & 8 & 7
\end{vmatrix}
\]

15. \[
\begin{vmatrix}
29 & 38 & 40 \\
24 & 32 & 34 \\
19 & 26 & 28
\end{vmatrix}
\]

Prove that the following lines are concurrent, Nos. 18-20:

18. \( x + 2y = 3 \)
   \( 3x + 4y = 5 \)
   \( 7x + 8y = 9 \)

19. \( 8x - 5y + 13 = 0 \)
   \( 4x - 7y + 11 = 0 \)
   \( x - 2y + 3 = 0 \)

20. \( 13x - 4y + 3 = 0 \)
   \( 3x + 2y = 1 = 0 \)
   \( 2x - 5y + 3 = 0 \)

USE OF DETERMINANTS

Write down in determinant-form the equation of the line through the two given points; expand and simplify it, Nos. 21-25:

21. \( (7, -6) \) \( (3, 2) \) \( (5, 3) \) \( (-2, -4) \) \( (-8, -1) \) \( (2, -5) \)

22. \( 3t + 1, 2t + 2 \)
   \( (6t + 7, 4t + 2) \)

23. \( \cos 6\theta, \sin 6\theta \)
   \( \cos 6\theta, \sin 6\theta \)

24. Find by using a determinant the equation of the chord joining the points \( t = p, t = q \) on the given curve, Nos. 26-29:

25. \( x : y = 2 : 1 \)

26. \( x : y : z = 3 : 3 : 2 \)

27. \( x : y : z = 2 : 2 : 1 \)

28. \( x : y : z = 3 : 1 : 2 \)

29. \( x : y : z = 2 : 3 : 1 \)

Prove that the three given points are collinear, Nos. 30-35:

30. \( (-2, 3) \)
   \( (5, -2) \)
   \( (19, -12) \)

31. \( (10, -5) \)
   \( (4, 11) \)
   \( (-3, 35) \)

32. \( (2p, 2q) \)
   \( (2p, 2q) \)
   \( (-ap, bq) \)

33. \( (cp, d) \)
   \( (cp, d) \)
   \( (0, c/p + c/q) \)

34. \( (x_1 + px_2 + y_1 + py_2, x_1 + px_2 + y_1 + py_2, x_1 + px_2 + y_1 + py_2, x_1 + px_2 + y_1 + py_2, x_1 + px_2 + y_1 + py_2) \)

35. \( (x + y, 1, q + r, 1, q + r) \)
   \( (r + q, 1) \)

36. Find the condition for the points \( (p^3, p), (q^3, q), (r^3, r) \) to be collinear if \( p, q, r \) are all unequal.

Evaluate the determinants, Nos. 37-39:

37. \[
\begin{vmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{vmatrix}
\]

38. \[
\begin{vmatrix}
1 & 0 & c \\
1 & 1 & 0 \\
0 & b & 0
\end{vmatrix}
\]

39. \[
\begin{vmatrix}
a & b & c \\
b & c & a \\
c & a & b
\end{vmatrix}
\]

Find the areas of the triangles whose vertices are given, Nos. 40, 41:

40. \( (pq, r) \)
   \( (q, p) \)
   \( (rp, q) \)

41. \( (q, r + p) \)
   \( (r + p, q) \)
   \( (p, q) \)

Find the areas of the triangles whose sides are given, Nos. 42, 43:

42. \( p = x + ap^2 \)
   \( q = x + aq^2 \)
   \( 2x + y + 2apq = 0 \)

43. \( y + gy = q + r \)
   \( x + r + p \)
   \( x + 2gy = p + q \)

Eliminate \( x : y : z \) from the given equations, Nos. 44-48:

44. \( x + 2y + az = 0 \)
   \( bx + y + 2az = 0 \)
   \( cx + cy + 3z = 0 \)

45. \( x + y + rz = 0 \)
   \( px + y + rz = 0 \)
   \( px + y + rz = 0 \)

46. \( x + p + z = 0 \)
   \( x + y + qz = 0 \)
   \( x + ry + x^2 = 0 \)

47. \( x + p + y = 2px \)
   \( x + q + y = 2qy \)
   \( x + px + py = (p + q)x \)

48. \( ax + by + cz = 0 \)
   \( bx + cy + az = 0 \)

49. If \( a_1, a_2, b_1, b_2, c_1, c_2 \) express \( x^2 : x : 1 \) in terms of \( a_1, b_1, c_1, a_2, b_2, c_2 \); hence eliminate \( x \).
Pairs of Lines [12.3]

12.3. Lines through the Origin. It was proved on p. 41 that the homogeneous function of the second degree in \(x\) and \(y\),
\[ax^2 + 2hxy + by^2\]
can be factorised if, and only if, \(h^2 > ab\).

If \(h^2 > ab\), there are constants \(l_1, m_1, l_2, m_2\) such that
\[ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)\]
and then the equation \(ax^2 + 2hxy + by^2 = 0\) represents two distinct lines
\[l_1x + m_1y = 0, l_2x + m_2y = 0,\]
where \(l_1, m_1, l_2, m_2, l_3 = a, l_1m_1 + l_2m_2 = 2h, m_1m_2 = b\).

If \(h^2 < ab\), the equation \(ax^2 + 2hxy + by^2 = 0\) represents two coincident lines.

If \(h^2 < ab\), there are no factors of \(ax^2 + 2hxy + by^2\); the locus \(ax^2 + 2hxy + by^2 = 0\) then consists of only one point, \(x = 0, y = 0\), see p. 41. For this reason, when speaking of the locus given by the general equation \(ax^2 + 2hxy + by^2 = 0\), it will be assumed that \(h^2 > ab\) and so the quadratic function \(ax^2 + 2hxy + by^2\) can be factorised, although the coefficients in the linear factors may not all be rational, see p. 222.

If \(a = 0\), the pair of lines is given by the equation, see p. 41,
\[a(ax^2 + 2hxy + by^2) = 0,\]
that is, \((ax + hy)^2 = (h^2 - ab)y^2;\)

hence the lines are given by the equations of the first degree,
\[ax + h\sqrt{(h^2 - ab)y^2} = 0, ax - h\sqrt{(h^2 - ab)y^2} = 0,\]
but in general it is best to avoid these forms and use symmetrical methods, as in Example 5.

Example 5. Find the coordinates of the orthocentre \(H\) of the triangle formed by the line-pair \(OB, OC\), \(ax^2 + 2hxy + by^2 = 0\), and the line \(BC, px + qy = 1\).

Let \(ax^2 + 2hxy + by^2 = 0\), and the line \(BC, px + qy = 1\), it is best to work with equal ratios as in 8.13.1, p. 119;
\[
\frac{x}{m_1} = \frac{y}{m_2} = \frac{px + qy}{pm_1 - q_l} = \frac{1}{pm_1 - q_l}.
\]
\[\therefore \text{the line from } B \text{ perpendicular to } OC, l_1x + m_1y = 0, \]
is
\[m_2x - l_1y - \ldots, \frac{m_2}{pm_1 - q_l} = \frac{l_1}{pm_1 - q_l} = \frac{l_1m_2}{pm_1 - q_l} = \frac{m_2}{pm_1 - q_l}.
\]
Also the line from \(O\) perpendicular to \(BC\) is \(x^2 - y^2 = 0\).

These lines meet at \(H\); therefore, using equal ratios, \(H\) is given by
\[
\frac{x}{m_1} = \frac{y}{m_2} = \frac{px + qy}{pm_1 - q_l} = \frac{l_1}{pm_1 - q_l} = \frac{l_1m_2}{pm_1 - q_l} = \frac{m_2}{pm_1 - q_l}.
\]
but as in 12.3, \(l_1 = a, l_2m_2 + l_1m_1 = 2h, m_1m_2 = b,\)
\[\therefore H \text{ is given by } \frac{x}{m_1} = \frac{y}{m_2} = \frac{a}{b^2 - 2hpq + aq^2}.
\]

12.4. Harmonic Division. If a segment \(AB\) of a line is divided internally at \(C\) in the ratio \(k : 1\) and is divided externally at \(D\) in the ratio \(-k : 1\), the points \(A, B\) are said to be separated harmonically by the points \(C, D\), see Fig. 101. The points \(C, D\) are called harmonic conjugates with respect to the points \(A, B\), and the segment \(AB\) is said to be divided harmonically at \(C\) and \(D\).

If \(\frac{AC}{DB} = -\frac{AD}{DB}\), then \(AC \cdot DB = AD \cdot CB\), \(\therefore \frac{CB}{DB} = -\frac{CA}{AD}\)

Thus if the points \(A, B\) are separated harmonically by the points \(C, D\), then \(C, D\) are separated harmonically by \(A, B\).

Fig. 101

Fig. 102

12.4.1 If a line cuts the line-pair \(OA, OB, ax^2 + 2hxy + by^2 = 0\), at \(P, Q\), and cuts the line-pair \(OA', OB', ax^2 + 2h'y' + b'y'^2 = 0\) at \(P', Q'\), then the points \(P, Q\) are separated harmonically by the points \(P', Q'\), if, and only if,
\[ab' + ba' = 2hh'.\]

Let \(P, Q\) be \((x_1, y_1), (x_2, y_2)\), then \(OP, OQ\) are \(x_1/t = y_1/y, x_2/t = y_2/y;\)
\[\therefore \text{the equation of the line-pair } OP, OQ \text{ is}
\]
\[(xy_1 - yx_1)(xy_2 - yx_2) = x_2y_1y_2 - xy(x_1y_2 + x_2y_1) + y^2x_2x_1 = 0.
\]
By hypothesis, this equation is equivalent to \(ax^2 + 2hxy + by^2 = 0\).
\[\therefore y_2' : (x_1y_2' + x_2y_1) : x_1x_2 : 2h : b \ldots\.\]
the point dividing \(PQ\) in ratio \(k : 1\) is \(\{(x_1 + kx_2)/(1 + k), (y_1 + ky_2)/(1 + k)\};\)
\[\text{this point lies on one of the lines of the line-pair } \text{OP, } \text{OQ} \text{ if}
\]
\[a(x_1 + kx_2)^2 + 2h'(x_1 + kx_2)(y_1 + ky_2) + b'y'(y_1 + ky_2) = 0.
\]
the roots of this quadratic in \(k\) are the values, see Fig. 102, of the ratios \(PP' : PQ, \text{PO}' : \text{PQ}'; \text{therefore } P, Q \text{ are separated harmonically by } P', Q'\) if, and only if, the sum of the roots of the quadratic is zero, that is, if the coefficient of \(k\) is zero.

Hence \(P, Q\) are separated harmonically by \(P', Q'\), if, and only if,
\[2a'x_2x_2' + 2h'(x_1y_2' + x_2y_1) + 2b'y_2y_2' = 0,
\]
and, from the ratios in equation (i), this condition becomes
\[2a'b' + 2h'(2h') + 2b'a = 0,
\]
that is,
\[a'b + b'a = 2hh'.
\]
12.4. If the line-pair $OA$, $OB$, $ax^2+2hxy+by^2=0$, and the line-pair $OA'$, $OB'$, $a'x^2+2h'xy+b'y^2=0$, meet one line $PQ$, which does not pass through $O$, in pairs of points separating one another harmonically, then they meet any other line, not passing through $O$, in pairs of points separating one another harmonically.

The line-pairs $OA$, $OB$ and $OA'$, $OB'$ are then said to separate one another harmonically.

By 12.4.1, p. 217, the condition, $ab'+ba'=2hh'$, for the line-pairs $OA$, $OB$ and $OA'$, $OB'$ to meet one transversal in pairs of points which separate one another harmonically is both necessary and sufficient. Hence if there is one transversal which cut harmonically, then $ab'+ba'=2hh'$, and this is a sufficient condition for any other transversal to be cut harmonically.

12.4.3. The equation of the pair of bisectors $OP$, $OQ$ of the angle between the pair of distinct lines $OA$, $OB$, $ax^2+2hxy+by^2=0$, is

$$\frac{x^2-y^2}{a-b} \frac{xy}{h}$$

First Method. The bisectors $OP$, $OQ$ of $\angle AOB$ are at right angles, therefore by 3.10.2, p. 41, the equation of the line-pair $OP$, $OQ$ is of the form, $a'x^2+2h'xy-a'y^2=0$.

Further, since the bisectors $OP$, $OQ$ of the vertical angle $AOC$ of $\angle AOB$ divide the base $AB$ internally and externally in the ratio of the sides, $OA : OB$, it follows that $OP$, $OQ$ separate harmonically $OA$, $OB$;

$\therefore$ $a(-a')+ba'=2hh'$, that is, $a'(b-a)-2hh'$;

$\therefore$ $a' = 2h' = h : (b-a)$.

The equation $a'(x^2-y^2)+2h'xy = 0$ of $OP$, $OQ$ can be written

$h(x^2-y^2)+(b-a)x = 0$, that is, $h(x^2-y^2)=(a-b)xy$.

Second Method. Let

$$ax^2+2hxy+by^2=b(y-x \tan \theta_1)(y-x \tan \theta_2), b \neq 0;$$

then $OA$, $OB$ make angles $\theta_1$, $\theta_2$ with $Ox$ given by

$$\tan \theta_1, \tan \theta_2 = -2h/b$$

and $\tan \theta_1, \tan \theta_2 = a/b$.

Let $(x_1, y_1)$ be a point on either of the angle-bisectors $OP$, $OQ$, then $y_1/x_1 = \tan \phi$, where $\phi = \frac{1}{2}(\theta_1 + \theta_2)$ or $\frac{1}{2}(\theta_1 + \theta_2) + 90^\circ$;

$\therefore$ $\tan 2\phi = \tan (\theta_1 + \theta_2) = 2 \tan \theta_1 \tan \theta_2 = 1 - \tan^2 \phi$.

$\therefore$ $2y_1/x_1 = -2h/b$;

$\therefore$ $x_1y_1 = -h/b$.

$\therefore$ the coordinates of any point either on $OP$ or on $OQ$ satisfy the equation, $h(x^2-y^2)=(a-b)xy$.

Note. A similar method can be used if $a=0$, taking $\theta_1$, $\theta_2$ as the angles which $OA$, $OB$ make with $Oy$.

12.4.1. Find the area of the triangle formed by the line $BC$, $5x+2y=6$, and the line-pair $OB$, $OC$, $3x^2-4xy-2y^2=0$.

Denote the equations of $OB$, $OC$ by $y-bx=0$, $y-cx=0$.

then $3x^2-4xy-2y^2 = -2(y-bx)(y-cx)$;

$\therefore$ $b+c = -2$ and $bc = -\frac{1}{2}$.

$BC$, $5x+2y=6$, meets $OB$, $y-bx=0$ where

$$x, y, 5x+2y = 6$$

$$1, b, 5+2b$$

$\therefore$ $D$ is the point $\left(\frac{6}{5+2b}, \frac{5b}{5+2b}\right)$; similarly $C$ is $\left(\frac{6}{5+2c}, \frac{5c}{5+2c}\right)$.

By 12.2.3, p. 212, the area of the triangle whose vertices are $(0, 0), (x_1, y_1), (x_2, y_2)$ is the numerical value of $\frac{1}{2}(x_1y_2-x_2y_1)$;

$\therefore$ area $\triangle ABC = \frac{3b(b-c)}{2(5+2b)(5+2c)}$.

But $(b-c)^2=(b+c)^2-4bc=4+6=10$, $\therefore$ $b+c = \pm \sqrt{10}$;

and $(5+2b)(5+2c)=25+10(b+c)+4bc=25-20-6=9$; $\therefore$ area $\triangle ABC = 18 \sqrt{10}$.

EXERCISE 49

1. Find the equations of the angle-bisectors of the line-pair $OB$, $OC$, $3x^2+3xy-y^2=0$. Prove $x-3y=2$ forms with $OB, OC$ an isosceles triangle.

2. Find the relation between $p$ and $q$ if the angle-bisectors of the line-pair $ax^2+2hxy-y^2=0$ are given by $x^2+y^2-y^2=0$.

3. Find the condition if $(ax+hy)(hx+by)-p(x^2-y^2)=0$ represents two lines at right angles.

4. Find the area of the triangle formed by the lines, $y=x-1$, $2x^2-xy-3y^2=0$.

5. Find the ratios in which the line joining $A(2, -3)$ to $B(-1, 2)$ is divided by the line-pair $x^2-2xy-y^2=0$.

6. A variable line passes through $A(1, -2)$ and cuts the line-pair $3x^2-xy-y^2=0$ at $P, Q$. Find the locus of a point such that $A, R$ separate harmonically $P, Q$.

7. If $ax^2-3xy+y^2=0$, $y=bx$ (or $y=mx, x=mz$), express the expression $\sqrt{\frac{1}{1+b^2}} - \sqrt{\frac{1}{1+m^2}}$. Interpret the result.

8. If $ax^2+2hxy+by^2=b(y-mx)(y-xm)$, express in terms of $a, h, b$ the equation $m^2+(y-x)(y-xm)=0$. Interpret the result.

9. Find the centroid of the triangle formed by the line $x+y=1$ and the line-pair $x^2-4xy-2y^2=0$. 


10. If \( P(x, y) \) belongs to the locus \( x^2 - xy - y^2 = 0 \), find the product of the lengths of the perpendiculars to the line \( P \). What can be deduced about the locus of \( P \)?

[11] Find the condition that the distance of \((h, k)\) from \( lx + my = 0 \) is equal to \( r \), and prove that each line of the line-pair whose equation is \((lx - hy) = r(2x^2 + y^2)\) touches the circle, centre \((h, k)\), radius \(r\).

12. Find the condition that the locus \((1 + c)x^2 + 4x^2 + (1 - c)y^2 - 0\) is a line-pair and find the value of \( h \) if it has the same angle-bisectors as \((1 + c)x^2 + 2hxy + (1 - c)y^2 = 0\).

[13] Prove that each line of the line-pair \( 11x^2 - 16xy - y^2 = 0 \) makes an angle 60° with the line \( 2x - y = 0 \).

14. Prove that the equation of any line-pair whose angle-bisectors are \( ax^2 + 2hxy + by^2 = 0 \) can be written \( a(x^2 - y^2) = my(x + hy) \).

15. If two of the lines \( y = ax, y = bx, y = cx, y = dx \) separate harmonically the other two, prove that one of the expressions, \( ab + cd, ac + bd, ad + bc \), equals the average of the other two.

16. Prove the median \( OD \) of the triangle \( OBC \) formed by \( B, y = mx + c \), and \( O, OC, ax^2 + 2hxy + by^2 = 0 \) is \( (a + bm)x + (b + bm)y = 0 \).

Prove that if any line meets \( OB, OC \) at \( H, K \), then \( HK \) is divided harmonically by \( OD \) and \( y = mx \). Interpret the result.

[17] Prove that the product of the lengths of the perpendiculars from \((x_1, y_1)\) to the line-pair \( ax^2 + 2hxy + by^2 = 0 \) is equal to \( (a(x_1^2 + 2hx_1y_1 + by_1^2))/\sqrt{(a - b)^2 + 4h^2} \).

18. If \( a, b, f, g, h \) are constant and \( l, m \) vary so that the lines \( ax^2 + 2hxy + by^2 = 0 \) are at right angles, prove the line \( lx + my = 1 \) passes through a fixed point; find its coordinates.

12.5. The general equation of the second degree in \( x \) and \( y \) is written in the form, see p. 54,

\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \]

In the special case when the left side of this equation can be expressed as the product of two functions of the first degree

\[ (l_1x + m_1y + n_1)(l_2x + m_2y + n_2), \]

the locus represented by the general equation is the pair of lines

\[ l_1x + m_1y + n_1 = 0, l_2x + m_2y + n_2 = 0. \]

If the left side can be factorised in this way, then

\[ (l_1x + m_1y)(l_2x + m_2y) = ax^2 + 2hxy + by^2; \]

therefore if \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) is the equation of a pair of lines, the equation of the lines parallel to them through the origin is \( ax^2 + 2hxy + by^2 = 0 \).

Example 7. Examine the locus whose equation is

\[ 6x^2 - 7xy - 6y^2 + x + 7y = 0. \]

Since \( 6x^2 - 7xy - 6y^2 = (2x + y)(3x - 6y) \), the equation can be written

\[ (2x + y + p)(3x - 6y + q) = r, \]

that is,

\[ 6x^2 - 7xy - 6y^2 + x + 7y + (2p + 3q) + (6p - 7q + r) = 0, \]

where \( p, q, r \) can and must be chosen so that

\[ 2p + 3q = 1 \quad \text{and} \quad 6p - 7q + r = 0. \]

From the first two equations, \( p = -1, q = 2 \); \( r = pq = -2 \).

\[ r = pq = -2, \]

the equation of the locus is \( (2x + y - 1)(3x - 5y - 2) = 0 \).

Hence, as on p. 172, the product of the distances of a point from the lines \( 2x + y - 1 = 0, 3x - 5y - 2 = 0 \) is constant; therefore the locus is a hyperbola with these asymptotes.

Note. This example illustrates the property that if the equation of a hyperbola is \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \), then the equation of its asymptotes is \( ax^2 + 2hxy + by^2 + 2gx + 2fy + k = 0 \), where \( k \) is chosen so that the equation represents a line-pair.

Example 8. For what value or values of \( k \) does

\[ 6x^2 - 11xy - 10y^2 + kx + 9y - 2 = 0 \]

represent a line-pair?

Since \( 6x^2 - 11xy - 10y^2 = (3x + 2y)(2x - 5y) \), the left side of the given equation has factors if, and only if, it can be written

\[ (3x + 2y + p)(2x - 5y + q) = 0, \]

that is, if \( 3x + 2y - k \) and \( 2x - 5y - q \) and \( pq = -2 \).

From the last two equations, \( 5p^2 + 9p + 4 = 0 \); \( p = -1, q = 2 \) or \( p = -\frac{4}{5}, q = \frac{3}{5} \); hence \( k = 4 \) or \( k = \frac{5}{3} \).

Check: \( 6x^2 - 11xy - 10y^2 + 4x + 9y - 2 = (3x + 2y - 1)(2x - 5y + 2); \)

\( 6x^2 - 11xy - 10y^2 + 5x - 9y - 2 = (3x + 2y - \frac{1}{2})(2x - 5y + \frac{3}{2}). \)

Example 9. Find the value of \( k \) for which

\[ kx^2 + 17xy + 6y^2 - 2x + 7y - 3 = 0 \]

represents a line-pair.

We cannot start with numerical factors of \( kx^2 + 17xy + 6y^2 \) since \( k \) is unknown; factorise the terms which do not contain \( x \).

Since \( 6y^2 - 7y - 3 = (3y - 1)(2y + 3) \), we assume the form

\( (3y - 1 + px)(2y + 3 + qz) = 0, \)

and choose \( p, q, k \) so that \( 2p + 3q = 17, 3p - q = -2, pq = k \);

hence,

\( p = 1, q = 5 \); \( k = pq = 5 \).

Check: \( 5x^2 + 17xy + 6y^2 - 2x + 7y - 3 = (x + 3y - 1)(5x + 2y + 3). \)
Example 10. Find the value of $k$ for which

$$x^2 - 10xy - 2y^2 + 18x + k = 0$$

represents a line-pair and then find their equations.

$$x^2 - 10xy - 2y^2 = (x - 5y)^2 - 27y^2 = (x - 5y)^2 - (3y \sqrt{3})^2;$$

$$\therefore x^2 - 10xy - 2y^2 = (x - 5 + 3 \sqrt{3}y)(x - 5 - 3 \sqrt{3}y).$$

It is now possible to proceed as in Example 8, but when, as here, the coefficients in the linear factors are not all rational, it is simpler to use a different method.

Solve for $x$ in terms of $y$: $x^2 - 2x(5y - 9) + (25y^2 - k) = 0;$$$
\therefore x = \frac{2(5y - 9) \pm \sqrt{(5y - 9)^2 - 4(25y^2 - k)}}{2} = 5y - 9 \pm \sqrt{27y^2 - 90y + 81 - k}.$$

This gives the equations of two lines if the quadratic function $27y^2 - 90y + 81 - k$ is of the form $(py + q)^2$; that is, if $45^2 = 27(81 - k)$; $75 = 81 - k$; $k = 6$.

If $k = 6$, $x = \frac{(5y - 9) \pm \sqrt{3(30y^2 - 30y + 25)}}{2} = 5y - 9 \pm \sqrt{9(2y - 3)^2} = 5y - 9 \pm 3\sqrt{3}y = 5y - 9 \pm 3\sqrt{3}y;$$$
\therefore x = -(5 + 3\sqrt{3}y) + (9 + 5\sqrt{3}) = 0 \text{ or } x = -(5 - 3\sqrt{3}y) + (9 - 5\sqrt{3}) = 0.$$

Note. This method can be used for finding the condition that

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a line-pair; a general discussion is given later, see p. 225.

12.6. This is a convenient opportunity for revising an important process described on p. 84 and illustrated by an example on p. 85.

If the lines $lx + my + n = 0$, $m'x + n'y + n'' = 0$, meets the locus

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

at $P$ and $Q$, the equation of $OP$, $OQ$, where $O$ is the origin, is

$$n^2(ax^2 + 2hxy + by^2) - 2n(gx + fy)(lx + my) + c(lx + my)^2 = 0.$$

The coordinates of $P$ and $Q$ satisfy any equation formed by a combination of the given equations.

Combine the equations so as to obtain a homogeneous equation of the second degree in $x$ and $y$, then the composite equation represents two lines through the origin and therefore represents the two lines $OP$, $OQ$.

Since $n \neq 0$, the equation $lx + my + n = 0$ is equivalent to

$$-(lx + my)/n = 1.$$

The required combination is therefore derived from

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

by multiplying the terms of the first degree, $2gx + 2fy$, by $-(lx + my)/n$ and multiplying the constant term $c$ by $\frac{1}{n^2}(lx + my)^2$.

Therefore the equation of the line-pair $OP$, $OQ$ is

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)(\frac{1}{n^2}(lx + my))^2 = 0,$$

that is,

$$n^2(ax^2 + 2hxy + by^2) - 2n(gx + fy)(lx + my) + c(lx + my)^2 = 0.$$

EXERCISE 50

Find the separate equations of the line-pairs, Nos. 1–4:

1. $x^2 + 2xy - 2x^2 - 2x + 5y - 3 = 0.$

2. $2x^2 + 2xy - y^2 - 8x + 4 = 0.$

3. $6x^2 + 5xy - 6y^2 - x - 3y - 2 = 0.$

4. $3x^2 + 30xy + 25y^2 - 12x + 20y + 4 = 0.$

Find $k$ if the equations, Nos. 5–11, represent line-pairs:

5. $x^2 + 2xy - y^2 - 13x + k = 0.$

6. $2x^2 + 3xy - 2y^2 - 5x + 10y + k = 0.$

7. $3x^2 - 7xy - 6y^2 - 5x + ky = 2 = 0.$

8. $x^2 + y^2 + 2x + ky - 45 = 0.$

9. $2x^2 + 5xy + ky^2 - x - 10y = 3 = 0.$

10. $3x^2 + kxy - 4y^2 + x + 14y - 10 = 0.$

11. $2xxy + 2x + 2fy + k = 0.$

12. Prove that $3x^2 - 12xy + 9y^2 - 2x - 3y - k = 0$ is a line-pair if $k < \frac{1}{2}$.

What can be said if: (i) $k = \frac{1}{2}$; (ii) $k > \frac{1}{2}$?

13. What locus is represented by $(x - 2y + 4)^2 + (2x + y - 7)^2 = k$, if: (i) $k = 0$; (ii) $k = 1$; (iii) $k = -1$?

14. Find the condition that the equation

$$(px^2 + qy^2)(px + qy + k^2 - 1) - (px + qy)^2 = 0$$

represents two perpendicular lines.

15. Prove that $(x - 1)^2 - 2(x - 1)(y + 2) - 3(y + 2)^2 = 0$ represents a pair of lines parallel to the pair $x^2 - 2xy - y^2 = 0$ and intersecting at the point $(1, -2)$.

16. Find the equation of the pair of lines which intersect at the point $(3, -1)$ and are parallel to the pair $x^2 + 2xy - 2y^2 = 0$.

17. Find the equation of the lines joining the origin O to the points of intersection $P$, $Q$ of $px + qy = 1$ with $2x + y = y^2 = 1$. Find the condition that: (i) $\angle POQ = 1$ rt. $\angle$; (ii) $px + qy = 1$ touches $3x^2 + 2xy - y^2 = 1$.

18. If $3x + 4y = 30$ cuts $16x^2 - 15y^2 = 36$ at $P$, $Q$, prove that the circle on $PQ$ as diameter passes through the origin.

19. If the line $x + 3y = 5$ cuts at $P$ and $Q$ the ellipse whose equation is $5x^2 + 12y^2 - 35y + 10 = 0$, prove that the circle on $PQ$ as diameter passes through the origin.

20. Prove that the hyperbola $(x - 2y)(x - 3y) + k(x + y - 1) = 0$ passes through two fixed points when $k$ varies; find their coordinates.

21. Prove that $x^2 - k^2y^2 - 4x + 4 = 0$ represents two lines which intersect at a fixed point when $k$ varies.
PAIRS OF LINES

22. The equation of a locus is \( x^2 + 2xy - 3y^2 - 2x - 10y - 3 = 0 \). Find (as on p. 90) its equation referred to parallel axes \( CX, CY \) through the point \( C(2, -1) \) and then describe the original locus.

23. The equation of a locus is \( x^2 - 8xy - 2y^2 + 14x - 20y + 31 = 0 \). Find its equation referred to parallel axes \( CX, CY \) through \( C(x_1, y_1) \) and show that \( x_1, y_1 \) can be chosen so that the equation becomes \( x^2 - 8xy + 2y^2 = 0 \).

[24] Prove that \( 3x^2 - 5xy - 2y^2 + x + 5y - 2 = 0 \) represents a line-pair intersecting at \( (\frac{1}{2}, \frac{1}{2}) \) and find the equation of their angle-bisectors.

25. \( PQ, lx + my - 1 = 0 \), is a variable line. If \( (3l + 2m + 5)(2l - 5m - 3) = 0 \), prove that \( PQ \) passes through one of two fixed points and find their coordinates.

[26] \( PQ, lx + my + 1 = 0 \), is a variable line such that \( l, m \) satisfy \( 4l^2 - 11lm - 3m^2 - 7l + 8m + 3 = 0 \). Prove that \( PQ \) passes through one of two fixed points and find their coordinates.

27. \( PQ, lx + my + n = 0 \), is a variable line such that \( l, m, n \) satisfy \( 2l^2 + 3lm - 2m^2 - 2ln + mn = 0 \). Prove that \( PQ \) either passes through a fixed point \( (h, k) \) or is parallel to a fixed line \( y = ax \), and find \( h, k, a \).

28. Prove that \( (x \cos \alpha + y \sin \alpha)^2 + 2px + 2py + c = 0 \) can be expressed in the form \( (x \cos \alpha + y \sin \alpha)^2 = k(x \sin \alpha - y \cos \alpha - p) \).

Find \( k \) in terms of \( f, g, a \).

If \( P \) is a point on the locus, find the relation between the lengths of the perpendiculars \( PN, PM \) from \( P \) to the lines \( x \cos \alpha + y \sin \alpha = p = 0 \), \( x \sin \alpha - y \cos \alpha = q = 0 \) and interpret the result geometrically.

29. If \( ab = h^2 \) and \( g/a = f/h \) and \( g^2 = ac \), prove that the equation \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) represents two coincident lines. If \( ab = h^2 \) and \( g/a = f/h \) and \( g^2 > ac \), prove the equation represents two parallel lines at distance \( 2\sqrt{g^2 - ac}/h \) apart.

30. Write down the conditions that \( Oz \) is the tangent at the origin \( O \) to the locus \( ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \).

If \( P \) and \( Q \) are variable points on this locus such that \( \angle POQ \) is a right angle.

(i) If this locus is not a rectangular hyperbola, prove that the normal at \( O \) meets \( PQ \) at a fixed point.

(ii) If this locus is a rectangular hyperbola, prove that \( PQ \) is parallel to the normal at \( O \).

CHAPTER 13
GENERAL PROCESSES

13.1. Differences between the various forms of the locus which is represented by the general equation of the second degree

\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \]

have been illustrated by examples. These include:

(i) a pair of distinct or coincident lines;
(ii) a parabola or ellipse or hyperbola;
(iii) the cases where the equation is satisfied by just one pair of values of \( x \) and \( y \) or by no value, as in \( x^2 + y^2 = 0 \) or \( x^2 + y^2 + 1 = 0 \).

It will be proved in this chapter that this is a complete classification of the loci represented by the general equation of the second degree, and tests which distinguish between the different forms will be obtained.

13.1.1. The locus represented by the general equation of the second degree is called a conic-section or, for short, a conic because a section of a complete right circular cone by a plane which does not pass through the vertex of the cone is either a parabola or ellipse or hyperbola; a section by a plane through the vertex of the cone is a pair of distinct or coincident lines or may be just one point, the vertex.

The proof itself of the general theorem on sections of a cone is of surprising simplicity, the difficulty lies in visualising the configuration. If the locus is a parabola or ellipse or hyperbola, it is called a proper conic; if it is a pair of distinct or coincident lines, it is called a degenerate conic.

In what follows, the vertex of the cone is denoted by \( V \) and its axis by \( VC \); the semi-vertical angle is denoted by \( \alpha \) and the angle which the plane of section makes with \( VC \) is denoted by \( \beta \), where \( \beta \leq 90^\circ \). If \( \beta = 90^\circ \), the section is a circle.

If \( P \) is any point on a given circular section, all points on the line \( VP \) produced both ways, lie on the complete cone and the line \( VP \) produced is called a generator of the cone.

A sphere is said to be inscribed in a cone if it is in contact with the cone along the circumference of a circle which by symmetry lies in a plane perpendicular to the axis of the cone; if also the sphere touches a given plane, it is called a focal sphere corresponding to the section of the cone by the plane.
13.4. If $P$ is any point on the curve of section and let the generator $P \ell$ meet $\ell$ at $Q$.

Let the plane cross the generator $\ell$ at some point $Q$.

In 13.5. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$.

13.6. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$. 

13.7. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$. 

13.8. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$. 

13.9. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$. 

13.10. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$. 

13.11. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$. 

13.12. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$. 

13.13. Draw the plane through the axis $OC$ of the cone perpendicular to $OC$.
EXERCISE 51

[The examples in this exercise refer to a right circular cone.]

1. The vertical angle of a cone is 120°; prove that a section making 45° with the axis is a rectangular hyperbola.

2. If the semi minor axis of an elliptic section of a cone is of length $b$ and if $r, r'$ are the radii of the corresponding focal spheres, prove that $b^2 = r'r$.

3. If $l, l'$ are the centres of the focal spheres, prove that the sphere whose diameter is $ll'$ passes through the auxiliary circle of the corresponding conic section.

4. $PP'$ is a variable diameter of a given elliptic section of a given cone, vertex $V$. Prove that $VPVP'$ is constant. State and prove the corresponding result for a given hyperbolic section.

5. If the semi latus rectum of a plane section of a cone, vertex $V$, semi vertical angle $a$, is $l$, prove that the length of the perpendicular from $V$ to the plane of section is $l \cot a$.

6. If a plane section of a cone, vertex $V$, meets a generator at right angles at $A$ and if the eccentricity of the section equals $e$, prove the latus rectum equals $2e \cdot VA$.

7. Prove that the locus of the vertex of a variable cone having a given ellipse, eccentricity $e$, major axis $AA'$, as a hyperbola of eccentricity $1/e$ having $A, A'$ as foci.

13.2. In discussing the reduction to standard form of

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

it is convenient to consider first the case where the terms of the second degree are not a perfect square, that is, $ab - h^2 < 0$.

The equation can be simplified by a change of origin.

Take the point $O_1(x_1, y_1)$ as new origin with axes $O_1X, O_1Y$ parallel to $Ox, Oy$. Let $(X, Y)$ be the coordinates referred to axes $O_1X, O_1Y$ of a point $P(x, y)$ referred to axes $Ox, Oy$;

then, as on p. 90, $x = X + x_1$ and $y = Y + y_1$.

Hence the equation of the locus referred to axes $O_1X, O_1Y$ is

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0;$$

this may be written

$$aX^2 + 2hXY + bY^2 + 2X(ax_1 + hy_1 + g) + 2Y(hx_1 + by_1 + f) + (x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0 . \quad (i)$$
13.2. If \( ab - h^2 = 0 \) and if \( \Delta > 0 \), division by \( -\Delta/(ab - h^2) \) gives
\[
pX^2 + 2kXY + qY^2 = 1
\]
where \( p = -a(ab - h^2)/\Delta, q = b(ab - h^2)/\Delta, k = -h(ab - h^2)/\Delta; \)
\[
pq - k^2 = (ab - h^2)/\Delta^2 + 0.
\]
The equation of the locus can now be reduced to the standard form
\[
ax^2 + by^2 = 1, \quad a \neq 0.
\]
by rotating the axes of reference.

13.2.2. If \( O_1 \xi, O_1 \eta \) are axes of reference which make angles 0, 90° with \( O_1 X \), and if the coordinates of \( P \) are \((X, Y)\) referred to \( O_1 X, O_1 Y \) and axes \((\xi, \eta)\) referred to \( O_1 \xi, O_1 \eta \), then
\[
X = \xi \cos \theta - \eta \sin \theta \quad \text{and} \quad Y = \xi \sin \theta + \eta \cos \theta.
\]
Draw \( PN, PN' \) perpendicular to \( O_1 X, O_1 \xi \).
\[
X = O_1 N = \text{projection of} \quad O_1 P \quad \text{on} \quad O_1 X
\]
\[
= \text{sum of projections of} \quad O_1 N, N' P \quad \text{on} \quad O_1 X,
\]
\[
X = \xi \cos \theta - \eta \sin \theta.
\]
\[
Y = NP = \text{projection of} \quad O_1 P \quad \text{on} \quad O_1 Y
\]
\[
= \text{sum of projections of} \quad O_1 N, N' P \quad \text{on} \quad O_1 Y,
\]
\[
Y = \xi \sin \theta + \eta \cos \theta.
\]
By rotating the axes of reference.

13.2.3. If, with the data of 13.2.1, axes \( O_1 \xi, O_1 \eta \) are chosen making angles 0, 90° with \( O_1 X, O_1 Y \), the locus whose equation is
\[
pX^2 + 2kXY + qY^2 = 1, \quad pq - k^2 = 0,
\]
is represented by the equation,
\[
p(\xi \cos \theta - \eta \sin \theta)^2 + 2k(\xi \cos \theta - \eta \sin \theta)(\xi \sin \theta + \eta \cos \theta)
\]
\[
+ q(\xi \sin \theta + \eta \cos \theta)^2 = 1.
\]
The coefficient of \( \xi \eta \) in this equation is
\[
2(q - p) \sin \theta \cos \theta + 2k(\cos \theta \sin \theta - \sin \theta \cos \theta), \quad \text{that is,} \quad (q - p) \sin \theta \cos \theta + 2k \cos \theta.
\]
If \( q - p \neq 0 \), the coefficient of \( \xi \eta \) is zero if \( \theta = 45°; \)
If \( q - p = 0 \), the coefficient of \( \xi \eta \) is zero if \( \theta \) is the acute angle determined uniquely by \( \tan 2\theta = 2k/(q - p) \).
If \( \theta \) is chosen in this way, the equation is of the form
\[
ax^2 + by^2 = 1
\]
where
\[
pX^2 + 2kXY + qY^2 = ax^2 + by^2.
\]
Further since \( pq - k^2 = 0, \)
\[
pX^2 + 2kXY + qY^2 \quad \text{is not a perfect square},
\]
\[
\therefore \quad \alpha \neq 0 \quad \text{and} \quad \beta \neq 0.
\]
If \( \alpha > 0 \) and \( \beta > 0 \), the equation represents an ellipse.
If \( \alpha < 0 \) and \( \beta < 0 \), there is no point belonging to the locus.
If \( \alpha < 0 \) and \( \beta > 0 \), the equation represents a hyperbola.
We proceed to interpret these conditions in terms of the coefficients of the general equation referred to axes \( Ox, Oy \).
13.2.5. There is another instructive method for obtaining the transformation from \( pX^2 + 2kXY + qY^2 = 1 \) to \( ax^2 + ay^2 = 1 \), to \( \alpha x^2 + \beta x^2 = 1 \).

If the circle \( X^2 + Y^2 = r^2 \) cuts the locus \( pX^2 + 2kXY + qY^2 = 1 \), the equation

\[
pX^2 + 2kXY + qY^2 = (X^2 + Y^2)/r^2,
\]

that is,

\[
(p - 1/r^2)X^2 + 2kXY + (q - 1/r^2)Y^2 = 0,
\]

represents the common chords which pass through the centre \( O \).

If this equation represents two coincident lines, then

\[
(p - 1/r^2)(q - 1/r^2) - k^2 = 0,
\]

and the lines coincide with a principal axis of the locus.

If \( r^2 = \sqrt{\alpha} \) is a root of the quadratic in \( 1/r^2 \),

\[
(p - 1/r^2)X^2 + 2kXY + (q - 1/r^2)Y^2 = (p - 1/r^2)(X^2 + Y^2)/r^2;
\]

the equation of the principal axis of length \( 2r \) is

\[
(p - 1/r^2)X + kY = 0.
\]

If the quadratic in \( 1/r^2 \) has two positive roots, the locus is an ellipse; if it has one positive root and one negative root, the locus is a hyperbola. The quadratic in \( 1/r^2 \) is the same as the quadratic in \( \lambda \) in 13.2.3, p. 231.

13.2.6. If \( a - h^2 = 0 \) and \( \Delta = 0 \), by 13.2, p. 229, the equation of the locus referred to axes \( O_1X, O_1Y \) is

\[ aX^2 + 2hXY + bY^2 = 0. \]

Therefore, by 12.3, p. 216, if \( a - h^2 < 0 \) and \( \Delta = 0 \), the locus is a pair of distinct lines intersecting at \( O \).

If \( a - h^2 > 0 \) and \( \Delta = 0 \), the locus consists of just one point \( O \).

13.2.7. Asymptotes of a hyperbola. If \( a - h^2 < 0 \), \( \alpha x^2 + \beta y^2 = -1 \) is the equation of a hyperbola whose asymptotes are given by the equation \( \alpha x^2 + \beta y^2 = -1 \); hence by 13.2.3, if \( a - h^2 < 0 \) and \( \Delta = 0 \), the equation of the asymptotes of a hyperbola given by the general equation

\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \]

is

\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c - c' = 0 \]

where \( c' \) is chosen so that the equation represents a pair of lines; therefore by 13.2.6 and 13.2,

\[
\begin{vmatrix}
g & f & c' \\
a & h & g \\
h & b & f \\
\end{vmatrix} = 0
\]

\[
\therefore \ c' = \frac{1}{a - h^2} = -\Delta; \quad \therefore \ c' = -\frac{\Delta}{(a - h^2)}.
\]

In particular, the asymptotes are at right angles if \( a + b = 0 \).

13.2.8. If the locus represented by the general equation is an ellipse or hyperbola, it is called a central conic.

It will be shown in 13.3 that, if \( a - h^2 = 0 \), the general equation does not represent either an ellipse or hyperbola. It follows from what has been proved above that \( a - h^2 = 0 \) and \( \Delta = 0 \) are necessary, but not sufficient, conditions for a central conic.
13.3. To complete the discussion of the general equation it is necessary to consider the case when the terms of the second degree form a perfect square, that is, \( ab - h^2 = 0 \).

13.3.1. If \( ab - h^2 = 0 \) and \( a \neq 0 \),
the general equation may be written, putting \( ab = h^2 \),
\[
a^2x^2 + 2ahxy + h^2y^2 + 2agy + 2fy + ao = 0,
\]
that is,
\[
(ax + hy)^2 + 2agy + 2fy + ao = 0.
\]
Also the expression \( abc + 2gh - af^2 - bg^2 - ch^2 \) for \( \Delta \) takes a simple form; \( abc + 2gh - af^2 = 0 \) and \( bg^2 = h^2y^2 \).
\[
\Delta = 2gh - af^2 - h^2y^2 = -(gh - af)^2/a.
\]
The general equation can be written in the form
\[
(ax + hy + \lambda)^2 + 2x(af + \lambda) + 2y(af - \lambda^2) + ac - \lambda^2 = 0,
\]
where, when possible, \( \lambda \) will be chosen so that the lines given by
\( ax + hy = \lambda, 2x(af - \lambda) + 2y(af + \lambda) = 0 \) are at right angles.
Choose \( \lambda \) so that
\[
\frac{ag - a\lambda}{a} = \frac{af - \lambda^2}{h} = \frac{h(af - \lambda^2) - a(af - \lambda^2)}{h^2 + a^2} = \frac{a(gh - af)}{h^2 + a^2} = \mu, \text{ say;}
\]
then \( a(af - \lambda) - h(af - \lambda^2) = 0 \).
\[
\lambda = a(af + f\lambda)/(a^2 + h^2).
\]
First suppose \( gh - af = 0 \), then \( \mu = 0 \) since \( a \neq 0 \).
The equation becomes
\[
(ax + hy + \lambda)^2 + 2x(af + \lambda) + 2y(af - \lambda^2) = 0,
\]
where \( v = \frac{1}{a}(ac - \lambda^2)/\mu \).
Take as new axes \( X = 0, Y = 0 \), the perpendicular lines
\( O_1Y, hx + ay + v = 0, \) and \( O_1X, ax + hy + \lambda = 0; \)
then the equation of the locus is of the form
\[
Y^2 - kX^2, b + h = 0.
\]
This represents a parabola having \( O_1X \) as axis and \( O_1Y \) as the tangent at the vertex.
Further since \( gh - af = 0 \), \( \Delta = - (gh - af)^2/a \neq 0 \).

Next suppose \( gh - af = 0 \), then \( \mu = 0 \) and \( g = \lambda \).
The equation becomes
\[
(ax + hy + \lambda)^2 = \lambda - ac.
\]
\( \because \) if \( gh - af = 0 \) and \( \lambda > 0 \), the locus is two parallel lines;
if \( gh - af = 0 \) and \( \lambda = 0 \), the locus is two coincident lines;
if \( gh - af = 0 \) and \( \lambda < 0 \), there is no point of the locus.
Further, since \( gh - af = 0 \), \( \Delta = -(gh - af)^2/a = 0 \).

13.3.2. If \( ab - h^2 = 0 \) and \( a = 0 \), then \( h = 0 \) and \( b = 0 \) because \( a, b, h \)
are not all zero.
Therefore the general equation becomes
\[
by^2 + 2gx + 2fy + c = 0, b \neq 0.
\]
For this form of the equation, \( \Delta = -bg^2 \).

13.3.3. In the discussion of the general equation of the second degree in 13.2 and 13.3, all possible cases, have been considered. It follows that:
(i) \( \Delta + 0 \) is a necessary condition for the general equation to represent a proper conic;
(ii) \( \Delta = 0 \) is a necessary condition for the general equation to represent two distinct or coincident lines.
Sufficient conditions have also been obtained; these can, however, be expressed in various ways.

13.3.4. If in a transformation from rectangular axes \( Ox, Oy \) to rectangular axes \( Ox_1, Oy_1 \), the transform of the expression \( ax^2 + 2hxy + by^2 \)
is \( a_1\xi^2 + 2h_1\xi\eta + b_1\eta^2 \), then
\[
a + b = a_1 + b_1 \quad \text{and} \quad ab - h^2 = a_1b_1 - h_1^2.
\]
If the coordinates of \( P \) are \( (x, y) \) referred to \( Ox, Oy \) and are \( (\xi, \eta) \) referred to \( Ox_1, Oy_1 \), then
\[
\begin{align*}
\xi^2 + \eta^2 &= OP^2 - \xi^2 + \eta^2; \\
\therefore \quad \text{for all values of} \xi, \\
ax^2 + 2hxy + by^2 &= \lambda(x^2 + y^2) \\
&= a_1\xi^2 + 2h_1\xi\eta + b_1\eta^2 - \lambda(\xi^2 + \eta^2).
\end{align*}
\]
that is, \( (a - \lambda)x^2 + 2hxy + (b - \lambda)y^2 = (a_1 - \lambda)\xi^2 + 2h_1\xi\eta + (b_1 - \lambda)\eta^2 \).
A value of \( \lambda \) which makes the left side of this identity a perfect square also makes the right side a perfect square; therefore
\[
(a - \lambda)(b - \lambda) = k = 0 \quad \text{and} \quad (a_1 - \lambda)(b_1 - \lambda) - k_1 = 0,
\]
that is, \( \lambda^2 - (a + b)\lambda + ab - h^2 = 0 \) and \( \lambda^2 - (a_1 + b_1)\lambda + a_1b_1 - h_1^2 = 0 \),
have the same roots. This proves what is required.

This property is expressed by the statement that \( a + b \) and \( ab - h^2 \)
are invariant under an orthogonal transformation; also \( a + b \) and \( ab - h^2 \)
are called invariants for such a transformation.
The condition that the value of an invariant is zero expresses a geometrical property. For example, if the axes remain rectangular and if \( S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) represents a proper conic, \( a + b = 0 \) is the condition for a rectangular hyperbola and \( ab - h^2 = 0 \) is the condition for a parabola.
Example 3. Examine the locus represented by the equation
\[(2x - 5y)^2 + 12(11x - 13y + 43) = 0.\]

The equation of the locus can be written in the form
\[(2x - 5y + 7)^2 = (132 - 4\lambda)x + (156 - 10\lambda)y + 0^2 - 516\]
where \(\lambda\) can be chosen so that the lines
\[2x - 5y = 0, \quad 4(\lambda - 33) + 10(5\lambda - 78)y = 0\]
are at right angles. This condition gives
\[8(\lambda - 33) + 10(5\lambda - 78)y = 0, \quad \text{that is,} \quad \lambda = 18.\]
\[\therefore \text{the equation of the locus can be written}\]
\[(2x - 5y + 18)^2 = -60x - 24y - 102 = -12(5x + 2y + 16).\]

This equation can be interpreted by writing it in the form
\[\frac{(2x - 5y + 18)^2}{(5x + 2y + 16)^2} = \frac{12}{29}\]
If \(PN\) and \(PM\) are the perpendiculars from a point \(P(x, y)\) of the locus to the lines \(AH, 2x - 5y + 18 = 0\), and \(AK, 5x + 2y + 16 = 0\), respectively,
\[PN^2 = (12/\sqrt{29})PM\]
where \(P(x, y)\) lies on that side of the line \(AK, 5x + 2y + 16 = 0\), for which
\(5x + 2y + 16 = 0\) is negative; therefore the locus lies entirely to the left of the
line \(AK\).

The locus is a parabola whose axis is \(AH, 2x - 5y + 18 = 0\), and whose
latus rectum equals \(12/\sqrt{29}\); the line \(AK, 5x + 2y + 16 = 0\), is the tangent
at the vertex \(A\), whose coordinates are given by the equations
\[2x - 5y + 18 = 0, \quad 5x + 2y + 16 = 0; \quad \therefore A = (-4, 2).\]

EXERCISE 52

Determine the nature of the loci represented by the equations in
Nos. 1-14. If the locus is a proper conic, find: (i) the equation of
its principal or major or transverse axis; (ii) its semi latus rectum.
1. \(8x^2 + 12xy + 17y^2 - 29 = 0.\)
2. \(x^2 - 6xy + y^2 + 1 = 0.\)
3. \(3x^2 - 2xy + 3y^2 + 1 = 0.\)
4. \(6x^2 - 60xy - 10y^2 - 78 = 0.\)
5. \(3x^2 + 5xy - 2y^2 - 31x + 15y - 22 = 0.\)
6. \(4x^2 - 4xy + y^2 - 14x - 18y + 31 = 0.\)
7. \(3x^2 - 8xy - 3y^2 - 44x - 8y + 3 = 0.\)
8. \(x^2 + xy + y^2 - 6x - 8y = 0.\)
9. \(0x^2 - 12xy + 4y^2 + 9x - 6y - 4 = 0.\)
10. \(3x^2 + 6xy + 3y^2 + 2x + 30y + 18 = 0.\)
11. \(10x^2 - 6xy + y^2 - 2y + 10 = 0.\)

12. \(x^2 + 2xy + 2y^2 - 2y + 10 = 0.\)
13. \(10x^2 + 12xy + 10y^2 - 8x - 8y + 1 = 0.\)
14. \(4x^2 - 24xy + 11y^2 - 40x + 70y + 96 = 0.\)

Examine the nature of the locus whose equation is given in Nos. 15-18,
for various values of \(\lambda\). Tabulate the results.
15. \((\lambda - 2)x^2 + (\lambda + 2)y^2 - (\lambda - 1)(\lambda - 3) = 0.\)
16. \((\lambda - 2)x^2 + (\lambda + 1)y^2 - (\lambda - 1)y = 2 = 0.\)
17. \(x^2 + 2xy + y^2 + 4x + 2(\lambda + 1)y + 4 = 0.\)
18. \(x^2 + 2xy + y^2 - 2x - 2y + \lambda = 0.\)

19. Find the conditions that \(ax^2 + 2hxy + by^2 + 2gx + 2fy - 0\) represents:
(i) two intersecting lines; (ii) two parallel lines; (iii) just one
point.
20. If \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\), \(ab - h^2 < 0\), represents
two lines, prove that the coordinates of their point of intersection are
\((x', y')\) where
\[\begin{align*}
\frac{(h'y - by)(ab - h^2)}{(h'y - by - af)(ab - h^2)} & = 0,
\end{align*}\]
and find the centre of the hyperbola, \(4xy - 3y^2 + 8x + 8y + 1 = 0\),
and find the value of \(c\) if it is congruent to \(4xy - 3y^2 = c\). Find \(p, q\) if it is
congruent to \(px^2 + qy^2 = 1.\)

21. Find the values of \(a, b\) if the central conics, given by
\(ax^2 - 12xy + 8y^2 = 5, \quad bx^2 - 12xy + 13y^2 = 10\),
are congruent.
22. If \(ax^2 + 2hxy + by^2 = 1\) is the equation of an ellipse, prove that its
area equals \(\pi\sqrt{\frac{c}{ab - h^2}}\).
23. Prove that the conditions that \(ax^2 + by^2 + c(x + y + d)^2 = 0\) is an
ellipse are \(a + b > c, \quad ab - c(a + b + 2d) < 0\).
24. If \(S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c\), where \(ab + h^2 > 0\),
explain why there is a value of \(c\) such that \(S + \lambda = 0\) represents
either two lines or just one point. Deduce that, with the usual notation, \(\Delta\) is an invariant of \(S\)
under an orthogonal transformation.
25. If in a transformation from rectangular axes \(Ox, Oy\) to rectangular
axes \(Ox', Oy', ax^2 + 2hxy + by^2 + 2gx + 2fy + c\) becomes
\[a_1x'^2 + 2h_1x'y' + b_1y'^2 + 2g_1x' + 2f_1y' + c_1,\]
prove \(g^2 + f^2 = q^4 + q^4.\)
26. If in a transformation from rectangular axes \(Ox, Oy\) to rectangular
axes \(Ox', Oy', ax^2 + 2hxy + by^2 + pxz^2 + qxy + cy^2\) becomes
\[a_2x'^2 + 2h_2x'y' + b_2y'^2 + p_2x'^2 + 2c_2x' + q_2y'^2,\]
what deduction can be made by considering the values of \(\lambda\) for which
\(ax^2 + 2hxy + by^2 + \lambda(px^2 + 2xy + qy^2)\) is the square of a function of \(x\) and \(y\).
Interpret the result geometrically.
27. If by using invariants the value of \(k\) if by a change of rectangular axes
\(Ox', Oy', ax_1^2 + 2h_1x'y' + b_1y'^2 + pxz^2 + qxy + cy'^2\) becomes
\(a_2x'^2 + 2h_2x'y' + b_2y'^2 + p_2x'^2 + 2c_2x' + q_2y'^2,\)
what deduction can be made by considering the values of \(\lambda\) for which
\(ax^2 + 2hxy + by^2 + \lambda(px^2 + 2xy + qy^2)\) is the square of a function of \(x\) and \(y\).
Interpret the result geometrically.
28. Find the nature of the parabola given by the equation
\(x^2 - 2xy + y^2 = c\), \(c \neq 0\), becomes \(\xi^2 = k.\)
13.4. The following substitutions and processes can be applied to the general equation of the second degree, but can be illustrated more simply by considering in the first instance the equation of a central conic \( px^2 + qy^2 = 1 \). Corresponding work with the general equation is discussed in the next chapter.

13.4.1. If \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) are given points, the line joining \( P_1 \) to \( P_2 \), see Fig. 107, p. 239, meets the central conic \( px^2 + qy^2 = 1 \), either at two distinct points \( P, P' \) or at two coincident points \( P, P' \) or does not meet it at any point.

13.4.2. Joachimsthal's Ratio Equation. If \( P \) is a point of intersection of the line joining \( P_1(x_1, y_1) \) to \( P_2(x_2, y_2) \) with the central conic \( px^2 + qy^2 = 1 \) and if \( P : PP' = k_2 : k_1 \), then the value of \( k_2 \) : \( k_1 \) are given by the quadratic,

\[
k_2^2(px_1^2 + qy_1^2 - 1) + 2k_1k_2(px_1x_2 + qy_1y_2 - 1) + k_1^2(px_1^2 + qy_1^2 - 1) = 0.
\]

By the ratio-formula, see p. 47, the coordinates of \( P \) are

\[
(k_2x_1 + k_1x_2) / (k_1 + k_2), \quad (k_2y_1 + k_1y_2) / (k_1 + k_2).
\]

Since \( P \) lies on the conic \( px^2 + qy^2 = 1 \), see Fig. 107,

\[
 p(k_2x_1 + k_1x_2)^2 + q(k_2y_1 + k_1y_2)^2 = (k_1 + k_2)^2;
\]

\[
 (k_1k_2x_1^2 + 2k_1k_2x_1x_2 + k_2^2x_2^2) + q(k_1k_2y_1^2 + 2k_1k_2y_1y_2 + k_2^2y_2^2) = (k_1^2 + 2k_1k_2 + k_2^2).
\]

Group together terms in \( k_1^2 \), terms in \( k_2^2 \), and terms in \( k_1k_2 \), this gives the required equation.

If this quadratic equation in \( k_2 : k_1 \) has two unequal roots, \( P_1P_2 \) meets the conic at distinct points \( P, P' \), see Fig. 107, p. 239. If the quadratic has two equal roots, \( P_1P_2 \) touches the conic at \( P \).

If the quadratic has no roots, \( P_1P_2 \) does not meet the conic.

13.4.3. If \( P_1(x_1, y_1) \) is a given point outside the central conic \( px^2 + qy^2 = 1 \), the equation of the pair of tangents from \( P_1 \) is

\[
(px_1^2 + qy_1^2 - 1)(px^2 + qy^2 - 1) = (px_1x + qy_1y - 1)^2.
\]

The ratios in which the line joining the fixed point \( P_1(x_1, y_1) \) to a variable point \( P_2(x_2, y_2) \) is divided by the roots of the ratio-equation in 13.4.2 are equal, see Fig. 107, p. 239. Therefore \( P_1P_2 \) touches \( px^2 + qy^2 = 1 \) if

\[
(px_1^2 + qy_1^2 - 1)(px_2^2 + qy_2^2 - 1) = (px_1x_2 + qy_1y_2 - 1)^2.
\]

Therefore the variable point \( P_2(x_2, y_2) \) lies on one or other of the tangents from the fixed point \( P_1(x_1, y_1) \) to \( px^2 + qy^2 = 1 \) if its coordinates satisfy the equation

\[
(px_1^2 + qy_1^2 - 1)(px^2 + qy^2 - 1) = (px_1x + qy_1y - 1)^2.
\]

13.4.4. If \( P_1(x_1, y_1) \) is a given point on the conic \( px^2 + qy^2 = 1 \), the equation of the tangent at \( P_1 \) is \( px_1x + qy_1y - 1 = 0 \).

Since \( P_1(x_1, y_1) \) lies on \( px^2 + qy^2 = 1 \), \( px_1^2 + qy_1^2 - 1 = 0 \); therefore one root of the ratio-equation in 13.4.2 is zero, it is the value of \( P : PP' = P_1P_2 \) when \( P \) is the same point as \( P_1 \), see Fig. 108.

If \( P_1P_2 \) meets the conic at no point except \( P_1 \), see Fig. 108, the second root of the ratio-equation is also zero;

\[
px_1x + qy_1y - 1 = 0.
\]

Therefore the variable point \( P_2(x_2, y_2) \) lies on the tangent at the fixed point \( P_1(x_1, y_1) \) if its coordinates satisfy the equation

\[
px_1x + qy_1y - 1 = 0.
\]

Note. This result was obtained by other methods on pp. 135 and 174.

![Fig. 107](image-url)

![Fig. 108](image-url)

13.4.5. If \( P_1(x_1, y_1) \) is a given point inside or outside the conic \( px^2 + qy^2 = 1 \) and if \( P_2(x_2, y_2) \) is a variable point such that \( P_1P_2 \) is divided harmonically by its points of intersection with the conic, then \( P_2 \) lies on the line \( px, x + qy, y - 1 = 0 \).

If \( P_1P_2 \) meets the conic at \( P, P' \), see Fig. 107, it is given that the values of \( P_1P : PP' \), \( P_1P' : PP' \), \( PP_2 \) are of the form \( k_1 - k_2 \), therefore the sum of the roots of the ratio-equation in 13.4.2 is zero;

\[
px_1x + qy_1y - 1 = 0.
\]

Therefore \( P_2(x_2, y_2) \) lies on the line \( px, x + qy, y - 1 = 0 \).

This line is called the polar of \( P_1(x_1, y_1) \) with respect to the conic \( px^2 + qy^2 = 1 \), and \( P_1 \) is called the pole of this line. This statement extends the meaning of the words polar and pole, previously defined (see pp. 147, 177) only if \( P_1 \) is outside the conic.

If \( P_1 \) is a point outside the given conic, a variable line through \( P_1 \) may not cut the conic, and in this case the polar of \( P_1(x_1, y_1) \) with respect to \( px^2 + qy^2 = 1 \), as just defined, is only part of the line \( px_1x + qy_1y - 1 = 0 \), but it is convenient to use the word polar to mean the whole of the line.
13.4.6. If the polar of $P_1(x_1, y_1)$ with respect to $px^2 + qy^2 = 1$ passes through $P_2(x_2, y_2)$, then the polar of $P_2$ passes through $P_1$, and $P_1, P_2$ are called conjugate points with respect to the conic.

The polar of $P_1$, $px_1x + qy_1y = 1$, passes through $P_2$ if $px_1x_2 + qy_1y_2 = 1$; this is the condition for the polar of $P_2$, $px_2x + qy_2y = 1$, to pass through $P_1$. Thus $(x_1, y_1), (x_2, y_2)$ are conjugate points if $px_1x_2 + qy_1y_2 = 1$.

13.4.7. If the pole $P_1(x_1, y_1)$ of $H_1K_1$, $l_1x + m_1y + n_1 = 0$, with respect to $px^2 + qy^2 = 1$ lies on $H_2K_2$, $l_2x + m_2y + n_2 = 0$, then the pole $P_2(x_2, y_2)$ of $H_2K_2$ lies on $H_1K_1$, and $H_1, K_1, H_2, K_2$ are called conjugate lines with respect to the conic.

By hypothesis, $px_1x + qy_1y = 1$ is equivalent to $l_1x + m_1y + n_1 = 0$,

\[ \therefore px_1x_2 + qy_1y_2 = 1 = l_1x_2 + m_1y_2 + n_1. \]

But $P_1(x_1, y_1)$ lies on $H_2K_2$ if $l_1x_1 + m_1y_1 + n_1 = 0$,

\[ \therefore l_1x_2 + m_1y_2 - n_1 = 0. \]

By symmetry, this is the condition for $P_2$ to lie on $H_1K_1$.

Thus $l_1x + m_1y + n_1 = 0$, $l_2x + m_2y + n_2 = 0$ are conjugate lines if $l_1l_2 + m_1m_2 - n_1n_2 = 0$.

13.4.8. If the tangents from a variable point $P(x_1, y_1)$ to the conic $px^2 + qy^2 = 1$, where $1/p + 1/q > 0$, are at right angles, the locus of $P$ is the circle $x^2 + y^2 = 1/p + 1/q$, called the director circle.

By 13.4.3, the tangents from $P$ are parallel to the line-pair through the origin given by $(px^2 + qy^2 - 1)(px + qy)^2 = 0$.

By 3.10.2, p. 41, this equation represents a perpendicular line-pair if

\[ p(px_1^2 + qy_1^2 - 1) = \frac{p^2x_2^2 + q^2y_2^2 - 1}{-1}. \]

That is,

\[ pqy_1^2 - pqx_1^2 + p^2 - q^2 = 0. \]

\[ \therefore P(x_1, y_1) \text{ lies on the circle } x^2 + y^2 = 1/p + 1/q, 1/p + 1/q > 0. \]

13.4.9. If $PT, PT'$ are the tangents from $P(x_1, y_1)$ to a conic $S, S'$, the angles $SPT, S'PT'$ are equal or supplementary.

By 11.2, p. 170, the equation of the conic whose foci are $S(ac, 0), S'(-ac, 0)$ can be written, $kx^2 + y^2 = ka^2$, where $k = 1 + e^2$.

By 13.4.3, $PT, PT'$ are parallel to the line-pair given by $(kx_2^2 + y_2^2 - 2kx_1x_2 + (y_2^2 - a^2)y_2 = 0$.

That is,

\[ \left( y_2^2 - a^2 \right)x_2^2 - 2kx_1x_2 + (y_2^2 - a^2) = 0. \]

\[ \therefore 12.4.3, p. 218, \text{ the angle-bisectors of } SPT \text{ are parallel to } (x^2 - y^2)x_2y_2 = 0. \]

Also $PS, PS'$ are parallel to $y_2 - (x_1 - ac)y = 0, y_2 - (x_2 + ac)y = 0$.

Also $PS, PS'$ are parallel to $(y_2 - x_1y_2) = a^2y_2^2 - 2k^2x_2y_2 + (x_2 - a^2)^2 = 0$.

The angle-bisectors of $SPT$ are parallel to $(x^2 + y^2)x_2y_2 + y_2^2x_2^2 - x_2^2 + a^2(1 - k) = 0$. 

EXERCISE 53

1. Find the ratios in which the join of $(-1, 4)$ to $(1, -2)$ is divided by the ellipse $11x^2 - 4xy + y^2 = 4$.

2. Find the ratios in which the join of $(-1, -12)$ to $(4, -2)$ is divided by the hyperbola $3x^2 - 3y^2 = 6x = 0$.

3. Find the condition that the line joining $(-1, -2)$ to $(h, k)$ is divided harmonically by its points of intersection with $x^2 - 6xy - 6y = 0$.

4. If the join of $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is divided in the ratio $k_1 : k_2$ by its points of intersection with $y^2 - 4ax = 0$, prove that

\[ k_1^2(y_2^2 - 4ax_2) + 2k_1k_2(y_2^2 - 2ax_2) + k_2^2(y_2^2 - 4ax_1) = 0. \]

5. If the tangents from a point $T$ to $y^2 - 4ax$ contain an angle of $90^\circ$, prove $T$ lies on the hyperbola $(3a + x)(3a - x) = y^2$.

6. $TP, TQ$ are the tangents from a fixed point $T$ to the variable conic $px^2 + qy^2 = 1$. Prove that the angle-bisectors of $\angle PTQ$ are fixed if $1/p - 1/q$ is constant and show that this is the condition that $T$ is the focus of the conic.

7. If $P$ is a variable point on a fixed line passing through the focus of the given parabola $y^2 = 4ax$, $PQ, PR$ are the tangents from $P$ to $y^2 = 4ax$. Prove that the angle-bisectors of $\angle QPR$ are fixed in direction.

8. Prove that the ratio of the hyperbola $2y = c$ is

\[ k_1^2(2x_2y_2 - c) + 2k_1k_2(x_1y_2 + x_2y_1 - c) + k_2^2(2x_1y_1 - c) = 0. \]

Find the equation of the pair of tangents from $(x_1, y_1)$ to $2xy = c$. 

13.5. If a step of directed length \( r \) is taken from \( V(x_1, y_1) \) in the direction make an angle \( \theta \) with \( Ox \), the point of arrival is
\[
(x_1 + r \cos \theta, y_1 + r \sin \theta).
\]
This point lies on the central conic
\[
ax^2 + 2hxy + by^2 = 1
\]
if
\[
a(x_1 + r \cos \theta)^2 + 2h(x_1 + r \cos \theta)(y_1 + r \sin \theta) + b(y_1 + r \sin \theta)^2 = 1,
\]
that is,
\[
r^2(a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) = 1.
\]

This equation is called the \( r \)-quadratic.

If the line drawn from \( V(x_1, y_1) \) in direction making angle \( \theta \) with \( Ox \) meets the conic at \( P, Q \) and if the directed lengths of \( VP, VQ \) are denoted by \( r_1, r_2 \), then
\[
r_1r_2 = (ax_1^2 + 2hxy_1 + by_1^2 - 1)/(a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta).
\]

13.5.1. If a chord \( PQ \) of \( ax^2 + 2hxy + by^2 = 1 \) is parallel to the diameter \( OH \), \( y = mx \), and if the middle-point \( V \) of \( PQ \) lies on the diameter \( OK \), \( y = mx \), then \( a + hm \pm mn \) = 0.

The sum of the roots \( r_1, r_2 \) of the \( r \)-quadratic in 13.5 is zero if \( V(x_1, y_1) \) is the middle-point of the chord \( PQ \) making angle \( \theta \) with \( Ox \).

\[
\cos \theta (ax_1 + hy_1) + \sin \theta (hx_1 + by_1) = 0, \quad \text{where} \quad \tan \theta = m.
\]

Since \( V(x_1, y_1) \) lies on \( y = mx \), \( y_1 = m \cdot x_1 \);
\[
\cos \theta (ax_1 + hy_1) + \sin \theta (hx_1 + by_1) = 0.
\]

This is the condition for the midpoint of chords parallel to \( y = mx \) to lie on \( y = mx \). The symmetry of this condition shows that if a chord parallel to a diameter \( OH \) is bisected by a diameter \( OK \), then chords parallel to \( OK \) are bisected by \( OH \); \( OH \), \( OK \) are then called conjugate diameters, see pp. 150, 180.

13.5.2. Newton's Theorem. If lines \( VP, VQ, VQ' \), drawn from a variable point \( V(x_1, y_1) \) in directions making fixed angles \( 0, 6 \) with \( Ox \), cut the conic \( px^2 + qy^2 = 1 \), centre \( O \), at \( P, Q \) and \( P', Q' \), then the value of the ratio \( VP \cdot VQ : VQ' \cdot VP' \) is constant.

By 13.5,
\[
VP \cdot VQ = \frac{p{x_1}^2 + q{y_1}^2 - 1}{p \cos^2 \theta + q \sin^2 \theta} \quad \text{and} \quad VP' \cdot VQ' = \frac{p{x_1}^2 + q{y_1}^2 - 1}{p \cos^2 \theta' + q \sin^2 \theta'}.
\]

This is constant.

Further, if the lines through the centre \( O \) making angles \( 0, 6 \) with \( Ox \) meet the conic at \( H, F \) and \( H', F' \), then \( OF = -OH, OP = -OH' \);
\[
VP \cdot VQ : VQ' \cdot VP' = OH \cdot OF' = OH' \cdot OF = OH^2 = OH'^2.
\]

If the conic is a hyperbola, the diameters parallel to \( VQ, VQ' \) may not meet the conic; a corresponding property can then be obtained by introducing their intersections with \( px^2 + qy^2 = 1 \).
12. A variable chord $PQ$ of $px^2 + qy^2 = 1$ is fixed in direction and meets $Ox$ at $M$. The perpendicular bisector of $PQ$ meets $Ox$ at $N$. Prove $OM : ON$ is constant.

13. The tangents at points $P, Q$ on a conic meet at $T_1$; a line parallel to $TQ$ cuts $PT, PQ$ at $T, R$ and cuts the conic at $H, K$. State a ratio equal to $T'H : T'R : T''P : T''Q$. Deduce $T''P = T''Q$.

14. A variable line cuts the fixed line-pair $ax^2 + 2hxy + by^2 = 0$ at $P, Q; R$ the mid-point of $PQ; PQ = 2c$, where $c$ is constant. Prove the locus of $R$ is $(ax + by)^2 + (hx + gy)^2 = c^2(h^2 - ab)$. (N)

15. A pair of variable perpendicular lines $K'P', K'Q'$ through a fixed point $K$ cuts $px^2 + qy^2 = 1$ at $P, Q$ and $P', Q'$. Prove that

\[ \frac{1}{(KP \cdot KQ)} + \frac{1}{(KP' \cdot KQ')} \]

is constant. (OC)

16. $V$ is the mid-point of a variable chord of fixed length $2c$ of the central conic $px^2 + qy^2 = 1$, prove that the locus of $V$ is the curve

\[ (p^2x^2 + q^2y^2)(px^2 + qy^2 - 1) + pqc^2(px^2 + qy^2) = 0. \]

17. From a variable point $P$ on the conic $px^2 + qy^2 = k$, centre $O$, tangents are drawn to $px^2 + qy^2 = 1$ and cut the diameter conjugate to $OP$ at $Q, R$. Prove $Q, R$ lie on $px^2 + qy^2 = k(h - k)$.

18. If $R$ is a point on the normal at $P(x_1, y_1)$ to the rectangular hyperbola $xy = c^2$, centre $O$, at distance $r$ from $P$, prove that the coordinates of $R$ are $(x_1 \pm r, \gamma, y_1 \pm r, \gamma, c, c)$ and deduce that the length of the normal chord at $P$ is $OP \sqrt{c^2}$.

19. $VT, VQ$ are the tangents to $V(h, k)$ to $px^2 + qy^2 = 1$. Prove that the mid-point of $PQ$ is \( \{h/(p^2x^2 + q^2y^2) - k/(p^2x^2 + q^2y^2) \} \). If $V$ varies so that the centroid of the triangle $VPT$ lies on $px^2 + qy^2 = 1$, prove $V$ lies on $px^2 + qy^2 = 4$ and deduce that $px^2 + qy^2 = 1$ is an ellipse if $V$ lies outside $px^2 + qy^2 = 1$. (OC)

20. The tangents from a variable point $P$ to $x^2/a^2 + y^2/b^2 = 1$ touch it at $Q_1(x_1, y_1), Q_2(x_2, y_2)$; $C$ is the fixed point $(c, 0), c^2 + a^2$. If $CQ_1, CQ_2$ are parallel to conjugate diameters, prove that:

(i) $b^2x_1x_2 + a^2y_1y_2 - 2c(x_1 + x_2) + a^2c^2 = 0;
(ii)$ the locus of $P$ is the similar ellipse

$ax^2 + ay^2 - 2cx(c^2 - 2b^2 + 2a^2) = 0$ if $c^2 < 2a^2$.

21. A variable chord $PQ$ of the given conic $px^2 + qy^2 = 1$ passes through a fixed point $K$. Prove that the mid-point of $PQ$ lies on a conic $S$. If $K$ moves on a fixed line, prove that the conic $S$ passes through two fixed points.

22. A focal chord $PSQ$ of a conic, foci $S, C$ and rectangular $SL$, be the corresponding directrix at $R$. Prove: (i) $PQ$ is divided harmonically by $S$ and $R$; (ii) $SP \cdot SQ = SL \cdot PQ$. (iii) If $OH, OK$ are semi-diameters parallel to focal chords $PSQ, P'SQ'$, then $PQ : P'Q' = OH : OK$. (OC)
CHAPTER 14
COORDINATES

14.1. Proofs of properties of the asymptotes of a general hyperbola can often be simplified by choosing the asymptotes as coordinate axes $Ox, Oy$, although these axes are then no longer at right angles. Similarly, in other geometrical properties there may be two prominent lines which, although not at right angles, it is convenient to take as coordinate axes, but if distances, other than those in the directions of the axes, and if sizes of angles are involved, it is usually best to choose rectangular axes.

14.1.1. Let $x'Ox, y'Oy$ be axes intersecting at the origin $O$ and let $y'Oy$ make an angle $\omega$ with $x'Ox$ where $\omega = 90^\circ$. The position of any point $P$ in the plane is determined by drawing $PM, PN$ parallel to $Ox, Oy$ to meet $Ox, Oy$ at $M, N$, see Fig. 109.

If $ON = x$ and $OM = y$, $x$ and $y$ are called the coordinates of $P$ referred to the oblique axes $x'Ox, y'Oy$, and $P$ is called the point $(x, y)$. The sign-conventions for the values of $x$ and $y$ are the same as for rectangular axes.

Fig. 109

Fig. 110

The coordinates of any point $P(x, y)$, and the equation of a locus $f(x, y)=0$, referred to oblique axes $Ox, Oy$ can be expressed in terms of the coordinates of $P$ referred to rectangular axes:

Take the axis $OX$ along $Ox$ and the axis $OY$ perpendicular to $OxX$ and let $P$ be the point $(X, Y)$ referred to rectangular axes $OX, OY$, see Fig. 110. Draw $PH, PK$ perpendicular to $OY, OX$.

Then $x = ON = OK - NK - X = Y \cot \omega$ and $y = NP = Y \cosec \omega$; also $X = OK = ON + NK = x + y \cos \omega$ and $Y = KP = y \sin \omega$.

These relations are true for all positions of $P$ in the plane.

The equations of a given line through the point of intersection of $L = px + qy + r = 0$, $L' = px + qy + r' = 0$, can be expressed in the form $L + kL' = 0$ for a suitable value of $k$. The proof is the same as for rectangular axes, see p. 83.

14.1.2. The equations $x = a$ and $y = b$ represent lines parallel to the axes $Oy$ and $Ox$ respectively.

The intercept form of the equation of a line and the equation of the line joining $(x_1, y_1), (x_2, y_2)$ are the same as for rectangular axes:

\[
\frac{x}{a} + \frac{y}{b} = 1; \quad \frac{x - x_1}{a - a_1} = \frac{y - y_1}{b - b_1} \quad \text{or} \quad \frac{x}{a_1} = \frac{y}{b_1} = \frac{1}{1}.
\]

The first equation is satisfied by $x = a, y = b$ and by $x = 0, y = 0$; the other equations are satisfied by $x = x_1, y = y_1$ and by $x = x_2, y = y_2$.

The equation of a line through the point of intersection of $L = px + qy + r = 0$, $L' = px + qy + r' = 0$.

14.1.3. If $Ox, Oy$ are oblique axes of reference and if $P(x, y)$ divides the line joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ in the ratio $m_2 : m_1$ where

\[
m_1 + m_2 = 0, \quad \text{then} \quad x = m_1 x_1 + m_2 x_2, \quad y = m_1 y_1 + m_2 y_2.
\]

The proof is precisely the same as for rectangular axes, see p. 47. The proofs of many of the formulas for rectangular axes depend only on this ratio-theorem, therefore all such formulas are unchanged when the axes are oblique. For example, the application of Joachimsthal's ratio-equation, see pp. 238, 239, for finding the equations of tangents and polars and tests for conjugate points and lines depend only on the ratio-theorem and therefore give results which hold unchanged for oblique axes; the same is true for relations between ratios of segments of the same line or of parallel lines. This is also true of the proof that the line-pairs $ax^2 + 2hxy + by^2 = 0$, $a'x'^2 + 2h'x'y' + b'y'^2 = 0$ separate one another harmonically if $ab' + a'b = 2h'h'$, see pp. 217, 218.

14.1.4. If $Ox, Oy$ are oblique axes, the equation of a central conic, centre $O$, is

\[ax^2 + 2hxy + by^2 = 1.\]

If $(x_1, y_1)$ is a variable point on the conic, so also is $(-x_1, -y_1)$; therefore in the general equation of the second degree, the coefficients of $x$ and $y$ are zero.
14.1.5. If \( \alpha \) is the angle between oblique axes \( Ox, Oy \), the gradient of \( AB, y = mx + c \), equals \( m = \tan \alpha \).

By 14.1.1, the equation of \( AB \) referred to rectangular axes \( OxX, OY \)

is \( Y \cos \alpha = m(X - Y \cot \alpha) - c \),

that is, \( Y(1 + m \cos \alpha) = mX \sin \alpha + c \sin \alpha \).

- if \( AB \) makes an angle \( \alpha \) with \( OxX \), tan \( \alpha = m \sin \omega \).
- Hence \( y = mx + c, y = m_x x + c_y \) represent parallel lines if \( m_1 = m_2 \) and \( c_1 = c_2 \). Hence also \( m_1x - c_1 = m_2x + c_2 = 0 \) represent parallel lines if \( c_1/c_2 = b_2/b_1 = c_1/c_2 \).

14.1.6. If \( \alpha \) is the angle between oblique axes \( Ox, Oy \), the lines

\[ y = m_1 x + c_1, y = m_2 x + c_2 \]

are at right angles if

\[ 1 + (m_1 x + m_2 x) \cos \alpha + m_1 m_2 \cos \alpha = 0. \]

The lines are at right angles if the product of their gradients equals \(-1\);

- the condition is \( m_1 m_2 \cos \alpha = -1 \);
- \( m_1 m_2 \sin \omega + (1 + m_1 \cos \alpha)(1 + m_2 \cos \alpha) = 0 \);
- \( 1 + (m_1 m_2 \sin \omega + m_1 m_2 \cos \alpha) \).

14.1.7. If \( \alpha \) is the angle between oblique axes \( Ox, Oy \), the equation

\[ ax^2 + 2hxy + by^2 = 0, h^2 > ab, b > 0 \]

represents two perpendicular lines if \( a + b - 2h \cos \alpha = 0 \).

\[ ax^2 + 2hxy + by^2 = 0 \]

if \( m_1 + m_2 = -2h/b \) and \( m_1 m_2 = a/b \). Therefore by 14.1.6 the equation represents perpendicular lines if \( 1 - (2h/b) \cos \alpha \).

Note. This test remains true if \( b = 0 \), but the proof must be modified.

14.1.8. If \( \alpha \) is the angle between oblique axes \( Ox, Oy \), the distance between the points \( P_1(x_1, y_1), P_2(x_2, y_2) \) is given by

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

With the notation of 14.1.1, if \( (X, Y), (X_2, Y_2) \) are the points \( P_1, P_2 \) referred to rectangular axes \( OX, OY \).

\[ d = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} \]

14.1.9. If \( \alpha \) is the angle between oblique axes \( Ox, Oy \), the equation of the circle, centre \( C(h, k) \), radius \( r \), is

\[ (x-h)^2 + (y-k)^2 + 2(x-h)(y-k) \cos \alpha = r^2. \]

If \( P_1(x_1, y_1) \) is a point on the circle, \( C_P^2 = r^2 \); therefore the result follows from 14.1.8.

In particular, the equation of the circle, centre the origin, radius \( r \), is

\[ x^2 + y^2 + 2xy \cos \alpha = r^2. \]

Example 1. Interpret the locus whose equation referred to oblique axes is

\[ x^2 - y^2 + 2xy - y^2 = 0. \]

Since \( xy + 2xy - y^2 = (x-y)(y-p) - (y-p), a \) change of origin, as on p. 115, makes interpretation easy. Take \( C(p, p) \) as the new origin and

\[ Ox', Oy' \parallel Ox, Oy \] as new axes.

![Diagram](image)

**Fig. 111**

Draw \( P'M'M', P'N'N' \) through \( P(x, y) \parallel Ox, Oy \) to meet \( Oy, Ox \) at \( M, N \) and \( Oy', Ox' \) at \( M', N' \).

Let \( (x', y') \) be the coordinates referred to \( Ox', Oy' \) of \( P(x, y) \). Then

\[ x' = CN' = MP - MM' = -y, \]

and \( y' = CM' = NP - NN' = y - p \).

Thus the locus is given by \( x' = y' = q + r \), and this may be written

\[ CN' = CM' = \text{constant}. \]

Hence, see 11.6.8, p. 184, if \( px + qy = \text{constant} \), the locus is a hyperbola with \( C(p, p) \) as centre and with asymptotes \( Ox', Oy' \parallel Ox, Oy \).

If \( px + qy = 0 \), the locus is the pair of lines \( Ox', Oy' \).

**EXERCISE 55**

In this exercise, the axes of reference \( Ox, Oy \) are oblique and the angle which \( Oy \) makes with \( Ox \) is denoted by \( \alpha \).

1. Interpret the loci whose equations are \( x+y = 0 \) and \( x-y = 0 \), and state the values of their gradients.

2. Find the equation of: (i) the line through \((a, 0) \) perpendicular to \( Ox \); (ii) the line through \((0, b) \) perpendicular to \( Oy \).

3. Prove that \( ax + by + m = 0 \) is perpendicular to \( Ox \) if \( m = \pm \cos \alpha \) and is perpendicular to \( Oy \) if \( m = \pm \cos \alpha \).

4. Find the equations of the lines through the point of intersection of \( ax + by + c_1 = 0, a_x x + b_y y + c_2 = 0 \), parallel to: (i) \( Ox \); (ii) \( Oy \).

5. Repeat No. 4 for lines perpendicular to: (i) \( Ox \); (ii) \( Oy \).

6. \( P_1, P_2 \) are the points \((x_1, y_1), (x_2, y_2) \); \( P_R, P_R \parallel Ox, Oy \). Use the cosine formula for \( \Delta P_1 P_2 \) to find \( P_R P_1 \).

7. A line cuts \( Ox, Oy \) at \( A, B \) and the mid-point of \( AB \) is \( (h, k) \); find the equation of \( AB \).

8. The line \( ax + by = 1 \) cuts \( Ox, Oy \) at \( A, B \); \( P \) and \( Q \) divide \( AB \) in the ratios \( k:1, -k:1 \); find the equations of \( OP, OQ \).
[14.1] COORDINATES

[9] Prove that \(a_1x + b_1y + c_1 = 0\), \(a_2x + b_2y + c_2 = 0\) make supplementary angles with Ox if \(a_1b_2 + a_2b_1 = 2a_1a_2 \cos \omega\).

[10] A, B are points on the positive half-lines, Ox, Oy such that \(AB = c\); \(\angle AOB = 60^\circ\); \(ABP\) is a parallelogram; \(Q\) divides \(AB\) in the ratio 2 : 1; \(PQ\) is parallel to the internal bisector of \(\angle AOB\). Find the coordinates of \(P\).

[11] A, B are the points \((2a, 0), (0, 3b)\); \(H\) is the mid-point of \(OA\); \(K\) divides \(OB\) in the ratio 2 : 1; \(HK\) meets \(AB\) at \(O\). Prove: (i) \(O\) is the point \((-2a, 6b)\); (ii) the mid-points of \(AK, BH, OA\) are collinear.

[12] Find the general equation of a proper conic which passes through the points \((h, k), (h, -k), (-h, k), (-h, -k)\), where \(hk \neq 0\), if the axes are oblique. What geometrical property connects the conic with the lines Ox, Oy?

[13] A is a fixed point; a variable line \(PAQ\) cuts \(Ox, Oy\) at \(P, Q\). Prove that the locus of the mid-point of \(PQ\) is a hyperbola. If \(P\) is the point on \(PQ\) such that \(AR\) and \(PQ\) have the same mid-point, prove that the locus of \(R\) is a hyperbola.

[14] A variable line passes through the fixed point \(C (a, b)\) and meets \(Ox, Oy\) at \(P, Q\). \(R\) divides \(PQ\) in the fixed ratio \(k : 1\). Prove the locus of \(R\) is given by \((1 + k)xy = kax + by\). Interpret the locus.

15. The base \(BC\) of a triangle \(ABC\) of given size slides along \(Ox\); \(Oy\) is taken parallel to \(BA\); \(D\) is a marked point on \(AC\); \(OD\) meets \(AB\) at \(P\). Prove \(P\) moves on a hyperbola; identify its asymptotes.

Use the transformation in 14.11, p. 246, for Nos. 16–19.

16. Prove that the area of the triangle whose vertices are \((0, 0), (\alpha, \beta), (\gamma, \delta)\) is \(\frac{1}{2} \sin \alpha \text{sin} \beta \text{sin} \gamma \text{sin} \delta\); hence find the area of the triangle whose vertices are \((x_1, y_1), (x_2, y_2), (x_3, y_3)\).

[17] Prove that the angle \(\theta\) between the lines \(y = m_1x + c_1, y = m_2x + c_2\) is given by \(\tan \theta = (m_1 - m_2) \sin \alpha / (1 + (m_1 + m_2) \cos \alpha + m_1m_2)\).

18. Prove that the length of the perpendicular from \((x_1, y_1)\) to the line \(ax + by + c = 0\) is \(\pm (a_1x_1 + b_1y_1 + c) / \sqrt{a^2 + b^2 - 2ab \cos \alpha}\).

19. Prove that \(x^2 + y^2 + 2xy \cos \omega + 2x + 2y = 0\) is the equation of a circle if \(\text{if}^2 + 2\text{if} \cos \omega + 2 = 0\) and find the coordinates of its centre.

[20] \(PH, PK\) are the perpendiculars from a variable point \(P(x, y)\) to \(Ox, Oy\). If \(HK = c\), where \(c\) is constant, prove that the locus of \(P\) is given by \(x^2 + y^2 + 2xy \cos \omega = c^2 \cos^2 \omega\). Interpret the equation.

21. \(OAB\) is a given triangle; \(P, Q\) are variable points on \(OB\) such that the length of \(PQ\) is constant; \(R\) is the mid-point of \(AQ\). Prove that \(PR\) passes through one of two fixed points.

14.2] OBLIQUE AXES

14.2. Some properties of the conic are proved most simply by using oblique axes.

14.21. The Parabola. If any point \(O\) on a parabola is taken as origin and if the diameter \(OH\) and the tangent \(OT\) are taken as axes \(Ox, Oy\), the equation of the parabola is of the form

\[y^2 = 4kx\]

By 14.1.1, the general equation of a parabola referred to oblique axes is

\[(lx + my)^2 + 2gx + 2fy + c = 0\]

Since \(OT, x = 0\), touches the curve at \(O (0, 0), f = 0\) and \(c = 0\); therefore for a proper parabola, \(g \neq 0\).

Since the diameter \(OH, y = 0\), meets the parabola at only one point, \(l = 0\); therefore the equation is of the form

\[m^2y^2 + 2gx = 0, m^2 = g, g \neq 0\]

that is,

\[y^2 = 4kx\]

This equation is equivalent to the property proved on p. 104:

The mid-point \(V\) of any chord \(PQ\) parallel to \(Oy\) lies on \(Ox\).

Also if the angle between \(OH\) and \(OT\) is \(\omega\) and if \(S\) and \(A\) are the focus and vertex,

\[PV^2 = 4SO \cdot OT\]

where \(SO = SA \cos^2 \omega\).

This relation is equivalent to \(y^2 = 4kx\) where \(k = SA \cos^2 \omega\).

The coordinates of any point on \(y^2 = 4kx\) can be taken as \((kt^2, 2kt)\), and as explained on p. 247, the equations of the chord \(t_1t_2\), of the tangent at \(t_1\), and of the polar of \((x_1, y_1)\) are the same as for rectangular axes; but the equation of the normal involves \(\omega\).

Example 2. \(OT\) is the tangent at \(O\) to a parabola; \(PQ\) is a chord parallel to \(OT\). The tangent at a point \(R\) meets the tangents at \(P, Q, O\) in \(P', Q', O'\). Prove \(O'\) is the mid-point of \(P'O'\).

With the notation and axes in Fig. 112, points of the parabola are given by \(x = kt^2, y = 2kt\).

With the notation and axes in Fig. 112, points of the parabola are given by \(x = kt^2, y = 2kt\).

\[P, Q, R, O\] can be taken as the points \(t = p, t = -p, t = r, t = 0\). By hypothesis, \(P', Q', O'\) are the poles of \(RP, RQ, RO\). Since the formulas for poles and polars are the same for oblique axes as for rectangular axes, it follows as in Example 7, p. 101, that \(P', Q', O'\) are the points,

\[P'(kxp, kp(r + p)), Q'(-kxp, k(r - p)), O'(0, kr)\]

\(O'\) is the mid-point of \(P'O'\).
14.2.2. Central Conics. If the conjugate diameters $H'OH$, $K'OK$ of a proper central conic are taken as axes $Oz, Oy$, the equation is of the form $px^2 + qy^2 = 1$.

![Image of a conic section](image)

Let $P(x_1, y_1)$ where $x_1 \neq 0$, $y_1 \neq 0$ be a point on the conic. Since $Oz, Oy$ are conjugate diameters, the mid-points $M, N$ of chords $PQ, PR$ parallel to $Oz, Oy$ lie on $Oy, Ox$.

\[ \therefore Q, R \text{ are the points } (-x_1, y_1), (x_1, -y_1). \]

Also if $POP'$ is a diameter, $P'$ is the point $(-x_1, -y_1)$.

Hence if the conic is given by the general equation, there are four relations between $x_1$ and $y_1$ of the form

\[ ax_1^2 + 2bx_1y_1 + cy_1^2 = d, \]

where either there is no minus sign or there are two minus signs.

\[ \therefore d = 0, e = 0, f = 0; \]

\[ \therefore \text{the general equation reduces to the form } ax^2 + by^2 = 0, \text{and, for a proper conic, this may be written } px^2 + qy^2 = 1. \]

If $p > 0$ and $q > 0$, the equation may be written $2x^2/a^2 + y^2/b^2 = 1$, where, see Fig. 113, $Ox, Oy$ meet the ellipse at $H, H'$ and $K, K'$ such that $OH = a, OK = b$.

If $p > 0$ and $q < 0$, the equation may be written $x^2/a^2 - y^2/b^2 = 1$, where, see Fig. 114, $Oz$ meets the hyperbola at $H, H'$ and $Oy$ meets the conjugate hyperbola $x^2/a^2 + y^2/b^2 = 1$ at $K, K'$ such that $OH = a, OK = b$.

The equation can be interpreted geometrically as follows:

with the data of Fig. 113,

\[ PN^2 = \frac{OQ^2 - OX^2}{OH^2}; \]

but

\[ OH^2 - OX^2 = (OH - ON)(OH + ON) = H'N \cdot NH; \]

\[ \therefore PN^2 = H'N \cdot NH = OK^2; \]

Similarly with the data of Fig. 114, where $K$ lies on the conjugate hyperbola, $PN^2 = OK^2 = OX^2/OH^2 - 1 = (H'N \cdot NH)/OH^2;

\[ \therefore PN^2 = H'N \cdot NH = -OK^2 = OH^2. \]

This is a special case of Newton's theorem, p. 243.

14.2.3. The test for conjugate diameters is the same for oblique axes as for rectangular axes:

If $OP, y = mx, x = m'y$, are conjugate diameters of $px^2 + qy^2 = 1$, referred to oblique axes, if $mm' = -p/q$.

If $(x_1, y_1)$ is any point on $OP$, then $y_1 = mx_1$, and the polar of $(x_1, y_1)$, $px_1^2 + qy_1^2 = 1$, is parallel to $OD, y = m'x$.

\[ \therefore m' = -(px_1)/(qy_1); \]

\[ \therefore mm' = -p/q. \]

Example 3. $OH, OK$ and $OP, OD$ are pairs of conjugate diameters of a central conic; the tangent at $H$ meets $OP, OD$ in $P_1, D_1$. Prove that:

(i) if the conic is an ellipse, $HP_1, HD_1 = -OK$;

(ii) if the conic is a hyperbola, $HP_1, HD_1 = OK$.

The equation of the conic referred to $OH, OK$ as axes $Ox, Oy$ is of the form $px^2 + qy^2 = 1, p > 0$;

\[ \therefore \text{the equation of the tangent at } H(1/\sqrt{p}, 0) \text{ is } x = 1/\sqrt{p}, \text{ see Figs. 113, 114.} \]

If $OP, OD$ are $y = mx, y = m'x$, then $mm' = -p/q$.

For the ellipse, $x = 1/\sqrt{p}$, $y = m/\sqrt{p}$; $HP_1 = m/\sqrt{p}$.

Similarly $HD_1 = m'/\sqrt{p}$; $HP_1, HD_1 = mm' = -1/q$.

Hence for an ellipse, see Fig. 113, $OK^2 = 1/q = -HP_1, HD_1$.

For a hyperbola, see Fig. 114, $OK^2 = -1/q = HP_1, HD_1$.

14.2.4. If the asymptotes of a general hyperbola are taken as oblique axes $Ox, Oy$, then by 11.6.8, p. 184, the equation of the hyperbola is of the form $xy = k^2$.

where with the usual notation $k^2 = \frac{1}{a^2 + b^2}$.

Points of the hyperbola are given by $x = kt, y = b/t$.

As explained on p. 247, the equations of a chord $t_1$, $t_2$, of the tangent at $t_1$, and of the polar of $(x_1, y_1)$ are the same as for rectangular axes.

Further, the method of 14.2.3 shows that $y = mx, y = m'x$ are conjugate diameters of $xy = k^2$ if $m + m' = 0$.

Example 4. $O, D$ are fixed points and $P$ is a variable point on a hyperbola, asymptotes $OE, OF$; $PC, PD$ meet $OQ, OR$ at $Q, R$. Prove length of $QR$ is constant.

Take $OP, OE$ as axes $Ox, Oy$, then points of the hyperbola are given by $x = kt, y = b/t$.

Let $O, D, P$ be given by $t = c, t = d, t = p$; then the equations of $PC, PD$ are

\[ x + ypc - k(p + c) = 0, \]

\[ y + ypd - k(p + d) = 0; \]

\[ \therefore Q, R \text{ are the points } \{k(p + c), b\}, \{k(p + d), b\}; \]

\[ QR = OR = OQ = k(p + d) - k(p + c) = k(d - c) = \text{constant}. \]
**EXERCISE 56**

(In this exercise, the axis Oy makes an angle \( \omega \) with OX.)

1. P is a variable point on \( xy = c^2 \). PQ, PR parallel to OX, Oy meet \( xy = c^2 \) at Q, R. Prove QR touches \( 4c^2 xy = (c^2 + \theta^2)^2 \).

2. The tangent at a point P of a hyperbola meets the asymptotes \( OE, OF \) at \( H, K \). Taking \( OE, OF \) as axes Ox, Oy, prove that:
   (i) \( HP = PK \)
   (ii) the area of \( \triangle OHK \) is constant.

3. P, Q are points on a hyperbola, whose asymptotes are \( OE, OF \); the tangent at \( P \) meets \( OE \) at \( P' \); the tangent at \( Q \) meets \( OF \) at \( Q' \). Prove that \( P'Q' \) is parallel to \( PQ \).

4. \( OBCD \) is a given parallelogram; \( OB = a, OD = b \). Q is a variable point on \( BC \); R is a point on \( OQ \) such that \( OR : OQ = EQ : BC \). If \( OB, OD \) are taken as axes Ox, Oy, prove that the equation of the locus of \( R \) is \( bx^2 + ay^2 = a^2y \). Interpret the locus. Prove that \( O \) lies on the locus and that the tangent at \( C \) bisects \( OB \).

5. P and Q are points on a hyperbola whose asymptotes are \( OE, OF; \) R is a point such that \( PR, QR \) are parallel to \( OE, OF \). Prove that the pole of \( PQ \) lies on \( OR \).

6. \( PP' \) is a diameter of the hyperbola \( xy = c^2 \); asymptotes \( OE, OF \); lines through \( P \) parallel to \( OE, OF \) meet the tangent at \( P \) in \( Q \) and \( R \). Prove that \( Q \) and \( R \) lie on \( xy + 3c^2 = 0 \).

7. \( OABC \) is a given parallelogram; \( OA = h, OC = k \); a variable line through \( B \) cuts \( OA, OC \) at \( P, Q \); \( AQ \) cuts \( CP \) at \( R \). If \( OA, OC \) are taken as axes Ox, Oy, prove that the locus of \( R \) is the ellipse,
   \[
   h^2x^2 + k^2y^2 + hkyx = hk(kx + hy).
   \]

8. \( QQ' \) is a variable chord of a parabola parallel to the tangent \( OT \) at a fixed point \( O \) taken as origin; the equation of the parabola is \( y^2 = 4cx \). P is a variable point on \( QQ' \) such that \( QP : PQ' = c^2 \), where \( c \) is constant. Prove that the locus of \( P \) is \( y^2 - 4cx + c^2 = 0 \).

9. \( H \) and \( H' \) are the extremities of the diameter of a central conic. With \( H'O \) as axis Ox, the equation of the conic is \( px^2 + gy^2 = 1 \). The tangent at \( H \) cuts the tangent at a variable point \( P \) in \( T \); \( HT \) meets \( HP \) at \( Q \). Prove that \( Q \) lies on the conic \( px^2 + 2gy^2 = 1 \).

10. Two hyperbolas have the same asymptotes \( OE, OF \); a variable tangent to one meets the other at \( P; \) \( PQR \) is a parallelogram with sides parallel to \( OE, OF \). Prove that the locus of \( R \) and \( R' \) consists of two hyperbolas.

11. Prove that the four points on the general hyperbola, \( x = kt, y = k/t \), given by \( t = t_1, t_2, t_3, t_4 \) are conocyclic if \( t_1t_2t_3t_4 = -1 \).

**GEOMETRICAL PROPERTIES**

12. A variable tangent to \( xy = c^2 \) meets the asymptotes \( OE, OF \) at \( Q, R \); \( QQ', RR' \) parallel to \( OF, OE \) meet the hyperbola at \( Q', R' \). Prove that \( R' \) touches the hyperbola \( 16xy = 26c^4 \).

13. \( POP', DOD' \) are conjugate diameters of an ellipse; the tangents at \( P, D \) meet in \( T \); a chord \( FR \) cuts \( DT \) at \( H; \) \( PR \) cuts \( OD \) at \( K \). Prove that \( TH : HD = OK : KD \).

14. \( Q, R \) are points on a parabola; a line parallel to the axis of the parabola meets the curve in \( O \) and meets the tangents at \( Q, R \) in \( H, K \) and meets \( QR \) in \( D \). Prove \( OD^2 = OH \cdot OK \).

15. \( ABCD \) is a quadrilateral; \( BA \) meets \( CD \) in \( O; \) \( AD \) meets \( BC \) in \( P; \) \( AC \) meets \( BD \) in \( Q \). \( OA = a, OB = b, OC = c, OD = d \). Prove that \( OP, OQ \) separate harmonically. \( OABC \), \( ODC \). [Take \( OA, OC \) as axes Ox, Oy.]

16. With the notation of No. 15, let \( M \) be the mid-point of \( OP \); draw \( MH, MK \) parallel to \( PA, PC \) to meet \( OAB \) at \( H, K \). Write down the coordinates of \( H, K \) and the equations of \( HM, KM \), with \( OA, OC \) as axes Ox, Oy.

17. \( PQR \) is a triangle whose sides touch a parabola; \( R' \) is the point of contact of \( PQ \); a fourth tangent cuts \( QR, RP, PQ \) in \( E, F, G \). Prove that \( EG : EF = QR : RP \).

18. The tangents at points \( P, Q \) on a parabola meet in \( T \); the tangent at \( R \) meets \( TP, TQ \) in \( H, K \); prove \( TH : HP = KR : RH \).

19. The fixed lines \( OAB, OC \) are taken as axes Ox, Oy; \( R \) is a variable point on \( OC \); \( A, B \) are fixed points; parallel lines \( AP, EQ \) meet \( BR, AR \) at \( P, Q \). If \( A, B, P, Q \) are the points \( (u, 0), (b, 0), (x_1, y_1), (x_2, y_2) \), prove \( x_2y_1 = -2x_1y_2 + b^2 \). Find the locus of \( Q \) if \( P \) moves on a fixed line.

14.3. The polar coordinates \( r, \theta \) which determine the position of a point in a plane were defined on p. 18. The pole or origin is denoted by \( O \) and the initial line is denoted by \( Ox \) unless otherwise stated. If \( Oy \) makes +90° with \( Ox \), the point \( (x, y) \) is the same as the point \( \{r, \theta \} \) if \( x = r \cos \theta, y = r \sin \theta \).

If the vectorial angle \( 0 \) is allowed to take any value from 0° to 360° and if the length \( r \) of the radius vector \( OP \) is measured by a directed (positive or negative) number, the coordinates, \( \{r_1, 0_1 \} \) and \( \{-r_1, \theta_1 \} \) denote the same point.

The polar equation of the half-line \( OP \) which makes the given angle \( \alpha \) with the initial line is \( \theta = \alpha \).

The polar equation of the circle, centre O, radius \( k \), is

\[ r = k. \]
The polar equation of a locus can be deduced from its \((x, y)\) equation for rectangular axes by writing \(r \cos \theta, r \sin \theta\) for \(x, y\). For example, the polar equation of the folium, \(x^2 + y^2 = axy\), is
\[ r^2 \cos^3 \theta + r^2 \sin^3 \theta - ar^2 \sin \theta \cos \theta = 0; \]
this reduces to
\[ r \cos (3 \theta - 3 \sin \theta) = a \sin 5 \cos \theta, \]
since \(0 = 0\) and \(0 = 90^\circ\) give \(r = 0\).

The polar equations corresponding to
\(6x + my + n = 0\) and \(x^2 + y^2 + 2px + 2fy + c = 0\) and \(y^2 - 4ax\), etc., can be written down in the same way, but it is often more convenient to use independent methods.

### 14.3.1. Equation of a Line

If \(OR\) is the perpendicular from \(O\) to a line \(AB\) and if the polar coordinates of \(R\) are \([p, \alpha]\), then the polar equation of \(AB\) is
\[ r \cos (\theta - \alpha) = p. \]

Let \(P[r, \theta]\) be a point on \(AB\);
\[ \angle ROA = \alpha, \angle POA = 0, \]
\[ :\angle POR = \theta - \alpha; \]
also \(OR = p, OP = r, \)
\[ : p = OP \cos (\theta - \alpha) = r \cos (\theta - \alpha). \]

Note. Expansion gives
\[ p = r \cos \theta \cos \alpha + r \sin \theta \sin \alpha = x \cos \alpha + y \sin \alpha \]
(perpendicular form).

### 14.3.2. Equation of a Circle

If the pole \(O\) is taken on a circle, diameter \(OA\), and if the polar coordinates of \(A\) are \([d, \alpha]\), then the polar equation of the circle is
\[ r = d \cos (\theta - \alpha). \]

Let \(P[r, \theta]\) be a point on the circle,
\[ \angle POA = \theta - \alpha \text{ and } \angle OPA = 1 \text{ rt. } \]
\[ : OP = OA \cos (\theta - \alpha), \]
that is,
\[ r = d \cos (\theta - \alpha). \]

In particular, if the diameter \(OA\) is taken as the initial line, then \(\alpha = 0\) and the equation of the circle is \(r = d \cos \theta\).

### 14.3.3. If a circle, centre \(C\), radius \(a\), does not pass through the pole \(O\) and if the polar coordinates of \(C\) are \([h, \alpha]\), then the polar equation of the circle is
\[ r^2 - 2rh \cos (\theta - \alpha) + h^2 - a^2 = 0, \]
\[ h^2 - a^2 = 0. \]

Let \(P[r, \theta]\) lie on the circle, then \(OC = h\) and \(\angle POC = \theta - \alpha\).

From \(\triangle POC, \) \(CP^2 = OP^2 + OC^2 - 2OP \cdot OC \cos POC, \)
\[ : a^2 = r^2 + h^2 - 2rh \cos (\theta - \alpha). \]

### 14.3.4. If a focus \(S\) of a conic is taken as pole or origin, and if the perpendicular \(SX\) from \(S\) to the corresponding directrix \(XZ\) is taken as initial line or \(x\)-axis, the equation of \(XZ\) is \(x - l/e = 0, \)
and the polar equation of the conic is
\[ ur = 1 + e \cos \theta \]
where \(l = \text{the semi latus rectum } SL\) and \(e = \text{the eccentricity}.\)
14.3.5. If the vectorial angles of the points $P, Q$ on the conic
\[ l/r = 1 + e \cos \theta \]
are $\alpha - \beta, \alpha + \beta$, where $\beta = 90^\circ$, the equation of the chord $PQ$ is
\[ l/r = e \cos \theta + \sec \beta \cos (\theta - \alpha). \]

Since $\beta = 90^\circ$, $PQ$ does not pass through the pole $S$.

The equation $l/r = p \cos \theta + q \cos (\theta - \alpha)$ where $p, q$ are constants is the equation of a line which does not pass through the pole and therefore the values of $p$ and $q$ can be chosen so that it is the line $PQ$. $P$ is given by $\theta = \alpha - \beta$, $l/r = 1 + e \cos (\alpha - \beta)$,
\[ 1 + e \cos (\alpha - \beta) = l/r = p \cos (\alpha - \beta) + q \cos \beta. \]

Similarly, $Q$ lies on the line if $p$ and $q$ satisfy the equation
\[ 1 + e \cos (\alpha + \beta) = l/r = p \cos (\alpha + \beta) + q \cos \beta. \]

By inspection, both equations are satisfied by $p = -e$, $q \cos \beta = 1$; therefore the substitutions $p = -e$, $q = \sec \beta$ give the equation of $PQ$.

14.3.6. If $\alpha$ is the vectorial angle of the point $P$ on the conic
\[ l/r = 1 + e \cos \theta, \]
the equation of the tangent at $P$ is
\[ l/r = e \cos \theta + \cos (\theta - \alpha). \]

In the equation of the chord joining the points given by $\theta = \alpha - \beta$, $\theta = \alpha + \beta$, suppose that $\beta \to 0$; then the limiting position of the chord is the tangent at the point $\theta = -\alpha$, and the limiting form of the equation is as stated.

14.3.7. If the polar equation, $l/r = m \sin \theta + -n = 0$, of a line is satisfied by $r = r_1$, $\theta = \theta_1$, it is also satisfied by $r = -r_1$, $\theta = \theta_0 + 180^\circ$, and therefore the polars of the points of intersection of two lines, whose polar equations are given, are found by solving the equations. But in general the solution of two polar equations does not give all the common points of the corresponding curves.

For example, $4l/r = 3 + \cos \theta$ is satisfied by $r = 1$, $\theta = 0^\circ$ but not by $r = -1$, $\theta = 180^\circ$; also $2l/r = 1 + 3 \cos \theta$ is satisfied by $r = 1$, $\theta = 0^\circ$ but not by $r = -1$, $\theta = 180^\circ$; therefore the point, $x = r \cos \theta = 1, y = r \sin \theta = 0$, lies both on the ellipse $4l/r = 3 + \cos \theta$ and on the hyperbola $2l/r = 1 + 3 \cos \theta$, but the only values of $r$ and $\theta$ which satisfy both equations are $r = 1/2, \theta = \cos^{-1} (\frac{1}{2})$.

14.3.8. If the tangents at $P$ and $Q$ to a conic, focus $S$, meet at $T$, then $ST$ is a bisector of $\angle PSQ$.

If $\alpha$ and $\beta$ are the vectorial angles of $P$ and $Q$ on $l/r = 1 + e \cos \theta$, by 14.3.8, $T$ is given by $l/r = e \cos \theta + \cos (\theta - \alpha), l/r = e \cos \theta + \cos (\theta + \beta)$,
\[ \therefore \cos (\theta - \alpha) = \cos (\theta + \beta), \therefore \theta = \frac{1}{2}(\alpha + \beta). \]

Therefore unless the conic is a hyperbola with $P, Q$ on different branches, $ST$ is the internal bisector of $\angle PSQ$. But if $P, Q$ are on different branches, with $SP$ and $QS$ produced making angles $\alpha, \beta$ with the initial line, $ST$ is the external bisector of $\angle PSQ$.

14.3.9. If, with $S$ as pole, the initial line $Sx$ is replaced by the initial line $Sx'$, where $Sx$ makes an angle $\gamma$ with $Sx'$, the point $P(r, \theta)$, initial line $Sx$, becomes $(r, \theta + \gamma)$, initial line $Sx'$, becomes $l/r = 1 + e \cos (\theta' - \gamma)$ where $\theta'$ is the vectorial angle measured from $Sx'$.

Similarly, each angle $\phi$ measured from $Sx$ is replaced by $\phi' - \gamma$ measured from $Sx'$. Thus the equation of the tangent at $\theta' = \alpha'$ to
\[ l/r = 1 + e \cos (\theta' - \gamma) \]
is $l/r = e \cos (\theta' - \gamma) + \cos (\theta' - \gamma - (\alpha' - \gamma)) = e \cos (\theta' - \gamma) + \cos (\theta' - \alpha')$. 

Example 5. $P$ and $Q$ are variable points on an ellipse, focus $S$, directrix $XZ$ such that $\angle PSQ$ is constant; the tangents at $P, Q$ meet in $T$. Prove that $T$ lies on a fixed conic, focus $S$, directrix $XZ$ and that $PQ$ touches a fixed conic, focus $S$, directrix $XZ$.

Denote the vectorial angles of $P, Q$ on $l/r = 1 + e \cos \theta$ by $\alpha, \beta, \gamma$; then $\beta$ is constant. The coordinates of $T$ are given by
\[ l/r = e \cos \theta + \cos (\theta - \beta), l/r = e \cos \theta + \cos (\theta + \beta), \]
\[ 0 = \alpha, l/r = e \cos \alpha + \cos \beta. \]

The coordinates of $T$ satisfy the equation, $l/r = e \cos \theta + \cos \beta$;

$T$ lies on the conic, $l / \sec \beta = r = 1 + e \sec \beta$.

By 14.3.4, this is a conic, focus $S$, eccentricity $e \sec \beta$, and having as directrix $x = (l / \sec \beta) / (l / \sec \beta) = l / \sec \beta$.

By 14.3.5, the equation of $PQ$ is $l/r = e \cos \theta + \sec \beta \cos (\theta - \alpha)$.

That is, $l / \cos \beta = r = (e \cos \beta) \cos \theta + \cos (\theta - \alpha)$;

$PQ$ touches the conic $l / \cos \beta = r = 1 + (e \cos \beta) \cos \theta$ at the point $\theta = \alpha$. By 14.3.4, this is a conic, focus $S$, eccentricity $e \cos \beta$, and having as directrix $x = (l / \cos \beta) / (l / \cos \beta) = l / \sec \beta$.

Example 6. $PQ$ is a focal chord of a conic, focus $S$, and semi latus rectum equal to $l$. Prove $SP : SQ = \frac{l}{2}$. $PQ$, where $SP, SQ, PQ$ denote signed lengths.

Let $x$ be the vectorial angle of the point $P$, where, if the conic is a hyperbola, $P$ lies on the branch enclosing $S$; then $P, Q$ may be taken as the points $(r_1, \alpha), (r_2, x + 180^\circ)$ on $l/r = 1 + e \cos \theta$, where $r_1 > 0$ and $r_2$ may be positive or negative.

Then $l/r_1 = 1 + e \cos \alpha$ and $l/r_2 = 1 + e \cos (\alpha + 180^\circ) = 1 - e \cos \alpha$.
\[ \therefore 1 / r_1 + 1 / r_2 = 2 / r_2. \]

If $P$ and $Q$ are on opposite branches of a hyperbola, $r_1 = +SP$ and $r_2 = -SQ$; also $S$ is outside the segment $PQ$;
\[ 2l/r = 1 + (SQ)^2 / (SQ . SP) , (SP . SQ) = PQ / (SP . SP). \]

Otherwise, $r_1 = -SP$ and $r_2 = +SQ$; also $S$ is inside the segment $PQ$;
\[ 2l/r = 1 + (SQ)^2 / (SQ + SP) , (SP . SQ) = PQ / (SP . SP). \]

14.3.9. If, with $S$ as pole, the initial line $Sx$ is replaced by an initial line $Sx'$, where $Sx$ makes an angle $\gamma$ with $Sx'$, the point $P(r, \theta)$, initial line $Sx$, becomes $(r, \theta + \gamma)$, initial line $Sx'$, where $\theta = \theta' - \gamma$.

Hence if the axis of a conic, focus $S$, makes an angle $\gamma$ with the initial line $Sx'$, the equation of the conic, $l/r = 1 + e \cos \theta$, initial line $Sx$, becomes $l/r = 1 + e \cos (\theta' - \gamma)$ where $\theta'$ is the vectorial angle measured from $Sx'$. 

Similarly, each angle $\phi$ measured from $Sx$ is replaced by $\phi' - \gamma$ measured from $Sx'$. Thus the equation of the tangent at $\theta' = \alpha'$ to $l/r = 1 + e \cos (\theta' - \gamma)$ is $l/r = e \cos (\theta' - \gamma) + \cos (\theta' - \gamma - (\alpha' - \gamma)) = e \cos (\theta' - \gamma) + \cos (\theta' - \alpha')$. 

14.3.10. GEOMETRICAL PROPERTIES
EXERCISE 57

[In this exercise, \( S \) denotes the focus of the given conic, \( e \) denotes its eccentricity and \( l \) denotes its semi latus rectum.]

Obtain the polar equations of the following loci referred to rectangular axes \( Ox, Oy, \) Nos. 1-6:

1. \( y^2 = 4ax + a^2. \)
2. \( y^2 = x^2. \)
3. \( x^2 + y^2 = 2ax + 2by. \)
4. \( x^2 - y^2 = 2a^2 xy. \)
5. \( x^2 + y^2 = x + y. \)
6. \( x^2 + y^2 = e^2 (a \cos \alpha + y \sin \alpha - a \cdot p^2). \)

Obtain the cartesian \((x, y)\) equations of the following loci:

7. \( r = c \cos \theta. \)
8. \( r = e \cos \theta. \)
9. \( r = c \sin \theta. \)
10. \( r = a + b \cos \theta. \)
11. \( r = a \sin \theta. \)
12. \( r = a + b \sin \theta. \)

13. If \( k > 0 \) and \( 90^\circ < \alpha < 180^\circ \), find the polar coordinates of the centre of the circle \( r = k \sin (\theta - x). \)

14. Find the equation of the common chord of the circles,

\( r = \cos (\theta - \alpha) \) and \( r = \cos (\theta - \beta). \)

15. Use a sketch of the circle \( r = \alpha \cos \theta \) to sketch the cardioid \( r = \alpha (1 + \cos \theta) \), taking values of \( \theta \) from 0° to 360°.

16. Write down the polar coordinates of the image of \( [x_1, y_1] \) in the line \( \theta = \alpha \). If the polar coordinates of \( x_1, y_1 \) in \( px + qy = 0 \) is \( (x_2, y_2) \), prove

\( p^2 - q^2 = (q^2 - p^2)x_1 - 2pqy_1, \)
\( p^2 + q^2 = (q^2 - p^2)y_1 - 2pqx_1. \)

17. Prove that the equations, \( l/r = 1 + e \cos \theta, \) \( l/r = -1 + e \cos \theta \) represent the same conic.

18. Prove that the circle \( r = \delta \) cuts the hyperbola \( 2/r = 1 + 3 \cos \theta \) at four points and find their polar coordinates.

19. \( 2/r = 3 \sin \theta - 4. \)
20. \( 2/r = 3 + 4 \cos \theta. \)
21. \( 4/r = 2 + 5 \sin \theta + 6 \cos \theta. \)

22. Prove that the lengths of the axes of the ellipse \( l/r = 1 + e \cos \theta, \) where \( e < 1 \), are \( 2l/(1 - e^2), \) \( 2l/(1 + e^2). \)

23. Show that the variable line given by \( 1/r = a \cos \theta + b \cos (\theta - x), \) where \( a, b \) are constant and \( x \) varies, touches a fixed conic and state its equation.

24. Repeat No. 23 for the variable line, \( 1/r = a \cos (\theta - \gamma) + b \cos (\theta - x), \) where \( a, b, \gamma \) are constant and \( x \) varies.

25. The vectorial angles of \( P, Q \) on \( l/r = 1 + e \cos \theta \) are \( x - \beta, a + \beta; \) prove that the equation of \( PQ \) is \( l/r = e \cos (\theta - \beta) + \sec \beta \cos (\theta - x). \)

26. Prove that the line \( l/r = p \cos \theta + q \sin \theta \) touches the conic \( l/r = 1 + e \cos \theta \) if \( (p - e)^2 + q^2 = 1. \)

27. Prove that the asymptotes of the hyperbola \( l/r = 1 + e \cos \theta, \) \( e > 1, \) are given by \( r \cos (\theta - \beta) = \sqrt{p^2 - 1} \), where \( \sin \beta = 1/e. \)

GEOMETRICAL PROPERTIES

28. \( O \) is a given point; \( P \) is a variable point on a given circle; \( OP \) is produced to \( Q \) so that \( OQ = k \cdot OP, \) where \( k \) is a constant. Find the locus of \( Q. \)

29. \( P \) and \( Q \) are points on a conic; the tangents at \( P, Q \) meet at \( T; \) \( PQ \) meets the directrix corresponding to the focus \( S \) at \( R. \) Prove that the angle \( RST \) is a right angle.

30. What is the equation of the directrix of \( l/r = 1 + e \cos (\theta - \gamma) \) corresponding to the pole \( S ? \) If this conic cuts \( l/r = 1 + e \cos \theta, \) explain why: \( (i) \) \( (l - r)/r = 2 \cos (\theta - \gamma) - 1; \) \( (ii) \) \( e \cos \theta = 0 \) is the equation of a common chord; \( (iii) \) this common chord passes through the point of intersection of the directrices corresponding to \( S. \)

31. Interpret the locus \( l/r = 1 + \sqrt{2} \cos \theta. \) Prove that perpendicular focal chords of a rectangular hyperbola are equal.

32. If \( P, Q \) are points on the parabola \( l/r = 1 + \cos \theta, \) the tangents at \( P, Q \) meet at \( T. \) Prove that \( ST = SP \cdot SQ. \)

33. If the tangents to a parabola at the points \( P, Q, \) \( R \) form the triangle \( P'Q'R', \) prove that \( SP \cdot SQ \cdot SR = SP \cdot SQ \cdot SR \cdot SY. \)

34. If \( S \) is a variable circle of given radius passes through the focus \( S \) of a given conic and cuts the conic at \( P_1, P_2, P_3, P_4; \) prove that \( SP_1 \cdot SP_2 \cdot SP_3 \cdot SP_4 \) is constant.

35. \( A' \) is the vertex of a hyperbola, focus \( S, \) on the branch which does not include \( S; \) the circle \( S \cdot A' \) as diameter cuts the hyperbola again at \( P \) and \( Q. \) Prove that \( SP = SU. \)

36. \( P \) is a variable point on the branch of the hyperbola \( l/r = 1 + \cos \theta, \) where \( e < 1, \) and cuts it at \( P_1, P_2, P_3, P_4. \) Prove

\[ 1/SP_1 + 1/SP_2 + 1/SP_3 + 1/SP_4 = 2/l. \]

37. If \( P, Q \) are points on the branch of the hyperbola \( l/r = 1 + e \cos \theta, \) where \( e > 1, \) not enclosing \( S; \) the circle \( SPQ \) cuts the hyperbola again at \( H, K. \) Prove \( 1/SH + 1/SK = 1/SP + 1/SQ = 2/l. \)

38. \( P, Q, R' \) are points on the parabola \( l/r = 1 + \cos \theta, \) given by \( \theta = 2x, 6 = 2z; \) \( P, Q, R \) are the poles of \( Q'E', R'P', Q'P'. \) Prove that \( P \) is the point given by \( r = 1 + \cos \beta \cos \gamma, \) \( \theta = \beta + \gamma; \) deduce that the circle \( r \cos \alpha \cos \beta \cos \gamma = -1 \cos (\theta - \beta - \gamma) \) passes through \( P, Q, R, \) and \( S. \)

39. \( P, Q \) are points on the parabola, \( 2r = 1 + \cos \theta, \) given by \( \theta = 2 \alpha, 6 = 2z; \) \( P, Q, R \) are the poles of \( Q'R, R'P, Q'P'. \) Prove that \( P \) is the point given by \( r = l \cos \beta \cos \gamma, \) \( \theta = \beta + \gamma; \) deduce that the circle \( r \cos \alpha \cos \beta \cos \gamma = -1 \cos (\theta - \beta - \gamma) \) passes through \( P, Q, R, \) and \( S. \)

40. \( PT \) is a variable tangent to a given circle, centre \( O, \) radius \( a; \) \( O \) is the mid-point of a given radius \( OA; \) \( Q \) is the image of \( O \) in \( FT. \) Find the polar equation of the locus of \( Q, \) with \( O \) as pole and \( OA \) as initial line.

41. The tangents to \( l/r = 1 + \cos \theta \) at the points \( r \sin \theta, r, x = \beta \) meet at \( \sin \gamma. \) Prove that \( 2 \cos \beta /r = 1/\sin \gamma = 2 \cos \beta /r = 2 \sin \gamma. \)

42. Prove that the locus of the foot of the perpendicular from the pole \( O \) to a variable tangent to \( r = 1 + \cos \theta \) is \( r^2 = \sin \theta. \)
14.4. If, with \( O \) as pole, the polar coordinates of a variable pair of points \( P[r, \theta], P'[r', \theta'] \) are connected by the relations,
\[
rr' = k^2, \quad \theta = \theta',
\]
where \( k \) is a positive constant, \( P, P' \) are called inverse points with respect to the circle, centre \( O \), radius \( k \); \( O \) is called the centre of inversion, and \( k \) is called the radius of inversion. It is customary to speak of inversion with respect to \( O \) if, as is usually the case, the actual value of the constant \( k \) is unimportant.

If \( P \) is the point \([r, \theta]\), the polar coordinates of the inverse point \( P' \) of \( P \) with respect to the pole \( O \) can be denoted by \([k^2/r, \theta]\); hence if, with \( O \) as pole, \( P \) is a variable point whose locus is given by the polar equation \( f(r, \theta) = 0 \), the polar equation of the locus of \( P' \) is given by \( f(k^2/r, \theta) = 0 \). The locus of \( P' \) is itself called the inverse of the locus of \( P \) with respect to \( O \). In particular, if \( P \) is a variable point on a given line \( OA \), then \( P' \) also lies on \( OA \); therefore the inverse with respect to \( O \) of the line \( OA \) is itself \( OA \).

14.4.1. If \( OA \) is a diameter of a given circle, the inverse of the circle with respect to \( O \) is a line \( A'P' \) perpendicular to \( OA \), cutting \( OA \) at the inverse of \( A' \).

Conversely if \( OA' \) is the perpendicular from \( O \) to a given line, the inverse of the line with respect to \( O \) is the circle on \( OA \) as diameter where \( A \) is the inverse of \( A' \).

Take the initial line along \( OA \); then the equation of the circle is
\[
r = a \cos \theta, \quad \text{where} \quad OA = a,
\]
and so the equation of the inverse of the circle is
\[
(k^2/r) = a \cos \theta,
\]
that is,
\[
r \cos \theta = k^2/a.
\]
Since \( OA' = k^2/a \), this is the line through \( A' \) perpendicular to \( OA \).

Conversely, if the initial line is along \( OA' \), the equation of the line is \( r \cos \theta = a' \), where \( OA' = a' \); therefore the equation of the inverse of the line is
\[
(k^2/r) = a' \cos \theta, \quad \text{that is}, \quad r = (k^2/a') \cos \theta.
\]
Since \( OA = k^2/a' \), this is the circle on \( OA \) as diameter.

Note. If \( C \) is the centre of the given circle, \( OC = \frac{1}{2}OA \); therefore the inverse \( C' \) of \( C \) is given by \( OC' = k^2/OC = 2k^2/OA = 2OA \); therefore the inverse of the centre of the circle is the image of \( O \) in the line \( A'P' \).

14.4.2. If \( O \) does not lie on a circle \( S \), centre \( C \), the inverse of \( S \) with respect to \( O \) is a circle \( S' \) whose centre \( D \) lies on \( OC \). \( S' \) does not pass through \( O \), and \( D \) is not the inverse of \( C \).

Take the initial line along \( OC \); then the equation of \( S' \) is
\[
r^2 - 2ar \cos \theta + c = 0, \quad c > 0,
\]
where \( OC = a \) and \( a^2 - c > 0 \).

... the equation of \( S' \) is
\[
k^4r^2 - 2ak^2/r \cos \theta + c = 0,
\]
that is,
\[
cr^2 - 2ak^2/r \cos \theta + k^4 = 0, \quad c > 0.
\]
By 14.3.3, this equation represents a circle which does not pass through the pole \( O \); also the polar coordinates of the centre \( D \) are \([ak^2/c, \theta] \). Therefore \( D \) lies on \( OC \) and \( OD = ak^2/c \).

... \( D \) is not the inverse of \( C \).

14.4.3. If the circles of a coaxal system have two distinct common points \( O, A \), their inverses with respect to \( O \) are concurrent lines intersecting at the inverse \( A' \) of \( A \).

By 14.4.1 the inverse of each circle is a line. Since each circle passes through \( A \), its inverse passes through the inverse \( A' \) of \( A \).

14.4.4. If a system of coaxal circles has two distinct limiting points \( O, L \), then the inverse with respect to \( O \) of the circles of the system are concentric circles whose common centre is the inverse \( L' \) of \( L \).

Take \( O \) as origin and \( OL \) as the initial line \( Ox \).

If \( O = 2x, c > 0 \), the equation of the radical axis is \( x - c = 0 \); also the equation of the point-circle \( O \) is \( x^2 + y^2 = 0 \).

... by 5.10.1, p. 70, the equation of any circle of the system is
\[
z^2 + y^2 + p(x - c) = 0, \quad \text{where} \quad p \text{ varies and} \quad c > 0;
\]
this equation may be written, \( r^2 + pr \cos \theta - pc = 0 \);

... the equation of its inverse is
\[
k^4r^2 + p(k^2/r) \cos \theta - k^4 = 0, \quad c > 0,
\]
that is
\[
r^2 = (k^2/c) \cos \theta - k^4/pc = 0.
\]
By 14.3.3, this is the equation of a circle whose centre is the fixed point \( L' \) whose polar coordinates are \([k^2/c, \theta] \);

... \( OL = (2x, (k^2/c) = k^2 \); ... \( L' \) is the inverse of \( L \).
14.4.5. Two curves $S_1, S_2$ cut at $P$; their inverses $S'_1, S'_2$ with respect to a point $O$ cut at the inverse point $P'$ of $P$; then the angle of intersection of $S_1, S_2$ at $P$ equals the angle of intersection of $S'_1, S'_2$ at $P'$.

Through $O$ draw a line cutting $S_1, S_2$ at $Q_1, Q_2$; then it cuts $S'_1, S'_2$ at the inverse points $Q'_1, Q'_2$ of $Q_1, Q_2$.

\[
\begin{align*}
Q_1PQ'_1 &= k^2 = OP \cdot OP' = OQ_2 \cdot OQ'_2; \\
\angle OQ_1Q'_2 &= \angle OQ_2Q'_1; \\
\angle OQ_1PQ'_1 &= \angle OQ_2PQ'_2; \\
\angle OPQ_1 &= \angle OP'Q'_1; \\
\angle OPQ_2 &= \angle OP'Q'_2.
\end{align*}
\]

When $OQ_1$ moves up to $OP$, the limiting positions of $PQ_1, PQ_2$ and $P'Q'_1, P'Q'_2$ are the tangents to $S_1, S_2$ at $P$ and the tangents to $S'_1, S'_2$ at $P'$; therefore the angle between the tangents to $S_1, S_2$ at $P$ equals the angle between the tangents to $S'_1, S'_2$ at $P'$.

In particular, if $S_1$ touches $S_2$ at $P$, then $S'_1$ touches $S'_2$ at $P'$.

14.4.6. If $S_1$ and $S_2$ are two orthogonal circles and if $S'_1, S'_2$ are their inverses with respect to a point $O$, then

(i) $S'_1, S'_2$ are orthogonal circles if $O$ is not on $S_1$ and $S_2$;
(ii) $S'_1, S'_2$ are perpendicular lines if $S_1$ cuts $S_2$ at $O$;
(iii) $S'_2$ is a diameter of the circle $S'_2$ if $O$ lies on $S_1$ but not on $S_2$.

These properties follow at once from 14.4.5 and 14.4.1, 14.4.2.

14.4.7. If the inverses of the points $P, Q$ with respect to a circle, centre $O$, radius $k$, are $P', Q'$ and if $OE, OF$ are the perpendiculars from $O$ to the lines $PQ, P'Q'$, then

\[
\begin{align*}
PQ \cdot P'Q' &= k^2; \\
PQ &= OP \cdot OQ. \\
k^2 &= OE \cdot OF.
\end{align*}
\]

As in 14.4.5, $\angle OPQ = \angle OQ'P'$; the triangles $OPQ, OQ'P'$ and the triangles $OEP, OFQ'$ are similar;

\[
\begin{align*}
OE &= OP \cdot PQ, \\
OP &= OQ \cdot OP', \\
k^2 &= OE \cdot OF.
\end{align*}
\]

EXERCISE 58

[In this exercise, $O$ denotes the origin or pole.]

1. Find the equation of the inverse of the line $2x + by - 8 = 0$ with respect to the circle $x^2 + y^2 = 9$.

2. Verify that the inverses with respect to $O$ of the circles $x^2 + y^2 + 2x - 4y = 0, x^2 + y^2 + 4x + 3y = 0$ are perpendicular lines.

3. If the points $(x, y), (x', y')$ are inverse points, with respect to the circle $x^2 + y^2 = k^2$, prove $x' = \frac{k^2x}{x^2 + y^2}, y' = \frac{k^2y}{x^2 + y^2}$.

4. If $OP$ meets the polar of $P$ with respect to $x^2 + y^2 = k^2$ at $Q$, prove that $P, Q$ are inverse points with respect to this circle.

5. Interpret the system of circles, $r^2 - 2ar \cos \theta + a^2 = p^2$, where $a$ is constant and $p$ varies. Prove that their inverses with respect to $O$ are circles of a coaxal system having $O$ as one limiting point.

6. Verify that the inverses of the perpendicular lines $r \cos (\theta - \alpha) = p, r \sin (\theta - \alpha) = q$ with respect to $O$ are orthogonal circles.

7. A variable line through $O$ meets the given circles $r = a \cos \theta, r = b \cos (\theta - \beta)$ again at $P, Q$. Prove that the locus of the mid-point of $PQ$ is a circle through $O$. What result is obtained by forming the inverse with respect to $O$ of this figure?

8. Prove that the inverses with respect to $O$ of parallel lines not passing through $O$ are circles touching at $O$. If $OABC$ is a parallelogram and if $OB$ meets $AC$ at $N$, describe the figure obtained by inversion with respect to $O$ and deduce a property from the relation $ON = \frac{1}{2}OB$.

9. (Ptolemy's Theorem.) $A, B, C$ are points in order on a line. By inversion with respect to a point $O$, deduce from the relation, $AB + BC = AC$, that if $OABC$ is a cyclic quadrilateral, then

\[OA', BC' + OC', AB' = OB', AC'.\]

10. Use the method of No. 9 to prove that if $O$ does not lie on the circle $ABC$, then $O, BC' + OC', AB' = OB', AC'$.

11. $O$ is a point on the circle $ABC$; $OL, OM, ON$ are the perpendiculars from $O$ to $BC, CA, AB$. Prove that one of the expressions $OC/OL, CA/OM, AB/ON$ equals the sum of the others. [Use 14.4.7, p. 264.]

12. (i) Prove that the inverse of the cardioid $r = a(1 + \cos \theta)$ with respect to its pole $O$ is a parabola having $O$ as focus.

(ii) If $OB$ is a diameter of a circle which touches the cardioid, $r = a(1 + \cos \theta)$, pole $O$, at a variable point $P$, prove that the locus of $R$ is a circle through $O$.

(iii) If $POQ$ is a chord of the cardioid $r = a(1 + \cos \theta)$, prove that the tangents at $P$ and $Q$ are at right angles.
CHAPTER 15

THE GENERAL EQUATION

15.1. The ratio-equation which gives the values of the ratios in which the line joining \( P_1(x_1, y_1) \) to \( P_2(x_2, y_2) \) is divided by its points of intersection with \( ax^2 + by^2 - 1 = 0 \) was obtained on p. 238. Precisely the same method can be used for

\[ S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \]

If \( P \) is a point of intersection of \( P_1, P_2 \) with \( S = 0 \) and if

\[ P \cdot P = \frac{P \cdot P}{kk_1}, \]

then

\[ a(k_1x_1 + k_2x_2)^2 + 2h(k_1x_1 + k_2x_2)(k_1y_1 + k_2y_2) + b(k_1y_1 + k_2y_2)^2 + 2g(k_1x_1 + k_2x_2)(k_1 + k_2) + 2f(k_1y_1 + k_2y_2)(k_1 + k_2) + c(k_1 + k_2)^2 = 0, \]

that is, \( k_2^2(ax_2^2 + 2hx_2y_2 + by_2^2 + 2gx_2 + 2fy_2 + c) + 2k_1k_2(ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c) + \frac{k_2^2}{k_1} = 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \)

This is called Joachimsthall’s ratio-equation.

15.1.1. The following abbreviated notation is not merely a labour-saving device for working with the general ratio-equation but also displays the symmetry of expressions which occur in the treatment of the general conic. Put

\[ S_{mn} = ax_2y_3 + by_2y_3 + cx_2y_3 + f(y_2 + y_3) + c. \]

The expression for \( S_{nn} \) is not altered if \( m \) and \( n \) are interchanged.

\[ S_{mn} = S_{nm}. \]

The coefficient of \( 2k_1k_2 \) in the ratio-equation is obtained by putting \( m = 1, n = 2 \) in \( S_{mn} \) and is denoted by \( S_{12} \) or by \( S_{21} \); the symmetry is illustrated by first collecting the coefficients of \( x_1 \) and \( y_1 \) and then collecting the coefficients of \( x_2 \) and \( y_2 \).

\[ S_{12} = x_1(ax_2 + hy_2 + f) + y_1(by_2 + f) + (gx_2 + fy_2 + c) = x_1(ax_2 + by_2 + f) + y_1(by_2 + f) + (gx_2 + fy_2 + c). \]

The coefficient of \( k_1^2 \) in the ratio-equation is obtained by putting \( m = 1, n = 1 \) in \( S_{mn} \) and is denoted by \( S_{11} \);

\[ S_{11} = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = x_1(ax_1 + hy_1 + f) + y_1(by_1 + f) + (gx_1 + fy_1 + c). \]

Similarly, the coefficient of \( k_2^2 \) in the ratio-equation is denoted by \( S_{22} \). With this notation, the ratio-equation is

\[ S_{11}k_1^2 + 2S_{12}k_1k_2 + S_{22}k_2^2 = 0. \]

Note. The condition \( S_{11} = 0 \) means that \( P_1(x_1, y_1) \) lies on \( S = 0 \).

15.1.2. The equation of the line-pair formed by the tangents from \( P_1(x_1, y_1) \) to the proper conic \( S = 0 \) is

\[ S_1S = S_2. \]

15.1.3. If \( P_1(x_1, y_1) \) is a point on the proper conic \( S = 0 \), the equation of the tangent at \( P_1 \) to \( S = 0 \) is

\[ S_1 = xu_1 + yu_1 + w_1 = xu + y + w = 0. \]

15.1.4. If the line joining \( P_1(x_1, y_1) \) to \( P_2(x_2, y_2) \) is divided harmonically by its intersections with the proper (or degenerate) conic \( S = 0 \), then

\[ S_1 = 0, \]

and conversely.

\( P_1, P_2 \) are then called conjugate points with respect to \( S = 0 \).

15.1.5. If \( P_1(x_1, y_1) \) is a point inside or outside the proper conic \( S = 0 \), the equation of the polar of \( P_1 \) with respect to \( S = 0 \) is

\[ S_1 = xu_1 + yu_1 + w_1 = xu + y + w = 0. \]

15.1.6. If the polar of \( P_1 \) with respect to the proper conic \( S = 0 \) passes through \( P_2 \), then the pole of \( P_2 \) passes through \( P_1 \).

15.1.7. If the line joining \( P_1(x_1, y_1) \) to \( P_2(x_2, y_2) \) is divided at \( P_3(x_3, y_3) \) in the ratio \( t_3 = 1 \), the equation of the polar of \( P_3 \) with respect to the proper conic \( S = 0 \) can be written in the form

\[ S_3 = S_1 + t_3S_2 = 0. \]

\( P_3(x_3, y_3) \) is given by \( x_3 = \frac{x_1 + t_3x_2}{1 + t_3}, y_3 = \frac{y_1 + t_3y_2}{1 + t_3} \).

\[ (x_1 + t_3x_2)(x + y + w) + (y_1 + t_3y_2)(x + y + w) = 1, \]

that is, \( (x_1 + y_1 + w)(x + y + w) + t_3(x_2y_2 + y_2w + w) = S_1 + t_3S_2 = 0. \)
15.2. The \( r \)-quadratic which gives the directed lengths from \( V(x_1, y_1) \) along a line in the direction making an angle \( \theta \) with \( Ox \) to its points of intersection with \( ax^2 + 2hxy + by^2 - 1 \) was obtained on p. 248. Precisely the same method can be used to obtain the \( r \)-quadratic for the conic \( S = 0 \) given by the general equation, and corresponding deductions can be made.

15.2.1. If the line drawn from \( V(x_1, y_1) \) in the direction making an angle \( \theta \) with \( Ox \) meets at \( P, Q \), the conic
\[
S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,
\]
then the directed lengths \( r_1, r_2 \) of \( VP, VQ \) are the roots of
\[
r^2(a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) + 2r(u_1 \cos \theta + v_1 \sin \theta) + S_{11} = 0.
\]
If a step of directed length \( r \) is taken from \( V(x_1, y_1) \) in the direction making an angle \( \theta \) with \( Ox \), the point of arrival is \((x_2 + r \cos \theta, y_2 + r \sin \theta)\); this point lies on \( S = 0 \) if
\[
a(x_1 + r \cos \theta)^2 + 2h(x_1 + r \cos \theta)(y_1 + r \sin \theta) + b(y_1 + r \sin \theta)^2 + 2g(x_1 + r \cos \theta) + 2fy_1 + c = 0,\]
that is,
\[
r^2(a \cos^2 \theta + 2h \sin \theta \cos \theta + b \sin^2 \theta) + 2r(u_1 \cos \theta + v_1 \sin \theta) + S_{11} = 0.
\]
This is equivalent to the stated form.

15.2.2. If \( V(x_1, y_1) \) is the mid-point of the chord \( PQ \) of \( S = 0 \), the equation of \( PQ \) is \( x_1y_1 + y_1y = x_1y + y_1y \). If \( V \) is the mid-point of \( PQ \), the sum of the directed lengths \( VP, VQ \) is zero; therefore, if \( PQ \) makes an angle \( \theta \) with \( Ox \), the sum of the roots of the corresponding \( r \)-quadratic is zero;
\[
\therefore u_1 \cos \theta + v_1 \sin \theta = 0, \quad \sin \theta = \cos \theta = u_1 = -v_1;
\]
but \( PVQ \) is parallel to the line \( x \sin \theta - y \cos \theta = 0 \); therefore, the equation of \( PVQ \) is \( x_1y_1 + y_1y = \ldots = x_1y_1 + y_1y \).

15.2.3. If \( C \) is the centre of the conic \( S = 0 \), \( \alpha \pm \beta \), then
(i) the equation of any diameter can be taken as \( u + tv = 0 \);
(ii) if \( HCH' \), \( u + tv = 0 \), and \( KCK' \), \( u + tv = 0 \), are conjugate diameters,
\[
bt' + h(t + t') + a = 0.
\]
(i) The centre \( C \) is the point of intersection of the lines \( u = 0, v = 0 \); see 15.1, p. 267; therefore the result follows from 6.5.3, p. 83.
(ii) If the mid-point \( (x_1, y_1) \) of a chord \( PQ \) lies on the diameter \( HCH' \), \( u + tv = 0 \), then \( u_1 + v_1 = 0 \).

The equation of \( PQ \) is \( x_1y_1 + y_1y = x_1y_1 + y_1y \) where \( u_1 = -v_1 \); therefore the diameter \( KCK' \) conjugate to \( HCH' \) is parallel to \( te - y = 0 \).

By hypothesis, \( KCK' \) is parallel to \((ax + by) + t'(hx + by) = 0\);
\[
\therefore t' = -(a + h')(h + h'), \quad bt' + h + h' + a = 0.
\]

15.2.4. The equation of the principal axes of the central conic \( S = 0 \), \( \alpha \pm \beta \), is
\[
h(u^2 - v^2) - (a - b)uv = 0.
\]
The principal axes of a central conic are the pair of conjugate diameters which are at right angles. Therefore it follows from the proof of 15.2.3 that \( HCH' \), \( u + tv = 0 \), is a principal axis if
\[
u + tv = (ax + by + g) + t(hx + by + f) = x(a + h) + y(b + h) + (a + f) = 0
\]
is perpendicular to the line \( tx - y = 0 \).
Therefore \( u + tv = 0 \) is a principal axis if \( t \) is given by
\[
t(a + h) - (b + h) = 0, \quad \text{that is}, \quad h^2 + (a - b)t = h.
\]
Therefore the equation of the principal axes is
\[
h(-u/v)^2 + (a - b)(-u/v) = 0, \quad \text{that is}, \quad h(u^2 - (a - b)uv - hv^2) = 0.
\]

15.2.5. The equation of the asymptotes of \( S = 0 \), \( h^2 > ab \), is
\[
bu^2 - 2huv + av^2 = 0.
\]
By 11.6.4 (l), p. 182, the limiting position of a variable pair of conjugate diameters of a hyperbola when they tend to coincidence is one or other of the asymptotes.

Therefore, by 15.2.3, \( u + tv = 0 \) is the equation of an asymptote if \( t \) is a root of the equation, \( h^2 + 2hx + a = 0 \).

Therefore the equation of the pair of asymptotes is
\[
h(-u/v)^2 + 2h(-u/v) + a = 0, \quad \text{that is}, \quad h(u^2 - 2huv + av^2) = 0.
\]
An alternative method of proof is indicated in Exercise 59, No. 10.

Example 1. \( P_1(x_1, y_1) \), \( P_2(x_2, y_2) \) are given general points; \( P \) is a variable point such that \( PP_1, PP_2 \) are conjugate lines with respect to the given conic \( S = 0 \). Prove that the equation of the locus of \( P \) is
\[
S_1x - S_2y - S_3z = 0.
\]
Let \( P \) be the point \((x_2, y_2)\). By hypothesis, the pole \( U \) of \( PP_1 \) lies on \( PP_2 \), see Fig. 123.

Let \( U \) divide \( PP_2 \) in the ratio \( 1:1 \), then by 13.1.7 the equation of the polar of \( U \) is \( S_{12} + S_{13} = 0 \).

But the polar of \( U \) passes through \( P(x_2, y_2) \) and \( P(x_1, y_1) \);
\[
\therefore S_{12}x_2 + S_{13}y_2 = 0 \quad \text{and} \quad S_{11}x_1 + tS_{14} = 0;
\]
eliminate \( t \), \( S_{12}x_2 + S_{13}y_2 = 0 \).

This is the condition that the coordinates of \( P(x_2, y_2) \) satisfy the equation \( S_{12}S_2 - S_2S_3 = 0 \).

For properties of this locus, see Exercise 59, Nos. 11, 15.
EXERCISE 59

[In this exercise, in addition to the notation on pp. 266-269, $S'$ denotes $a'x^2 + 2b'xy + b'y^2 + 2f'x + 2f'y + c'; u', v', w'$ have similar meanings.]

1. State the relations if $P_1(x_1, y_1)$ lies on: (i) $S = 0$; (ii) $S_1 = 0$; (iii) $S_1 = 0$; (iv) $S_1 + kS_2 = 0$, where $k$ is a constant.

2. If $P_3(x_2, y_2)$ and $P_4(x_2, y_2)$ are points on $S = 0$, prove that the locus whose equation is $S_1 + S_2 + tS_3$ passes through $P_2$. Deduce that $S_1 + S_2 = S_1$ is the equation of the chord $P_1P_2$.

3. State the equation of the polar of $P_1(x_1, y_1)$ with respect to the conic $S = kS'$, where $k$ is a constant.

4. If $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are conjugate points with respect to $S = 0$ and with respect to $S' = 0$, prove that they are conjugate points with respect to $S + kS' = 0$, where $k$ is a constant.

5. Prove that: (i) the chord $PQ$ of $S = 0$ whose mid-point is $V(x_1, y_1)$ is parallel to the polar of $V$ with respect to $S = 0$; (ii) the equation of $PQ$ is $S_1 = S_1$.

6. If $S = 0$ is a central conic, find the equation of the diameter conjugate to: (i) $u = 0$; (ii) $v = 0$.

7. $C(x_1, y_1)$ is the centre of $S = ax^2 + 2bxy + by^2 + 2fx + 2fy + c = 0$, $ab + bc = 0$. Prove that the equation of the conic referred to axes $CX, CY$, parallel to $Ox, Oy$ is $aX^2 + 2bXY + bY^2 + c_1 = 0$.

8. Find the centre $C$ of the conic, $2x^2 - xy - 3y^2 + 11x + 10y = 11 = 0$ and find its equation referred to axes $CX, CY$ parallel to $Ox, Oy$.

9. Prove that the conic $S = ax^2 + 2bxy + by^2 + 2fx + 2fy + c = 0$, $ab + bc = 0$, can be expressed in the form $bu^2 - 2huv + wu^2 + w^2 + u^2 = 0$.

10. State the conditions that both roots of the $r$-quadratic in 15.2.1, p. 288, tend to infinity. Hence prove that the asymptotes of $S = 0$ are given by $bu^2 - 2huv + wu^2 = 0$.

11. With the data of Example 1, p. 286, if the polar of $P_1$ cuts $S = 0$ at $D, E$, prove that the locus of $P$ passes through $D$ and $E$. Prove further that, if $P_1, P_2$ are conjugate points with respect to $S = 0$, the locus of $P$ is the pair of lines formed by the polars of $P_1, P_2$.

12. $P_3(x_2, y_2)$ is a point outside the conic $S = 0$. Find the value of the constant $k$ if the locus whose equation is $S' = kS - S_1^2 = 0$ passes through $P_3$, and interpret the result.

15.8] ABBREVIATED NOTATION

13. If $S = 0$ is the equation of a central conic, prove that the equation of the diameter parallel to the tangent at $(x_1, y_1)$ in $S_1 = \Delta/(ab - h^2)$.

14. $P_1Q_1R_1$ and $P_2Q_2R_2$ are parallel lines through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ which meet the conic $S = 0$ at $Q_1, R_1$ and $Q_2, R_2$. Prove that $P_1Q_1 : P_2Q_2 = P_1R_1 : P_2R_2 = S_1 : S_2$.

15. Find the value of the constant $k$ if $P_1$ lies on the curve whose equation is $S' = kS + S_2; S_2 = 0$. Prove that if $P_1$ lies on $S' = 0$, then $P_1$ also lies on $S' = 0$ and interpret the equation.

16. $P_1(x_1, y_1), P_2(x_2, y_2)$ are given points on the line $DE$. Prove that the pole of $DE$ with respect to the conic $S = 0$ is given by the equations $S = 0, S_1 = 0$. If $S = 0, S' = 0$ are given conics, prove that the equation of the locus of the pole of $DE$ with respect to the conic $S + bS' = 0$, where $t$ varies, is $S_1S_2 - S_2S_1^2 = 0$.

17. If $S = 0, S' = 0$ are the equations of two given conics, prove that the equation of the locus of the centre of the conic $S + tS' = 0$, where $t$ varies, is $u^2 - u'v' = 0$.

18. If the tangents from a variable point $T$ to the central conic $S = 0, ab + bc = 0$, are at right angles, prove that the equation of the locus of $T$ is $(a + b)S = u^2 + v^2$.

What is the corresponding property if $ab = h^2$?

19. $P$ is a variable point on the line joining the given points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$; $Q$ is a point such that $P$ and $Q$ are conjugate points with respect to each of the conics, $S = 0, S' = 0$. Prove that the equation of the locus of $Q$ is $S_1S_2 - S_2S_1^2 = 0$.

20. The poles of a variable point $P$ with respect to each of the central conics $S = 0, S' = 0$ are at right angles. Find the equation of the locus of $P$ and prove that it passes through the centre of each of the conics.

15.3. As noted on p. 229.

$$\Delta = a, h, g \equiv h, b, f \equiv g, f, c \equiv a, h, g$$

By introducing the following standard notation,

$$A = bc - f^2, B = ca - g^2, C = ab - h^2, F = gh - af, G = hf - bg, H = fg - ch,$$

the expansions of the above three forms of $\Delta$ may be written

$$\Delta = aA + hH + gG = hH + bB + fF = gG + fF + cC.$$  (i)

$A, B, C, F, G, H$ are called the cofactors of $a, b, c, f, g, h$ in $\Delta$. 
15.3.1. The cofactors of the elements of a row of a determinant were defined in 15.3, p. 271, in such a way that the value of the determinant is given by multiplying each element of a row by its cofactor and then taking the sum of these products for the row. The following property of cofactors is important.

If the elements of a row are multiplied by the cofactors of the corresponding elements of another row, the sum of these products is zero.

For example if in \( \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \) the elements \( h, b, f \) of the second row are multiplied by the cofactors \( A, H, G \) of the first row, their sum is zero:

\[
hA + bH + fG = h b f = 0 \quad \text{because two rows are identical.}
\]

The following relations can be verified in the same way:

\[
0 = aH + bB + gF = hG + bF + fC = gA + fH + eG
\]

\[
- aG + hF + gC = hA + bH + fG = gH + fB + eF.
\]

(i) If the line \( lx + my + n = 0 \) touches the proper conic \( S = 0 \) with respect to \( S = 0 \), \( (\Delta = 0) \), then

\[
\Sigma = A l^2 + 2Hlm + Bm^2 + Gln + 2Fmn + Cn^2 = 0,
\]

where \( A, B, C, F, G, H \) are the cofactors of \( a, b, c, f, g, h \) in \( \Delta \).

If the point of contact of \( lx + my + n = 0 \) with \( S = 0 \) is \( (x_1, y_1) \),
\[
(x_1 + h y_1 + g) (h x_1 + b y_1 + f) + (g x_1 + f y_1 + c) = 0
\]

is equivalent to \( lx + my + n = 0 \); also \( l x_1 + m y_1 + n = 0 \);

\[
\frac{ax_1 + by_1 + g}{m} \cdot \frac{hx_1 + by_1 + f}{n} \cdot \frac{gx_1 + fy_1 + c}{n} = k, \quad \text{say},
\]

where by 12.2.4, p. 212, \( k = 0 \) because \( \Delta = 0 \).

\[
\therefore \frac{ax_1 + by_1 + g}{m} \cdot \frac{hx_1 + by_1 + f}{n} \cdot \frac{gx_1 + fy_1 + c}{n} = k = 0 \quad \therefore \text{(i)}
\]

and

\[
\frac{ax_1 + by_1 + g}{m} \cdot \frac{hx_1 + by_1 + f}{n} = 0 \quad \therefore \text{(ii)}
\]

and

\[
\frac{ax_1 + by_1 + g}{m} \cdot \frac{hx_1 + by_1 + f}{n} \cdot \frac{gx_1 + fy_1 + c}{n} = 0 \quad \therefore \text{Multiply (i), (ii), (iii) by \( A, H, G \) and add; then, by 15.3(i) and 15.3.1(i)}
\]

\[
\Delta x_1 - k(Al + Bm + Gn) = 0 \quad \text{and} \quad y_1 - k(Gl + Fm + Cn) = 0.
\]

Similarly, multiply (i), (ii), (iii) by \( H, B, F \) and add; also by \( G, F, C \); then

\[
\Delta y_1 - k(Hl + Bm + Fn) = 0 \quad \text{and} \quad \Delta - k(Gl + Fm + Cn) = 0.
\]

But \( lx_1 + my_1 + n = 0 \) and \( \Delta = 0 \) and \( k = 0 \),

\[
\therefore l(Al + Bm + Gn) + m(Hl + Bm + Fn) + n(Gl + Fm + Cn) = 0;
\]

\[
A l^2 + 2Hlm + Bm^2 + 2Gl + 2Fmn + Cn^2 = 0.
\]

15.3.3. (i) The pole \( (x_1, y_1) \) of the line \( lx + my + n = 0 \) of general position with respect to the proper conic \( S = 0 \) is given by

\[
x_1 = (Al + Bm + Gn)/(Gl + Fm + Cn),
\]

\[
y_1 = (Hl + Bm + Fn)/(Gl + Fm + Cn).
\]

(ii) If the lines \( lx + my + n = 0 \) and \( l'x + m'y + n' = 0 \) are conjugate lines with respect to the proper conic \( S = 0 \), then

\[
(Al + Bm + Gn) + m(Hl + Bm + Fn) + n(Gl + Fm + Cn) = 0.
\]

(i) Since the form of the equation of the polar of \( (x_1, y_1) \) with respect to \( S = 0 \) is the same as the form of the equation of the tangent, the values of \( x_1, y_1 \) are given by the algebra used in 15.3.2.

By hypothesis, \( (x_1, y_1) \) is the pole of \( lx + my + n = 0 \) with respect to \( S = 0 \); \( x_1, y_1 \) lies on \( l'x + m'y + n' = 0 \);

\[
\therefore l(x_1 + m'y_1 + n') = 0;
\]

by (i) \( (Al + Bm + Gn) + m(Hl + Bm + Fn) + n(Gl + Fm + Cn) = 0 \).

The converse follows by reversing the argument.

15.3.4. (i) The equation of the director circle of the central conic \( S = 0 \) is

\[
C(x^2 + y^2) - 2Gx - 2Fy + (A + B) = 0.
\]

(ii) The equation of the director circle \( S = 0 \) is

\[
2Gx + 2Fy - (A + B) = 0.
\]

(i) By hypothesis, \( C = ab - k^2 = 0 \). If \( T(x_1, y_1) \) lies on the director circle, the tangents from \( T \) to \( S = 0 \) are at right angles.

If \( lx + my + n = 0 \) is a tangent from \( (x_1, y_1) \) to \( S = 0 \),

\[
A l^2 + 2Hlm + Bm^2 + 2Gl + 2Fmn + Cn^2 = 0.
\]

Elimination of \( n \) gives a homogeneous equation of the second degree in \( l \) and \( m \).

\[
A l^2 + 2Hlm + Bm^2 - 2Gl - 2Fmn + Cn^2 = 0.
\]

(i) If the values of \( l/m \) given by this quadratic are \( l_1/m_1, l_2/m_2 \), the gradients of the tangents from \( T(x_1, y_1) \) are \( -l_1/m_1, -l_2/m_2 \),

\[
(-l_1/m_1) \cdot (-l_2/m_2) = 1; \quad (l_1/m_1) \cdot (l_2/m_2) = -1.
\]

The quadratic equation (i) can be written

\[
(A - 2Gx + Cx^2)/m^2 + (B - 2Fy + Cy^2)/m = 0;
\]

\[
\text{product of roots} = (B - 2Fy + Cy^2)/(A - 2Gx + Cx^2);
\]

\[
(A - 2Gx + Cx^2)/(A - 2Gx + Cx^2) = 1.
\]

Therefore the locus of \( T \) is given by \( (B - 2Fy + Cy^2) = (A - 2Gx + Cx^2) \), that is,

\[
C^2 + Cy^2 - 2Fx + A = 0.
\]

(ii) By hypothesis, \( C = ab - k^2 = 0 \). If \( T(x_1, y_1) \) lies on the directrix, the tangent from \( T \) to \( S = 0 \) are at right angles.

Therefore by exactly the same algebra as in (i), with \( C = 0 \), it follows that the locus of \( T \) is given by \( -2Gx - 2Fy + A = 0 \).
15.3.5 (i) The coordinates of the focus of a central conic \( S = 0 \) are given by the equations,
\[
C(x^2 - y^2) - 2Gx + 2Fy + A - B = 0, \quad Cxy - Fx - Gy + H = 0.
\]
(ii) The coordinates of the focus of a parabola \( S = 0 \) are given by
\[
2Gx - 2Fy = A - B, \quad Fx - Gy = H = 0.
\]
(iii) It follows from 10.2, p. 160, and 11.7.1, p. 186, that any pair of perpendicular lines intersecting at a focus \((x_1, y_1)\) are conjugate lines. Therefore the lines \(lx + my = (lx_1 + my_1) = 0\), \(mx - ly = (mx_1 - ly_1) = 0\) are conjugate;
\[
\begin{align*}
&: \text{by 15.3.3}, \quad A1m + H(m^2 - l^2) - Blm + C(lx_1 + my_1)(mx_1 - ly_1) \\
&\quad - G[(lx_1 - ly_1) + m(x_1 + my_1)] - F'(m(x_1 + my_1) - l(x_1 + my_1)) = 0; \\
&\quad (m^2 - l^2)(Ca_1y_1 - Fx_1 - Gy_1 + H) \\
&\quad \quad + \text{Im}(C(x_1^2 - y_1^2) - 2Gx_1 + 2Fy_1 + A - B) = 0.
\end{align*}
\]
Since this is true for all values of \(m, l\), the coordinates of a focus \((x_1, y_1)\) satisfy the equations,
\[
Cxy - Fx - Gy + H = 0, \quad C(x^2 - y^2) - 2Gx + 2Fy + A - B = 0.
\]
(ii) If \(S = 0\) is a parabola, \(C = ab - h^2 = 0\); therefore by 7.5.5, 7.6.7, p. 97, the equations connecting the coordinates of a focus become
\[
-2Gx - 2Fy + A - B = 0.
\]
15.3.6. The following properties of cofactors are important:
If \(A, B, C, F, G, H\) are the cofactors of \(a, b, c, f, g, h\) in \(\Delta\), then the cofactors of \(A, A, B, C, F, G, H, H, A\) in \(\Delta'\) are
\[
\begin{array}{cccc}
A & H & G \\
B & F & G \\
G & F & C
\end{array}
\]
equal respectively \(a\Delta, b\Delta, c\Delta, f\Delta, g\Delta, h\Delta\); also \(\Delta' = \Delta^2\).
By 15.3.1, \(ah + hB + gF = 0\) and \(aG + hF + gC = 0\);
\[
: \text{solving for } a, g, h: f = \frac{ah}{B-C} = \frac{gh}{A-F}.
\]
Also \(BC - F^2 = (ab - h^2) - (gh - af)^2\), \(B = \frac{ab - c - ah}{c - h}\), \(c\Delta = h\Delta, \Delta' = c\Delta\).
\[
A = (BC - F^2) + H(FC - CH) + G(HF - BG) = \Deltaa + \DeltaH = \DeltaG = \Deltaa + \DeltaB + \DeltaH + \DeltaG = \Deltaa.
\]
Similarly, \(AB - H^2 = cB\Delta\).
Further \(\Delta' = (BC - F^2) + H(FC - CH) + G(HF - BG)
\[
\begin{align*}
\Deltaa + \DeltaH = \DeltaG = \Deltaa + \DeltaB + \DeltaH + \DeltaG = \Deltaa.
\end{align*}
\]
If the quadratic function, \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c\), can be factorised, then \(\Delta = 0\); therefore, with a change of letters, if the quadratic function, \(A1^2 + 2H1m + Bm^2 + 2G1n + 2F1m + Cn^2\), can be factorised, \(\Delta' = 0\).

15.3.7 [TANGENT AND POLAR]

Example 2. Prove that the equation of the directrices of the central conic \(S = 0\) is given by \(S + \lambda T = 0\), where
\[
T = C(x^2 + y^2) - 2Gx - 2Fy + (A + B)
\]
and \(\lambda\) is a root of the equation \(\lambda C - \lambda(a + b) + 1 = 0\).
If the axes of reference are changed from \(Ox, Oy\) to the principal axes \(O'X, O'Y\) of the central conic \(S = 0\), then \(k_jS = X'^2 + Y'^2 + p - 1, X'^2 + p, p + 0\), where \(X, Y\) are linear functions of \(x, y\), and \(k_j\) is a constant.
Also by 15.3.4 and by 13.4.8, p. 240,
\[
k_jT = X'^2 + Y'^2 - (x'^2 + p), \quad \text{where } k_j \text{ is a constant}.
\]

Also the equations of the directrices referred to \(O'X, O'Y\) are
\[
X = \pm x\sqrt{(x^2 - p)} = D, \quad \text{that is, } D = (1 - p/x^2)x^2 - a^2 = 0;
\]
\[
\therefore \quad D' = k_jT - pk_jS;
\]
\[
\therefore \quad \text{the equation of the directrices is } S + \lambda T = 0, \text{ referred to axes } Ox, Oy. \quad \text{Since the directrices are parallel, } (a + \lambda C)x^2 + 2(k_j + \lambda )y^2 \text{ is a perfect square};
\]
\[
\therefore \quad (a + \lambda C)(b + \lambda C) = b_1; \quad \therefore \lambda C^2 + \lambda C(a + b) = b - ab = -G;
\]
but \(C = 0\).
\[
\therefore \quad \lambda C^2 + \lambda C(a + b) + 1 = 0.
\]
Note. Since \(a + b)^2 - 4C = (a + b)^2 - 4ab + 4h^2 = (a - b)^2 + 4h^2 \geq 0\), the quadratic, \(\lambda C^2 + \lambda(a + b) + 1 = 0\), has two roots \(\lambda = \lambda_1, \lambda = \lambda_2\).
If the equation of the directrices is \(S + \lambda_1 T = 0\), it is natural to inquire what is the meaning of \(S + \lambda_2 T = 0\).

\(S + \lambda_2 T = 0\) is equivalent to \((1 - p/x^2)x^2 - a^2 = 0\), where the terms of second degree form the perfect square \(x^2\). Similarly, by eliminating \(x^2\), it follows that the terms of the second degree form the perfect square \(y^2\) in the function
\[
p = k_j; S - pk_jT = (x^2 - p)y^2 - p^2; \quad S + \lambda_2 T = 0 \text{ is equivalent to } (x^2 - p)y^2 + p^2 = 0.
\]
Since \(p^2 > 0\) and \(p + 0\), there are no points whose coordinates satisfy \((x^2 - p)y^2 + p^2 = 0\); therefore there is no locus \(S + \lambda_2 T = 0\).

EXERCISE 69
[The standard notation, pp. 260-274, is used in this exercise.]
1. \(P\) is a point on \(L = lx + my + n = 0\); prove the points of contact of the tangents from \(P\) to \(S = 0\) and \(S - L^2 = 0\) are collinear.
2. If \(lx + my + n = 0\) is a variable line such that
\[
A' = Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0
\]
where \(A, B, C, F, G, H\) are constants, prove that the line touches a fixed rectangular hyperbola if \(C(A + B) = F^2 + G^2\).
3. Find the equation of the directrix of \((2x - y)^2 + 22x - 6y + 24 = 0\).
A variable circle, whose centre \((a, b)\) is given, cuts the given conic \(S = 0\) at \(P, Q\); prove that the locus of the mid-point of \(PQ\) is the rectangular hyperbola given by \(v(x-a) - u(y-b) = 0\).

5. The polar of a variable point \(T\) with respect to \(S = 0\) cuts \(S = 0\) at \(P, Q\); \(\angle POQ\) is a right angle; prove that the locus of \(T\) is given by

\[c(x^2 + y^2) - 2(gu + f)u + (a + b)u^2 = 0.\]

6. Prove that the conics, \(x - y = 1\) and \(x - y = -1\), each other and find their other common tangents.

7. Prove that \(lx + my + n = 0\) touches a conic having \((a, b), (a', b')\) as feet if \((la + mb + n)(la' + mb' + n) = \lambda(l^2 + m^2),\) where \(\lambda\) is a constant. What is the geometrical significance of \(\lambda?\)

8. If a variable conic touches the four given lines, \(lx + my + n = 0, px + qy + r = 0,\) prove that its focus lies on the rectangular hyperbola, \((lx + my + n)(px + qy + r) = (px + qy + r)^2 = (a + b)(a' + b')\). This is a corollary of \(l^2 + m^2 = 1\).

9. Prove that an axis \(lx + my + 1 = 0\) of the central conic \(S = 0\) can be expressed in the form \(u + vz = 0,\) where \((a + bt) = (at + b)\) and deduce that \(a + bm + n = bl + m = gl + fm.\)

10. Prove that the conic, \(3x^2 + 3y^2 = 12xy + 24x + 16y + 16,\) has one focus at \(O,\) find the equation of the corresponding directrix.

11. Four given pairs of lines are conjugate with respect to a variable central conic; prove that the centre lies on a fixed line.

12. Prove that \(lx + my + n = 0\) touches a conic inscribed in the triangle whose vertices are \(D(x_1, y_1), E(x_2, y_2), F(x_3, y_3),\)

\[p(x_1 + my_1 + n) + q(x_2 + my_2 + n) + r(x_3 + my_3 + n) = 0,\]

where \(p, q, r\) are given non-zero constants and that the point of contact with \(EF\) divides \(EF\) in the ratio \(q : r\).

13. Discuss the special case of Example 2, p. 275, when the roots of the quadratic, \(\lambda^2 + 2\lambda(a + b) + 1 = 0,\) are equal.

14. If the line \(lx + my + n = 0\) touches the parabola whose focus is \((a, b)\) and whose directrix is \(px + qy + r = 0,\) prove that

\[(px + qy + r)^2 - 2(py + qn) + (a + b) = 0.\]

15. If \(x \cos \psi + y \sin \psi = 1, x \cos \psi + y \cos \psi = p,\) are a variable pair of parallel tangents to the given central conic \(S = 0,\) prove that

\[C(p_1 + p_2) = 2(F \sin \psi + G \cos \psi)\]

and

\[C(p_1 + p_2) = A \cos \psi + B \sin \psi = 2E \sin \psi \cos \psi.\]

(ii) If \(\lambda, \mu\) are constants, the parallel lines

\[x \cos \psi + y \sin \psi = \lambda p_1 + \mu p_2,\]

\[x \cos \psi + y \cos \psi = \lambda p_2 + \mu p_1,\]

touch a fixed conic with asymptotes parallel to those of \(S = 0.\)

16. Interpret with reference to the conic \(S = 0\) the identity

\[(lx + my + n)^2 - (x - x_1)^2 + (y - y_1)^2 = \lambda S.\]

Prove that:

(i) \((l^2 - m^2)(a - b) + lm = \lambda;\)

(ii) if \(lx + my + n = D,\) then \(u/l = v/m = w/n = D/l;\)

(iii) \(A(l^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm) = \lambda;\)

15.4. HOMOGENEOUS COORDINATES

15.4.1. The following two properties are related so that points and lines in either are interchangeable with lines and points in the other; this relation is called duality.

(i) If \(P_1(x_1, y_1)\) is a variable point such that

\[ax_1 + by_1 + c = 0,\]

where \(a, b, c\) are constant, then \(P_1\) lies on a fixed line.

(ii) If \(Q_1, R_1, l_1x + m_1y + n_1 = 0,\) is a variable line such that

\[a_l + b_m + c_n = 0,\]

where \(a, b, c\) are constant, then \(Q_1, R_1\) passes through a fixed point.

Duality becomes more apparent if homogeneous point-coordinates are introduced. The coordinates of a point, instead of being called \((x, y, z),\) can be called \((x/z, y/z)\) where \(z 
 0,\) and the point is then denoted in homogeneous Cartesian coordinates by \((x, y, z)\) where only the ratios \(x : y : z\) are relevant. The equation of a locus becomes homogeneous in \((x, y, z);\) for example, the line \(l(x/z) + m(y/z) + n = 0\) is written

\[lx + my + nz = 0,\]

and the conic

\[a(x/z)^2 + b(y/z)^2 + 2h(x/z)(y/z) + 2f(x/z) + 2g(y/z) + c = 0\]

is written

\[ax^2 + by^2 + cz^2 + 2fyz + 2gzy + 2hxy = 0.\]

If it is desired to obtain an equation between the actual coordinates \((x, y, z),\) it is merely necessary to put \(z = 1,\) but then in general equations cease to be homogeneous and this causes a loss of symmetry which may be undesirable.

15.4.2. The line whose equation is \(lx + my + nz = 0\) is determined by the ratios \(l : m : n.\) Just as \((x, y, z)\) are called homogeneous coordinates of a point, so \([l, m, n]\) are called homogeneous coordinates of a line; in such cases, only the ratios \(x : y : z\) and \(l : m : n\) are relevant, and we speak of the point \((x, y, z)\) and the line \([l, m, n]\) where, to avoid ambiguity, coordinates of a line are enclosed in square brackets.

Thus the variable point \((x, y, z)\) lies on the given line \([l_1, m_1, n_1]\) if

\[l_1x + m_1y + n_1z = 0;\]

the equation \(L_1 = 0\) connects coordinates of collinear points on the line \([l_1, m_1, n_1].\) Also the variable line \([l, m, n]\) passes through the given point \((x_1, y_1, z_1)\) if

\[x_1 = lx + my + nz = 0;\]

the equation \(x_1 = 0\) connects coordinates of concurrent lines through the point \((x_1, y_1, z_1).\)

If a variable point lies on a given curve it is said to trace out the curve, which is then called the locus of the point. If a variable line touches a given curve it is said to envelop the curve, which is then called the envelope of the line. Thus the words locus and envelope are dual terms.
A curve can be determined by the equation which connects the coordinates of points on the curve, called the \textit{point-equation} of the curve; or it can be determined by the equation which connects the coordinates of tangents to the curve, called the \textit{line-equation} of the curve. Thus a given conic may be determined by the point-equation
\[ S = ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0 \]
or by the line-equation, see 15.3.2,
\[ \Sigma = A^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \]
where \(A, B, C, F, G, H\) are the co-factors of \(a, b, c, f, g, h\) in \(\Delta\). In the first case the conic is regarded as a locus and in the second case as an \textit{envelope}.

15.4.3. The use of homogeneous Cartesian coordinates has the further advantage that the introduction of new phrases often makes it unnecessary to qualify general enunciations.

The word 'infinity' has no meaning in isolation, but useful meanings can be assigned to phrases containing it. There is no such thing as a point on \(Ox\) at an infinite distance from \(O\), but a precise meaning can be given to the statement that a point tends to infinity along a line or curve, see p. 111.

The \textit{parallel} lines whose equations are
\[ lx + my + nz = 0, \quad lx + my + n'z = 0, \quad n + n', \]
have no common \textit{point}, but solution of the equations gives
\[ lx + my = 0, \quad z = 0, \quad \text{that is,} \quad x : y : z = m : -l : 0. \]

It is permissible to say that these \textit{parallel} lines meet at the 'point at infinity (\(m, -l, 0\))' and that this point lies on the line whose equation is \(z = 0\), called 'the line at infinity', provided it is understood that these phrases are no more than verbal descriptions of the algebraic relations set out above. The use of such phrases secures generality in the enunciation of many geometrical properties. For example:

(i) Any two lines meet in a point; in particular, two lines parallel to \(Ox\) meet at the point at infinity (1, 0, 0).

(ii) If \(l, m, n\) are not all zero, there is a line whose coordinates are \([l, m, n]\); in particular, \([0, 0, 1]\) is the line at infinity, \(z = 0\).

(iii) The equation of the tangent to \(x^2/a^2 - y^2/b^2 - z^2 = 0\) at \((x_1, y_1, z_1)\) is \(xx_1/a^2 - yy_1/b^2 - zz_1 = 0\); in particular, the tangents at the points at infinity on the hyperbola, \((a, 0, 0)\), \((-a, 0, 0)\), are given by \(x/a - y/b = 0, \; x/a + y/b = 0\), that is, the asymptotes, see 11.3.3, p. 176.

Similarly, the equation of the polar of the centre \(O\) \((0, 0, 1)\) of the conic \(px^2 + qy^2 + z^2 = 0, \; pq + 0\), is \(z = 0\), that is, the line at infinity.

15.4. Example 3. Prove that the polar of the fixed point \(D\) \((f, g, l)\) with respect to the variable central conic, \(ax^2 + 2fxy + by^2 + cz^2 = 0, \; abc + 0\), where \(t\) varies and \(a, b, c\) are constant, passes through a fixed point.

By 15.1.5, the equation of the polar of \(D(f, g, l)\) is
\[ af + tl(gx + fy) + bg + cz = 0; \]
\[ \therefore \text{the point whose coordinates are given by} \]
\[ af + bg + cz = 0 \quad \text{and} \quad gx + fy = 0. \]

lies on the polar of \(D\) for all values of \(t\).

The equations give
\[ x = \frac{y}{-g}, \; \frac{z}{-f}; \quad \text{that is,} \]
\[ x : y : z = -q : q : qf^2 - bg^2. \]

\[ \therefore \text{if} \; af^2 - bg^2 = 0, \text{the polar passes through a fixed ordinary point; and} \]
\[ af^2 - bg^2 = 0, \text{the polar passes through a fixed point at infinity, which means it is fixed in direction.} \]

The condition \(af^2 - bg^2 = 0\) can be interpreted geometrically:

By 15.2.5, if \(E\) is the fixed point \((f, -g, 1)\) and \(af^2 - bg^2 = 0\), the lines \(OE, \; OB\) given by \(x^2 + by^2 + cz^2 = 0\) are conjugate diameters of the conic \(ax^2 + 2fxy + by^2 + cz^2 = 0\) for all values of \(t\); therefore the polar of \(D\) is parallel to the fixed diameter \(OB\).

15.4.4. If the coordinates \([l, m, n]\) of the variable line, \(lx + my + n = 0\), are connected by a homogeneous equation of the second degree,
\[ \Sigma = A^2 + 2Hlm + Bm^2 + 2Fmn + 2Gnl + Cn^2 = 0, \]
there are in general two positions of the line which passes through a given point \(P(x_1, y_1)\) given by the values of \(l : m : n\) which satisfy the equations,
\[ lx_1 + my_1 + n = 0 \quad \text{and} \quad \Sigma = 0. \]

If, however, \(P\) lies on the curve which is enveloped by \(lx + my + n = 0\), the values of \(l : m : n\) are equal.

The values of the ratio \(l : m\) are given by the quadratic
\[ A^2 + 2Hlm + Bm^2 - 2(Fm + Gl)(lx_1 + my_1) + C(lx_1 + my_1)^2 = 0, \]
that is,
\[ l^2(A - 2Cx_1 + Cx_1^2) + m^2(B - 2Fy_1 + Cy_1^2) + 2lm(Fx_1 - Gy_1 + Cy_1)x_1 = 0. \]

Therefore the equation which connects \(x_1, y_1\) if \(P(x_1, y_1)\) lies on the envelope is given by the condition that the roots of this quadratic in \(lm\) are equal.

Alternatively, it has been proved, see 15.3.2, that the line \(lx + my + n = 0\) envelopes the conic whose point-equation is
\[ S = ax^2 + 2hxy + by^2 + 2gxz + 2hxy + cy = 0 \]
if, by 15.3.6, p. 274, the values of \(a, b, c, f, g, h\) are given by
\[ \lambda a = BC - F^2, \lambda b = CA - G^2, \lambda c = AB - H^2, \lambda d = GH - AF, \lambda q = HF - BG, \lambda m = FG - CH, \]
provided that \(\Delta^2 = \Delta' = ABC + 2FGH - AF^2 - BG^2 - CH^2 + 0. \)
THE GENERAL EQUATION

Example 4. The coordinates \([l, m, n]\) of a variable line \(QR, lx + my + n = 0\), satisfy the equation \(2l^2 + 5m^2 - 3n^2 = 0\). Find the envelope of \(QR\).

**First Method.** \(QR\) passes through the point \(P(x_1, y_1)\) if \(l : m : n\) can be chosen so that

\[
lx_1 + my_1 + n = 0
\]

and

\[
2l^2 + 5m^2 - 3n^2 = 0.
\]

This is possible if \(l : m : n\) can be chosen so that

\[
2l^2 + 5m^2 - 3(n_1 + my_1)^2 = 0.
\]

This is a quadratic in \(l : m\):

\[
(2 - 3x_1^2)l^2 - 6x_1y_1ml + (5 - 8y_1^2)n^2 = 0.
\]

The quadratic has no roots if \(9x_1^2y_1^2 < (2 - 3x_1^2)(5 - 8y_1^2)\),

that is, if

\[
15x_1^2 + 6y_1^2 - 10 < 0.
\]

This is the condition for \(P(x_1, y_1)\) to lie inside the ellipse

\[
15x^2 + 6y^2 - 10 = 0,
\]

see Fig. 124.

![Diagram](image)

**Fig. 124**

The quadratic has two unequal roots if \(15x_1^2 + 6y_1^2 - 10 > 0\); therefore two distinct lines satisfying the given conditions pass through \(P(x_1, y_1)\) if \(P\) lies outside the ellipse \(15x^2 + 6y^2 - 10 = 0\).

The quadratic has two equal roots if \(15x_1^2 + 6y_1^2 - 10 = 0\); therefore there is just one line (described as two coincident lines) passing through \(P(x_1, y_1)\) and satisfying the given conditions if \(P\) lies on the ellipse \(15x^2 + 6y^2 - 10 = 0\), and this line touches the ellipse at \(P\). Therefore the envelope of \(QR\) is the ellipse

\[
15x^2 + 6y^2 - 10 = 0.
\]

**Second Method.** With the notation of 15.4.4,

\[
A = 2, B = 5, C = 3, F = G = H = 0;
\]

\[
\lambda \Delta = BC - F^2 = -15, \lambda b = CA - G^2 = -6, \lambda \Delta = AB - H^2 = 10,
\]

\[
\lambda f = GH - AF = 0, 2\lambda g = HF - BG = 0, \lambda \Delta = FG - CH = 0;
\]

\[
\Delta = 0 \text{ and } a : b : c = -15 : -6 : 10 \text{ and } f - g = h = 0;
\]

the equation of the envelope of \(QR\) is

\[
15x^2 + 6y^2 - 10 = 0.
\]

Conversely, if \(lx + my + n = 0\) touches \(15x^2 + 6y^2 - 10 = 0\) at \((x_0, y_0)\),

\[
lx_0 + my_0 + n = 0 \text{ is equivalent to } 15x_0^2 + 6y_0^2 - 10 = 0;
\]

\[
15x^2 + 6y^2 - 10 = 0;
\]

\[
: 2l^2 + 5m^2 - 3n^2 = 0.
\]

This example has been worked so as to illustrate the general methods in 15.4.4, but the result can be written down at once by using 3.4.2, p. 142.

**ENVELOPES**

15.4.5. By 15.4.4, each line belonging to the collection of lines, whose coordinates \([l, m, n]\) satisfy the equation

\[
2l^2 + 5m^2 + 3n^2 = 0
\]

is a tangent to a fixed conic. This collection of lines is called a conic-envelope. \(\Sigma = 0\) is called the equation of the conic-envelope, and the lines are said to envelop the conic.

Similarly by 15.4.2, each line belonging to the collection of lines, whose coordinates \([l, m, n]\) satisfy the equation \(x_1 = xl + y_1m + z_1n = 0\) passes through the fixed point \((x_1, y_1, z_1)\). This collection of concurrent lines is called a point-envelope and \(x_1l + y_1m + z_1n = 0\) is called the equation of the point-envelope; the lines may be regarded as the tangents to the point-circle whose centre is \((x_1, y_1, z_1)\) and whose point-equation is \((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 0\) and so envelop this point-circle. The phrase, equation of a point-envelope, is usually shortened to the form, equation of a point, but this abbreviation fails to suggest that in fact the equation defines a collection of lines through the point.

15.4.6. If the ratios of the coordinates \([l, m, n]\) of a variable line \(lx + my + nz = 0\) are the ratios of functions of the first degree of a parameter \(t\), then the line passes through a fixed point.

By hypothesis, \(l : m : n = (a + at) : (b + bt) : (c + ct)\),

where \(a : b : c\) is not equivalent to \(a' : b' : c'\);

therefore the equation of the variable line can be written

\[
(ax + by + cz) + (a'x' + b'y' + c'z') = 0;
\]

the line passes through the fixed point whose coordinates are given by

\[
ax + by + cz = 0, a'x' + b'y' + c'z' = 0;
\]

that is,

\[
x : y : z = (ax + by + cz) : (a'x' + b'y' + c'z') : (a'x' + b'y' + c'z') = (a'b' - a'b).
\]

If \(a'b' - a'b = 0\), the variable line passes through a fixed ordinary point.

If \(a'b' - a'b = 0\), that is, if \(a'b = a'b\), the variable line passes through a fixed point at infinity, which means the line is fixed in direction.

In either case, the collection of lines is called a point-envelope.

15.4.7. The collection of lines whose coordinates \([l, m, n]\) are given by

\[l : m : n = (a_1t^2 + 2b_1t + c_1) : (a_2t^2 + 2b_2t + c_2) : (a_3t^2 + 2b_3t + c_3)\]

where \(t\) is a parameter, is in general a conic-envelope.

The variable line \(lx + my + nz = 0\) passes through the point \((x_1, y_1, z_1)\) if

\[
lx_1 + my_1 + nz_1 = 2[(a_1x_1 + a_2y_1 + a_3z_1)(a_2x_1 + a_3y_1 + a_2z_1) + 2[(a_2x_1 + a_3y_1 + a_3z_1)(a_3x_1 + a_3y_1 + a_3z_1)] = 0.
\]

Hence as in 15.4.4, \((x_1, y_1, z_1)\) lies on the curve enveloped by \(lx + my + nz = 0\) if the roots of this quadratic are equal; therefore the point-equation of the envelope is

\[
(b_2 + b_3 + b_2b_3) = (a_1x + a_2y + a_3z)(c_1t + c_2y + c_3z).
\]

In general, this is the point-equation of a conic, and the collection of the tangents to the conic is a conic-envelope.
Example 5. If the product of the lengths of the perpendiculars from the fixed points \( H, K \) to a variable line QR is constant, prove the envelope of QR is a conic having \( H, K \) as axes.

Choose rectangular axes so that \( H, K \) are the points \((c, 0), (-c, 0)\). Denote the equation of QR by \( lx + my + n = 0 \), then the lengths of the perpendiculars from \( H, K \) to QR are

\[
\left(\frac{cl}{\sqrt{l^2 + m^2}}\right)\sqrt{l^2 + m^2}, \left(\frac{-cl}{\sqrt{l^2 + m^2}}\right)\sqrt{l^2 + m^2};
\]

\[
\therefore \quad (l^2 + m^2)k = l^2 + m^2, \text{ where } k \text{ is constant.}
\]

Therefore QR envelopes a conic whose envelope-equation is

\[
(k + c^2)x^2 + kmx + n^2 = 0.
\]

The equations of any pair of perpendicular lines through \( H \) are of the form, \( px + qy - pq = 0 \), \( qx + py - qe = 0 \). By 15.3.3, p. 273, these lines are conjugate with respect to the conic because

\[
(k + c^2)pq + k(-q) + (-1)(-pe)(-qe) = 0,
\]

for all values of \( p, q \); therefore, \( H \), and similarly \( K \), is a focus of the conic.

Example 6. \( O \) is a fixed point; \( P \) is a variable point on a fixed line \( AC \). Prove that the envelope of the line \( PQ \) perpendicular to \( PQ \) is a parabola.

Take \( O \) as origin and \( OX \) perpendicular to \( AC \) so that the equation of \( AC \) is \( x = a \), and \( P \) is \((a, t)\) where \( t \) varies.

The equation of \( OP \) is

\[
x(t - a) - y = 0.
\]

The equation of \( PQ \) is

\[
ax + ty = a^2 + t^2.
\]

There are two positions of the line \( PQ \) passing through a given point \( Q(x_1, y_1) \), if there are two unequal roots \( t_1, t_2 \), of the quadratic equation,

\[
t^2 + ty_1 - (a^2 - ax_1) = 0.
\]

Hence as in 15.4.4, \( Q(x_1, y_1) \) lies on the envelope of \( PQ \) if the roots of the quadratic are equal; therefore the envelope of \( PQ \) is the parabola whose point-equation is

\[
y^2 = 4(a^2 - ax).
\]

This result can be obtained geometrically as follows.

Consider the parabola having \( O \) as focus and \( AC \) as tangent at its vertex \( A \); then the foot \( P \) of the perpendicular from the focus \( O \) to any tangent \( PQ \) of this parabola lies on \( AC \); see 7.9.4, p. 97.

EXERCISE 61

1. If \( l : m : n = (t - 2) : (1 - 3t) : 4 \), where \( t \) varies, prove that \( lx + my + n = 0 \) passes through a fixed point; find its coordinates.

[2] Repeat No. 1 if \( l : m : n = (2t + 3) : (3t + 5) : -2 \).

3. Identify the point-envelopes whose equations are:
   (i) \((t - m)^2 - n^2 = 0\);
   (ii) \(l - 3m = 0\);
   (iii) \(2l^2 + 3lm - 2m^2 = 0\);
   (iv) \(m^2 - n^2 = 0\);
   (v) \(6l^2 - lm - 2m^2 - n^2 = 0\).

4. Find the envelope of \( y = mx + a/4 \), where \( m \) varies.

5. Find the envelope of \( xy + y = 1 \) if \( y \) and \( g \) vary subject to the condition \( y + g = 1 \). Integrate the result geometrically.

6. If \( a \), \( b \) are constant and \( t \) varies, find the envelope of:
   (i) \((1 + t^2)x + 2ty + a + 2tyb = 1 + t^4\);
   (ii) \((1 + t^4)x + 4ty + a - 2tyb = 1 - t^4\).

7. Find the \((x, y)\) equation of the envelope of \( lx + my + n = 0 \) if \( l : m : n \) vary subject to:
   (i) \(2l^2 + 3lm - 4n^2 = 0\);
   (ii) \(l^2 - 3lm - 2m^2 = 0\).

8. A variable parabola \( y^2 = 4ax \) meets the fixed parabola \( x^2 = 4ay \) at the origin and at \( P \). Prove that the tangent at \( P \) to \( y^2 = 4ax \) envelopes the parabola \( x^2 + 2ay = 0 \).

9. \( P \) is a variable point on the fixed line \( x + y + a = 0 \). Find the envelope of the perpendicular from \( P \) to its polar with respect to the parabola \( x^2 = 2ay \).

10. \( P, Q \) are variable points and \( K \) is a fixed point on the parabola \( x = at^2 \), \( y = 2at \), given by \( t = p, t = q, t = k \). If \( \angle PKQ \) is a right angle, prove that \( PQ \) passes through the fixed point \( (at^4, 2at^2) \).

11. \( P, Q \) are the feet of the perpendiculars from the fixed points \( A(a, 0) \), \( B(-a, 0) \) to a variable line \( PQ \). If \( AP^2 - BQ^2 = c^2 \), where \( c \) is constant, prove \( PQ \) envelopes \( 4ax + y^2 = c^2(x^2 + 4ax) \).

12. The normals to \( y^2 = 4ax \) at \( P(ap^2, 2ap) \), \( Q(aq^2, 2aq) \) meet at \( R(ar^2, 2ar) \). Prove that \( p, q \) are the roots of \( t^2 + rt + s = 0 \). If \( R \) varies, prove that \( PQ \) passes through a fixed point.

13. \( P, Q \) are variable points \( (ap_1^2, 2ap_1), (aq_1^2, 2aq_1) \) such that \( p_1^2 + q_1^2 + y^2 = k^2 \), where \( a, k \) are constant. Find the envelope of \( PQ \).

14. \( P, Q \) are variable points \( (ap_2^2, 2ap_2), (aq_2^2, 2aq_2) \); \( C \) is the fixed point \((c, 0)\). \( \angle PCQ = 1 \). \( C \), prove the equation of the envelope of \( PQ \) is \( ax^2 + (c + a)y^2 - 2c(x + 2ax) + c^2 = 0 \).

15. \( PP' \) is a diameter of \( x^2 + y^2 = c^2 \); the tangent at \( P \) meets \( Ox, Oy \) at \( Q, R \). Prove that \( PQ = PR \) the envelope \( 2xy + c^2 = 0 \).

16. The equation of a variable conic is \( (p + t)x^2 + (q + t)y^2 = 1 \), where \( t \) varies and \( p, q \) are constant. What can be said about the polar of the fixed point \((h, k)\) if:
   (i) \( p = q; \) (ii) \( p = q \)?

17. The mid-point of a variable chord \( PQ \) of \( px^2 + qy^2 = 1 \) lies on the fixed line \( ax + by - 1 = 0 \). Prove that the equation of the envelope of \( PQ \) is \((bpx - cry)^2 + 4pq(ax + by - 1)^2 = 0 \).

18. A variable chord \( PQ \) of \((x + f)(y + g) = fg - c \) subtends a right angle at \( O \); prove the envelope of \( PQ \) is \((fx - gy)^2 = 4c(fx + fy + c) \).
19. Prove that the envelope of the polar of the fixed point \( (b, k) \) with respect to the conic \( z^2/(a^2+t) + y^2/(b^2+t) = 1 \), \( a^2 + b^2 \), where \( t \) varies is a parabola.

20. \( P \) is a variable point on the ellipse \( z^2/a^2 + y^2/b^2 = 1 \), major axis \( AA' \); \( P, A, P' \) meet the tangents at \( A, A' \) in \( Q, Q' \). Prove that the envelope of \( QQ' \) is \( z^2/a^2 + y^2/b^2 = 1 \).

21. \( (x, y) \) is the focus of a parabola and \( px + qy + r = 0 \) is the tangent at the vertex. Prove that the envelope-equation of the parabola is \( (px_1 + qy_1 + r)(px + qy + r) = (px_1 + qy_1 + r)(pl + gm) \).

15.5. If the ratios of the coordinates \( [l, m, n] \) of a variable line \( lx + my + nz = 0 \) are the ratios of functions of the third or higher degree of a parameter \( t \), the envelope of the line can be found by a method similar to that in 15.4.7 by expressing the condition that two roots of the equation in \( t \) are equal. This is done either by working with symmetrical functions of the roots or by a calculus method.

15.5.1. If the equation of a variable line is

\[ xf(t) + yg(t) + h(t) = 0, \]

where \( f(t) \), \( g(t) \), \( h(t) \) are functions of \( t \), the point of contact of the line with its envelope is its point of intersection with the line

\[ xf'(t) + yg'(t) + h'(t) = 0, \]

where \( f'(t) \), \( g'(t) \), \( h'(t) \) denote the derivatives of \( f(t) \), \( g(t) \), \( h(t) \).

**Fig. 126**

**Fig. 127**

The values \( t = t_1, t = t_1 + 3t \) give two positions of the variable line, intersecting at \( P \) and touching the envelope of the line at \( Q \) and \( Q' \), see Fig. 126; the equations of \( PQ, PQ' \) are

\[ xf(t_1) + yg(t_1) + h(t_1) = 0 \quad \text{.. (i)} \]

\[ xf(t_1 + 3t) + yg(t_1 + 3t) + h(t_1 + 3t) = 0 \quad \text{.. (ii)} \]

therefore \( P \) also lies on the line \( PR \) whose equation is

\[ xf(t_1 + 3t) - f(t_1) + (g(t_1 + 3t) - g(t_1)) + h(t_1 + 3t) - h(t_1) = 0. \]

15.5] Envelopes

In general, the line \( PR \) does not touch the envelope; its limiting position, corresponding to \( t = t_1 \), when \( 3t \rightarrow 0 \), is given by the equation

\[ xf(t_1) + yg(t_1) + h(t_1) = 0 \quad \text{.. (iii)} \]

When \( 3t \rightarrow 0 \), the point \( Q \) is regarded as fixed and the points \( Q', P \) tend to coincidence with \( Q \), and the line \( PQ' \) tends to coincidence with \( PQ \) which touches the envelope at \( Q \), see Fig. 127. Therefore the point of contact \( Q \) of the line given by \( t = t_1 \) with the envelope is determined by the equations (i), (iii) of the tangent at \( Q \) and the limiting position of the line \( PR \).

The coordinates of the point of contact in terms of \( t \) may be regarded as parametric equations of the envelope; the \((x, y)\) equation of the envelope is obtained by eliminating \( t \) from (i) and (iii).

**Example 7.** Find the equation of the envelope of the normal to the parabola \( y^2 = 4ax \) as a variable point \((at^2, 2at)\).

The equation of the normal is \( y + ax = 2at + a^2 \).

**First Method.** By 15.5.1, the coordinates of the point of contact of the normal with its envelope are given by the equations:

\[ \frac{y + ax = 2at + a^2}{x = 2a + 3at^2}, \]

\[ y = -2at^2. \]

The \((x, y)\) equation of the envelope is given by eliminating \( t \),

\[ 27ay^2 = 4(x - 2a)^3. \]

**Second Method.** The normals at the points \( t = t_1, t = t_1 + 3t \) pass through an arbitrary point \((x_1, y_1)\) if \( t_1, t_2, t_2 \) are the roots of

\[ ax^2 + (2a - x_1)a - y_1 = 0. \]

If \( t_1 = t_2 \), then \( t_1 = t_2 = -(t_1 + t_2) = -2t_2 = -2t_2 + 2t_2 \) and \( 2t_2 - 2t_2 = 0 \) and \( y_1/a = -2t_2^2 \). Therefore \( x = 2a + 3at^2, y = -2at^2 \).

8. Find parametric equations of the envelope of the normal to the ellipse \( x^2/a^2 + y^2/b^2 = 1 \) at a variable point \((a \cos \theta, b \sin \theta)\).

The equation of the normal is \( ax \cos \theta - by \sin \theta = a^2 - b^2 \).

The simplest method is to isolate the term in \( y \) and then isolate the term in \( x \) before differentiation with respect to \( \theta \).

Isolate the term in \( y, ax \cos \theta - by = (a^2 - b^2) \sin \theta; \)

therefore

\[ by = a^2 - b^2 \cos^2 \theta : - \sin^2 \theta : 1. \]
EXERCISE 32

Find (i) parametric equations, (ii) the \((x, y)\) equation, of the envelope of the variable line whose equation is given, parameter \(t\).

1. \(y = tx + at^4\).
2. \(x^2 + y^2 = a\).
3. \(x(3 + t^2) + 2yt - 3a = 0\).
4. \(y = tx + at^4\).
5. \((x/a) \sec t + (y/b) \cos t = 1\).
6. \(x \cos t + y \sin t = a\).
7. \(x \sqrt{(\cos t)} + y \sqrt{(\sin t)} = a\).
8. \(x \sec t + y \tan t = a\).

Prove that the equation of the normal to \(x^2/a^2 - y^2/b^2 = 1\) at the point \((a \sec t, b \tan t)\) is \(ax \cos t + by \cot t - a^2 + b^2\) and find the coordinates of its point of contact with its envelope when \(t\) varies.

10. \(PN, PM\) are perpendiculars from a variable point \(P\) on \(x^2/a^2 + y^2/b^2 = 1\) to \(Ox, Oy\). Prove that the envelope of \(MN\) is \((x/a)^2 + (y/b)^2 = 1\).

11. \(P, Q, R\) are points on \(x = at^2, y = 2at\), given by \(t = p, t = q, t = r\); \(PR\) passes through the focus and the normals at \(P, Q\) meet on the curve. Prove: (i) \(pq = 2\) and \(pr = -1\); (ii) if \(p\) varies, the equation of the envelope of \(QR\) is \(y^2 = 3ax\).

12. \(P, Q, R\) are points on \(x = at^2, y = 2at\) given by \(t = p, t = q, t = r\); \(PQ, PR\) are tangents to \(y^2 = 4ax\). Prove that \(a, q, r\) are the roots of \(b^2 + 2b(2a - 2x) + x^2 = 0\). If \(p\) varies, prove \(QR\) envelops \(2x - b)^2 = 4abx^2\).

13. \(Q_1, Q_2, Q_3\) are the feet of the normals from a variable point on the fixed line \(x + 2y = 2a\) to \(y^2 = 4ax\). Prove that the sides of the triangle \(Q_1Q_2Q_3\) touch \((y + 2ax)^2 = 16ax\).

14. Prove that the equation of the normal to the curve \(x = at, y = at^2\) at the point \(t\) is \(x/t + 3yt = 2(t^4 + 1)\) and find parametric equations for its envelope.

15. \(P, Q\) are variable points on \(x = at^2, y = 2at\), given by \(t = p, t = q\). If the point dividing \(PQ\) in the ratio \(k : 1\) lies on \(x^2 = 4by\), prove \(a(k^2 + p^2) = 2(k^2 + p)(k + 1)\). Hence prove that if \(PQ\) is divided harmonically by its points of intersection with \(x^2 = 4by\), the envelope of \(PQ\) is the rectangular hyperbola, \(xy = 2ab\).

16. A variable conic has \(Ox\) as an asymptote and touches \(Oy\) and passes through the fixed point \(A(h, k)\). Prove that the envelope of the other asymptote is a conic whose asymptotes are \(Oy\) and a line parallel to \(Ox\) and touching \(OA\) at \(A\).

17. A variable hyperbola \(xy = p^2\) meets \(y^2(x - 1) = ax^2\) at \(P\). Prove that the normal to \(xy = p^2\) at \(P\) envelopes \(y^2 = 4ax\).

18. \(P, Q, R\) are variable points on \(x : y = 1 = (1 - t^2) : 2bt : (1 + t^4)\) given by \(t = p, t = q, t = r\); \(PQ, PR\) pass through fixed general points \((ka, 0)\), \((kb, 0)\). Prove: (i) \(pq(1 + k) = k - 1, pr(k + 1) = k - 1\); (ii) \(QR\) envelops the curve \(x^2/a^2 + (1 - bk)^2/[1 - (k^2)(1 - k^4)]^2 = 1\).

CHAPTER 16

SYSTEMS OF CONICS

16.1. The following abbreviations will be used in this chapter.

\(L = lx + my + n, L' = lx + my + n', L_1 = lx + my + n + 1, \) etc.

Thus \(L = 0, L' = 0, L_1 = 0\), etc., denote the equations of lines.

\(S = 0, S' = 0, \) denote the equations of proper or degenerate conics.

\(S = ax^2 + 2bxy + by^2 + 2px + 2qy + c, \)

\(S' = a'x^2 + 2b'xy + b'y^2 + 2p'x + 2q'y + c', \) etc.

16.1.1. Since by a change of axes, the equations of a proper conic can be written either as \(px^2 + qy^2 = 1\) or \(y^2 = px\), two proper conics or a proper conic and degenerate conic cannot meet at more than four points. Further, two line-pairs cannot have more than four points in common unless they have a line in common. If they have three points of a line in common, they have all points of that line in common. Therefore if two proper or degenerate conics have just four points in common, no three of these points are collinear and there is only one conic which passes through five points, no four of which are collinear.

16.1.2. If the conics \(S = 0, S' = 0\) meet in just four points \(A, B, C, D\), then:

(i) \(S + kS' = 0, k\) constant, is a conic through \(A, B, C, D); \)

(ii) the equation of any other conic \(S'' = 0\) through \(A, B, C, D\)

is of the form \(S + kS' = 0\).

(i) Since the coordinates of each of the points \(A, B, C, D\) satisfy the equations \(S = 0\) and \(S' = 0\), they also satisfy the equation \(S + kS' = 0\) for all values of \(k\).

\(\therefore\) the conic \(S + kS' = 0\) passes through \(A, B, C, D\).

(ii) Since \(S = 0, S' = 0\) do not meet in more than four points, no three of the points \(A, B, C, D\) are collinear.

Take on \(S' = 0\) a point \(E(x_1, y_1)\) which does not lie either on \(S = 0\), or on \(S' = 0\), then \(S_{11} + S_{11}' = 0\); \(S_{11}' + kS_{11} = 0\); \(S_{11} + kS_{11}' = 0\); \(S_{11} + kS_{11}' = 0\).

\(\therefore\) there is a value \(k, k\) such that \(S_{11} + kS_{11}' = 0\).

\(\therefore\) for this value \(k\) of \(k\), the conic \(S + kS' = 0\) passes through \(E\); also by (i) the conic \(S + kS' = 0\) passes through \(A, B, C, D\).

Further, since no three of the points \(A, B, C, D\) are collinear, no four of the points \(A, B, C, D, E\) are collinear; therefore there is only one conic through \(A, B, C, D, E\), and so the conic \(S + kS' = 0\) is the same as the conic \(S' = 0\).
16.1.3. The special cases of the \(S + kS'\) property in 16.1.2 when \(S' = 0\) is a line-pair and when both \(S = 0\) and \(S' = 0\) are line-pairs are important, see Fig. 128 and Fig. 129.

If the lines \(AB, L_1 = 0,\) and \(CD, L_2 = 0,\) meet the proper conic \(S = 0\) at \(A, B\) and \(C, D,\) then \(S'' = S + kL_1 . L_2 = 0,\) \(k\) constant, is the equation of a conic through \(A, B, C, D,\)

Conversely, the equation of any conic through \(A, B, C, D\) can be expressed in the form, \(S + kL_1 . L_2 = 0,\) \(k\) constant.

![Fig. 128](image)

![Fig. 129](image)

If the proper conic \(S = 0\) is replaced by the line-pair \(L_3 . L_4 = 0,\) this property may be stated as follows:

If the line-pair \(L_3 . L_4 = 0\) meets the line-pair \(L_2 . L_3 = 0\) at \(A, B, C, D,\) then
\[
S'' = L_2 . L_4 + kL_2 . \quad . L_3 = 0, \quad k \text{ constant,}
\]
is the equation of a conic through \(A, B, C, D,\) and conversely.

16.1.4. If the line \(L_2 = 0\) meets the proper conic \(S = 0\) at \(A, B,\) then
\[
S'' = S + kL_1 . L_2 = 0, \quad k \text{ constant,}
\]
is the equation of a conic touching \(S = 0\) at \(A, B,\) and conversely.

If in Fig. 128, \(C\) tends to \(D\) and \(D\) tends to \(B\) along the conic \(S = 0,\) then \(L_4\) tends to coincidence with \(L_1,\) and the limiting position of the conic \(S'' = 0\) touches the conic \(S = 0\) at \(A, B;\) also the limiting form of the equation \(S + kL_1 . L_3 = 0\) is \(S + kL_1 . L_2 = 0,\) see Fig. 130.

The conic \(S'' = S + kL_1 . L_2 = 0\) is said to have double contact with the conic \(S = 0\) and \(L_1 = 0\) is called the chord of contact.

![Fig. 130](image)

![Fig. 131](image)

Similarly, if the line \(L_1 = 0\) meets the line-pair \(L_2 . L_3 = 0\) at \(A, B,\) then
\[
S'' = L_2 . L_3 + kL_2 . L_3 = 0, \quad k \text{ constant,}
\]
is the equation of a conic touching \(L_3 = 0, L_4 = 0\) at \(A, B,\) see Fig. 131, and conversely.

16.1.5. If the line \(AE, T = 0,\) touches the proper conic \(S = 0\) at \(A\) and if the line \(L = 0\) cuts \(S = 0\) at \(C, D,\) see Fig. 132,
\[
S'' = S + kT . L = 0, \quad k \text{ constant,}
\]
is the equation of a conic which touches \(S = 0\) at \(A\) and cuts \(S = 0\) at \(C, D,\) and conversely.

This follows from 16.1.3 by supposing that \(B\) tends to \(A\) along \(S = 0,\) in Fig. 128.

If \(AC, AD, CD\) are the lines \(L_2 = 0, L_4 = 0, L = 0,\) and if \(AE\) is the line \(T = 0,\) see Fig. 133,
\[
S'' = L_2 . L_4 + kT . L = 0, \quad k \text{ constant,}
\]
is the equation of a conic which touches \(AE\) at \(A\) and cuts \(L_2 = 0, L_4 = 0\) at \(C, D,\) and conversely.

This follows from 16.1.3 by supposing that \(B\) tends to \(A\) along \(L_1 = 0,\) in Fig 129.

![Fig. 132](image)

![Fig. 133](image)

![Fig. 134](image)

16.1.6. Three-point Contact. If in Fig. 128, \(B\) tends to \(A\) and \(C\) tends to \(A\) along the proper conic \(S = 0,\) the limiting position of the conic \(S'' = 0\) is said to intersect \(S = 0\) at three coincident points, coincident with \(A,\) and is said to have three-point contact with the proper conic \(S = 0\) at \(A;\) thus the limiting position of \(S'' = 0\) both touches and crosses \(S = 0\) at \(A,\) see Fig. 134.

If the line \(AE, T = 0,\) touches the proper conic \(S = 0\) at \(A\) and if the line \(AD, L = 0,\) meets \(S = 0\) again at \(D,\) see Fig. 134,
\[
S'' = S + kT . L = 0, \quad k \text{ constant,}
\]
is the equation of a conic which has three-point contact with \(S = 0\) at \(A\) and cuts \(S = 0\) again at \(D,\) and conversely.

The values of \(x\) and \(y\) given by the equations
\[
S = 0, S + kT . L = 0
\]
also satisfy \(T . L = 0\) that is, either \(T = 0\) or \(L = 0.\)

The roots of \(T = 0, S = 0\) are the \((x, y)\) coordinates of \(A,\) repeated; the roots of \(L = 0, S = 0\) are the \((x, y)\) coordinates of \(A\) and of \(D.\)
Thus the solution gives the coordinates of \(A\) three times and the coordinates of \(D\) once. \(S'' = 0\) does not meet \(S = 0\) at any other point.
16.1.7. Circle of Curvature. The circle which has three-point contact with a proper conic \( S = 0 \) at \( A \) is called the circle of curvature of the conic at \( A \); its centre is called the centre of curvature and its radius is called the radius of curvature at \( A \).

**Example 1.** Find the coordinates of the centre of curvature of the parabola \( S = y^2 - 4ax = 0 \) at the point \( P(at^2, 2at) \).

The equation of the tangent at \( P(at^2, 2at) \) is \( x - ty + at^2 = 0 \), and \( x + my - at^2 - 2mat = 0 \) is the equation of a line through \( P \).

- the conic \( k(y^2 - 4ax) + (x - ty + at^2)(x + my - at^2 - 2mat) = 0 \)
- has three-point contact with \( S = 0 \) at \( P \), and is a circle if \( 1 = k - mt \) and \( m = 1 \), that is, \( m = 1 \) and \( k = 1 + t^2 \).

- the equation of the centre of curvature at \( P(at^2, 2at) \) is

\[
(1 + t^2)(y^2 - 4ax) + (x - ty + at^2)(x + my - at^2 - 2mat) = 0,
\]

that is,

\[
x^2 + y^2 - 2ax(3t^2 + 2) + 4at^2y - 3at^4 = 0.
\]

- the centre of curvature at \( P \) is the point \( (3at^2 + 2, -2at^2) \).

It should be noticed that this is the point of contact of the normal to the parabola at a variable point \( P \) with its envelope, see Example 7, p. 285. Thus the centre of the circle which has three-point contact with the parabola at \( P \), the centre of curvature at \( P \), is the limiting position of the point of intersection of the normal at \( P \) with the normal at a point \( Q \) when \( Q \) tends to \( P \) along the curve. This is true in general for any curve; the locus of the centres of curvature is called the evolute, and the tangents to the evolute are normals to the original curve.

16.1.8. Four-point Contact. If in Fig. 132, p. 289, \( C \) and \( D \) tend to \( A \) along the proper conic \( S = 0 \), the limiting position of the conic \( S'' = 0 \), which touches \( S = 0 \) at \( A \), is said to intersect \( S = 0 \) at four coincident points, coincident with \( A \), and is said to have four-point contact with the proper conic \( S = 0 \) at \( A \); this limiting position of \( S'' = 0 \) touches \( S = 0 \) but does not cross \( S = 0 \) at \( A \) or meet it again.

If \( AE, T = 0 \), touches the proper conic \( S = 0 \) at \( A \) see Fig. 135,

\[
S'' = S + kT^2 = 0, \quad k \text{ constant},
\]

is the equation of a conic which has four-point contact with \( S = 0 \) at \( A \), and conversely.

The values of \( x \) and \( y \) given by \( S = 0 \), \( S + kT^2 = 0 \) also satisfy \( T^2 = 0 \). The roots of \( T^2 = 0 \), \( S = 0 \) are the \( (x, y) \) coordinates of \( A \) four times, and \( S'' = 0 \) meets \( S = 0 \) at no point except \( A \).

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16.1 INTERSECTING CONICS

**Example 2.** Find the equation of the conic which passes through the five points \( A(1, 2), B(-2, 1), C(-1, -3), D(2, -1), E(4, 4) \).

The equations of \( AB \) and \( CD \) are respectively

\[
x - 3y + 5 = 0 \quad \text{and} \quad 2x - 3y - 7 = 0.
\]

The equations of \( AD \) and \( BC \) are respectively

\[
x + 2y - 5 = 0 \quad \text{and} \quad 2x + y - 7 = 0.
\]

- the equation of any conic through \( A, B, C, D \) can be taken as

\[
S = (x - 3y + 5)(2x - 3y - 7) + (3x + y - 5)(4x + y - 7) = 0;
\]

- \( S = 0 \) passes through \( E(4, 4) \) if \( 3, 11 + k, 11, 27 = 0 \), that is, \( k = -\frac{1}{3} \).

- the equation of the (unique) conic through \( A, B, C, D, E \) is

\[
9(x - 3y + 5)(2x - 3y - 7) - (3x + y - 5)(4x + y - 7) = 0.
\]

**EXERCISE 63**

Describe the systems of conics in Nos. 1–8, where \( k \) varies:

1. \( kxy + (x - y + 1)(2x + y - 2) = 0 \).
2. \( k(x - a)(x - b) + (y - c)(y - d) = 0 \).
3. \( k(y^2 - 4ax) + (x - ty + at^2)(2x - ty + at^2 + 2at, t^2 = 0 \).
4. \( k(x^2 + by^2 - 1) + (axr + byr - 1)(axr + byr - 1) = 0 \).
5. \( k(x^2 + y^2 - 1) + (x \cos \phi + y \sin \phi - b)(x \cos \phi + y \sin \phi - b) = 0 \).
6. \( k(x^2 - c^2) + (x + y)(x - y) = 0 \).
7. \( k(x^2 - y^2)(x^2 - y^2) + (x + y)(x - y)(x + y)(x - y) = 0 \).
8. \( k(x^2 + y^2 - 1) + (x \cos \phi + y \sin \phi - b)(x \cos \phi + y \sin \phi - b) = 0 \).

Find the equation of the conic through the given points, Nos. 9, 10:

9. \( (2, 0), (0, -1), (-3, 0), (0, -4), (-2, -3) \).
10. \( (2, -4), (5, 3), (4, 1), (1, -6) \).

11. Find the equation of the rectangular hyperbola which passes through the common points of \( (2x - y)(y - 5y) = 1 \), \( 7x^2 + 3y^2 = 10 \).

12. Find the equation of the rectangular hyperbola which passes through the points \( (2, 1), (-1, -1), (-1, -2), (3, 3) \).

13. Find the equations of the two parabolas which pass through the points \( (-3, 0), (-1, 0), (1, 2), (2, 6) \).
14. If the conic \( px^2 + qy^2 = 1 \) meets the parabola \( (2x + by)^2 + 2px = 0 \) at \( A, B, C, D \), find the other parabola through \( A, B, C, D \).

15. Find the equation of the parabola which touches \( Ox, Oy \) at their points of intersection with \( x + 2y = 1 = 0 \).

16. Find the equation of the circle which can be expressed in the form \( x^2 + y^2 + 1 + k(x - b)^2 = 0 \). Interpret the result.

17. Find the conditions if the equation of a circle through the origin is of the form,

\[
k(x^2 + y^2 - 1) + (x \cos \phi + y \sin \phi - b)(x \cos \phi + y \sin \phi - b) = 0.
\]
18. Find the conditions if \((ax + by - 1)(cx + dy - 1) + kxy = 0\) is:
   (i) a circle; (ii) a rectangular hyperbola; (iii) a parabola.

19. Find the equation of the circle which passes through the origin and
   the points of intersection of \(x + 2y = 5\) with \(y^2 = x - 3\).

[20] If \(x^2 + y^2 = 1\) meets \((y - a)^2 = 4k(x - b)\) at \(A, B, C, D\), prove
   \(A, B, C, D\) lie on a circle and find its equation.

21. Prove that the common points of the conics
   \[x^2 - 6xy + 5y^2 + 9x - 5y - 10 = 0,\]
   \[7x^2 + 2xy + y^2 - 5x - y - 2 = 0,\]
   lie on a circle and find its equation.

22. Find the equation of the circle through the origin and the points of
   intersection of \(kx + my = a\) with \(y^2 = 4ax\) and show that it cuts the parabola
   again at \((4am^2, 4am^2)\).

23. Prove that the equation of the circumcircle of the triangle formed
   by the lines \(ax^2 + 2hxy + by^2 = 0, x + y = 1\) is
   \[(a + b - 2h)(x^2 + y^2) + 2x + 2by - (a - b - 2h)y = 0.\]

24. A circle touches \(x^2/a^2 + y^2/b^2 = 1\) at \((a \cos \theta, b \sin \theta)\) and passes
   through the focus \(S(a, 0)\) and the centre \(O\), prove \(\sin \theta = \cos \theta\).

25. Find the equations of the three pairs of common chords of
   \[x^2 + y^2 = 17\] and \[x^2 + xy + y^2 = 21.\]

26. Prove that \(x + y = 1 = 0, x + 3y - 1 = 0\) and \(x - y + 1 = 0, x - 2y + 1 = 0\)
   are two pairs of common chords of the two conics,
   \[2xy + y^2 - 4x - y = 0, 2xy + 3y^2 - 7y + 2 = 0.\]

27. Find the equation of the pair of common chords through the origin
   of \(x^2 + 2hxy + by^2 = 1, x^2 + y^2 = r^2\). Find the condition that the pair is a
   coincident line-pair and interpret the answer for
   the ellipse \(8x^2 + 4xy + 5y^2 = 25\) and the circle \(x^2 + y^2 = r^2\).

28. Find the equation of the conic which touches \(Oa, Ob\) at their points
   of intersection with \(ax + by = 1\) and touches \(ax + dy = 1\).

29. The lines joining \(P(x \cos \theta, b \sin \theta)\) on \(x^2/a^2 + y^2/b^2 = 1\) to the points
   \((ak, 0), (-ak, 0)\) meet the ellipse again at \(Q\) and \(R\). Prove that the pole
   of \(QR\) is \(-a \cos \theta, b \sin \theta(k^2 + 1)/(k^2 - 1)\).

30. Two parabolas intersect at four conyclic points; prove that the
   axes of the parabolas are at right angles.

31. Prove that the equation of the locus of the centre of curvature at a
   variable point \(P\) on the parabola \(y^2 = 4ax\) is
   \[4(x - 2a)^2 = 27ay^2.\]

32. \(S\) is the focus of the parabola \(y^2 = 4ax\); prove that the length
   of the radius of curvature at a point \(P\) on \(y^2 = 4ax\) is \(2\sqrt{(SP)^2/a}\).

16.2

16.2.1. If a circle cuts a proper conic \(S = 0\) at \(A, B, C, D\), then \(AB\) and \(CD\) make supplementary angles with a principal axis of \(S = 0\), and conversely.

Take a principal axis as \(x\)-axis, then the origin can be chosen so that either \(S = px^2 + qy^2 - 1\) or \(S = y^2 - 4ax.

Let \(AB, CD\) make angles \(\phi, \phi\) with \(Ox\), where \(\phi, \phi\) have values from \(0^\circ\) to \(180^\circ\), then the equations of \(AB, CD\) are of the form
   \[x \sin \phi - y \cos \phi + c = 0, x \sin \phi - y \cos \phi + d = 0;\]
   therefore since \(A, B, C, D\) lie on a circle, there is a value of \(k\) such that
   \[kS + (x \sin \phi - y \cos \phi + c)(x \sin \phi - y \cos \phi + d) = 0.\]
   This represents a circle. For this value of \(k\), the coefficient of \(xy\) is zero;
   \[\sin \theta \cos \phi + \cos \theta \sin \phi = 0, \sin (\theta + \phi) = 0.\]
   \[\theta + \phi = 180^\circ.\]

Conversely, if \(\theta + \phi = 180^\circ\), the coefficient of \(xy\) in (i) is zero, and the
value of \(k\) can be chosen so that the coefficients of \(x^2\) and \(y^2\) are equal.

16.2.2. If a circle cuts the ellipse \(a^2x^2 + b^2y^2 = 1\) at points whose
eccentric angles are \(\phi_1, \phi_2, \phi_3, \phi_4\), then \(\phi_1 + \phi_2 + \phi_3 + \phi_4\) is a multiple of \(180^\circ\) and conversely.

The equations of the chords joining the pairs of points \(\phi_1, \phi_2\) and
\(\phi_3, \phi_4\) are
   \[(x/a) \cos \frac{1}{2}(\phi_1 + \phi_2) + (y/b) \sin \frac{1}{2}(\phi_1 + \phi_2) = \cos \frac{1}{2}(\phi_1 - \phi_2),\]
   \[(x/a) \cos \frac{1}{2}(\phi_3 + \phi_4) + (y/b) \sin \frac{1}{2}(\phi_3 + \phi_4) = \cos \frac{1}{2}(\phi_3 - \phi_4),\]
   hence, as in 16.2.1,
   \[\cos \frac{1}{2}(\phi_1 + \phi_2) \sin \frac{1}{2}(\phi_3 + \phi_4) + \cos \frac{1}{2}(\phi_1 + \phi_2) \sin \frac{1}{2}(\phi_3 + \phi_4) = 0,\]
   \[\sin \frac{1}{2}(\phi_1 + \phi_2 + \phi_3 + \phi_4) = 0;\]
   \[\frac{1}{2}(\phi_1 + \phi_2 + \phi_3 + \phi_4)\] is a multiple of \(180^\circ\).

The converse follows as in 16.2.1.

Example 3. The line \(x \cos \theta + y \sin \theta = p\) meets \(S = ax^2 + by^2 - 1 = 0\) at
\(K, H\); the circle on HK as diameter, \(S = 0\), meets \(S = 0\) again at \(P, Q\). Find
the equation of \(PQ\).

Since the coefficient of \(xy\) in \(S = 0\) is zero,
\[S = k(x^2 + y^2 - 1) = (x \cos \theta + y \sin \theta)(x \cos \theta - y \sin \theta) = 0;\]
the equation of \(PQ\) is \(x \cos \theta - y \sin \theta = 0.\)

If \((x_1, y_1)\) is the centre of the circle \(S = 0,\)
   \[x_1 : y_1 = \text{ratio of coefficients of } x \text{ and } y \text{ in } S';\]
   \[x_1 : y_1 = - \cos \theta(p - q) + \sin \theta(p + q);\]
    Since \((x_1, y_1)\) is the mid-point of \(HK\), the pole of \((x_1, y_1)\) with respect to
\(S = 0\) is \(x_1, x_1 + y_1 y - 1 = 0,\) is parallel to \(HK, x \cos \theta + y \sin \theta = p = 0,\)
   \[x_1 : y_1 = \cos \theta = 0;\]
   \[x_1 : y_1 = - (p + q) - (p - q) = b - a;\]
   \[q = p(b + a)/(b - a) = 0.\]

   \[\text{the equation of } PQ \text{ is } x \cos \theta - y \sin \theta - p(b + a)/(b - a) = 0.\]
Example 4. If the circle \( S = x^2+y^2-r^2=0 \) has double contact with \( S = ax^2+2hxy+by^2-1=0, h \neq 0 \), prove that the chord of contact \( DE \) is a principal axis of \( S = 0 \), and find its equation and its length. Apply the results to the hyperbola \( 4x^2-6xy-3y^2=1 \).

There is a value of \( k \) such that
\[
(a^2 + 2hax + by^2 - 1) - (lx + my + n) = k^2(x^2 + y^2 - r^2),
\]
where \( l + m + n = 0 \) is the equation of \( DE \).

Equate coefficients, then \( h - m = 0, \therefore l = h, m = 0; \) also \( n = 0, \therefore n = 0; \) and \( k^2 = 1 + n^2 = 1, \therefore r = 1/\sqrt{k} \).

\( DE \) passes through the common centre \( O \) of \( S' = 0 \) and \( S = 0 \).

\( DE \) is perpendicular to the tangents at \( D, E \) to the circle \( S' = 0, \) that is, to the tangents at \( D, E \) to \( S = 0 \). Therefore \( DOE \) is a principal axis of \( S = 0 \).

Further \( (a-k)x^2 + 2hxy + (b-k)y^2 = (l+my+n)^2 = k^2(x^2 + y^2 - r^2), \)

\( \therefore \) since \( l + m + n = 0 \), \( a-k : h = t^2 ; im = n = m \) and \( (a-k)(b-k) = h^2 \).

Therefore the equation of a principal axis is \( (a-k)x + hy = 0 \), where \( k \) is given by \( (a-k)(b-k) = h^2 \); also \( DE = 2r = 2/\sqrt{k} \).

In particular, the equation of a principal axis of \( 5x^2 - 6xy - 3y^2 = 1 \) is
\[
(5-k)x^2 - 3y^2 = 1, \quad \text{where} (5-k)(5-k) = 9,
\]

\( k = 0 \) gives the transverse axis \( x + 3y = 0 \), length \( 2/\sqrt{6} \); there is no value of \( r \) for which \( r^2 = k^2 \).

16.2.3. If two proper or degenerate rectangular hyperbolas, \( S = 0 \), \( S' = 0 \), intersect at \( A, B, C, D \), then every conic through \( A, B, C, D \) is a proper or degenerate rectangular hyperbola.

By 13.2.7, p. 232, if \( S = 0 \), \( S' = 0 \), given by general equations of the second degree, are rectangular hyperbolas, \( a + b = 0 \) and \( a' + b' = 0 \).

The terms of the second degree of the equation \( S + kS' = 0 \) are
\[
(a + ka')x^2 + 2(k+kh')xy + (b + kb')y^2
\]
where \( (a + ka') + (b + kb') = (a + b) + k(a' + b') = 0 \).

\( \therefore \) the conic \( S + kS' = 0 \) is a rectangular hyperbola for all values of \( k \).

16.2.4. If \( A, B, C \) are points on a rectangular hyperbola \( S = 0 \), the orthocentre of the triangle \( ABC \) lies on the curve.

Let the perpendicular from \( A \) to \( BC \) meet \( S = 0 \) again at \( D \); then the line-pair \( AD, BC \) is a degenerate rectangular hyperbola meeting the rectangular hyperbola \( S = 0 \) at \( A, B, C, D \); therefore the line-pair \( AB, CD \) is a degenerate rectangular hyperbola.

\( \therefore CD \) is perpendicular to \( AB; \therefore D \) is the orthocentre of \( \triangle ABC \).
16.3. The following abbreviations are used in this section.

\[ \Sigma = A1^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm, \]
\[ \Sigma' = A'1^2 + B'm^2 + C'n^2 + 2F'm'n + 2G'n'l + 2H'lm, \]

Also \( x_1 = x_1 + y_1 + m + n \) or \( x = x_1 + y_1 + m + z_1 n, \) etc.,
and \( x' = x' + y' + m + n \) or \( x' = x' + y' + n + z'n, \) etc.

With this notation, \( \Sigma = 0, \) \( \Sigma' = 0, \) etc., are the equations of conic-envelopes, which determine the collections of lines touching given conics: also \( x_1 = 0, \) \( x_1 = 0, \) etc., are the equations of point-envelope which determines the collection of lines passing through the given point \( (x_1, y_1), \) or \( (x_1, y_1, z_1), \) and similarly for \( x_1 = 0, \) \( y_1 = 0, \) etc.

16.3.1. If the quadratic function of \( l, m, n, \) denoted by \( \Sigma \) can be factorised, the conic-envelope \( \Sigma = 0 \) is degenerate.

If \( \Sigma = (x_1 + y_1 + m + z_1 n)(x_1 + y_1 + m + z_1 n) = x_1 = 0, \) \( \Sigma = 0 \) is the equation of a point-envelope pair (the dual of a line-locus pair or line-pair) which determines the collection of lines which pass through one or other of the points \( (x_1, y_1, z_1), \) \( (x_2, y_2, z_2). \)

If two tangents can be drawn from the point \( (x_1, y_1, z_1) \) to the proper conic whose envelope-equation is \( \Sigma = 0, \) there are two lines whose coordinates are given by \( \Sigma = 0, x_1 = 0, \) and then the conic-envelope \( \Sigma = 0 \) and the point-envelope \( x_1 = 0 \) have two lines in common.

The duals of the properties in 16.1.1, p. 287, are as follows:

Two proper conic-envelopes \( \Sigma = 0 \) or a proper conic-envelope \( \Sigma = 0 \) and a degenerate conic-envelope \( \Sigma = x_1 = 0, x_2 = 0 \) cannot have more than four lines in common. Further, two point-envelope pairs \( \Sigma = x_1 = 0, \Sigma = x_2 = 0, x_3 = 0 \) cannot have more than four lines in common unless they have a point-envelope in common. Therefore if two proper or degenerate conic-envelopes have just four lines in common, no three of the lines are concurrent; hence five lines, no four of which are concurrent, determine uniquely a conic-envelope which is degenerate if three of the lines are concurrent.

The duals of the properties in 16.1.2, 16.1.3, 16.1.5, pp. 287-289, will now be stated and can be proved by use of dual arguments. It is left to the reader as an exercise to state the duals of 16.1.4, 16.1.6, 16.1.8.

16.3.2. If the conics, whose envelope-equations are \( \Sigma = 0, \Sigma' = 0, \) have just four tangents \( a, b, c, d \) in common, then

(i) \( \Sigma + k\Sigma' = 0, k \) constant, is the envelope-equation of a conic touching \( b, c, d, \)

(ii) The envelope-equation \( \Sigma'' = 0 \) of any other conic touching \( a, b, c, d \) is of the form \( \Sigma + k\Sigma = 0, k \) constant.
16.3.1. If the point-envelope pair \( x_1 \cdot z_2 = 0 \) and the proper conic-envelope \( \Sigma = 0 \) have just four lines \( a, b, c, d \) in common, 
\[
\Sigma'' = \Sigma + k \alpha_1, \quad \alpha_1 = 0, \quad k \text{ constant,}
\]
is the envelope-equation of a conic touching \( a, b, c, d \), see Fig. 137. Consequently, the envelope-equation of any conic touching \( a, b, c, d \) can be expressed in the form, \( \Sigma + kx_1 \cdot z_2 = 0, \) constant.

If the point-envelope pair \( x_1 \cdot z_2 = 0 \) and the point-envelope \( x_3 \cdot z_4 = 0 \) have the lines \( a, b, c, d \) in common, see Fig. 138,
\[
\Sigma'' = \alpha_2 + k \alpha_4, \quad \alpha_4 = 0, \quad k \text{ constant,}
\]
is the envelope-equation of a conic touching \( a, b, c, d \), and conversely.

**Fig. 137**

It is left to the reader to consider the limiting forms of these results when \( x_3 \) tends to \( x_1 \), that is, when \( (x_2, y_2, z_2) \) tends to coincidence with \( (x_1, y_1, z_1) \), which form the duals of the properties in 16.1.4.

16.3.4. If the proper conic-envelope \( \Sigma = 0 \) and the point-envelope \( x_1 = 0 \) have just one line \( a \) (two coincident lines) in common and if \( \Sigma = 0 \) and the point-envelope \( x_3 = 0 \) have the lines \( c, d \) in common, see Fig. 139,
\[
\Sigma'' = \Sigma + k \alpha_1, \quad \alpha_1 = 0, \quad k \text{ constant,}
\]
is the envelope-equation of a conic touching \( a \) at \( (x_1, y_1, z_1) \) and touching the lines \( c, d \), and conversely.

**Fig. 139**

If the point-envelope pair \( x_3 \cdot z_4 = 0 \) and the point-envelope \( x_2 = 0 \) have just one line \( a \) (two coincident lines) in common and if \( x_3 \cdot z_4 = 0 \) and the point-envelope \( x_2 = 0 \) have the lines \( c, d \) in common, see Fig. 140,
\[
\Sigma'' = \alpha_3 + k \alpha_4, \quad \alpha_4 = 0, \quad k \text{ constant,}
\]
is the envelope-equation of a conic touching \( a \) at \( (x_1, y_1, z_1) \) and touching the lines \( c, d \), and conversely.

**Fig. 140**

16.4. The notation for the envelope-equation \( \Sigma = 0 \) corresponding to the notation on p. 266 for the point-equation \( S = 0 \) is obtained by replacing \( x, y, z \) by \( l, m, n \) and the coefficients \( a, b, c, f, g, h \) by their co-factors \( A, B, C, F, G, H \) in \( \Delta \). Thus, by 15.3.3 (ii), p. 273,
\[
\Sigma_{12} = A_1l_2 + B_1m_2 + C_1n_2 + F(l_2m_2 + m_2n_2) + G(n_2 l_2 + n_2 l_2) + H(l_2m_2 + l_2n_2) = 0
\]
is the condition that \( [l_1, m_1, n_1], [l_2, m_2, n_2] \) are conjugate lines with respect to the conic whose envelope-equation is \( \Sigma = 0 \), and
\[
\Sigma_{11} = A_1l_2 + \ldots + 2Fm_1n_1 + \ldots = 0
\]
means that the line \( [l_1, m_1, n_1] \) touches the conic \( \Sigma = 0 \).

Also, by 15.3.2, 15.3.3, pp. 272, 273,
\[
\Sigma_1 = \Sigma(A_1l_1 + B_1m_1 + C_1n_1) + m(0 + B_1m_1 + C_1n_1) + n(0 + C_1n_1 + A_1l_1) + 0 = 0
\]
is the equation of the point-envelope which determines the contact point of \( [l_1, m_1, n_1] \) if \( [l_1, m_1, n_1] \) touches the conic \( \Sigma = 0 \) and determines the pole of \( [l_1, m_1, n_1] \) if \( [l_1, m_1, n_1] \) is a chord of the conic \( \Sigma = 0 \).

**Example 6.** If \( [l_1, m_1, n_1], [l_2, m_2, n_2] \) are chords \( P_1Q_1, P_2Q_2 \) of a conic whose envelope-equation is \( \Sigma = 0 \), prove that the tangents \( T_1P_1, T_1Q_1, T_2P_2, T_2Q_2 \), and the chords \( P_1Q_1, P_2Q_2 \) touch a conic whose envelope-equation is
\[
\Sigma_{12} = \Sigma_1 - \Sigma_2 = 0
\]
By 16.4, the equations of the point-envelopes determined by the points \( T_1 \) and \( T_2 \) are \( \Sigma_1 = 0 \) and \( \Sigma_2 = 0 \);

\[
\Sigma'' = \Sigma + k \Sigma_1, \quad \Sigma_1 = 0
\]
is the envelope equation of a conic which touches \( T_1P_1, T_1Q_1, T_2P_2, T_2Q_2 \). Further \( P_1Q_1, [l_1, m_1, n_1] \) is a tangent of \( \Sigma'' = 0 \) if \( k \) is chosen so that
\[
\Sigma_{11} + k \Sigma_{12} = 0, \quad \Sigma_{12} = 0, \quad \Sigma_2 = 0; \quad \Sigma'' = 0; \quad \Sigma_{11} = 0,
\]
is a tangent. By 16.3.3, for each value of the constant \( k \),
\[
\Sigma'' = \Sigma + k \Sigma_1, \quad \Sigma_1 = 0
\]
is the envelope equation of a conic which touches \( T_1P_1, T_1Q_1, T_2P_2, T_2Q_2 \). Further \( P_1Q_1, [l_1, m_1, n_1] \) is a tangent of \( \Sigma'' = 0 \) if \( k \) is chosen so that
\[
\Sigma_1 + k \Sigma_{12} = 0, \quad \Sigma_2 = 0, \quad \Sigma_3 = 0; \quad \Sigma'' = 0.
\]
by symmetry \( P_2Q_2 \) is also a tangent.

**Example 7.** The conics of a system touch four given lines. Prove that the director circles of the conics form a coaxial system.

Let the equation of two conics of the system be denoted by \( \Sigma = 0 \) and \( \Sigma' = 0 \); then the envelope-equation of any conic of the system can be expressed in the form \( \Sigma + k \Sigma' = 0 \), where \( k \) is a constant.

By 16.3.4, p. 273, the point-equation of the director circle of \( \Sigma + k \Sigma' = 0 \) is
\[
(Cx^2 + Gy^2 - 2Gx - 2Cy + A + B) + k(C''x^2 + C''y^2 - 2G''x - 2G''y + A' + B') = 0
\]
When \( k \) varies, this gives the equations of the circles of a coaxial system.

The radius axis of the system is given by the value of \( k \) for which \( C + kC'' = 0 \).
EXERCISE 65

Interpret the envelope-equation for a system of conics, when \( k \) varies, in Nos. 1-4:

1. \( (l - 2m + n)(2l + m + n) + (k - 2m - n)(l + 3m - n) = 0. \)
2. \( (l - an)(l - bn) + k(m - an)(m - bn) = 0. \)
3. \( (l - 2n)(m - 3n) + k(l + m + n)^2 = 0. \)
4. \( (2l + n)(3l + n) + k(4m + n) = 0. \)

[5] Find the envelope-equation of the conic which touches the five lines \( x + 2y + 1 = 0, \ 2x - y - 1 = 0, \ 2x - y + 1 = 0, \ x + 3y - 1 = 0, \ 4x + 4y + 1 = 0. \) \[Use \ \text{Ex. 3, p. \( 39 \) and dual of method for Example 5, p. \( 291 \).}\]

6. Find the envelope-equation of the parabola which touches the four lines \( x + 2y = 7, \ 2x - y = 4, \ 3x + 2y = 20, \ 4x - y = 1. \)

[7] Prove that the envelope-equation of the conic which touches \( Ox, Oy \) at their points of intersection with \( ax + by = 1 \) and touches \( cx + dy = 1 \) is \( (l + an)(m + bn) - (a-c)(b-d)n^2 = 0. \)

8. Prove that: (i) the coordinates of the centre of the conic given by the general envelope-equation \( \Sigma = 0 \) are \( x = GC, \ y = F/C; \) (ii) the centres of conics which touch four given lines are collinear.

[9] The envelope-equations of two given conics are

\[ \Sigma = px^2 + qy^2 + rn^2 = 0, \ \Sigma' = p'x^2 + q'y^2 + r'n^2 = 0 \]

and a system of conics is determined by \( \Sigma + k\Sigma' = 0 \), for any value of \( k \).

Prove that: (i) the poles of the line \( ax + by + c = 0 \) with respect to conics of the system lie on the line \( \frac{ax' + by'}{a + b} = \frac{cy' + dy'}{c + d} = 0; \) (ii) the poles of the point \( (a, b, c) \) touch the conic given by the envelope-equation, \( (aq - bq')(r - r') + c(p - p')/b = 0; \)

10. A variable chord \( PQ \) of \( S = ax^2 + 2kxy + by^2 + 2xy + 2y + c = 0 \) subtends a right angle at the origin \( O; \) prove that \( PQ \) touches the conic given by the envelope-equation

\[ \Sigma' = (l + m)^2 - 2pl - 2km + (a + b)n^2 = 0. \]

If \( S = 0 \) is a variable conic through four given points, prove that \( \Sigma' = 0 \) touches four fixed lines.

11. \( OAB \) is an asymptote of each of two hyperbolas; \( CA, OB \) are their other common tangents. Show that rectangular axes \( Ox, Oy \) can be chosen so that values \( k = k_1, k = k_2 \) in the envelope-equation

\[ k(l + m) + (a + n)(b + n) = 0 \]

determine the hyperbolas. Deduce that \( C \) lies on the radical axis of their director circles.

12. \( ABC \) is a given triangle; \( H, K \) are given general points. \( \Sigma = 0, \ \Sigma' = 0 \) are the envelope-equations of two conics which touch \( HC, KA \) and \( KB, KA \) respectively. Prove that the common tangents which do not pass through \( A \) meet on \( BC \).

GEOMETRICAL PROPERTIES

16.5.1. A variable conic passes through four given points \( A, B, C, D \) of general position. The pairs of sides \( AB, CD \) and \( AC, BD \) and \( AD, BC \) of the quadrangle \( ABCD \) meet in \( O, E, F \). Prove that the locus of the centre of the conic is a conic passing through \( O, E, F \) and the mid-points of the six sides of the quadrangle \( ABCD \).

Take \( OAB, OCD \) as oblique axes \( Ox, Oy \) and the points \( A, B, C, D \)

as \( (1/a, 0, 0), (1/b, 0, 0), (0, 1/c, 0), (0, 1/d). \) Then the equations of \( AC, BD \)

are \( ax + cy = 1, bx + dy = 1 \), and so the equation of the variable conic is of the form \( S = kxy + (ax + cy - 1)(bx + dy - 1) = 0. \)

Therefore the centre of \( S = 0 \) is given by the equations,

\[ \frac{\partial S}{\partial x} = ky + a(bx + dy - 1) + b(ax + cy - 1) = 0, \]

\[ \frac{\partial S}{\partial y} = kx + c(bx + dy - 1) + d(ax + cy - 1) = 0. \]

Hence, eliminating \( k \), the equation of the centre-locus is

\[ S' = (ax - cy)(bx + dy - 1) + (bx - dy)(ax + cy - 1) = 0. \]

The centre-locus \( S' = 0 \) cuts \( y = 0 \) where \( ax(bx - 1) + bx(ax - 1) = 0 \),

that is, at \( O(0, 0) \) and at \( \{1/(a + 1/b), 0\} \), the mid-point of \( AB \).

Hence by symmetry the centre-locus also passes through \( E, F \) and through the mid-points of the other five sides of the quadrangle.

16.5.2. (Pascal's Theorem). If \( ABCDEF \) is a hexagon whose vertices lie on a proper conic \( S = 0 \), then the points of intersection \( L, M, N \) of pairs of opposite sides, \( AB, DE \)

and \( BC, EF \) and \( CD, FA \) respectively, are collinear.

The following notation is convenient:

If \( lx + my + n = 0 \) is the equation of any line \( PQ \), then \( (PQ) \) means a selected multiple of \( lx + my + n \).

Multiples can be selected so that \( S = (AD) \cdot (EF) + (AF) \cdot (ED) \)

and

\[ S = (AD) \cdot (BC) + (AB) \cdot (CD); \]

\[ \cdot (AD), (EF) - (AD), (BC) = (AB), (CD) - (AF), (ED) = S', \] say.

\[ S' = (AD) \cdot (EF) - (BC), (EF) - (BC) = 0 \]

is a line-pair formed by the line \( AD \) and a line through the point of intersection \( M \) of \( EF, BC \).

\[ S' = (AD) \cdot (CD) - (AF), (ED) = 0 \]

is a line-pair formed by the line \( AD \) and a line through \( M; \) but this line-pair passes through the points \( A, L \) where \( AB \) meets \( AF, ED \) and through the points \( N, D \) where \( CD \) meets \( AF, ED \), and so it is the line-pair \( AD, LN \);

\[ \cdot \]
EXERCISE 66

1. A circle touches \( x = a^2, y = 2ax \) at \( P \) and cuts it again at \( Q, R \) and passes through the focus \( S(a, 0) \); \( P, Q, R \) are the points \( t = p, t = q, t = r \). Prove: (i) \( \sqrt{t} = 2), q + r = 2p \); (ii) \( \sqrt{t} \cdot \sqrt{r} = 4a \cdot \sqrt{r} \).

2. \( A, B, C, D \) are fixed points on a given conic; \( PH, PK, PM, PN \) are the perpendiculars from a variable point \( P \) on the conic to \( AB, CD, AD, BC \), respectively. Prove that \( PH \cdot PK : PM \cdot PN \) is constant.

3. A circle \( S = 0 \) has double contact with \( S = ax^2 + by^2 - 1 = 0 \) at \( H, K \); \( P \) is a variable point on \( S = 0 \). Prove that the tangent from \( P \) to \( S = 0 \) varies as the distance of \( P \) from \( HK \).

4. A circle of variable radius \( r \) passes through the foci \((\pm a, 0)\) of \( S = \frac{x^2}{a^2} + \frac{y^2}{(a^2-1)} - a^2 = 0 \) and meets \( S = 0 \) at \( M, N, P, Q \), where \( MN \) is parallel to \( PQ \); \( MN, PQ \) meet \( X = 0 \) at \( H, K \). Prove that, if \( O \) is the origin: (i) \( OH \cdot OK \) is constant; (ii) \( HK = 2r(1 - \frac{1}{a^2}) \).

5. Two parabolas touch \( Ox \) at \( 0 \) and cut again at \( P, Q \). Prove that \( OP, OQ \) are separated harmonically by the lines through \( O \) parallel to the axes of the parabolas.

6. A variable circle through the origin touches \( S = ax^2 + by^2 - 1 = 0 \) at \( P \) and cuts it again at \( Q, R \); prove that the locus of the pole of \( QR \) with respect to \( S = 0 \) is \( (a^2x^2 + b^2y^2)z = (a - b)(a^2x^2 + b^2y^2) \).

7. \( A, B, C, D \) are fixed points on a circle; prove that the axis of a variable conic through \( A, B, C, D \) have fixed directions.

8. \( PP' \) is a chord of \( S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \) perpendicular to \( Ox \); \( QT \) is the tangent from a point \( Q \) on \( S = 0 \) to the circle which touches \( S = 0 \) at \( P \) and \( P' \); \( QK \) is the perpendicular from \( Q \) to \( PP' \). Prove that \( QT = c \), \( QK \).

9. The lines \( L_1 = 0 \) and \( L_2 = 0 \) meet the conic \( S = 0 \) at \( P_1, Q_1 \) and \( P_2, Q_2 \); the conics \( S = 0, S' = 0 \) touch \( S = 0 \) at \( Q_1, Q_2, P_1, P_2 \), respectively. Prove that \( P_1Q_1, P_2Q_2 \) and one pair of common chords of \( S = 0, S' = 0 \) are concurrent. State the dual property.

10. A circle has double contact with \( y^2 = 4ax \) and meets \( y = 0 \) at \( H, K \); the perpendiculars at \( H, K \) to \( y = 0 \) meet \( y^2 = 4ax \) at \( H', K' \). Prove that \( HH' + KK' = HK \).

11. A variable conic \( S = 0 \) passes through the fixed points \( A (a, 0), B (-a, 0) \), and has double contact with a given conic \( S = 0 \) at \( P \) and \( Q \). Prove that \( PQ \) passes through one of two fixed points \( E, F \) which separate \( A, B \) harmonically.

GEOMETRICAL PROPERTIES

12. Two conics cut at \( A, B, C, D \); explain why their equations \( S = 0 \) and \( S' = 0 \) can be taken so that \( S - S' = L_1, L_2 \), where \( L_1 = 0, L_2 = 0 \) represent lines. Prove that the conic \( k(S - S') - \frac{1}{2}(kL_1^2 + L_2^2) = 0 \) has double contact with \( S = 0 \) and \( S' = 0 \). Find the chords of contact.

13. A proper conic \( S = 0 \) touches the sides \( BC, CA, AB \) of \( \triangle ABC \) at \( P, Q, R \). Prove that with the notation of 16.5.2, p. 301, the function \( S \) can be expressed as \( (PQ) \cdot (AB) + (QR) \cdot (PC) \). Deduce that the points of intersection of \( QR, RP, PQ \) with \( BC, CA, AB \) respectively are collinear.

14. A parabola touches rectangular axes \( Ox, Oy \) at \( (a, 0), (0, b) \). Prove that its axis is parallel to \( bx - ay = 0 \) and deduce that its directrix is \( ax + by = 0 \) and its focus is \((\frac{a^2b}{a^2 + b^2}, \frac{ab^2}{a^2 + b^2})\).

15. If \( D \) is the orthocentre of \( \triangle ABC \), prove that each conic through \( A, B, C \) is a rectangular hyperbola and that the centre of the conics is the nine-point circle of \( \triangle ABC \).

16. If the conic \( S = ax^2 + 2kxy + by^2 + 2px + 2qy = 0 \) meets the \( A, B, C \) and the conic \( S' = ax^2 + 2kyx + bx^2 + 2fy + 2gy + 0 \), prove that the conic \( a(x^2 - 2(y + f)) = x(2y + f)/2 \), \( (2x + by)(ax - 2y) = 0 \) passes through \( A, B, C \) but not through \( O \). Prove also that any conic through \( A, B, C \), not passing through \( O \), is given by \( S + kS - S' = 0 \).

17. If the conic \( ax^2 + by^2 + x = 0 \) meets \( px^2 + qy^2 = 0 \) at \( A, B, C \) and \( O \), find the equation of the circle \( ABC \).

18. Find the condition that \( ax^2 + 2kxy + by^2 + 2px + 2qy + c = 0 \) meets \( Ox, Oy \) at four concyclic points \( A, B, C, D \). If the condition is satisfied, find the equation of the circle \( ABC \).

19. Use the notation and method which is the dual of that used for Pascal's theorem in 16.5.2, p. 301, to prove Brianchon's theorem: If the sides of the hexagon \( abedf \) touch a proper conic \( \Sigma = 0 \) the joins of pairs of opposite vertices \( ab, de, bc, ef \) are concurrent.

20. The equation of a conic referred to conjugate diameters \( POP' \), \( DOD' \) as oblique axes \( Ox, Oy \) is \( S = ax^2 + by^2 - 1 = 0 \); \( QR, QR' \) are chords of \( S = 0 \) cutting \( POP' \) at \( H, H' \), where \( HO = HH' \). \( T \) is the pole of \( RR' \). Prove that the mid-point of \( QT \) lies on \( DOD' \).

21. A variable circle passes through the origin \( O \) and cuts \( S = x^2 + y^2 - a^2 = 0 \) at \( M, N, P, Q \). If \( MN \) passes through the fixed point \( (b, k) \), prove that the envelope of \( PQ \) is given by \( \Sigma = a^2(b^2 + 2km) - 2bk + 2km = 0 \).

22. \( H, P, Q, R \) are conyclic points on an ellipse; the three circles through \( H \) which touch the ellipse at \( P, Q, R \) respectively meet it again at \( P', Q', R' \). Use 16.2.2, p. 293, to prove that \( H, P', Q', R' \) are conyclic.

23. Three conics have double contact with a fourth conic. Prove that six of their common chords are the sides of a quadrangle.
16.6. Concurrent Normals. The feet $P_1, P_2, P_3, P_4$ of the normals from a point $G(h, k)$ to the central conic $S = px^2 + qy^2 - 1 = 0$ lie on the rectangular hyperbola, $S' = (p - q)y - px + kq = 0$. A method for finding the chord $P_3P_4$ when the chord $P_1P_2$ is given was described in Example 5, p. 285.

16.6.1. (i) If $\phi_1, \phi_2, \phi_3, \phi_4$ are the eccentric angles of the feet $P_1, P_2, P_3, P_4$ of the normals from $G(h, k)$ to $S = px^2 + qy^2 - 1 = 0$, then $\phi_1 + \phi_2 + \phi_3 + \phi_4$ equals an odd multiple of $180^\circ$.

(ii) If the circle $P_1P_2P_3$ meets $S = 0$ again at $Q$, then $QP_4$ is a diameter of $S = 0$.

(i) The equations of $P_1P_2, P_2P_3, P_3P_4$ are

$L = (x/a) \cos \frac{1}{3}(\phi_1 + \phi_2) + (y/b) \sin \frac{1}{3}(\phi_1 + \phi_2) - \cos \frac{1}{3}(\phi_1 - \phi_2) = 0,$

$L' = (x/a) \cos \frac{1}{3}(\phi_3 + \phi_4) + (y/b) \sin \frac{1}{3}(\phi_3 + \phi_4) - \cos \frac{1}{3}(\phi_3 - \phi_4) = 0;$

$t$ can be chosen so that $t(x/a^2 + y/b^2 - 1) + L + L' = 0$ is equivalent to $xy(1/a^2 + 1/b^2) - kx/a^2 + hy/b^2 = 0;$

$t(x/a^2 + 1/a^2) \cos \frac{1}{3}(\phi_1 + \phi_2) \cos \frac{1}{3}(\phi_3 + \phi_4) = 0,$

and $t(x/b^2 + 1/b^2) \sin \frac{1}{3}(\phi_1 + \phi_2) \sin \frac{1}{3}(\phi_3 + \phi_4) = 0;$

$
\cos \frac{1}{3}(\phi_1 + \phi_2) \cos \frac{1}{3}(\phi_3 + \phi_4) - \sin \frac{1}{3}(\phi_1 + \phi_2) \sin \frac{1}{3}(\phi_3 + \phi_4) = 0,$

$
\cos \frac{1}{3}(\phi_1 + \phi_2) + \phi_3 + \phi_4 = \text{odd multiple of } 180^\circ.$

(ii) Let $\theta$ be the eccentric angle of $Q$.

Then by 16.2.2, p. 293, $\phi_1 + \phi_2 + \phi_3 + \phi_4 = 0$ is even multiple of $180^\circ$;

$\theta = \phi_4 - 180^\circ$; $\quad QP_4$ is a diameter of $S = 0$.

Note. The property (ii) holds also for a hyperbola, see Exercise 67, No. 7.

Example 8. $P_1, P_2, P_3, P_4$ are the feet of the normals to the central conic $S = px^2 + qy^2 - 1 = 0$ from the point $P_1(x_1, y_1)$ on $S = 0$. Prove that the equation of the circle $P_1P_2P_3P_4$ is $p(x^2 + y^2) - p^2x + x - py - qy = 0$.

The feet of the normals from $(x_1, y_1)$ to $S = 0$ lie on the hyperbola $S' = (p - q)y - px + kq = 0$, that is, $p(x_1x + y_1y) = q(x_1y - y_1x)$; since $(x_1, y_1)$ lies on $S = 0$, the equation of $S = 0$ can be written $px^2 + qy^2 - px_1x - qy_1y = 0$, that is, $p(x - x_1)(x + x_1) = -q(y - y_1)(y + y_1)$; the points common to $S = 0, S = 0$, excluding $P_1(x_1, y_1)$, satisfy the equation $p^2x + px_1x = -q^2y + qy_1y$.

For all values of $h, k$, the points $P_1, P_2, P_3, P_4$ lie on the locus $k(px_1x + qy_1y - 1) - [p^2x + px_1x + q^2y + qy_1y] = 0$.

This equation represents a circle if $k$ is chosen so that $p^2x + qy^2 - 1 = 0$, that is, $k = p + q$.

The equation of the circle $P_1P_2P_3P_4$ is

$$(p + q)(px^2 + qy^2 - 1) - (p^2x + qy^2 + p^2x + px_1x + q^2y + qy_1y) = 0,$$

that is, $p^2x + qy^2 - p^2x + qy_1y = 0$.

EXERCISE 67

1. Prove that the feet of the normals to $xy = c^2$ from $(h, k)$ lie on $S'(x_1x - y_1y - kx + ky = 0$ and are the feet of the normals from $(-\frac{1}{2}h, -\frac{1}{2}k)$ to $S' = 0$ if $h^2 = 4c^2$.

2. $F, H, K$ are the feet of the normals to $xy = c^2$ from a point $G$; $EF$ is the line $kx + my + c = 0$; prove that $HK$ is the line $mx - ly + c(l^2 - m^2) = 0$ and find the coordinates of $G$.

3. $E, F, H, K$ are the feet of the normals from a point $G$ to the hyperbola $x = at, y = ct$; prove that the centroid of the triangle $EFG$ lies on the diameter parallel to $GE$.

4. A circle of variable radius with fixed centre $(3a, 3b)$ cuts the hyperbola $x = at^2, y = 3bt$ at $P_1, P_2, P_3, Q$, where $Q$ is given by $t = a$. Prove the centroid $G$ of $\Delta P_1P_2P_3$ is $(3a - kq, 2b - kq)$ and find the locus of $G$.

5. The normals to $ax^2 + by^2 = 1$ at $P_1, P_2, P_3, P_4$ are concurrent. If $P_1P_2$ is a variable chord through the fixed point $c, d$, prove that $P_1P_3$ envelopes the parabola $(ax + by + 1)^2 = 4abcy$.

6. If the normals to $px^2 + qy^2 = 1$ at $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are concurrent, prove that $\Delta x_1y_2 + \Delta x_2y_1 = \Delta x_3y_4 + \Delta x_4y_3 = 0$.

7. If the normals to $x/a - y/b = 1$ at $t_1, t_2, t_3, t_4$ are concurrent and the circle $t_1t_2t_3t_4$ meets the hyperbola again at $t_5, t_6, t_7, t_8$, prove: (i) $t_1t_2t_3t_4 = -1$ and $t_1t_2t_3t_4 = 0$; (ii) $t_1 = t_4$ and the chord $t_1t_2$ is a diameter of the hyperbola.

8. $C, D, E, F$ are the feet of the normals to $ax^2 + by^2 = 1$ from $(h, k)$; $(c, d)$ and $(e, f)$ are the mid-points of $CD$ and $EF$. Prove that $h = (c - a)(e + f)/5$ and $k = (a - b)(d + f)/5$.

9. The normals to $ax^2 + by^2 = 1$ at variable points $P_1, P_3$ meet at $G$; $P_4P_5$ is parallel to the fixed line $x + cy = 0$. Prove that the locus of $G$ is $ab(x + cy)(x - y) = c(a - b)(a^2 - b^2)/(a^2 + b^2)$.

10. Find the equation of the circle which passes through the three points of intersection of $y = x$ and $(x - c)(y - b) = 0$.

11. $P_1, P_2, P_3, P_4$ are points on $x = ct, y = ct$; given by $t = t_1, t_2, t_3, t_4$. Prove that, if $t_5 = 1/(t_1t_2t_3), t_6 = 1/(t_1t_2t_4), t_7 = 1/(t_2t_3t_4), t_8 = 1/(t_1t_3t_4)$, the equation of the circle $P_1P_2P_3P_4$ is $x^2 + y^2 - c\sum t_1x - c\sum t_1y = 0$.

12. A circle meets $x = at, y = bt$ at points $A, B, C, D$ given by $t = a, b, c, d$; prove that $abcd = 1$. Two other circles are drawn, the first meeting the hyperbola at $A, B, P, Q$ and the second meeting the hyperbola at $C, D, R, S$. Prove that $P, Q, R, S$ are concyclic.
13. If \( P_1, P_2, P_3 \) are the feet of the normals to \( y^2 = 4ax \) from \( G(h, k) \), find the equation of the conic through the origin \( O \) and through \( P_1, P_2, P_3 \), and prove it is an ellipse having \( OG \) as a diameter.

14. If \( KF, KQ \) are the tangents to \( y^2 = 4ax \) from the point \( K(a, ak) \), prove that the equation of circle \( KFPQ \) is \( x^2 + y^2 = 2(k^2 + ak)x - 2ay = 0 \). What can be said about the system of circles \( KFPQ \) if \( k \) varies, \( k^2 > 4a^2 \) ?

15. Examine the case \( k^2 = 4a^2 \).

16. The normal to \( y^2 = 4ax \) at \( P(at^2, 2at) \) meets \( y = 0 \) at \( Q \); the circle through \( P, Q, R \) and the origin meets \( y^2 = 4ax \) again at \( Q, R \). Prove: (i) \( t^2 \geq 8 \); (ii) the equation of \( QR \) is \( 2x + ty + 4a = 0 \).

17. The normals to \( x = at^2, y = 2at \) at points \( P, Q, R \) given by \( t = p, t = q, t = r \) are concurrent. Prove that the perpendicular to \( P \) to \( QR \) is a normal to \( x = at^2 - 12a, y = 4at \) at the point \( t = -1 \).

17. If the normals to \( x = at^2, y = 2at \) at \( t = t_1, t = t_2, t = t_3 \) form an equilateral triangle \( PQR \), prove:

(i) \( \Sigma t_i = -3t_i t_j \) and \( \Sigma t_i t_j = -3 \);

(ii) the locus of the centroid of \( \triangle PQR \) is \( 3x^2 = 2a(x - 3a) \). [See Example 8, p. 304.]

18. From a variable point \( P \) on \( px^2 + y^2 = 1 \), normals \( PQ_1, PQ_2, PQ_3 \) are drawn to the conic. Prove that the locus of the centre of the circle \( Q_1Q_2Q_3 \) is \( px^2 + y^2 = \frac{4}{3}y \). [See Example 8, p. 304.]

19. If the intersections of the parabola \( x = at^2, y = 2at \) with the circle \( x^2 + y^2 + 2px + qy + r = 0 \) are given by the equation \((t - 1)^2(t - 2) = 0\), express \( t_1, t_2, t_3 \) in terms of \( a, t_1 \). Hence find the coordinates of the centre of \( y^2 = 4ax \) at the point \( (at_1^2, 2at_1) \).

20. The normals to \( ax^2 + by^2 = 1 \) at \( P_1, P_2, P_3, P_4 \) are concurrent; \( P_4 \) is \((x_4, y_4)\). Prove that the perpendicular to \( P_1P_2 \) from their poles passes through the point \( (a - b)x_4y_4, (a - b)y_4x_4 \). Deduce that the perpendiculars to \( P_1P_2, P_2P_3, P_3P_4 \) from their poles concur at a point on \( k^2x^2 + ay^2 = (a - b)^2(ab) \).

21. Points \( P_1, P_2, P_3 \) on \( x = at^2, y = 2at \) are given by the roots \( t_1, t_2, t_3 \) of \( \Delta^2 - px^2 + qy^2 + r = 0 \); \( T_1, T_2, T_3 \) are the poles of \( P_1P_2, P_2P_3, P_3P_1 \). Prove that the equation of the circle \( T_1T_2T_3 \) is \( x^2 + y^2 = a(1 - q)x + a(r - p)y + a^2q = 0 \).

Verify that the circle passes through the points of the parabola.

22. Prove that two parabolas can be drawn to pass through the feet of the normals from \( (h, k) \) to \( px^2 + y^2 = 1 \), where \( p > 0, q > 0 \) and that their axes meet at \( \{q^2(p^2 - p^2), p^2(k^2 - q^2)\} \).

23. If \( P \) is a point on the parabola \( y^2 = a(x + a) \); \( O \) is the origin. The circle on \( OP \) as diameter meets the normals from \( (h, k) \) to \( px^2 + y^2 = 1 \), where \( p > 0, q > 0 \) and that their axes meet at \( \{q^2(p^2 + p^2), p^2(k^2 + q^2)\} \).
17.1. If the parametric equations of a curve are simple, the general shape of the curve can be sketched quickly by calculating mentally approximate values of \(x\) and \(y\) when \(t\) increases from \(-\infty\) to \(+\infty\) and noting any key-points.

**Example 2.** Sketch the curve discussed in Example 1
\[ x : y : a = 6t : (3 + t)(3 - t) : 1, \quad (a > 0), \]
and examine the nature of the tangent at the origin.

The curve is symmetrical about the origin because if \(t = t_1\) gives \((x_1, y_1)\), then \(-t = t_2\) gives \((-x_1, -y_1)\). The curve meets \(Ox\) at the points \(t = -3, \ t = 0, \ t = 3\). The equation of the tangent at \(t = t_1\), see Example 1, is \((t_1^2 - 9)x + 2y = 4at_1^2\). In particular, the tangent at \(O, t = 0\), is \(3x - 2y = 0\).

![Fig. 142](image)

Also the tangent is parallel to \(Ox\) at \(t = -\sqrt{3}\) and at \(t = \sqrt{3}\). If \(t\) is numerically large, \(x = 6t, \ y = -t^3\); this shows how the curve behaves when \(t \rightarrow -\infty\) and when \(t \rightarrow +\infty\). For \(-3 < t < 0, x < 0\) and \(y < 0\), and \(y\) is a minimum at \(t = -\sqrt{3}\). Hence the curve is shaped as in Fig. 142, which is drawn to show how points corresponding to values of \(t\) when \(t\) increases from \(-\infty\) to \(+\infty\).

The gradient of the tangent at any point \(t\) is \(14 - t^2\); therefore, as \(t\) increases from \(-\infty\) to 0, the gradient increases steadily from \(-\infty\) to \(14\), and, as \(t\) increases from 0 to \(+\infty\), the gradient decreases steadily from \(14\) to \(-\infty\).

Therefore the value of the gradient is a maximum at \(t = 0\). Before reaching \(t = 0\), the direction of the tangent is rotating counterclockwise; after passing the point \(t = 0\), it is rotating clockwise. Hence in passing through the point \(t = 0\), the curve crosses over from one side of the tangent, \(3x - 2y = 0\), to the other.

The point \(O, t = 0\), is called a point of inflexion and the tangent, \(3x - 2y = 0\), at \(O\) is called an **inflexional tangent**.

17.1.2. A point on a curve at which the value of the gradient is either a maximum or minimum is called a point of inflexion.

In Fig. 142, the tangent, \(3x - 2y\), at the point of inflexion \(O\) meets the curve where \(18t = 2(9 - t^2)\) which gives \(t = 0\), that is at three coincident points; this is a necessary but not a sufficient condition for a point of inflexion, see Example 3.

**Example 3.** Examine the curve
\[ x : y : a = 3t^2 + 2 : 2t^3 + 1, \quad (a > 0), \]
and find the nature of the tangent at \(t = t_1\) if \(t_1\) gives \((x_1, y_1)\), then \(t = -t_1\) gives \((-x_1, -y_1)\).

Also \(x = a(t^2 + 2) \geq 2a\), no part of the curve lies to the left of the line \(x = 2a\).

![Fig. 143](image)

Let \(lx + my + nz = 0\) cut the curve at \(t = t_1, t = t_2, t = t_3\), then \(t_1, t_2, t_3\) are the roots of \(l(3t^2 + 2) + m(2t^3 + n) = 0\), that is, \(2ml^2 + 3lt^2 + (2m + n) = 0\).

\(\text{Therefore, if } t_1, t_2, t_3 \text{ are collinear points on the curve,}\)

\[ \begin{align*}
2m & = 3l, \\
2l & = n - 3t_1t_2 - 2mt_1t_2 \\
0 & = 3t_1t_2 + 2mt_1t_2
\end{align*} \]

therefore this condition is satisfied by all values of \(t_1\) if \(t_1 + t_2 = 0\)
and \(t_1t_2 = 0\), that is, \(t_1 = t_2 = 0\).

The point \(t = 0\) is \(A(2a, 0)\); therefore if \(P\) is any other point on the curve, the line \(AP\) cuts the curve at two coincident points, coincident with \(A\), and cuts the curve again at \(P\). But the line \(y = 0\) meets the curve only where \(t^2 = 0\), that is, at three coincident points, coincident with \(A\); the name, tangent at \(A\), is therefore reserved for the line \(y = 0\).

If \(t_1 + t_2 = 0\), the condition for collinearity gives \(2t_1t_2 + t_1^2 + t_2^2 = 0\), that is, \(t_2 = -t_1^2\) since \(t_1 + t_2 = 0\);

\(t_1\) the tangent at the point \(t = t_1(0, +0)\) meets the curve again at the point \(t = -t_1^2\); therefore the tangent at \(t = t_1\) is obtained by substituting \(t_1\) in the tangent, \(x = a(t^2 + 2)\), this gives \(t_1x - y = a(t_1^2 + 2a) = 0\).

This form of the equation of the tangent includes also the equation \(y = 0\) when \(t_1 = 0\).

The gradient of the tangent at any point \(t\) equals \(t\), and this has neither a maximum nor a minimum value as \(t\) increases from \(-\infty\) to \(+\infty\); therefore there is no point of inflexion on the curve although the tangent at \(A\) meets the curve at three coincident points.

The point \(A(2a, 0)\) is called a **cusp**.
Example 4. Examine the curve
\[ x : y : \alpha = t : (t^2 - 1) : 1, (t > 0). \]

If \( t = t_1 \) gives the point \((x_1, y_1)\), then \( t = -1 \) gives \((x_2, -y_2)\); it is therefore sufficient to obtain the branch of the curve corresponding to values of \( t \) from 0 to \( +\infty \) and combine it with the image of this branch in \( \mathbb{R}^2 \) corresponding to values of \( t \) from 0 to \(-\infty \).

Since \( x = at^2 \), no point of the curve lies to the left of \( Oy \).
Since \( y = at(t^2 - 1) \), the curve meets \( Ox \) at \((0, 0)\) and \( A(a, 0) \).

When \( t \) is small and positive, \( x = at^2 \), \( y = -at \); \( y = -\sqrt{ax} \); hence the tangent at \( O \) is \( x = 0 \).

For \( 0 < t < 1 \), \( y \) is negative; for \( t > 1 \), \( y \) is positive.

When \( t \) is large and positive, \( x = at^2 \), \( y = at^3 \); \( y = +\sqrt{ax} \).

These considerations show that the branch of the curve corresponding to values of \( t \) from 0 to \( +\infty \) is shaped roughly as in Fig. 144 (i).

The branch corresponding to values of \( t \) from 0 to \(-\infty \) is the image of Fig. 144 (i) in \( \mathbb{R}^2 \), see Fig. 144 (ii). Therefore the combination of these branches in Fig. 144 (iii) indicates the form of the complete curve.

![Diagram](image)

**Fig. 144**

The points where \( tx + ny + ma = 0 \) meets the curve are given by

\[ u^2 + m(t^2 - 1) + n = 0, \]

that is, \( m^2 + u^2 - mt + n = 0 \).

:: if the points \( t_1, t_2, t_3 \) are collinear, \( t^2_1 + t^2_2 + t^2_3 = -1 \);

also \( m : l : n = 1 : -\Sigma t_i : -t_1t_2t_3 \);

::: the line of collinearity is

\[ L = x\Sigma t_i - y + at_1t_2t_3 = 0. \]

Conversely, as on p. 397, if \( \Sigma t_i = -1 \), the points \( t_1, t_2, t_3 \) lie on \( L \).

If \( t_2 = t_3 = 0 \), the test for collinearity gives

\[ 2t_1t_3 + t_3^2 = 1, \]

that is, \( t_3 = -1/(1 + t_1^2)t_1 \), since \( t_1 > 0 \);

:: the tangent at \( t = 1 \) meets the curve again where \( t = -1/(1 + t_1^2)t_1 \);

:: the tangent at \( t = t_3 \) is obtained by substituting \( t_1, t_2 = -1/(1 + t_1^2)t_1 \) for \( t_1, t_2, t_3 \) in \( L = 0 \); this gives

\[ x(3t_1^2 - 1) - 2t_1y - at_1^2(1 + t_1^2) = 0. \]

This is also true when \( t_1 = 0 \) because the tangent at \( O \) is \( x = 0 \).

17.1. A cusp may be regarded as the limiting form of a node when the tangents to the two branches forming the node tend to coincide. But there is an important difference between the nature of a point of inflexion and the nature of a cusp or node.

As in Example 4, the curve \( x = at^2, y = at(t^2 - c^2), c > 0 \), has a node at the point \( (ac^2, 0) \) given by \( t = c \) and \( t = -c \). The limiting form of the curve when \( c \to 0 \) is given by \( x = at^2, y = at^3 \); this corresponds to the curve in Fig. 143 with \( A \) as origin, which has a cusp at \( A \). Thus the cusp in Fig. 143 is the limiting form of the node in Fig. 144 when \( ac^2 \to 0 \).

Since a cubic equation has either one root or three roots not necessarily unequal, an arbitrary line has either one intersection or three intersections with a cubic curve. Any line through the cusp \( A \) in Fig. 143 or through the node \( A \) in Fig. 144, other than a tangent at \( A \), meets the curve again at just one point \( t = t_3 \) (not repeated); therefore intersection at \( A \) for each curve counts as two points. For this reason, a cusp or node on a cubic curve is called a double point.

A point of inflexion is not a double point. Each line through the point of inflexion \( O \) on the curve in Fig. 142, other than the inflexional tangent at \( O \), meets the curve again at two distinct points and so its intersection with the curve at \( A \) counts as only one point.

17.2. It is often convenient to use a calculus method for finding the equation of the tangent to a curve.
17.2.1. If the equation of a curve is given in the form
\[ y = f(x), \]
the gradient of the tangent at the point \((x_1, y_1)\) on the curve is equal to the value of \(\frac{dy}{dx}\) when \(x = x_1\); this is denoted by \(f'(x_1)\) where \(f'(x)\) is the derivative of \(f(x)\).

Therefore the equation of the tangent at \((x_1, y_1)\) can be written
\[ y - y_1 = f'(x_1)(x - x_1). \]

17.2.2. If the equation of a curve is given in the form
\[ x : y : a = t : t^3 : (1 - t^4), \]
the equation of the tangent at the point \(t = t_1\) is
\[
\begin{vmatrix}
f(t_1) & g(t_1) & h(t_1) \\
f'(t_1) & g'(t_1) & h'(t_1)
\end{vmatrix} = 0.
\]
The equation of the chord joining the points \(t = t_1, t = t_1 + k\) is
\[
\begin{vmatrix}
x & y & a \\
f(t_1) & g(t_1) & h(t_1) \\
f(t_1 + k) & g(t_1 + k) & h(t_1 + k)
\end{vmatrix} = 0.
\]
The third row of this determinant can be replaced by
\[
\begin{vmatrix}
(f(t_1 + k) - f(t_1))/k & (g(t_1 + k) - g(t_1))/k & (h(t_1 + k) - h(t_1))/k
\end{vmatrix}.
\]
When \(k \to 0\), the limiting position of this chord is the tangent at the point \(t = t_1\), and the limiting values of these elements in the third row are \(f'(t_1), g'(t_1), h'(t_1)\), giving the required result.

Example 5. Find the equation of the tangent at the point \(t = t_1\) on the curve, \(x : y : a = t : t^3 : (1 - t^4)\).

By 17.2.2, the equation of the tangent at \(t = t_1\) is
\[
\begin{vmatrix}
x & y & a \\
1 & 1 & 1 - t_1^4 \\
2t_1 & 3t_1^2 & -4t_1^3 + t_1^4
\end{vmatrix} = 0.
\]

Expansion gives
\[
x(-4t_1^2 + 2t_1 + 2t_1^3) + y(1 - t_1^4 + 4t_1^3) + a(2t_1^2 - t_1^4) = 0,
\]
that is,
\[
2t_1x(1 + t_1^4) - y(1 + 3t_1^4) - at_1^4 = 0.
\]

Example 6. Find parametric equations for the curve, whose \((x, y)\) equation is \(x^2 + ay^2 = a^2y^2 = 0\).

The line through the origin, \(y = tx\), meets the cubic curve where \(x^2 + ay^2 = 0\), that is, \(x^2(1 + t^2) = 0\). Therefore points of the curve are given by
\[
x : y : a = t^3 : t : (1 + t).
\]

Note. This method is effective because the origin is a double point. For \(t\) small, \(x^2 \approx ay^2\); therefore the origin is a cusp.

17.2.3. Examine the curves, \(x : y : a = t : t^3 : (1 - t^4), (a > 0)\).

If \(t = t_1\) gives the point \((x_1, y_1)\), then \(t = -t_1\) gives the point \((-x_1, y_1)\), therefore the curve is symmetrical about \(Oy\).

If \(2x + my + nt = 0\) does not pass through \(O\), then \(m = 0\) and, if it meets the curve at the points \(t_1, t_2, t_3, t_4\), these values of \(t\) are the roots of \(\Delta + mt^3 + nt^2 - k = 0\), that is, \(a^2 - mt^2 - k = 0\).

\[
\begin{vmatrix}
f(t_1) & g(t_1) & h(t_1) \\
f'(t_1) & g'(t_1) & h'(t_1)
\end{vmatrix} = 0.
\]

If the points \(t_1, t_2, t_3\) lie on a line not passing through \(O\),
\[
t_1 + t_2 + t_3 - 1/(f(t_1)g(t_2)h(t_3)) = 0.
\]

Conversely, if \(t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 - 1 = 0\) and if \(t_4\) is chosen so that \(t_1 + t_2 + t_3 - t_4 = 0\), then, as on p. 307, the points \(t_1, t_2, t_3, t_4\) lie on the line, \(x^2y^2z^2 - \Sigma t_i^2 + \alpha = 0\), where summation extends to \(t_1, t_2, t_3, t_4\).

It is impossible to choose \(t_1, t_2, t_3\) so that for all values of \(t_4\)
\[
t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 - 1 = 0;
\]
therefore there is no double point except possibly at \(O, t = 0\).

If \(x_1 = t_1 = t_2\), then \(3t^4 = 1\); therefore \(t = 0\) gives two points of inflexion because they are not double points.

If \(t_1 = t_2\), then, since \(\Sigma t = 0\) and \(t_1t_2t_3 = -1, t_1 + t_2 = -2t_4\) and \(t_4 = -1/t_1^2\), therefore the tangent at \(t = t_1\) meets the curve again at two points \(t_1, t_2\) given by the equation \(\Delta - 2t_1^2 - 1/t_1^2 = 0\).

The equation of the tangent at any point \(t\), see Example 5, is
\[
2tx(1 + t^4) - y(1 + 3t^4) = at^4.
\]

The limiting form of the equation of the tangent when \(t \to 1\) gives the asymptote \(BAB\), \(x^2 + ay^2 = 0\), which meets the curve again at \(D, D'\), given by \(x^2t^2 = -1\).

Similarly, \(t \to -1\) gives the asymptote \(OCA\), \(x^2 + ay^2 = 0\), which meets the curve again at \(E, E'\), given by \(x^2t^2 = -1\).

Points near \(O\) are given by small values and by large values of \(t\).

If \(t = 0\), \(x \approx t, y \approx t^3\); \(x \approx t^3, y \approx t^3, x \approx t^3\).

Therefore there are two branches of the curve near \(O\), one of which is shaped like the parabola \(y = x^2\) and the other like the cubic curve \(y^2 = -x^2\) which has a cusp at \(O\), see Fig. 145.

A line of general position through the origin, \(y = tx\), meets the curve again at just one point \(t = k\) (not repeated) and therefore its intersection with the curve at \(O\) counts as three points, and \(O\) is called a multiple point of the curve.
EXERCISE 68

1. Find the equation of the tangent to \( y = x^2(x - 1) \) at (2, 4) and the coordinates of the point where it meets the curve again.

2. Find parametric equations, parameter \( t \), for \( 3y^2 = 2x^3 + x^2 \); find the equation of the tangent at \( t = 1 \) and the value of \( t \) which gives its other point of intersection with the curve. (L)

3. Find the equation of the tangent to the curve \( x = 3t^2, y = 2t^3 \) at the point \( P, t = \sqrt{2} \). Find the parameter of the point \( Q \) where this tangent meets the curve again and prove \( PQ \) is normal at \( Q \). (N)

4. If \( \Sigma t^2 + 3t = 6 \), prove that the points \( t_1, t_2, t_3 \) on the curve given by \( x = 2a^2t^3, y = a(t^2 - 4) \) are collinear and find the equation of the line of collinearity.

5. If \( \Sigma t^2 + 4 = -4 \), prove that the points \( t_1, t_2, t_3 \) on the curve \( x = 3a^2t^3, y = a(t^2 - 4) \) are collinear and find the equation of the line of collinearity.

6. If \( t_1 + t_2 + t_3 = 1 \), prove that the points \( t_1, t_2, t_3 \) on the curve \( x = a + t : t^2 : (1 - t^2) \), are collinear.

7. Find the points of inflection on the curve given by \( x = 3a^2t^3, y = a(t^2 - 4) \). Find the point of inflection at \( x = 1 \) and the value of \( t \) when \( t \to \infty \). Sketch the curve.

8. Find the points of inflation on \( x : y = 1 : (t^2 - 3) : t : t^2 \). Is there a double point on the curve? Sketch the curve.

9. Find the condition for the collinearity of the points \( t_1, t_2, t_3 \) on \( x : y = a - t^2 : t^2 : (t^2 + 3) : t : t^2 \) (\( a > 0 \)). Find the nature of the double point (if any) on the curve. Sketch the curve.

10. Prove that \( x = a(t^2 - 3) : (t^2 - 1) : 1, (a > 0) \), has a double point but has no point of inflection.

11. Prove that \( x = y : 1 = (t^2 - 3) : 7(t^2 - 1)(t - 2) : (14t - 12) \) has one double point and one point of inflation; find these points.

12. Prove that \( x : y = a = 1 : (t^2 - 1) : (t^2 + 1), (a > 0) \), has a double point and find the equations of the tangents at this point. Find the limiting form of the equation of the tangent at the point when \( t \to \infty \). Interpret the result and sketch the curve.

13. Find the equation of the tangent to \( x : y = a = t : t^2 : (1 + t^2), (a > 0) \), at the point \( t = 1 \) and deduce that \( 2x + 3y - a = 0 \) is the equation of an asymptote. Prove that the curve has a node at \( Q \). Sketch the curve.

14. Find the condition for the collinearity of the points \( t_1, t_2, t_3 \) on \( x : y = a(t^2 - 1) : (t^2 - 1) : t : t^2 \) (\( a > 0 \)). Is there a point of inflation or a double point on the curve? Find the limiting form of the equation of the tangent at the point \( t \) when \( t \to 0 \). Sketch the curve.

15. Prove that the pole of a tangent to \( x : y = a = t^2 : (1 + t^2) \) with respect to \( x^2 + y^2 = b^2 \) lies on \( 4a^2(x + y)^2 + 2a^2y = 0 \).

16. Find the values of \( t \) for the points of inflection on the curve \( x = 3t^2, y = 2t^3, (a > 0) \), called the Witch of Agnesi. Show that the curve has an asymptote. Sketch the curve.

17. Find the value of \( c \) if there is a point of inflection at the point \( t = 2 \) on the curve \( x = 1 : (1 - t^2) : (1 - t^2) : c - t^2(1 + t^2) \).

18. Use the substitution \( x = t^2 \) to obtain parametric equations for the curve \( y = a - t^2 : a - t^2 \). Find the tangent at \( t = t_1 \) meets the curve again at \( t = t_2 \). Find \( t_2 \) if the tangents at \( t = t_1 \) and \( t = t_2 \) meet the curve.

19. If \( x + y = 0 \), find the points of intersection at the points \( P, P', P'' \) and the tangents at \( P, P', P'' \) meet the curve again at \( Q, Q', Q'' \). Prove that \( Q, Q', Q'' \) lie on \( x^3 - 2x^2 + y^3 - 3x^2y + 3xy^2 + 2y^3 = 0 \).

20. Find the double point and the equations of the tangents at this point \( x = 1 : (t^2 - 1) : (t^2 - 2t) \). Find also the equation of the asymptote.

21. Find the points of inflection on \( x : y = 1 : (t^2 - 1) : (t^2 - 1) \). Prove that the tangent at \( t = t_1 \) meets the curve at points given by \( t_1^4 = 2t_1 + 1 \).

22. If \( 2x^2 + 4y^2 + 2tx^2 + 6ty^2 = 0 \), prove that \( t_1, t_2, t_3 \) are collinear points on \( x : y = a = 1 : (t^2 - 1) : (t^2 - 1) : (t^2) \). Find the double point and point of inflation on the curve.

23. If \( t_1, t_2, t_3 \) are collinear points on \( x = 1 : (t^2 - 1) : (t^2 - 1) : (t^2 - 1) : (t^2 - 1) : (t^2) \), prove that the line of collinearity is \( x^4 + y^4 + z^4 = 0 \). Find the double point and the equations of the tangents at this point. Find the equation of the asymptote and sketch the curve.

24. Prove that from a point \( P, t = t_1 \), on \( x : y = 1 : (t^2 + 1) : (t^2 + 1) : t \), two tangents can be drawn to the curve, outer than the tangent at \( P \), and that their points of contact are given by \( t^4 - 2t + (1 - t^2) = 0 \). Prove that there are three points of inflation which lie on a line parallel to \( Ox \). Find the equation of the asymptote. (N)

25. If \( P, Q \) are points on \( x = 3t, y = 2t^2 \) such that \( PQ \) is the tangent to the curve and the normal at \( Q \). Prove that \( Q \) is one of the points \( (1/3, \pm 1/\sqrt{2}) \).

26. If the points \( t_1, t_2, t_3 \), on \( x = 1 : (t^2 - 1) : (t^2 + 1) : t \), called a lemniscate, lie on a circle, prove that \( t_1 t_2 t_3 = 1 \) and that the circle is given by \( (x^2 + y^2)(2t_1 + 2t_2 + 2t_3 - 2) - y \Sigma t_i + 1 = 0 \). Prove that the circle has three points of contact with the lemniscate at \( t = t_1 \) meets the curve again at \( t = t_1/t_2 \) and that its equation is \( 3t_1(1 + t_1^2)(2x^2 + y^2) = t_1^2(3 + t_1^2)x - (1 + 3t_1^2)y + t_1^3 = 0 \).
17.3. Tests for collinearity of points on a curve are useful for proving geometrical properties of systems of tangents.

Example 8. A line HK meets at $P, Q, R$ the curve

$$x : y : a = t : t^2 : 1 + p^3, (a > 0),$$

to the Folium of Descartes.

The tangents at $P, Q, R$ meet the curve again at $P', Q', R'$. Prove that:

(i) $P', Q', R'$ lie on a line $HK'$; (ii) there are four positions of $HK$ which give the same line $HK'$.

To sketch the curve, use the method of Example 7. There are branches through $O$ shaped like $x^2 = ay$ and $y^2 = ax$.

Any line $y = mx$ through $O$ meets the curve at one other point, the point $t$.

By 17.2.2, the equation of the tangent at the point $t$ is

$$x(t^2 - 2t) + y(1 - 2t^3) + at^2 = 0;$$

then the limiting form of this equation when $t \to -1$ gives the asymptote,

$$3x + 3y + a = 0;$$

(i) Let $P, Q, R$ and $P', Q', R'$ be given by $t = p, q, r$ and $t = p', q', r'$; let the equation of $HK$ be $x + y + a = 0$.

Then

$$\begin{align*}
\text{the roots of the equation } & x + mt^2 + (1 + p^3) = 0, \\
\therefore & pqm = -1;
\end{align*}$$

also $\frac{1}{t^2} + \frac{1}{t^3} + \frac{1}{t^4} = \Sigma t^2 = 0$, and $m = -\frac{1}{t^2} - \frac{1}{3t} = -\Sigma t$;

$$\therefore \text{the equation of } HK \text{ is } x \Sigma t + y \Sigma t + a = 0;$$

Conversely, if $t_1 t_2 t_3 = -1$, the points $t = t_1, t = t_2, t = t_3$ are collinear because the line $x \Sigma t^2 + y \Sigma t + a = 0$ meets the curve where

$$\ell t_1 t_2 t_3 - \ell t^2 + \ell t^2 = 0;$$

that is, where $\ell^2 - \ell t_1 t_2 t_3 = (t_1 t_2 t_3 - 1) t^2 = 0$.

By the condition for collinearity, since $P, Q', R'$ are tangents,

$$\begin{align*}
p q' & = -1, q r' = -1, q r = -1; \\
\therefore & \frac{1}{p q r} = -1/3; \\
\therefore & t = r = 1/3. \\
\therefore & \text{the points } P', Q', R' \text{ are collinear.}
\end{align*}$$

Let $P, Q_1, R_1$ be the points, $t = -p, t = q, t = -q$, then $P, Q_1, R_1$ are collinear because $n(-q)(-q) = pq = -1$.

Also the tangent at $Q_1$ meets the curve again at the point given by $t = -1/(q + q) = -1/(2q^2) = -1/2$.

Similarly the tangent at $R_1$ meets the curve again at $R'$.

The line $PQ_1R_1$ gives rise to the line $P'Q'R'$, that is, $HK'$.

Similarly the lines $Q_1R_1P$ and $R_1P_1Q$ also give rise to $HK'$.

**Exercise 69**

1. If the tangent to $x : y : a = t : t^3 : 1$ at a variable point $P$ meets the curve again at $Q$, prove $PQ$ is divided in constant ratios by the asymptotes.

2. $P, Q$ are variable points on $x : y : a = t^3 : t^2 : 1$ such that $PQ$ subtends a right angle at the origin. Prove that the tangents at $P$ and $Q$ meet on the parabola $4x^2 = a(3y - a)$.

3. Find parametric equations for the curve $ay^2 = x^2 - a$. If $P, Q$ are points on the curve such that $PQ$ subtends a right angle at the origin, prove that the tangents at $P, Q$ meet the curve $x^2 = 2ay^2$.

4. $P, Q$ are variable points on the Folium, $x : y : a = t : t^2 : (1 + t^2)$. If $PQ$ subtends a right angle at the origin, prove that it passes through a fixed point.

5. $PQ$ is a variable chord of the Cissoid, $x : y : a = t : t^3 : 1 + t^2$, subtending a right angle at the origin. Prove: (i) the mid-point of $PQ$ lies on a fixed line; (ii) the tangents at $P, Q$ meet on a fixed circle.

6. Prove $3x + x = 2x + t$ is a normal to $x = t$, $y = t^2$ and that no normal crosses $Oy$ between $(0, 4/3 \sqrt{3})$, $(0, -4/3 \sqrt{3})$.

7. Find the condition that the points $t = t_1, t = t_2, t = t_3$ on the curve, $x : y : a = (t - 1)^2 : (t - 2)^2 : 1$, are collinear. A variable line passes through $(a, 0)$ and meets the curve again at $P_1, P_2$, prove that the locus of the mid-point of $P_1 P_2$ is $x : y : a = t : t^2 : (t^2 + 2)^2$.

8. $PQ$ is a variable chord of $x : y : a = t^3 : 3t : 1$ parallel to a fixed line $y = kx$. Prove the mid-point of $PQ$ lies on $8t^2 y^2 = 27a^4 (3y - k^2 x)$.

9. If the tangents to $x : y : a = (t^2 - 1)^2 : (t^2 - 1)^3 : (1 + t^3)$ at variable points $P, Q$ meet on the curve, prove the envelope of $PQ$ is $x^2 : 3(t - y)(y - 2a) = 0$.

10. If $t_1, t_2, t_3$ are the roots of $t^3 + 3t^2 + c = 0$, prove the tangents to $x : y : a = t^3 : 1$ at $t_1, t_2, t_3$ are concurrent. Find the points of concurrence.

11. If the tangents to $x : y : a = (t^2 - 1) : (t^2 - 1) : t^3$ at $P, Q$ meet on the curve, prove the envelope of $PQ$ is $x^2 : 3(t^2 - a)(y - 2a) = 0$.

12. Find the condition that the tangent to $x : y : a = 3t : 3t^3 : (1 + t^4)$ at the point $t$ passes through $(b, k)$. If the tangent to $4x^2 = a^2$ at $(2ap, 2ap^2)$ cuts the cubic curve at $t = t_1, t_2, t_3$, prove the tangents at $t_1, t_2, t_3$ meet at the point $\{ -ap(t^2 + 3) \}^{-1} = 2\sqrt{a} - ap(t^2 + 3)$.

13. Prove the points $t_1, t_2, t_3$ on $x : y : 1 = (t^2 - 1) : 2 : (t^2 + 1)^2$ are collinear if $(2t_1 t_2 t_3)^2 + 3t_1 t_2 t_3 + 3 = 0$. If the points of contact of the three (real) tangents from a point $P$ to the curve are collinear, prove that $P$ lies on the circle, $x^2 + y^2 - 4z = 0$. 
14. Prove that the points $t_1$, $t_2$, $t_3$ on $x : y : a = t^2 : t^4 : 1$ are collinear if $\Sigma t_i t_2 = -1$. The tangents at the points $P_1$, $P_2$ meet at the point $N$, $t = n$, on the curve $P_1 P_2$ meets the curve again at $M$, $t = n^2$, prove $mn = -1$. If $Q_1$, $Q_2$, are the points of contact of the tangents from $M$ to the curve, prove that $N$ lies on $Q_1 Q_2$.

15. A line cuts the sides $BC$, $CA$, $AB$ of $\Delta ABC$ at $L$, $M$, $N$; the points $A$, $B$, $C$, $L$, $M$, $N$ lie on $x : y : 1 = t : t^2 : t^3$ and are given by $t = a, b, c, 1, m, n$. Prove: (i) $a + b + c = 0$; (ii) tangents at $A$, $L$ meet on the curve.

16. Prove that the points $t_1$, $t_2$, $t_3$ on $x = at^2$, $y = at^3$ are collinear if $\Sigma t_i t_2 = 0$. The chord PQ subtends a right angle at the origin O; $PQ$ meets the curve again at $R$; the tangent at $S$ passes through $R$. Prove $OP$, $OQ$, $OR$ are the bisectors of $\angle xOS$.

17. The points of contact $P$, $Q$, $R$ of the tangents from $A(h, k)$ to the curve, $x = at^2$, $y = at^3$ are given by $t = p, q, r$. Prove $p, q, r$ are the roots of $2a^2 - 3k^2 + h = 0$. If $AP$, $AQ$, $AR$ meet the curve again at $L$, $M$, $N$, prove the tangents at $L$, $M$, $N$ concur at $(-8k, -2k)$.

18. $A, B, C$ are non-collinear points on $x = at^2$, $y = at^3$. $BC$, $CA$, $AB$ meet the curve again at $L$, $M$, $N$; $AC, BO, CO$ meet the curve again at $A'$, $B'$, $C'$. Prove $LA'$, $MB'$, $NC'$ concur at a point on the curve.

19. If the line $lx + my + n = 0$ meets the cubic curve, $x : y : 1 = t : t^2 : (t + 1)$, at points $t = t_1$, $t = t_2$, $t = t_3$, find in terms of $l, m, n, a, b, c$ the equations whose roots are $-t_1$, $-t_2$, $-t_3$. Find the equation in $t$ whose roots give the points of intersection of the conic $ax^2 + 2bxy + by^2 + 2px + 2qy + c = 0$ with the cubic curve. Hence show that there is a conic touching the cubic curve at the points $-t_1$, $-t_2$, $-t_3$, and that its equation is $(l^2 + 4m^2)x^2 - 2(2 + 2m)xy + (m^2 + 4l^2)y^2 - 2lx - 2my + n + 1 = 0$.

20. $P_1$, $P_2$, $P_3$ are collinear points on $x : y : 1 = (t^3 - 3t) : (t^3 - 3t) : (2t - 3)$ given by $t = 1$, $t_2$, $t_3$; prove that $t = (2t_3 - 3(t_1 + t_3) + 9) / (2t_3 - 3(t_1 + t_2)) + S$. Find the values of $t_1$ and $t_2$ for which $t_3$ is indeterminate and interpret the result. Prove that the tangents at $P_1$ and $P_2$ meet at a point on the curves if $2t_2 - 3(t_1 + t_2) + 4 = 0$. Interpret the result obtained by putting $t_3 = t_1$.

21. The tangent to $x : y : 1 = 1 : 1 : 1$ at the point $t$ meets $Ox$, $Oy$ at $P$, $Q$. Prove that area $\Delta OPQ$ is not greater than $3 \sqrt{3}/8$.

22. Prove that the points on the curve, $x = at^2 + 1$, $y = at^3 - 1$, given by $t = 1$, $t_2$, $t_3$, $t_4$, are collinear if $t_1^2 + t_2^2 + t_3^2 = -1$ and $2l_1 t_2 = 0$. Prove also that from an arbitrary point on the four curvilinear lines can be drawn to touch the curve and that their points of contact are collinear. Prove also that $x + y = 0$, $x - y = 0$ are tangents at the origin and that there are two asymptotes parallel to $Oy$. 

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**Quick Revision Papers**

**Q.R. 29**

1. Find the condition that $ax^2 + 2bxy + by^2 = 0$ represents the coincident line-pair $(ax + by)^2 = 0$.

2. Find the area of the triangle whose vertices are the points $(q + r, qr), (r + p, rp), (p + q, pq)$.

3. If $a$, $b$, $c$ are constant and if $t$ varies, prove that the angle-bisectors of the line-pair $(ax^2 + 2bxy + by^2) + t(x^2 + y^2) = 0$ are fixed lines.

4. Find the value of $k$ if $2x^2 + xy - 6y^2 - 3z = 13y - k$ represents a pair of lines and then find their point of intersection.

5. Find the equation of the conic $3x^2 + 2xy + 2y^2 + 2x - 6y + 1 = 0$ referred to parallel axes through its centre.

6. Find the pole of $x - 3y = 1$ with respect to $2x^2 + 5y^2 = 40$.

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**Q.R. 30**

1. If $p$, $q$, $r$ are unequal, find the condition that the points $(ap^3, ap^2), (aq^3, aq^2), (ar^3, ar^2)$ lie on a line.

2. Find the value of $h$ if the line-pair $3x^2 + 2hxy + y^2 = 0$ separates harmonically the line-pair $1x^2 - xy - 2y^2 = 0$.

3. The line-pair $px^2 + 2qxy + ry^2 = 0$ through the vertex of the parabola $x = at^2, y = 2at$ meets it again at $t = t_1, t_2$. Find in terms of $a, p, q, r$ the values of $t_1 - t_2, t_1 t_2$ and the equation of chord $t_1 t_2$.

4. Find the equation of the pair of lines through the point $(2, -1)$ parallel to the line-pair $x^2 - 3xy - 6y^2 = 0$.

5. What can be said about a variable line, $lx + my + n = 0$, if $(2l - m - 3n)(l + 5m - 2n) = 0$?

6. Find $p, q, r$ if $(3, -2)$ is the centre of $x^2 + 5xy - 2y^2 + px + qy + 1 = 0$.

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**Q.R. 31**

1. If $ax^2 + 2bxy + by^2 = 0$ is the line-pair $(y - px)(x + qy) = 0$, express in terms of $a, b, c$ the line-pair $(x + py)(x + qy) = 0$. Interpret the result.

2. Write down in ratio-form the coordinates of the point of intersection of the lines $ax + by + c = 0$, $px + qy + r = 0$. What happens if $aq - bp = 0$?

3. Find the ratio of the line which the line joining $(1, -2)$ to $(-1, 1)$ is divided by the pair of lines $x^2 - 2xy - y^2 = 0$.

4. Find $k$ if $kx^2 - 5xy - 6y^2 - 10x - 11y = 4$ represents a line-pair.

5. Find the equations and lengths of the axes of the ellipse $3x^2 + 4xy + 2y^2 = 4$.

6. State the equations of the auxiliary and director circles of $7x^2 + 5y^2 = 4$. 

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**Ch. 1-13**

Quick Revision Papers
Q.R. 32

1. Find the value of $c$ if the lines $x + 2y + c = 0$, $x - 2y - 3 = 0$, $2x + 15y + 2 = 0$ concur at a point $P$ and then find the equation of $OP$.

2. Find the angle-bisectors of the line-pair $7x^2 + 12xy + 2y^2 = 0$.

3. Find the coordinates of the centroid of the triangle whose sides are the pair of lines $2x^2 + 6xy - y^2 = 0$ and the line $2x - y = 11$.

4. Find the relation between $p$, $q$, $a$, $b$, $h$ if the lines given by

$$ax^2 + 2hxy + by^2 - 1 = 0$$

and $ax^2 + 2kxy + by^2 - 1 = 0$ are at right angles.

5. Find asymptotes and transverse axis of $2x^2 + 5xy - 3y^2 + 5x - 5y - 6 = 0$.

6. Find the equation of the diameter of $2x^2 - 3xy - 5y^2 = 1$ conjugate to $2x + y = 0$ and the coordinates of the mid-point of the chord $2x + y = 12$.

Q.R. 33

1. Find the equation of the chord joining the points $t = p$, $t = q$ on $x/a : y/b : 1 = 2t : (1 + t^2) : (1 - t^2)$ and the equation of the diameter bisecting this chord.

2. Find the condition that the lines $ax + by + c = 0$, $bx + cy + a = 0$, $cx + ay + b = 0$ are concurrent.

3. State the conditions for the conic $ax^2 + 2hxy + by^2 = 1$ to be congruent to the conic $px^2 + 2kxy + qy^2 = 1$.

4. What locus is represented by $(3x - 4y)^2 + k(4x + 3y)^2 = 1$ if:
   (i) $k = 0$; (ii) $k = -4$; (iii) $k = 1$; (iv) $k = -1$; (v) $k = 0$.

5. Find the equation and length of transverse axis of $3x^2 + 12xy - 2y^2 = 1$.

6. Find the conditions that $(x - h) + m(y - k) = 0$, $m(x - h) - l(y - k) = 0$ are conjugate lines with respect to $px^2 + qy^2 = 1$ for all values of $l$, $m$. Explain why $(h, k)$ is then a focus of the conic.

Q.R. 34

1. If $x = k \cos \theta + h \sin \theta$, $y = p \cos \theta + q \sin \theta$, prove by using a determinant, or otherwise, that the points given by $0 = x$, $0 = y$, $0 = \gamma$ are collinear.

2. Find the perpendicular line-pair which separates harmonically the line-pair $16x^2 + 8xy - 12y^2 = 0$, and state what it represents.

3. The lines $OH$, $OK$ make with the line $y = px$ an angle $\tan^{-1} \frac{p}{2}$. Find the equation which represents the line-pair $OH$, $OK$.

4. If $3x^2 - 5y^2 = 2m^2 - 3n + 2m + 2 = 0$, prove that the variable line $bx + my = 1$ passes through one of two fixed points.

5. Express the equation $(3x + 2y)^2 = 16x - 11y + 24$ in the form $(3x + 2y)^2 = a(2x - 3y + r)$ and interpret the result.

6. State the equation of the pair of tangents to $px^2 + qy^2 = 1$ from $P(r \cos \theta, r \sin \theta)$. Find the condition they are at right angles and the length of the tangent from $P$ to the auxiliary circle if $p > q > 0$. 

Q.R. 35

1. $Ox$, $Oy$ are oblique axes, angle $\alpha$. Interpret the loci given by:
   (i) $x(a + y)/(a - y) = 1$; (ii) $x \cos \theta + y \cos \alpha = p$;
   (iii) $a/(y^2 + b(y^2 + a^2) = q$.

2. Find $l \cdot m \cdot n$ if $bl + my + n = 0$ and $5x + 11y + 12 = 0$ are conjugate lines with respect to $3x^2 + 2y^2 = 12$ and $3x^2 - 4y^2 = 1$.

3. Find the equation of the axes of $15x^2 + xy - 2y^2 = 50x + 12y + 10$.

4. The polar coordinates of $P_1$, $P_2$ are $(r_1, \theta_1)$, $(r_2, \theta_2)$. State the area of $\triangle OP_1 P_2$ and express the result in Cartesian coordinates.

5. With the notation on p. 266, interpret the loci: (i) $S_1 = 0$ if $S_1 + 0$; (ii) $S_1 = 0$ if $S_1 + 0$. Find $l$ if $(x, y)$ lies on the locus $S_1 = l$. $S$.

6. Find the envelope of the line $x \sec \theta - y \tan \theta = a$, where $\theta$ varies.

Q.R. 36

1. $Ox$, $Oy$ are oblique axes, angle $\alpha$. Find the equation of the line through $(2, 0)$ perpendicular to: (i) $y = 0$; (ii) $3x - y = 1$.

2. Find the equations of the axes of $24x^2 - 4xy + 21y^2 = 20$, and their lengths.

3. Find the polar coordinates of the centre $O$ of the circle $r = \cos \theta - 30^\circ$ and the polar equation of the diameter perpendicular to $OC$.

4. Show that the inverse of the line $3x - 4y - 2 = 0$ with respect to the circle $x^2 + y^2 = 4$ is a circle. Find the length of its diameter.

5. Find the equation of the tangent to $x^2 + 3xy - 5y^2 + 5x - 30y + 50 = 0$ at $(-3, 1)$ and the equation of the parallel tangent.

6. Find the condition that $ax^2 + by^2 + 2xy = 0$ represents: (i) an ellipse; (ii) a hyperbola; (iii) a parabola. Obtain parametric equations for the locus.

Q.R. 37

1. Find the coordinates of the centre of a circle whose equation is $x^2 + 2xy \cos \alpha + y^2 + 2y + 3 = 0$, referred to oblique axes.

2. Find the condition in terms of $p$, $q$, $r$, $x$, $y$ for the equations $x^2 + y^2 + r^2 = 0$, $x^2 + my + n = 0$ to give equal values of $t : m : n$.

3. A line through $V(0, \frac{3}{2})$ meets $x^2 - 2xy - y^2 = 4$ at $P$, $Q$ and makes the angle $\tan^{-1} \frac{3}{2}$ with $Ox$. Find: (i) $FP$, $FQ$; (ii) $PQ$.

4. State in detail the locus of the point whose polar coordinates are $(r, \theta)$ if $r = \cos \theta + \cos \theta$, where $a, b$, $\gamma$ are constants; $(a > 0, b > 0, 0 < \gamma < 90^\circ)$.

5. Show that the principal axes of $S = 0$ are parallel to $\beta(a^2 - x^2) = (\beta - b)xy$.

6. With the notation of p. 267, prove that the feet of the normals to $S = 0$ from $(b, k)$ lie on $a(y - k) = x(x - k)$. Interpret the locus.
Q.R. 38

1. \( P, Q \) are variable points \((c \rho, c \phi), (a \mu, a \nu)\). Find the envelope of \(PQ\) if \(a^2 - c^2 - k^2 = 0\), where \(k\) is constant.

2. If the tangent at \(P(kx, 2k)\) to \(y^2 = 4x\), oblique axes \(Ox, Oy\), angle \(\alpha\), is perpendicular to \(Ox\), find \(t\). State the significance of \(P\).

3. Find the equation and length of the transverse axis of \(x^2 + 4xy - 12a^2 = 0\), \(P, Q\) are points on \((1 + \frac{1}{2} \cos \theta) = 3\) given by \(\theta = 30^\circ, 90^\circ = 60^\circ\). Find: (i) the equations of \(PQ\) and the tangent at \(P\); (ii) the polar coordinates of the pole of \(PQ\).

4. With the notation of p. 266 if \(P(x_1, y_1)\) lies on the locus \(kS - S_1S_2\) where \(k\) is constant, prove \(P_2(x_2, y_2)\) also lies on it and state the condition that \(P_2(x_2, y_2)\) then lies on it.

5. With the notation of p. 267, prove \(h \mu = aw = 0\) is the diameter conjugate to \(\nu = 0\). Find the diameter conjugate to \(v = 0\).

Q.R. 39

1. \( P_1, P_2 \) are the points \((x_1, y_1), (x_2, y_2)\) referred to oblique axes \(Ox, Oy\), angle \(\alpha\). Find: (i) area \(\Delta OP_1P_2\); (ii) length of perpendicular from \(O\) to \(P_1P_2\).

2. Find \(a\) if \(\sin \theta(x^2 - y^2) + 2xy \cos \theta = 0\) is congruent to \(\theta = c^2\).

3. Prove that the inverses with respect to \(x^2 + y^2 = k^2\) of \(x^2 + y^2 + 2x - 3y = 0, x^2 + y^2 - 3x - 2y = 0\) are perpendicular lines. What does this show?

4. Find the equation of the axis, the coordinates of the vertex and the length of the latus rectum of the parabola \((x + 3y)^2 = 12x - 11y - 36\).

5. With the notation of p. 267, find the pole with respect to \(S - 0\) of \(\nu + k\mu = 0, k\) constant, \(k \neq 0\). What happens when \(k \to 0\)?

6. Find the \((x, y)\) equation of the envelope of the variable line \(lx + my + n = 0\) if \(l^2 + 2lm - m^2 - in - n^2 = 0\).

Q.R. 40

1. Prove that the angle \(\theta\) between the tangents from \(P(h, k)\) to \(y^2 = 4ax\) is given by \((h + a)^2 \tan \theta = -k^2 - kh\). What happens if \(h = -a\)?

2. The equation of a conic \(px^2 + qy^2 = 1\), oblique axes \(Ox, Oy\), angle \(\alpha\). Find \(a\) \(= b\) if \(ax^2 + 2bxy + by^2 = 0\) represents the principal axes.

3. State in detail the envelope of the line whose polar equation is \(\alpha = \phi\) \(= c\) \(= 0\), \(\phi\) \(= c\) \(= 0\), \(\phi\) \(= c\) \(= 0\). Find the parametric equations for the line whose equation is \(ax \sin \theta + by \tan \theta = k^2 - b^2\), where \(b\) varies.

4. With the notation of p. 266, identify the loci:

   (i) \(S_1, S_1\); (ii) \(S_1 + S_2 = S_2 \) where \(S_1 = S_2\); (iii) \(SS_1 = S_1\).

5. Find parametric equations for the line whose equation is \(lx = my = k = 0\), \(k = 0\) \(= b\) \(= 0\), where \(b\) varies.

6. If \(x + y = 0, \mu = 0, g \mu = 0\) are conjugate lines of the conic whose line-equation is \(f^2 + 6am - 3m^2 + 2an = 0\) for all values of \(a, t\), find \(h\) and \(k\), and interpret the result.

Q.R. 41

1. Find \(k\) if \(2x^2 + 5xy + 2y^2 + 18x + 18y = k\) represents two lines.

2. If the feet of the normals from \(h, k\) to \(x = 2at, y = 2at\) are given by \(t = t_1, t_2, t_3\), find \(h, k\) and \(l\) in terms of: (i) \(t_1, t_2, t_3\); (ii) \(t_1, t_2, t_3, t_4\).

3. Find the condition that \((ax + by)^2 + 2xyz + (y - nx + c)(y - mx + d) = 0\) represents a circle. Deduce a geometrical property.

4. Find the hyperbola which touches \(Ox, Oy\) at \((2, 0)\) and \((0, 3)\) and has one asymptote parallel to \(3x = 4y\). Find the asymptotes.

5. If \(k(y^2 - 4x^2) + (ax + 2kxy + by)^2 = \lambda(x^2 + my + n), \) find \(l, m, n\) in terms of \(a, b, c, h\). State the significance of the line \(kx + my + n = 0\).

6. Prove two pairs of common chords of \(x^2 + 4xy + 2y^2 - 6x + 4y - 3 = 0\) and \(x^2 + y^2 = 1\) coincide. Find the other pair.

Q.R. 42

1. Interpret relative to \(px^2 + qy^2 = 1\) the condition that \(r = \pm c\) are the roots of \(p(h + r \cos \theta)^2 + q(k + r \sin \theta)^2 = 1\).

2. Find the equations of the asymptotes and the values of \(t\) which give the vertex of the conic, \(x = 2t(t + 1), y = t(t - 1)\).

3. Find the equation of the conic which passes through the points \((1, 2), (2, 1), (3, 1), (-1, 1), (-2, 3)\).

4. State in detail the relations between the conics given by: (i) point-equations, \(S = 0, S_1 = S_2; \) (ii) line-equations, \(S = 0, S_1 = S_2, S_2; \) (iii) where the tangent to \(y = ax^2\) at \((1, 1)\) cuts the curve.

5. Find the equation of the parabola which has four-point contact with \(x^2 + y^2 = 2^2\) at \((a, c, 0, a)\) \(= \tan 0\).

Q.R. 43

1. Prove \(y = 4x\) cuts \(4x^2 + y^2 = 4x + 4\) orthogonally.

2. The feet of the normals from \(h, k\) to \(ax + by = 1\) are given by \((x_1, y_1), r = 1, 2, 3, 4\). Prove \(Sx = -2bh(a - b)\). What is the value of \(S_2y^2\)?

3. Prove that the equation of the directrix of the parabola, \(x = h^2 + 2mt, y = pt^2 + 2ft, \) is \(lx + my + m^2 + y^2 = 0\).

4. If \(x^2 + xy - 4y^2 = 0\) meets \(x^2 + y^2 + x - 3y = 0\) at \(A, B\) in addition to the origin \(O\), find the equation of \(AB\).

5. \(S = 0, S' = 0\) are given conics; \(P(x_1, y_1)\) is a given point. Prove the polar of \(P\) with respect to the variable conic \(S + kS^2 = 0\) concur.

6. The circle which has three-point contact with \(x = y = ax = t^2, t^2 = (1 + t^2)\) at \(t = t_1, t_2\) meets the curve again at \(t = t_2\). Prove \(t_2 = -\frac{1}{t_1}\).
Q.R. 44

1. Find the nature of the locus, \(x = (at + b)/(t + c), y = (at' + b')/(t' + c')\), if: (i) \(c = c'\); (ii) \(c = c'\).

2. Prove \(12x^2 + 7xy - 12y^2 = 55x - 10y\) is congruent to \(x^2 - y^2 - 4\).

3. \(P, Q\) are variable points on \(ax^2 + 2kxy + by^2 + 2px + 2qy = 0\) such that \(LP\) and \(MQ\) cut the normal at \(O\) at a fixed point.

4. \(P_1(x_1, y_1), P_2(x_2, y_2)\) are fixed, \(P_3(x, y)\) varies so that for some value of \(k\) the polar of \(P_3\) to \(O\) is \((kx_1 + x_2)/(1 - k), (ky_1 + y_2)/(1 + k)\) with respect to a fixed conic \(S = 0\) is \(P_3P_2P_1\). Prove \(P_3\) lies on \(S_{x_1y_1} = S_{x_2y_2}\). Interpret this result.

5. If \((ax^2 + 2kxy + by^2 + (cx + dy + 1)(dx + my) = 0\) is a circle, find \(l, m\) in terms of \(a, b, c, d, k\). What does \(lx + my = 0\) then represent?

6. Find the equation of the tangent to \(x : y = 1 = (t + 2) : (t^2 + 2) : (t^3 - t)\) at the point \(t\) and the equations of the asymptotes.

Q.R. 45

1. Find the condition that \((ap^2, bp^2), (aq^2, bq^2), (ar^2, br^2)\) are collinear.

2. If \(y^2 = 4ax\) meets \(x^2 + 2fy + c = 0\) at \(P, Q, R\), prove \(O, P, Q, R\) lie on a circle and find its equation.

3. Find the condition that the origin lies on one of the asymptotes of the hyperbola \(ax^2 + 2kxy + by^2 + 2px + 2qy + c = 0\).

4. Find the line-equation of a variable conic touching \(x + y = 3, 2x + y = 5, x + 2y = 1, x - 2y = 3, x = 0\), and the \((x, y)\) equation of its directrix.

5. Find the conditions that the normals to \(ax^2 + by^2 = 1\) at its points of intersection with \(lx + my = 1, px + qy = 1\) are concurrent.

6. Prove the circle of curvature of \(ax^2 + 2kxy + by^2 + 2px + 2qy = 0\) at the origin is \((af^2 - 2gfh + bd^2)(x^2 + y^2) + 2(f^2 + g^2)(fx + fy) = 0\).

Q.R. 46

1. Find the value of \(k\) if \(x^2 + y^2 = k(x - y - 1)^2\) represents a parabola and interpret the result in detail.

2. \(px + qy = 1\) cuts \(S = ax^2 + 2kxy + by^2 + 2px + 2qy + c = 0\) at \(A, B\); \(OA, OB\) cut \(S = 0\) again at \(C, D\). Find the equation of \(CD\).

3. Find the equation of the polar of \((x_0, y_0)\) with respect to \(S_1xS_2y\).

4. \(x \cos \alpha + y \sin \alpha = p = 0\) meets \(ax^2 + by^2 = 1\) at \(H, K\). Find the equation of the circle on \(HK\) as diameter.

5. If \(S = 0, S' = 0\) are conics having \(L_1 = 0, L_2 = 0\) as one pair of common chords, prove there is a conic touching \(S = 0\) where it is cut by \(L_1 + kL_2 = 0\) and touching \(S' = 0\) where it is cut by \(L_1 - kL_2 = 0\).

6. Find the tangent to \(x : y : 1 = t : 1 : 1 + t^2\) at the point \(t\). Prove the tangents at \(t = \pm 1\) are parallel to \(x + y = 0\) and that \(x^2 + y^2 \leq \frac{1}{2}\). Examine the form of the curve at the origin and sketch the curve.

Test Paper 23

1. Prove that the line whose equation is \(lx + my + n = 0\) forms with the pair of lines given by \((lx + my)^4 = 3(lx - my)\) an equilateral triangle.

2. The normal to the parabola \(y^2 = 4ax\) at a variable point \(P\) meets the curve again at \(Q\); \(R\) divides \(PQ\) so that \(PR : RQ = 1 : 3\). Prove that the equation of the locus of \(R\) is \(y^2 = a(x - 3a)\).

3. Find the coordinates of the centre and the equation of the transverse axis of the hyperbola \(x^2 + 4xy + y^2 - 2x + 2y = 0\).

4. A variable chord of \(y^2 = 4ax\) passes through the point \((h, k)\). Prove that the locus of the mid-point of the chord is the parabola, \(y^2 - 2ax - ky + 2ah = 0\), and find the coordinates of the vertex of this parabola.

5. The perpendicular from a point \(P\) to its polar with respect to the conic \(px^2 + qy^2 = 1\) meets \(Ox, Oy\) at \(O, \theta\); \(N, N'\) are the feet of the perpendiculars from \(R\) to \(Ox, Oy\). Prove that \(\theta - ON' = ON' = ON = -p : q\).

6. \(PQ\) is a diameter of the auxiliary circle of an ellipse. Prove that the tangents from \(P\) and \(Q\) to the ellipse either are parallel or intersect on a directrix.

Test Paper 24

1. Prove that the angle between the pair of lines given by \(x^2(1 + 2 \cos \theta) + 4xy \sin \theta + y^2(1 - 2 \cos \theta) = 0\) is independent of the value of \(\theta\). Find the equation of the locus of the centre of the circle which touches the lines, if \(\theta\) is given.

2. If \(x + y = c\) is a normal to \(x^2 + y^2 + b^2 = 1\) and to \(x^2/p^2 + y^2/q^2 = 1\), prove that \((a^2 + b^2)(p^2 + q^2) = (p^2 - q^2)(a^2 + b^2)\).

3. Find the value of \(k\) if \(x^2 + 2axy + by^2 + xk + 2 = 0\) represents a pair of lines.

4. The pole \(T\) of a chord \(PQ\) of the parabola \(y^2 = 4ax\) lies on the line \(x + 2a = 0\). Prove that the vertex of the parabola is the orthocentre of the triangle \(TPQ\).

5. \(PN\) is the perpendicular from a variable point \(P\) to a given line \(AB\); \(PT\) is the tangent from \(P\) to a given circle. If \(PT = c\), \(PN\), where \(c\) is constant, prove the locus of \(P\) is a conic of eccentricity \(c\).

6. \(P, Q\) are the points \((kp^2, 2kp), (kp^2, 2kp)\) on the parabola \(y^2 = 4ax\), referred to oblique axes \(Ox, Oy\); the tangents at \(P, Q\) and chord \(PQ\) meet the diameter through \(O\) at \(P', Q', R\). Prove \(OP' = OQ' = OR\).
Test Paper 25

1. Find $g, f, c$ if $6x^2 - 7xy - 3y^2 + 2xz + 2fy + c = 0$ represents two lines which intersect at the point $(1, -2)$.

2. A variable chord $PQ$ of the parabola $y^2 = 4ax$ passes through the point of intersection of the directrix and axis. Prove that the normals at $P, Q$ meet on a fixed parabola whose vertex is at the focus.

3. $T(h, k)$ is a point on the director circle of $x^2/a^2 + y^2/b^2 = 1$. $N(h', k')$ is the foot of the perpendicular from $T$ to its polar with respect to the ellipse. Prove that $(a^2 - b^2)k' = a'k$ and $(a^2 - b^2)h' = b'k$.

4. The normal to $x^2/a^2 - y^2/b^2 = 1$ at $P$ meets the transverse axis at $G$ and $G$ is the midpoint of $Gy$. (i) Prove that $OZ$ is inversely proportional to the distance of $O$ from the tangent at $P$. (ii) If $a = b$, prove that $Z$ coincides with $P$. (OC)

5. The mid-point of a variable chord $PQ$ of the general hyperbola $xy = c^2$, referred to oblique axes, lies on a fixed line parallel to one asymptote; prove the pole of $PQ$ lies on a fixed line parallel to the other.

6. The polar equation of a given circle is $x^2 + 2x + 3y = 0$. Find the polar equation of the locus of the pole of a line $OPQ$ if: (i) $OP = OQ + P Q$; (ii) $1/OP = 1/OP + 1/OQ$.

Test Paper 26

1. The point $(x_1, y_1)$ is the image of $(h, k)$ in the line $px + qy = 0$. Prove $(p^2 + q^2)x_1 = (q^2 - p^2)h - 2pqk$ and $(p^2 + q^2)y_1 = (p^2 - q^2)k - 2pqh$. Find the equation of the image of the line $y = mx$ in $px + qy = 0$.

2. The tangents at the points $H, K, P, Q$ on an ellipse of eccentricity $e$ form a square with sides of length $c$ and with vertices on the axes of the ellipse. Prove that the normals at $H, K, P, Q$ form a square with sides of length $a\sqrt{2 - e^2}$.

3. A circle whose centre lies on $x^2/a^2 + y^2/b^2 = 1$ touches two conjugate diameters. Prove its radius equals $ab/\sqrt{(a^2 + b^2)}$.

4. $ST, ST'$ are the perpendiculars from the foci $S, S'$ of $x^2/a^2 + y^2/b^2 = 1$ to the tangent at $(a\cos, b\sin)$. Prove that the $y$-coordinate of the centre of the circle on $Y'$ as diameter is $a'\cos \sin \theta$. Find the radius. Prove the circle touches $SS'$ if $(a^2 - b^2)\sin \theta = b^2$.

5. Prove $z^2 + 2xy + 8z + 7y = 0$, $z^2 + 4xy - 2y^2 + 14z - 4y + 1 = 0$ represent conics with the same centre $C$. Find their equations referred to parallel axes $CX, CY$ and prove that $X^2 + 2XY - Y^2 = 0$ is the equation of their common conjugate diameters.

6. $PQ$ is a variable chord of $px + qy = 0$. If $\angle POQ$ is a right angle, prove the locus of the pole of $PQ$ is an ellipse unless $p + q \leq 0$.

Test Paper 27

1. Find the coordinates of the points on $4x^2 + 9y^2 = 1$ at which the tangents are parallel to $8x = 9y$. (L)

2. The normal to $x^2/a^2 - y^2/b^2 = 1$ at $P$ meets the conjugate axis at $Q$. $OP$ is the perpendicular to the tangent at $P$. Prove $OQ \cdot OP = a^2$.

3. The tangents from a variable point $P$ to $x^2/a^2 + y^2/b^2 = 1$ are separated harmonically by the lines joining $P$ to the fixed points $(c, 0), (-c, 0)$. Prove that the locus of $P$ is $x^2 + y^2/(x^2 + y^2) = a^2$.

4. If the focus of the parabola $x = at^2, y = 2at$ is the orthocentre of the triangle $PQR$ whose vertices are given by $t = p, t = q, t = r$, prove $p + q + r = p + q + r = 3$. Deduce that the sides of the triangle $PQR$ touch the circle $(x - 3a)^2 + y^2 = 4a^2$.

5. $P$ lies on an ellipse, foici $S, S'$; $T, T'$ are points on $SP$ produced beyond $P$ and on $PS'$ produced beyond $S'$. Prove the ellipse through $P$ having $T, T'$ as focus cuts the first ellipse orthogonally at $P$. (N)

6. $Ox, Oy$ are oblique axes, angle $\alpha$; $Ox\alpha, Oy\alpha$ are oblique axes, angle $\alpha'$. If the transform of $ax^2 + 2bxy + by^2$ is $\alpha x^2 - 2b\alpha y^2 + b\alpha^2$, prove $(a^2 + b^2)h + bh = \lambda(x^2 + 2xy + y^2)$ and $(a^2 + b^2)h + bh = \lambda(x + 2xy + y^2)$ and $(a^2 + b^2)h + bh = \lambda(x^2 + 2xy + y^2)$ are perfect squares for the same values of $\lambda$. Deduce that:

(i) $(a + b - 2\cos \alpha) / \sin \omega = (a + b + 2\cos \alpha / \sin \omega)^2$;

(ii) $(a - b) \sin \omega = (a + b) / \sin \omega$.  

Test Paper 28

1. $OAB$ is a given triangle; $P, Q$ are variable points on $OA, OB$ such that $AP = BQ$. Prove that the locus of the mid-point of $PQ$ is a line parallel to the bisector of $\angle AOB$.

2. A, B, C are the points $(x, 0), (0, y, 0)$; $PQ$ is the diameter of $x^2/a^2 + y^2/b^2 = 1$ parallel to $AC$. Prove that $PQ = AC$.

3. Prove that the fixed line $HK, ax + by + c = 0$, with respect to the conic $x^2/(p+q) + y^2/(q+c) = 1$, where $a, b, c$ are constant and $t$ varies, lies on a fixed line perpendicular to $HK$.

4. $S$ and $S'$ are the foci of a conic; $S'$ is the perpendicular to the tangent at $P$; $Q$ is the point of contact of the other tangent from $Y$. Prove that $SQ$ is parallel to $SP$.

5. The equation of a hyperbola is $x^2 - y^2 = b^2$, oblique axes $Ox, Oy$ at angle $\omega$. Prove the equations of the tangents and normal at $(k, \omega)$ are $x + t\omega = 2k$ and $x + t\omega = \omega j\omega + \omega (t - j\omega)$. Use this to infinity along the major axis that if the tangents to a hyperbola at $Q, R$ meet at $P$, then (i) $\angle QSP = \angle RSP$ and (ii) the angle between $PQ$ and $PR$ produced equals each of the angles $PSQ, PSR$.

(iii) $SP^2 = SQ \cdot SR$. 

Test Paper 29
Test Papers 29–36 (Ch. 1–17)

Test Paper 29

1. Prove that the normals to \( x^2 + ay^2 = 2ax \) at \((x_1, y_1), (x_2, y_2)\) meet at the point whose abscissa is \(-x_1y_1(x_1 + x_2) / (8a^2)\). (S)

2. \( AB \) is a given diameter of a circle, centre \( O \); \( T \) is the pole of a variable chord \( BP \); \( OP \) cuts \( AT \) at \( Q \). Prove the locus of \( Q \) is an ellipse, focus \( O \), directrix \( BT \), eccentricity \( \frac{1}{2} \).

3. The tangent at \( P \) to \( x^2/a^2 + y^2/b^2 = 1 \) meets the asymptotes at \( H, K \); the normal at \( P \) meets \( Ox, Oy \) at \( G, g \). Prove that the points of \( G, g, H, K \) lie on a circle.

4. The mid-point of a variable chord \( PQ \) of \( x^2/a^2 + y^2/b^2 = 1 \) lies on the fixed line \( HK, px + qy + r = 0, r \neq 0 \). Prove that the envelope of \( PQ \) is a parabola whose directrix is \( a^2px + b^2qy + r(a^2 + b^2) = 0 \).

5. A circle, centre \( C \), cuts \( xy = c^2 \) at \( P_1, P_2, P_3, P_4 \); the normals at \( P_1, P_2, P_3, P_4 \) meet \( xy = c^2 \) again at \( Q_1, Q_2, Q_3, Q_4 \). Prove: (i) the mean centre of \( P_1, P_2, P_3, P_4 \) is the mid-point of \( OQ \); (ii) \( Q_1, Q_2, Q_3, Q_4 \) are concyclic.

6. A variable conic touches \( S = 0 \) where it is out by the variable line \( x + \lambda y = 0 \) and passes through the fixed point \((x_0, y_0)\). Prove the equation of the locus of its centre is \((S_1 - w)^2 = S_1(S - w)\). [Notation as on p. 367.]

Test Paper 30

1. A variable tangent to \( x^2 + y^2 = 2ax - 2ay - a^2 \) cuts \( Ox, Oy \) in \( P, Q \). Prove the locus of the mid-point of \( PQ \) is \( x^2 + y^2 = a^2 \).

2. If \( a \cos \alpha + y \sin \alpha = p \) is a common tangent of \( x^2/a^2 + y^2/b^2 = 1 \) and \( x^2/c^2 + y^2/d^2 = 1 \), prove \( a^2 - c^2 \cos^2 \alpha = d^2 - b^2 \sin^2 \alpha \). If the meets of the four common tangents are concyclic, prove \( a^2 + b^2 - c^2 = d^2 \). (OC)

3. Prove that \( x^2 + y^2 = 2x, y = x^2 - 2 \) represents a parabola whose vertex is given by \( t = -1 \), with \( \frac{1}{2}, -\frac{1}{2} \) as focus and \( x - y = -3 \) as directrix.

4. If the focus of \( x^2/a^2 + y^2/b^2 = 1, x^2/c^2 + (y - k)^2/d^2 = 1 \), \( a > b > 0, d > c > 0 \), lie on a circle and if \( x^2 - c^2 > a^2 - b^2 \), prove \( k^2 = d^2 - a^2 + b^2 - a^2 \) and that the equation of the circle is \( x^2 + y^2 - 2ky = a^2 - b^2 \).

5. \( U \) is the pole of a given chord \( AB \) of a given circle, centre \( O \). If the distance from \( AB \) of a variable point \( P \) is equal to the tangent from \( P \) to the circle, prove that the locus of \( P \) is a parabola whose focus is the mid-point of \( OAB \).

6. Prove that the tangent to \( x = a(t \cos \theta - \sin \theta), y = a \sin \theta \) at the point \( t \) is \( x^2 - y^2 = 2a(x \cos \theta - y \sin \theta) \). Prove that the equation of the locus of the foot of the perpendicular from \( O \) to the tangent is \( (x^2 + y^2)^2 = 4a^2(x \cos \alpha - y \sin \alpha) \). [Notation as in p. 397.]

Test Paper 31

1. Prove that the conics whose equations are \( ax^2 - by^2 = 1 \) and \( ax^2 + 2bxy + by^2 + (a + b)(x - a) - c = 0 \) cut orthogonally.

2. If \( a^2 = 2b^2 \), prove there is a value of \( b \) such that the tangent at \( (a \cos \theta, b \sin \theta) \) to \( x^2/a^2 + y^2/b^2 = 1 \) cuts the normal at \( (a \cos \theta, -b \sin \theta) \) on \( Oy \). (C)

3. With the notation of p. 267, prove the chord \( PQ \) of \( S = 0 \) whose mid-point is \((x_0, y_0) \) is parallel to \( c_0x + c_1y = 0 \) and the diameter bisecting \( PQ \) is \( 0, x + y = 0 \). Deduce that the axes of \( S = 0 \) are \( h = (x^2 - y^2) = (a - b) \mu \).

4. A circle through the origin cuts \( x^2 + 2hxy + y^2 + 2gx + 2fy + c = 0 \) at \( A, B, C, D \). If \( AB \) is \( kx + my + n = 0 \), find the equation of \( CD \).

5. Prove that the equation of the circumcircle of the triangle formed by the tangents to \( y^2 = 2x \) at \((t_1^2, -2t_1), (t_2^2, -2t_2), (t_3^2, -2t_3) \), where \( t_1, t_2, t_3 \) are the roots of \( a^2 + b^2 + x^2 + x(1 + t^2) = 0 \) is \( a(x^2 + y^2) - (a + e)x - (b - d)y + e = 0 \).

6. Prove the tangents to \( px^2 + qy^2 = 1 \) at the feet of the normals from \( G(h, k) \) touch the parabola given by \( \Sigma x^2 = (p - q)x + qy(4x + h) = 0 \). Show that \( G \) lies on the directrix of the parabola and that the point of \( G \) with respect to \( px^2 + qy^2 = 1 \) touches the parabola.

Test Paper 32

1. The normal to \( x^2/a^2 + y^2/b^2 = 1 \) at a point \( P \) in the first quadrant meets \( z = 0 \) at \( g; \) \( PQ \) is a diameter. If \( c > 1/\sqrt{2} \), prove a position of \( P \) exists such that \( OQ \) passes through the focus \((-ae, 0)\).

2. Find the centre and equation of the asymptotes and transverse axis of the hyperbola \( 3x^2 + 8xy + 2y^2 = 10x + 12y - 4 \).

3. If the polar of \( (x_0, y_0) \) with respect to \( px^2 + qy^2 = 1 \) is the chord \( QR \), prove the mid-point of \( QR \) is \((k, \lambda x_0)\), where \( k(px_0^2 + qy_0^2) = 1 \). If \( P \) varies so that \( QR = 2c \), where \( c \) is constant, prove the equation of the locus of \( P \) is \( (px_0^2 + qy_0^2 - 1)(px^2 + qy^2) = c^2 pq(px_0^2 + qy_0^2) \).

4. \( OH \) is the tangent at a given point \( O \) on a conic. The tangents from a variable point \( P \) to the conic meet \( OH \) in \( Q, R \). If \( OQ \), \( OR \) is constant, prove the locus of \( P \) is a line parallel to \( OH \).

5. Prove the equation of the normal to \( x = n \cos t, t = \cos nt, y = n \sin t, n > 1 \), at the point \( t = \cos \frac{n}{2} + (n^2 - 1)n \sin \frac{n}{2} = n(-1) \cos \frac{n}{2}, \frac{n}{2} \). If \( n \) is an even integer, prove the normals at the point \( t = t_1, t = t_1 + n, n \), intersect at right angles on the circle \( x^2 + y^2 = (n - 1)^2 \). (C)

6. Prove the locus of \( z = at + 2b + c, a = at + 2b + c, t \) is a parabola. If \( t = t_1, t_1, t_1, t_1 \), \( t_1 \) are concyclic points on the locus, prove that \((a^2 + a^2) (t_1 + t_2 + t_3 + t_4 + 4(\alpha + a, b) = 0 \).

Deduce that the vertex of the parabola is given by \( t = -(\alpha + a, b)/(a^2 + a^2) \).
Test Paper 33

1. Prove the circle \( x^2 + y^2 - 2kx + (k - 2a)^2 = 0 \) touches \( y^2 = 4ax \) if \( k > 2a > 0 \). If the circle passes through the focus and touches \( y^2 = 4ax \) at \( P \), prove its radius equals the distance of \( P \) from the directrix. (C)

2. If \( \cos \alpha + y \sin \alpha = p \) cuts \( x^2 + 2hxy - y^2 = 0 \) at \( A, B \), prove that the equation of the circle \( OAB \), where \( O \) is the origin, is \((\cos 2x + h \sin 2x)(x^2 + y^2) - 2p(k \cos \alpha - \sin \alpha) = 0\). State the coordinates of the midpoint of \( AB \). (N)

3. Find the equation of the directrix of the parabola, with axis parallel to \( x + 2y = 0 \), touching \( Ox \) at \( (3, 0) \) and passing through \( (0, 2) \).

4. If \( S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \) and \( S' = x^2 + y^2 - r^2 \), prove the locus of the centre of \( S + kS' = 0 \), where \( k \) varies, is a rectangular hyperbola through the feet of the normals from the origin to \( S = 0 \).

5. If the tangents from a variable point \( T \) to a parabola intersect a constant length on the tangent at a given point \( O \), prove the locus of \( T \) is an equal parabola. [Use oblique axes.]

6. Find the condition that the line \( (x - x_1) + m(y - y_1) = 0 \) touches the parabola given by \( x = a^2 + 2bt, y = a^2 + 2dt \). Deduce that the equation of the directrix is \( ax + cy + b^2 + d^2 = 0 \).

Test Paper 34

1. \( x \cos \alpha + y \sin \alpha = p \) and \( x \cos \beta - y \sin \beta = q \) cut \( ax^2 + by^2 = 1 \) at \( A, B, C, D \). Prove: (i) the conics through \( A, B, C, D \) have parallel axes; (ii) there is in general just one rectangular hyperbola through \( A, B, C, D \). Under what conditions is every conic through \( A, B, C, D \) a rectangular hyperbola?

2. POP', QOQ' are diameters of \( x^2/a^2 + y^2/b^2 = 1 \) such that \( PQ' \) passes through the fixed point \( (h, 0) \). Prove that the envelope of \( PQ \) is \( x^2/a^2 + y^2/b^2 = 1 \).

3. Prove the locus of the pole with respect to \( px + qy - 1 \) of a variable tangent to the Apollonius hyperbola \( (p - q)x^2 + (p + q)y^2 = 4pr - 4qy \) is the parabola \( y^2 = 2px + 2qy - q(x^2 + y^2) \).

4. If the normals to \( y^2 = 4ax \) at \( P, Q, R \) are concurrent, prove the perpendiculars to \( QR \), \( RP \), \( PQ \) from their poles are concurrent.

5. A given line \( ax + by - 1 = 0 \) cuts oblique axes \( Ox, Oy \) at \( A, B \). Prove that the points of contact of the tangents from a given point \( T(h, k) \) to a variable conic touching \( OA, OB \) at \( A, B \) lie on the conic \( 2x^2(a + bx - 1) = (ax + by - 1)(ax + by) \).

6. \( T \) is the pole with respect to an ellipse, foci \( S, S' \), of a line which meets the directrices corresponding to \( S, S' \) at \( R, S' \). BS, BS' meet at \( U \). Prove \( TU \) is a diameter of the circle \( 7S'S' \). If \( T \) varies on a fixed line \( HK \), prove the locus of \( U \) is a hyperbola.

Test Paper 35

1. Prove the mean centre of the feet of the four normals from \( G \) to a rectangular hyperbola, centre \( C \), divides \( GC \) in the ratio \( 1 : 3 \).

2. \( PM \) is the perpendicular from \( P \) on \( y^2 = 4ax \), focus \( S \), to a chord \( QR \) perpendicular to \( Ox \). A coaxal system of circles is determined by the circle touching \( y^2 = 4ax \) at \( Q, R \) and the point-circle \( M \). Prove the radical axis of \( Q, R \) is the tangent at \( P \) and the second limiting line lies on \( SP \), produced if necessary. (OC)

3. \( ax^2 + 2hxy + by^2 = 0 \) represents \( OH, QK, C \) is \( (f, g) \). Interpret the locus \( (ax^2 + 2hxy + by^2) + [f(x - f)^2 - 2g(x - f)(y + g)] + [f(y - g)^2] = 0 \).

4. Prove the normals to \( 4x^2 + 9y^2 = 36 \) at its points of intersection with \( 3x + 2y = 18 \) and \( 8x + 15y = -20 \) are concurrent.

5. The points \( P_1, P_2, P_3, P_4 \) on \( xy = c^2 \) form a triangle and its orthocentre; \( T_{12} \) is the pole of \( P_1, P_3, P_3, P_4, P_4 \) and similarly for \( T_{24}, T_{32}, T_{43} \) etc. Prove the circles with diameters \( T_{12}T_{14}, T_{23}T_{24}, T_{34}T_{32}, T_{41}T_{43}, T_{14} \) touch at \( 0 \).

6. Prove the envelope equation of a conic touching \( Ox \) at \( O \) can be taken as \( \Sigma = a^2b^2 + 2bma + c^2 - a = 0 \). A variable conic has four-point contact at \( O \) with \( \Sigma = 0 \), prove its foci lie on the curve \( (x^2 + y^2)(y^2 - Fx) = -Dxy \).

Test Paper 36

1. Prove there are points \( P, P' \) on \( x^2/a^2 + y^2/b^2 = 1 \), \( a^2 > b^2 \), foci \( S, S' \), such that \( SPS'P' \) is a rectangle and that its area is \( 2b^2 \).

2. \( ABD_1, BHD_2, CHD_3 \) are the altitudes of \( \triangle ABC \); \( BC, CA, AB \) are given by \( x = a + by + c, y = b + cx + 0 \), \( r = 1, 2, 3 \). Prove that \( H \) is given by \( \lambda_1t_1, \mu_1t_1, \lambda_2t_2, \mu_2t_2, \mu_3t_3, \nu_3t_3, \) where \( \lambda_1, \mu_1, \lambda_2, \mu_2, \mu_3, \nu_3 \) and the meets of \( BC, DA, DB, DB, AD, BD \) lie on \( \lambda_1t_1, \mu_1t_1, \lambda_2t_2, \mu_2t_2, \mu_3t_3, \nu_3t_3 \).

3. \( \Sigma = 0, \Sigma' = 0 \) are the envelope equations of two conics. Use Ex. 10, No. 15, p. 271, to interpret the envelope \( \Sigma, \Sigma' \), \( \Sigma, \Sigma \).

4. If \( S = ax^2 + ay^2 + 2c = 0, a^2 + b^2 < c \), cuts orthogonally at four points, prove \( x(ax + q)/a^2 + y(by + p)/b^2 = 0 \) is a conic of the system \( S + kS' = 0 \); deduce that \( g = 0 \) or \( f = 0 \). Prove that if \( g = 0 \) and \( a^2 + b^2 < 2c \), \( S = 0 \) is of the form \( x^2/a^2 + 2by^2 = 2b^2 + 2y^2 - c = 0 \). (OC)

5. If \( x^2/a^2 + y^2/b^2 = 1, a > b > 0, x^2/a^2 - y^2/b^2 = 1, c > d > 0 \), have \( (a, b, 0) \) as foci, prove they cut orthogonally at \( P(ae, bd, 0) \). Prove by orthogonal projection that \( GP \) divides the area of the first quadrant of the ellipse in the ratio \( \alpha = (x + z) \) where \( \tan \alpha = \alpha \).

6. \( G \) is the centroid of \( \triangle ABC \); \( A' \) is the mid-point of \( BC \); the circles \( OA'A', BA'B, CA'C \) cut again at \( H, K \). Prove by inversion that \( A'K, A'H, A'K \) touch the circles \( A'BG, A'CG, A'HK \).
FORMULAS AND EQUATIONS

7. Nature of the Locus $S=0$.
(i) If $ab>h^2$ and $(a+b)\Delta<0$, $S=0$ is an ellipse.
(ii) If $ab<h^2$ and $\Delta<0$, $S=0$ is a hyperbola, which is
    rectangular if $a+b=0$.
(iii) If $ab=h^2$ and $\Delta=0$, $S=0$ is a parabola.
(iv) If $\Delta=0$, $S=0$ is either a pair of lines intersecting or
    parallel or coincident) or the locus contains just one
    point or no point at all.

8. Special Points and Lines of the Proper Conic $S=0$.
(i) If $ab+h^2$, centre of $S=0$ is given by

$$u=\frac{1}{2}\frac{a}{\partial S}\frac{a}{\partial x}+\frac{b}{\partial y}+g=0,$$

$$v=\frac{1}{2}\frac{b}{\partial S}\frac{b}{\partial y}+h=0.$$  

(ii) If $ab+h^2$, the equation of the principal axis of $S=0$ is

$$h(a^2-x^2)-(a-b)w=0.$$  

(iii) If $ab+h^2$, the equation of the asymptote of $S=0$ is

$$S-\Delta(ab-h^2)=0$$  

or $bu^2=2hu+aw^2=0$.  

(iv) If $ab+h^2$, the coordinates of the foci of $S=0$ satisfy the
    equations,

$$G(x^2+y^2)-2Gy+A-B=0,$$

$$Cx^2-Ax-By+H=0.$$  

(v) If $ab+h^2$, the equations of the directrices of the parabola $S=0$
    is $2Gz=2Fy+(A+B)=0$.

8. Tangents and Polars of the Proper Conic $S=0$.
(i) If $S=0$ divides the segment from $P_2$ to $P_3$ in ratio $k_2:k_3$

$$S_1=x_2+y_2+1, x_2+y_2+1.$$  

(ii) The equation of the tangent to $S=0$ at $P_3$ and the equation of
    the polar of $P_3$ with respect to $S=0$ is

$$S_1=x_2+y_2+w_2=x_2+y_2+w.$$  

(iii) The polar of $P_3(1,1,1)$ with respect to $S=0$ is

$$S_1=x_2+y_2+w_2=x_2+y_2+w.$$  

(iv) $P_1$ and $P_2$ are conjugates points with respect to $S=0$ if

$$S_1=x_2+y_2+w_2=x_2+y_2+w.$$  

(v) The equation of the pair of tangents from $P_1$ to $S=0$ is

$$S_1=S_1.$$  

Formulas and Equations (Ch. 12-17)

1. Equation of Line

2. Area of Triangle

3. Elimination of $x:y:o$ from

$$a_1x+b_1y+c_1x=0 \quad a_2x+b_2y+c_2x=0 \quad a_3x+b_3y+c_3x=0.$$  

4. $ax^2+2hxy+by^2=0$ separate harmonically

$$ax^2+2hxy+by^2=0$$

5. The equation of the angle-bisectors of $ax^2+2hxy+by^2=0$ is

$$\left(x^2-y^2\right)/(a-b)=xy/h.$$  

6. The section of a right circular cone of vertical angle $2\alpha$ by a
plane making an angle $\beta$ with the axis of the cone is a conic
whose eccentricity equals $\sqrt{\cos\beta}$.  

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(i) The lengths of steps from \( P_1 \) to points of \( S = 0 \) in direction making an angle \( \theta \) with \( Ox \) are the roots of
\[
r^4(a \cos^4 \theta + 2b \sin \theta \cos \theta + b \sin^2 \theta) + 2r(a \cos^3 \theta \cos \theta + c \sin \theta) + \Sigma_{12} = 0.
\]

(ii) \( u + tv = 0, u + tv'v = 0 \) are conjugate diameters of \( S = 0 \) if
\[
y = \frac{mx}{y} = \frac{\Sigma_{1}}{\Sigma_{2}}x^2 \text{ and } y = \frac{\Sigma_{2}}{\Sigma_{1}}x^2 \text{ are conjugate diameters of } S = 0.
\]

(iii) \( l = m + n = 0, l + m \gamma + n = 0 \) are conjugate lines with respect to \( S = 0 \) if
\[
\Sigma_{12} = 0.
\]

(iv) If \( P_1, P_2 \) are fixed and if \( P \) varies so that \( PP_1, PP_2 \) are conjugate lines with respect to \( S = 0 \), the equation of the locus of \( P \) is
\[
[S_1y - S_2]x = S_1x - S_2y = 0.
\]

11. The equation of the director circle of \( S = 0, ab + h^2 = 0 \), is
\[
(Cx^2 + dy^2 - 2Gx - 2Py + A + B = 0)
\]
If \( S = px^2 + qy^2 + 1 = 0 \), the director circle is \( x^2 + y^2 = 1/p + 1/q \).

12. Invariants.
If \( x \) is a change from rectangular axes \( Ox, Oy \) to rectangular axes \( Ox, Oy \), the transform of \( ax^2 + 2kxy + by^2 \) is
\[
a_1x^2 + 2k_1xy + b_1y^2, \text{ then } a = a_1 + b_1 \text{ and } ab - h^2 = a_1 - b_1.
\]

13. Oblique axes \( Ox, Oy \), where \( \omega \text{ is } \omega \).
(i) The gradient of \( y = mx + c = \omega (1 + m \cos \omega) \).
(ii) The lines \( y = m_1x + c_1, y - m_2x + c_2 \) are at right angles if
\[
1 + (m_1c_2 + m_2c_1) \cos \omega - m_1m_2 = 0.
\]

(iii) The pair of lines \( ax^2 + 2kxy + by^2 = 0 \) are at right angles if
\[
a + b + 2k \cos \omega = 0.
\]

(iv) \( P_1P_2 = (x_1 - x_2)^2 + (y_2 - y_1)^2 + 2(x_1 - x_2)(y_2 - y_1) \cos \omega \).

(v) The equation of the circle, centre \( O \), radius \( r \), is
\[
x^2 + y^2 + 2kxy \cos \omega = r^2.
\]

(vi) The equation of a parabola, lateral rectangle \( 4a \), referred to a diameter as axis \( Ox \) and tangent at \( O \) as axis \( Oy \) is
\[
y^2 = 4ax, \text{ where } k = -a \cos \omega.
\]

(vii) The equation of a central conic referred to a pair of conjugate diameters as axes \( Ox, Oy \) is \( px^2 + qy^2 = 1 \).
Any other pair of conjugate diameters is given by \( y = \frac{mx}{n} \), \( y = \frac{\Sigma_{1}}{\Sigma_{2}}x^2 \text{ where } m^2n - \Sigma_{1} = -\Sigma_{2} \).

(viii) The equation of a general hyperbola referred to its asymptotes as axes \( Ox, Oy \) is \( xy = b^2 + 4(a^2 + b^3) \).

14. Polar Coordinates \( (r, \theta) \).
(i) Perpendicular form of the equation of a line,
\[
r \cos (\theta - \alpha) = p.
\]

(ii) Forms of the equation of a circle,
\[
y = d \cos \theta; \ r = d \cos (\theta - \alpha); \ r^2 - 2r \cos (\alpha + \theta) = \alpha^2.
\]

(iii) Equation of conic, pole at focus, initial line along axis, is
\[
1 - r \cos (1 + \epsilon \cos \theta) = 0.
\]

(iv) Equation of chord joining points \( l = r \cos (1 + \epsilon \cos \theta) \) given by
\[
[l^2 + \epsilon^2 \cos ^2 \theta + 2\epsilon \cos \theta \cos \epsilon \sin \theta = 0;
\]
Equation of tangent at \( \theta = \alpha \) is \( l = r \cos \theta = 0 \text{ cos } \cos \theta \).

(v) Equation of conic, pole at focus, if axis makes angle \( \gamma \) with initial line, is
\[
l = r \cos (1 + \epsilon \cos \theta).
\]

15. Inversion with respect to centre \( x^2 + y^2 = k^2 \), centre \( O \).
(i) If \( F \) is an inversion point, \( F' \) is an inverse point, \( \tau^2 = k^2 \).

(ii) Equation of inverse of focus \( f(r, 0) = 0 \) is \( f(r^2, \theta) = 0 \).

(iii) If \( P, P' \) and \( Q, Q' \) are pairs of inverse points and if \( OE, OF \) are the perpendiculars to \( O \) from \( PQ, P'Q' \), then
\[
PQ : P'Q' = k^2 : OP' ; OQ' = OP ; OQ : k^2 = OE : OF.
\]

(i) If \( \Sigma = 0 \), the \( (x, y) \) equation of the envelope of \( bx + my + n = 0 \) is \( S = 0 \).

(ii) The point of contact of the variable line
\[
xf(t) + yg(t) + h(t) = 0
\]
with its envelope is its intersection with
\[
xf'(t) + yg'(t) + h'(t) = 0
\]

(iii) The equation of the normals to \( y^2 = 4ax \) is given by
\[
x = 2a + 3at, y = -2at^2.
\]

(iv) The envelope of the normals to \( x^2 + y^2 = 1 \) is given by
\[
x^2 + 2 \text{ sin } \alpha y + y^2 = 0
\]

17. Systems of Conics through given Points; Point-equations.
(i) \( S + kS = 0 \) passes through the common points of \( S = 0, S' = 0 \).

(ii) \( S + kL_1L_2 = 0 \) passes through the points where \( L_1 = 0 \), \( L_2 = 0 \) meet \( S = 0 \).

(iii) \( L_1L_2 + L_4L_0 = 0 \) passes through the points where the line-pair \( L_1L_2 = 0 \) meets the line-pair \( L_4L_0 = 0 \).

(iv) \( S + kL_4 = 0 \) has double contact with \( S = 0 \) where it is cut by \( L_4 = 0 \) and four-point contact if \( L_4 = 0 \) touches \( S = 0 \).

(v) \( L_1L_2 + L_4L_0 = 0 \) touches \( L_1 = 0 \), \( L_2 = 0 \) where \( L_4 = 0 \) touches them.

(vi) \( T = 0 \) touches \( S = 0 \) at \( A \) and if \( L = 0 \) cuts \( S = 0 \) at \( C, D \), \( S + kTL = 0 \) touches \( S = 0 \) at \( A \) and passes through \( C, D \), and has three-point contact with \( S = 0 \) at \( A \) if \( C \) coincides with \( A \).

(vii) \( L_1 = 0, L_2 = 0 \) touch at \( A, L_1L_2 + kTL = 0 \) touches \( T = 0 \) at \( A \) and passes through the points of intersection of \( L_1 = 0 \) and \( L_2 = 0 \).

(i) \( \Sigma + k^2 \Sigma' = 0 \) touches the common tangents of \( \Sigma = 0, \Sigma' = 0 \).
(ii) \( \Sigma + k^2 \Sigma_1 = 0 \) touches the tangents from \( P_1 \) and \( P_2 \) to \( \Sigma = 0 \).
(iii) \( a^2 x^2 + k \times \Sigma = 0 \) touches the lines \( P_1 P_3, P_1 P_4, P_2 P_3, P_2 P_4 \).
(iv) If \( P_1 \) lies on the line \( P_3 P_4, x_1 x_2 + k \times \Sigma = 0 \) touches \( P_3 P_4 \) at \( P_1 \) and touches \( P_2 P_4, P_3 P_4 \).
(v) If \( l_1 x + m_1 y + n_1 = 0, l_2 x + m_2 y + n_2 = 0 \) cut \( S = 0 \) at \( P_1, Q_1 \) and \( P_2, Q_2 \), then \( P_1 Q_1 \) and \( P_2 Q_2 \) and the tangents at \( P_1, Q_1 \) and \( P_2, Q_2 \) touch the conic whose envelope-equation is \( \Sigma_1 + \Sigma - \Sigma_2 = 0 \).

[This is the dual of the locus in (iv), p. 334, \( S_1 \Sigma - S_2 \Sigma = 0 \).]

19. Applications of the \( S + k \Sigma \) Theorem.

(i) Any pair of common chords of a circle and conic make supplementary angles with a principal axis of the conic.
(ii) If \( \phi_1, \phi_2, \phi_3, \phi_4 \) are the eccentric angles of the points of intersection of an ellipse with a circle, then \( \phi_1 + \phi_2 + \phi_3 + \phi_4 = \text{a multiple of} 360^\circ \).
(iii) If \( S = 0 \) and \( S' = 0 \) are rectangular hyperbolas, then \( S + k S' = 0 \) is also a rectangular hyperbola. 

Note. A perpendicular line-pair is a degenerate rectangular hyperbola.

(iv) The feet \( P_1, P_2, P_3, P_4 \) of the normals to \( S = p x^2 + q y^2 - 1 = 0 \) from \( (h, k) \) lie on the rectangular hyperbola,

\[
pxy + qxy = 0,
\]

called Apollonius' hyperbola.

If the equation of \( P_1 P_2 \) is \( px' x + qy' y = 1 = 0 \), the polar of \( (x', y') \), the equation of \( P_2 P_4 \) is \( px' x + qy' y + 1 = 0 \).

If the circle \( P_1 P_2 P_3 \) meets \( S = 0 \) again at \( Q_1 \), then \( Q_1 P_1 \) is a diameter of \( S = 0 \).

(v) The locus of the centre of a variable circle through four fixed points \( A, B, C, D \) is a conic which passes through the mid-points of the six sides and through the three diagonal points of the quadrangle \( ABCD \).

(vi) Pascal's Theorem. If the vertices \( A, B, C, D, E, F \) of a hexagon lie on a conic, the points of intersection of pairs of opposite sides, \( AB, DE, BC, EF, CD, FA \), are collinear.

20. The equation of the tangent at \( t = t_1 \) to the curve given by

\[
\frac{x}{x(t)} + \frac{y}{y(t)} + \frac{a}{h(t)} = 0.
\]

\[
\begin{align*}
x(t) & \quad y(t) & \quad a(t) & \quad h(t) \\
\frac{f(t)}{g(t)} & \quad \frac{g(t)}{h(t)}
\end{align*}
\]

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PART I Pages i-xv
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PART I

Exercise 1
1. 8; 12: -1; -3; -51; -23; 2; (60, 47), (-40, -28), (8, 8).
2. -25, 20; 215; -20; 16; -391; (-10, 12), (25, 40).
3. 112; -4; 31; -11; 6, 22; (-9, 6); (26, -8); 12.
4. 16; 16; 4 (or 6); 10; a = -11, b = -11, c = 3 or 7, d = -2 or 12, t = 1 or 2.
5. -6; 14; 3 (or 6); 2, 7, 14; a = -14, b = -24, c = 1 or 8, d = -2 or 11, t = 1 or 14.

Exercise 2
1. 8x + 3y = 1. 2. 4y = 15x - 3x². 3. 7x + 4y = 0. 4. 5x + 3y + 15 = 0.
5. Lines parallel to Oy. 6. Lines parallel to Ox; Ox. 7. a = 13; y = 12x.
8. -3x² + 2, -5. 9. 1, 31, -2; (4, 0). 10. -3. 7. 11. (3, -4), (-2, 6).
12. a = ±b; (5, 0), (-5, 0); (9, 5), (0, -5); circle.
13. t = -1/m or 2m + 1/m.

Exercise 3
1. 0, 4. 2. 0, 3. 3. 0, 5. 4. 0, -4. 5. 0, -4. 6. 0, -5.
13. -2a, 2b. 14. 2c, 2d. 15. m + n, -5, -k. 16. (5, 11). 17. (0, 3).
21. (x² + r cos 6, y + r sin 6).

Exercise 4
1. 5. 2. 5. 3. 10. 4. 5. 5. 13. 6. 3 times 5. 7. r.
8. (t + 1)/t. 9. 2 times 5. 11. x² - 4y = 7. 12. 4x² + 4y² = 1.
13. x² + 4y² = 0. 14. x² + y² + 6x - 8y = 11.
15. x² + y² - 4x + 10y + 13 = 0. 16. 5x² + 9y² + 12x + 42y + 17 = 0.
17. x² + y² - 2x - y = 1. 18. x² + y² = 2x. 19. x² + y² + 2bx + 2by = 0.
20. x² + y² - 4x - 6y = 4. 21. x² + y² + 2x + 4y = 0.
22. 5; x² + y² + 4x - 8y = 5.

The answers to Nos. 23-37 give the centre and radius of the circle.
23. (0, 0); -3. 24. (0, 0); 21. 25. (1, 0); 0. 26. (2, -1); 3.
27. (-3, 4); 14. 28. (-5, 0); 17. 29. (0, 3); 2. 30. (1, 4); 21.
31. (-5, 1); 6. 32. ( -2, -1); 21. 33. (-5, -2); 14. 34. (1, -1); 13.
35. (-3, 1); 5 times 2. 36. (a, 0); a. 37. (0, -b); b. 38. 2. 39. 4.
40. 5; inside.
Exercise 5

1. (3 - √3, 6). 2. (−1 − 2√2, −2 − 2√2). 3. (2 + 3√2, 3 − 3√2).
4. 5. (7, 0), (−5, −4). 6. (−53, 54), (−25, −14).
15. 4 + r cos θ, r sin θ; r² = 8r cos θ + 12 = 0, 3, 4, 3, −3, −4.
16. 1 + r cos φ, r sin φ; x² + 2r cos φ = −3 = 0, 12, −2, ± 2√3).
17. r = 4 cos θ.
18. 19. r cos θ                                           20. x = 2 cos x − r sin x, y = p sin x + r cos x.

Exercise 6

1. (−h, k), (−h, k), (h, k). 4. x = −1/k; line parallel to Oy.
5. 6. circle centre {−d(1 + k²)/(1 − k²), 0}, radius 2k(1 − k²).
7. r² = 4a² cosec² θ; OA². 8. 10a² − 6ax, (10a² − 6ax), 0. 9. (a, 0), (b, 0), (0, c), (0, −c).

Exercise 7

1. x = 10, y = 3, 3x = 10y, x'(10 + y) = 1 = 1.
2. x = 10, y = 7, 7x = 4y, x + y = 1 = 1.
3. (9, 2), (−3, −6); 2x = 9y, 2x = −3y, 5x = 3y, 5x = −9y, −2y + 3y = 1, x = 9y − 3y = 1 = 1.
4. (−3, 2), (20, −6); x = 5, y = −3; 2x + 5y = 0, 8x = −5y, 8x = 5y, 3x = 10y, 20y + 8 = 1, x = 5 + 8y = 1.
5. (5, 2), (−3, −4); −3x + 2y = 2, x = 5y = 8 = 1 = 1.
6. (−2, −7), (11, −8); x + y + 7 = 1, x + y + 8 = 1 = 0.
7. x = −y, 2x = −3y, 5x − 3y, 5x − 3y, x = −2y, E, K, F, L, G, H.
8. y = 3x. 11. 4y + 3x = 0. 12. 2y + 5x = 0.
13. 7y − 2x = 0.
14. 0, 1. 15. 0, 6. 16. 3, 7. 17. x = 2y − 1 = 1. 18. x + 3y = 17.
19. 3x + 4y = −2. 20. 2x + 9y = 8. 21. 2x − 4y = 16.
22. 2x − 4y = 15.
23. 2x + y = 2ax + ay; x = −y + 1 = 0; 2x − 4y = 10.
24. 2ax sin θ = (a² − b²) sin θ + 0, bx θ + ay sin θ = ab.
25. 3x = 2y = 4y = −11 = 11. 26. 4, 4x − y = 10 = 10.
27. 2ax = 4x = 8.
28. 2. 9a = 7a = 29. 29. 1 = 1 = 2. 30. x = 2y = −3 = 5.
31. 2(1/2m₁); 2x = 3x₁, 2y = 3y₁, 3x₁ + 3y₁ = 0.
32. −1/(1 + t²), x + y = 2(1 + t²). 33. 35. 4. 34. 5.
36. 37. 1/2. 38. 39. 2/x. 40. Parallel to Oxy.
41. Join of O to (4, 2). 42. Join of O to (2, −3).
43. Intercepts on axes, 4, −7. 44. Intercepts on axes, −3, −5.

Exercise 8

1. x = y − 1 = 1. 2. 3x + 2y = −3 = 1. 3. 32 − 3y = 0 = 1.
4. 8x = 3y = 0. 5. 6x = 2y = −15 = 6. 6x + 2y = 15 = 15.
7. a(x = 2, y = b) = 0 = 0. 8. (x + 4)² + (y − 4)² = 0 = 0. 9. − ± 2 = 2.
13. (3, cos 0, y sin 0). 14. (x − h)/x = 3, (y − k)/y = 1.
15. ± 2 = 2, 2 ± 2 = 18, ± 2 = 1.
16. 2x + 4y = 16, 2x + 4y = 16, ± 2 = 3, ± 2 = 1, 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1 = 1.
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Exercise 12

1. 13. 2. 7. 3. 14. 4. 17. 5. 20. 6. 17. 7. 20.
8. 19. 4. 5. 10. 11. 11. $\frac{m_1 - m_2}{(m_1 + 1)(m_2 + 1)}$.
12. $x + y + b = 10, x + a - y = 30, x = y = 0$.
13. $\frac{e}{4}(x_1 + x_2 - x_3 + x_4)$.
14. $\pm \frac{e}{4}(p - q)(p + q - r)(q - r)$. 15. $+ \frac{a}{4}(r - q)(q - r)(p - q)$.

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Exercise 13

1. $x + 2y = 0, x - 2y = 0$. 2. $x = 0, x - 3y = 0$. 3. $x = 0, y = 0$.
4. $2x - y = 0, 3x - 2y = 0$. 5. $2x + y = 0$ (rep.).
6. $x + b = 0, y - a = 0$. 7. $x = y = 0$.
8. $x = y = 0$, $x = y = 0$. 9. $y = 0,
10. \sqrt{13}$.
11. 12. $\frac{a}{b}, \frac{b}{a}$. 13. $\frac{1}{2}$. 14. $\frac{1}{2}$. 15. $\frac{1}{2}$. 16. $\frac{1}{2}$. 17. $\frac{1}{2}$. 18. $\frac{1}{2}$.
19. $\frac{1}{2}$. 20. $\frac{1}{2}$. 21. $\frac{1}{2}$. 22. $\frac{1}{2}$. 23. $\frac{1}{2}$.
24. $\frac{1}{2}$.
25. $\frac{1}{2}$.

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Exercise 14

1. $x + y = 0, x + y = 0$. 2. $x = 0, x - 3y = 0$. 3. $x = 0, y = 0$.
4. $2x - y = 0, 3x - 2y = 0$. 5. $2x + y = 0$ (rep.).
6. $x + b = 0, y - a = 0$. 7. $x = y = 0$.
8. $x = y = 0$, $x = y = 0$. 9. $y = 0,
10. \sqrt{13}$.
11. 12. $\frac{a}{b}, \frac{b}{a}$. 13. $\frac{1}{2}$. 14. $\frac{1}{2}$. 15. $\frac{1}{2}$. 16. $\frac{1}{2}$. 17. $\frac{1}{2}$. 18. $\frac{1}{2}$.
19. $\frac{1}{2}$. 20. $\frac{1}{2}$. 21. $\frac{1}{2}$. 22. $\frac{1}{2}$. 23. $\frac{1}{2}$.
24. $\frac{1}{2}$.
25. $\frac{1}{2}$.

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Exercise 15

1. $\frac{a}{b}$. 2. $\frac{a}{b}$. 3. $\frac{a}{b}$. 4. $\frac{a}{b}$. 5. $\frac{a}{b}$.
6. $\frac{a}{b}$. 7. $\frac{a}{b}$. 8. $\frac{a}{b}$. 9. $\frac{a}{b}$.
10. $\frac{a}{b}$. 11. $\frac{a}{b}$. 12. $\frac{a}{b}$. 13. $\frac{a}{b}$.
14. $\frac{a}{b}$. 15. $\frac{a}{b}$. 16. $\frac{a}{b}$. 17. $\frac{a}{b}$.
18. $\frac{a}{b}$.

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Exercise 16

1. $x^3 + y^3 - 2x + 4y + 1 = 0, x^3 + y^3 + 4x + 4 = 0$.
2. $2x^2 + 3x - 1 = 0$. 3. $x^3 + y^3 - 5xy - 2y = 0$.
4. $x^3 + y^3 + 5x - 2y = 0$. 5. $x^3 + y^3 + 7x + 3y + 12 = 0$.
6. $x^3 + y^3 + 2x + 2y = 0$. 7. $x^3 + y^3 + 4x + 4y = 0$.
8. $x^3 + y^3 + 2x + 2y = 0$. 9. $x^3 + y^3 + 2x + 2y = 0$.
10. $x^3 + y^3 + 2x + 2y = 0$. 11. $x^3 + y^3 + 2x + 2y = 0$.
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Exercise 19

5. \(x^2 + 3x + 4 = 0\)  6. \(x^2 + 7x + 12 = 0\)  7. \(12x^2 + 3x - 1 = 0\)  8. \(x^2 - 5x + 6 = 0\)

Exercise 20

1. \(4x^2 + 3y^2 = 1\)  2. \(2x^2 + 3y = 0\)  3. \(x^2 + 4y^2 = 1\)

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Exercise 21

1. \(y = 2x^2 - 3x + 1\)  2. \(y = x^2 - 2x + 1\)  3. \(y = x^2 + 3x + 2\)

Exercise 22

1. \(y = x + 3\)  2. \(y = -x + 2\)  3. \(y = x^2 - 4x + 3\)

Exercise 23

1. \(x + 2y = 0\)  2. \(x - y = 0\)  3. \(x^2 + y^2 = 1\)

Exercise 24

1. \(x^2 + y^2 = 1\)  2. \(x^2 + y^2 = 4\)  3. \(x^2 + y^2 = 9\)

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Exercise 25

1. \(x^2 + y^2 = 1\)  2. \(x^2 + y^2 = 4\)  3. \(x^2 + y^2 = 9\)

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Exercise 26

1. \(x + y = 1\)  2. \(x^2 + y^2 = 4\)  3. \(x^2 + y^2 = 9\)
**Exercise 28**

1. $6x - 5y = 24$.  
2. $-(18, 3)$.  
3. $(8, 8)$.  
4. $x - 4y = -24$.  
5. $3x - 2y = -2$.  
10. $(ab^2, -2a^2b)$, $(-2ab^2, a)b)$.  
22. $c/(ap_b) - c/(ap_b)$.  
23. $(-\frac{1}{4}a, 0)$, $6a$.

**Exercise 29**

1. $(1\frac{1}{2}, 1\frac{1}{2})$.  
2. $(\frac{3}{2}, 1\frac{1}{2})$.  
3. $\frac{1}{2}h^2(a - h, k)$.  
4. $3x + y = 3$.  
5. $3y = 4$.  
6. $x - y = -1$.  
8. $y^2(25 - a) = 4b^2x$.  
9. $(c, 0)$, $(c + \frac{1}{2}a, 0)$.

**Exercise 30**

1. $1, 3, -4, 2$.  
2. $(\frac{1}{2}, 1)$, $(12\frac{1}{2}, -7)$.  
3. $(33a - 30a)$, $x^2 + y^2 - 35ax + 15ay = 0$.  
6. $h = a(p^2 + pq + q^2 + 2)$, $k = -ap(p + q)$.  
17. $(-2a, 0)$.  
18. $(-a, 0)$.

**Exercise 31**

1. $xy = -15$, $(0, 0)$, $x + y = 0$.  
2. $x(2y - 1) = 8$, $(0, -1)$, $x = y + 1$.  
3. $(x - 3)(y + 1) = -6$, $(3, -1)$, $x + y = 2$.  
4. $(x + 4)(y - 5) = -20$, $(4, 0)$, $x + y = 5$.  
5. $(x + f)(y + g) = fg$, $(x - g, y - f)$, $x - y = g - f$.  
6. $y^2 - x^2 = h^2 (0, 0)$, $x = 0$.  
7. $(2x - y - 7)(x + 2y + 4) = -63$, $(2, -3)$, $3x + y = 3$.  
8. $(3x + 4y - 4)(4x - 3y + 3) = 48$, $(0, 1)$, $x - y = 7$.  
9. $(ax + by + c)(bx - ay + d)$, $-ad + bd + ac + (-b^2)(x^2 + b^2) + (ax - by)(x + ad + c)$.  
10. $x = 0$, $y = 0$, $(0, 0)$, $x = (4)$, $y = (4, 4)$.  
11. $x = 2$, $y = 3$, $(2, 3)$, $(1, 2)$, $(3, 4)$.  
12. $x = -4$, $y = 1$, $(-4, 1)$, $(-2, 3)$, $(-6, -1)$.  
13. $x = 3$, $y = -2$, $(3, -2)$, $(0, -5)$.  
14. $x = 5$, $y = 2$, $(5, 2)$, $(8, 5)$, $(2, 1)$.  
15. $x = -2$, $y = 1$, $(-2, 1)$, $(0, 3)$, $(-4, -1)$.

**Exercise 32**

1. $(3, 3)$.  
2. $(2, 3)$, $(-3, -2)$, $(2, 3)$, $(rep.)$.  
2. $\frac{1}{2}$, $-3$, $-k$ $(rep.)$.  
3. $3x + y = 48$, $3x + 8y = -65$.  
4. $3x + y = 4$, $x - 3y = 5\frac{1}{2}$.  
5. $x + y = 2$, $x - y = 0$.  
6. $x = 4$, $y = 4c$, $x + 8y = 16c = 0$.  
7. $x = 8$, $x = y = 2$.  
8. $x + 2y = 3$, $2x = y + 4$.  
9. $x = 4 - 3y$, $4x + 5y = 40$.  
10. $x = 3y + 3$, $2x = y - 4$.  
12. $-\frac{1}{2}b^2a^2$, $\pm \sqrt{(1 - m)}$.  
13. $4x^2 + y^2$, $(\frac{1}{2}a, b)$.  
14. $1 - m = \frac{c - b}{c - b}$.  
15. $t_2 = (ct_1 - b)/(ct_1 - c)$.  
16. $t_1 = (ct_1 - b)/(ct_1 - c)$.  
17. $1 - m = \frac{c - b}{c - b}$.  
18. $t_2 = (ct_1 - b)/(ct_1 - c)$.  
19. $t_1 = (ct_1 - b)/(ct_1 - c)$.  
20. $t_2 = (ct_1 - b)/(ct_1 - c)$.

**Exercise 33**

1. $5x + 3y = -60$.  
2. $(-\frac{1}{6}, 9\frac{1}{2})$, $(2\frac{2}{3}, 1\frac{1}{2})$.  
3. $(3, 8)$, $(-4, -6)$.  
4. $3x + y = 12$, $3x + 4y = -24$.  
5. $(2x^2g/p + q)$, $2x^2p + g (p + q)$.  
7. $4m + n = 8$.  
8. $(\frac{1}{2}, \frac{1}{2})$.  
9. $2x = 3y$, $3x = 2y$.  
14. $2e^2(h + m) = 4xy$.  
15. $r^x \cos 2\theta + 2x(h \cos \theta - k \sin \theta) + h^2 - k^2 - a^2 + b^2 = 0$.  
16. $k \sin \theta = 0$.

**Exercise 34**

1. $\frac{1}{2}$, $\frac{1}{2}$, $(p + q + r + 1)/pqr$, $(\frac{1}{2})pq(p + r) + (p + q + r)$.
2. $(-\frac{1}{2}a, -\frac{1}{2}a)$.  
23. $2x/k, h/k$.  
35. $\frac{1}{2}h^2 - \frac{1}{2}b^2 - \frac{1}{2}k^2 = -c^2$.

**Exercise 35**

1. $0, 0)$.  
2. $x = 0$, $y = 0, 2, 1\frac{1}{2}$.  
3. $(0, 0)$.  
4. $x = 0$, $y = \pm \frac{1}{2}\sqrt{5}$.  
4. $x = 4, y = -4, x = 1, 6, 4$.  
5. $(-2, -1)$, $x = -2, y = -11$.  
6. $(-2, 3)$, $x = -2, y = 3, 2, \frac{1}{2}\sqrt{3}$.  
7. $(-2, -1)$, $x = -4, y = -1$, $15, 6$.  
8. $(-3, -1)$, $y = -1, x = -3, 2, \sqrt{5}$.  
2. $(3, 3)$, $x = 2, y = 3, 2, \sqrt{3}, 2$.  
10. $(-\frac{1}{2}, \frac{1}{2})$, $y = 1$, $x = -\frac{1}{2}, 2\sqrt{3}, 2$.  
11. $(\frac{3}{2}, \frac{3}{2})$, $x = -\frac{3}{2}, y = -\frac{3}{2}$, $\pm \frac{1}{2}\sqrt{20}$, $\pm \frac{1}{2}\sqrt{6}$.  
12. $(i)$ Ellipse; major axis $x = -y/a$, minor axis $y = -f/b$. (ii) One point $(g/a, -f/b)$. (iii) No point.

**Exercise 36**

1. $\pm (\sqrt{3}, 1, -\frac{1}{2} \pm \sqrt{3} / 2)$.  
26. $(\pm \frac{1}{2} \pm \sqrt{3} / 2, \sqrt{3} / 2, \pm \sqrt{3} / 2)$.  
27. $p = \pm a \sqrt{2}$, $q = \pm a \sqrt{2}$.  
28. $\pm a \sqrt{2}$.  
29. Directrices $x = \pm a / \sqrt{2}$.  
30. $\pm a \sqrt{2}$.  
31. $\pm a \sqrt{2}$.
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Exercise 37

1. \(5x - 2y = 12, \ 2x + 6y = -1\).
2. \(3x - y = -8, \ x + 3y = 7\).
3. \((-\frac{1}{2}, 1)\).
4. \((2, -1)\).
5. \(9x - 8y = \pm 50\).  
6. \(x + y = \pm 2\frac{1}{2}\).  
7. \(8x - 9y = \pm 12\).
8. \(3x - 3y = \pm 1\).
9. \((-3, 3)\).
10. \(3x + y = 3, \ 3x - 5y = -9\).
11. \(4x + 6y = -25, \ 3x + 8y = 25\).
12. \(q = (k - a)/(p + k + a)\).
13. \(r = (q + a + 1)/(q + a + 2)\).
14. \((\pm a^2 + b^2, \pm b^2)/\sqrt{(a^2 + b^2)}\).
15. \(m^2 + n^2 = 0\).
16. \((0, 0)/\sqrt{(a^2 + b^2)}\).
17. \(h^2 + k^2 = a^2 + b^2\).

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Exercise 38

1. \(4x + 27y = 36\).  
2. \(6x - 10y = -1\).
3. \((-3, 3)\).  
4. \((1, 4)\), \((7, 2)\), \((7, 3)\).
5. \((1, 0)/\sqrt{(a^2 + b^2)}\).
6. \((a, b)/\sqrt{(a^2 + b^2)}\).
7. \(x^2 + y^2 < 1\).
8. \(x^2 + y^2 < 1\).
9. \(p + g = q = -1\).
10. \((\pm a, \pm b)\).

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Exercise 39

1. \(\frac{4}{5}, 8x - 15y = -61\).
2. \((4, -3)\).  
3. \(p + g = -1\).
4. \((\pm a, \pm b)\).
5. \((\pm a, \pm b)\).
6. \((\pm a, \pm b)\).
7. \((\pm a, \pm b)\).
8. \((\pm a, \pm b)\).
9. \((\pm a, \pm b)\).

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Exercise 41

1. \(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \pm \frac{3}{2}\).
2. \(\frac{1}{2} (\pm 2, 0)\).
3. \(3x - 2y = 0\).
4. \(\frac{1}{2} (\pm 2, 0)\).
5. \(\frac{1}{2} x - \frac{3}{2} y = 0\).
6. \(\frac{1}{2} x + \frac{3}{2} y = 0\).
7. \(\frac{1}{2} x - \frac{3}{2} y = 0\).
8. \(\frac{1}{2} x + \frac{3}{2} y = 0\).
9. \(\frac{1}{2} x - \frac{3}{2} y = 0\).
10. \(\frac{1}{2} x + \frac{3}{2} y = 0\).
11. \(\frac{1}{2} x - \frac{3}{2} y = 0\).
12. \(\frac{1}{2} x + \frac{3}{2} y = 0\).
13. \(\frac{1}{2} x - \frac{3}{2} y = 0\).
14. \(\frac{1}{2} x + \frac{3}{2} y = 0\).

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Exercise 43

6. \(\sqrt{(4x + 3y = -5, \ 2x - 3y = 7)}\).
7. \(x^2 + y^2 = a^2 + b^2\).
8. \(30^\circ, \cos^{-1}(\frac{1}{2})\).
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Q.R. 1. 1. 3, -10. 3. 5. 2. 6. 2x + 3y - 7, 3x - 2y = -22.
   4. 3x - 4y = 15.
   5. 3x - 2y = 11.
   6. x = 5, y = 7; 2x - 3y = 29, 3x + 2y = 29 = 0.
   7. 5x + 2y = 4.
   8. 5. 5. 6. Inside.

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Q.R. 5. 2. -9. 4. (3 + 3\sqrt{3})/2, -7. 5. 45°. 6. 3k - 4k = 15.
   7. 3k - 4k = 15.
   8. 2k - 2k = 0.
   9. 2k - 2k = 0.
   10. 2k - 2k = 0.
   11. 2k - 2k = 0.
   12. 2k - 2k = 0.
   13. 2k - 2k = 0.

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Q.R. 9. 1. -1, 2. h + 2k = 5(2a - 1). 3. 2 = e^p.
   4. (0, 0), (-\frac{3}{2}, 6). 5. (\frac{1}{2}, -\frac{1}{2}). 6. 7 or -13.
   7. 8x - 6y = -5.
   8. 3x + pqy - c(p + q) = 0.
   9. \frac{1}{2}, \frac{1}{2}.
   10. 5x + 4y = 10.
   11. 5x + 4y = 10.
   12. 5x + 4y = 10.
   13. 5x + 4y = 10.
   14. 5x + 4y = 10.

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   3. 2. 4. h cos x^2 + k sin x^2 = p = \pm r.
   5. 7x + 8y = 18.
   6. 16(x^2 + y^2) + 18x - 36y + 11 = 0.
   7. 18x + 8y = 18.
   8. 18x + 8y = 18.
   9. 18x + 8y = 18.
   10. 18x + 8y = 18.
   11. 18x + 8y = 18.
   12. 18x + 8y = 18.
   13. 18x + 8y = 18.

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Q.R. 17. 1. 2x - 3y = 12; (2, 3, -2). 2. -4. 3. \pm 5; (1, 3/2, 4/3).
   4. x + 4y = 2a + a/4. 5. x + y + 2a - h = 0.
   6. (0, 9), (4, 1).
   7. -2. 8. -1. 9. \pm 2.
   10. x + 2y + 3a = 0, 4x - 8y = 9a.
   11. 1. \pm 1; 15. 2. x = -y.
   3. 3x - 2y = 7. 4. -2, -6, 14.
   5. \{pq, a(p + q)\}.
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Paper 1. 1. \((a^2 - b^2) \sin 20\). 2. \((9, -9); 32\frac{1}{2}; \frac{3}{4}; \frac{1}{2}\).
   3. \((1\frac{1}{4}, \frac{3}{4})\).
   4. \((-2, -5); (-4, -15\frac{1}{2})\).
   5. \(x^2 + y^2 - 3xy + 4y = 0; 2y = t, 2y + 9t = 0\).
   6. \((-a, -b)\).

Paper 2. 1. \(y = 2x; 4\sqrt{5}\).
   2. \(-\frac{1}{2}; \frac{1}{2}; -4\frac{1}{2}\).
   3. \(0, -1); (2, 2); (3, -3); (4, -4); (5, 5); (6, 1); (7, 0); (8, -1); (9, 2).
   4. \(y = 2x; 90^\circ\).
   5. \(\pm 1/\sqrt{3}; x^2 + y^2 = 4x + 2y/\sqrt{3} - 3\).
   6. \((1, 0); (3, 0)\).

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Paper 3. 1. \(3\sqrt{5}; 19\frac{3}{4}\).
   2. \(2x + y = 5\).
   3. \(y = c, 2cx + (a + b)y = c(a + b)\).
   4. \((-12, -5); (13, 6); (1, 1)\).

Paper 4. 1. \(7x + 2y = -9, 4x + 5y = 9, x - y = -6\).
   2. \(\frac{1}{2}\).
   3. \((-a, -b); (4, -4\frac{1}{2}); (-a, -b); (4, -4\frac{1}{2}); 57 : 58\).
   4. \(x^2 + y^2 + x - y = 8 = 0; 5x + 3y = 16\).
   5. \(x^2 + y^2 = 12x + 8y + 30 = 0\).

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Paper 5. 1. \(x = -17y, 7x - 5y = -2\).
   2. \(x = 2\).
   3. \(1: 7, 7: 1, -5: 1; -1, 5, 6\).

Paper 6. 1. \(4x - y = -3, 2x - y = 0; (-1\frac{1}{4}, -3)\).
   2. \((-1, 2); 8; (8, -1), (1, -2)\).
   4. \(x^2 + y^2 = 4x + 4y - 3\).
   5. \(2\sqrt{6}; x^2 + y^2 = 4x - 2y, x^2 + y^2 + 4x - 2y = 0\).

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Paper 7. 2. \((-\frac{1}{2}, -\frac{3}{4}); (\frac{1}{2}, \frac{3}{4}); (b - dh) = (bc - add) = x\).
   4. \(19x^2 + 19y^2 - 44xy - 40y = 5\).
   5. \(x^2 + y^2 = 2ax^2 + 2by^2 - 2k^2x - k^2\).

Paper 8. 1. \(3x + 2y = 36; 3\sqrt{15}\).
   2. \(\frac{1}{2}\).
   3. \(x_2y_3 - x_3y_2 = k_4, y_3x_2 - y_2x_3 = k_3, k_4 - k_3, ratios\) are \(k_3: k_4: k_2: k_1; k_3: k_2: k_1; product = 1\).
   4. \((3, 1)\).
   5. \((\frac{1}{2}) cos \alpha/\rho, \frac{1}{2}k^2 \sin \alpha/\rho\).

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Paper 9. 1. \(4x - y = 7, x + 4y = -11\).
   2. \((-2, 5); (3, 5); 1\frac{1}{2}\).
   3. \((-\frac{1}{2}; -\frac{1}{2}); 1\frac{1}{2}\).

Paper 10. 1. \((-14, 20)\).
   2. \(x^2 + y^2 + 10x - 20y + 25 = 0, x^2 + y^2 - 6x - 4y + 9 = 0\).
   3. \(1 : 2\).
   6. \((3x - 5y - 7)(3x + 3y + 39) = 560; x + 4y = 23 = 0\).

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Paper 11. 2. \((4x - 3y)^2 + 30x + 40y - 25 = 0\).

Paper 12. 1. \((-5, -1), (2, 2), (7, 3); 20\).
   2. \(3 = 2, -1 = 2\).

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Paper 13. 2. \(p + q = 2\).
   4. \((4, 1); (4, 1\frac{1}{2})\).
   5. \((6x + y = 19, 6x - y = 29)\).
   5. \((3a, 0), (3\frac{1}{2}a, 0)\).
   6. \(b = \pm c \sqrt{2}\).

Paper 14. 2. \(a(\cos \beta + \cos \alpha), a(\cos \beta - \cos \alpha)\).
   4. \(x + by + ac = 0\).
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PART II

Exercise 48

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1. 1; 2. 8; 3. 16; 4. 4; 5. 49; 6. 35; 7. 1; 8. 1; 9. 7; 10. 4.

11. 12. −1. 13. 0. 14. 0. 15. −15. 16. 0. 17. −3.

21. 4x + 5y = −2. 22. x − y = 2. 23. 2x + 5y = −21. 24. 2x − 3y = 8.

25. \[(\sin^2 - \cos^2 \theta) \div (\sin^2 + \cos^2 \theta) \div 3 = 1.0. \]

26. x + py = q + p.

27. 2px + y(p + q) = −2a. 28. (1 − pq)x + (p + q)y = a + pq.

29. \[(1 + pq)(a + b + c) \div (p + q) = p + q. \]

30. p + q + r = 0. 37. 6.

32. abc. 39. a + b + c = 0. 40. \[(a^2 + b^2 + c^2) \div 2(a - b)(b - c)(c - a). \]

41. \[\frac{1}{2}(a + b)(a - b)(c - a). \]

42. \[\frac{1}{2}a^2(b - c). \]

43. \[\frac{1}{2}(a + b)(b - c)(a - b). \]

44. bc + ca + ab + 1 = 0. 45. 2pqr − q - r = p + q + r.

46. \[(a + b)(a - b)(c - a). \]

48. \[\frac{1}{2}y - a + b - c = 0. \]

49. \[(b_1c_2 - a_1c_3) \div (c_1a_2 - a_1c_3) \div (b_2a_3 - b_1a_3). \]

50. \[(b_1c_2 - a_1c_3)^2 = (b_1c_2 - a_1c_3)(a_2b_3 - b_1a_3). \]

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Exercise 49

1. 3x^2 - 8xy = 9y^2 = 0. 2. pq = -1. 3. \(k(x + b) = 0. \)

4. \((\sqrt{4} + 1)/40. \)

5. \(x^2 + 2x = 1; 3 - x^2 = 2. \)

6. \(8x + 3y = 0. \)

7. \((2x^2 = 5y^2 + 2x^2)/\sqrt{73}. \)

8. \(2x^2 - 3xy + ay^2 = 0. \)

9. \(0, \sqrt{3}. \)

10. \(\sqrt{5}. \)

11. \(-\sqrt{3}. \)

12. \(\sqrt{3} < c < \sqrt{3}. \)

13. \(\{2 + (a + b), 3/(a + b)\}. \)

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Exercise 50

1. \(x + 2y = 3, x - y = -1. \)

2. \(3x = y = 2, x + y = 2. \)

3. \(3x - 2y = 2, 2x + 3y = 1. \)

4. \(3x - 2y = 2. \)

5. \(-6. \)

6. \(-12. \)

7. \(-7 \div 18. \)

8. \(15. \)

9. \(-3. \)

10. \(-4 \div 5. \)

11. \(2y + 2h. \)

12. \(i) 2x - 3y = 1, (ii) no point. \)

13. \(i) point-circle at (3, 2); (ii) centre, centre (2, 3), radius 1/\sqrt{5}. \)

14. \(a^2 + b^2 - 1 = (x + b)^2 + (y + c)^2 - 1 > 0. \)

15. \((x - 3)^2 = 2(x - 3)(y + 3) - 2(y + 3) = 0. \)

16. \(3x^2 + 2y^2 = 0. \)

17. \(x^2 + y^2 = (x + y)^2. \)

18. \(x^2 = 2y^2 = 2. \)

19. \(x^2 + pq - 3q^2 = 3b. \)

20. \((\frac{1}{2}, \frac{1}{2}) \pm \sqrt{2} \cdot (2, 0). \)

21. \((a, b) \div (3, 0). \)

22. \(X^3 + 2X^2 + 3X + 2 = 0. \)

23. \(X^3 - 3X^2 - 2X + 1 = 0. \)

24. \(x^2 + 14xy - 7y^2 - 18x + 4y + 2 = 0. \)

25. \((-\frac{3}{2}, \frac{1}{2}); (\frac{3}{2}, -\frac{3}{2}). \)

26. \(1, -3; (1, \frac{1}{2}). \)

27. \(-1, -2, -3. \)

28. \(k = 2(f \cos \alpha - g \sin \alpha); \) \(PN^2 = k \cdot PM. \)

29. \(c = 0, g = 0. \)

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Exercise 52

1. Ellipse; \(x + 2y = 0\); \(\frac{1}{2}\). 2. Hyperbola; \(x - y = 0\); \(\frac{1}{2}\sqrt{2}\). 3. No point.
4. Hyperbola; \(2x + 3y = 0\); \(\frac{2}{\sqrt{3}}\). 5. Line-pair, \(3x - y = -2, x + 2y = 11\).
6. Parabola; \(2x - y = 1\); \(\sqrt{5}\). 7. Rect. hyperbola; \(x + 2y = -6\); \(\sqrt{5}\).
8. Ellipse; \(x + y = 3\); \(\frac{1}{2}\sqrt{3}\). 9. Parallels, \(3x - 2y = -4, 3x - 2y = 1\).
10. Parabola; \(3x + 2y = 8\); \((7\sqrt{2})/6\). 11. One point, \((3, 10)\).
12. No point. 13. Ellipse; \(2x + 2y = 1\); \(\sqrt{8}\).
14. Hyperbola; \(3x - 4y = 10\); \(\frac{1}{2}\sqrt{3}\).
15. \(\lambda < -2\) and \(-2 < \lambda < 3\), ellipse; \(-2 < \lambda < 1\) and \(1 < \lambda < 2\), hyperbola;
\(\lambda > 3\), no point; \(\lambda = \pm 2\), two parallel lines through \(O\); \(\lambda = -3\), one point.
16. \(-2 < \lambda < 3\), hyperbola; \(-2 < \lambda < 2\), ellipse; \(-2 < \lambda < 3\), hyperbola; \(-2 < \lambda < 2\), two parallel lines; \(\lambda = -2\), one point.
17. \(\lambda < -1\) and \(\lambda > 1\), hyperbola; \(\lambda = -1\), parabola; \(-1 < \lambda < 1\), ellipse;
\(\lambda = 1\), two coincident lines.
18. \(-2 < \lambda < -2\) and \(-2 < \lambda < -1\) and \(\lambda > 1\), hyperbola; \(-1 < \lambda < 1\), parabola;
\(-2 < \lambda < -2\), one coincident line; \(\lambda = 1\), two coincident lines.

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Exercise 53

1. \(\frac{1}{3}, 7 : 1\). 2. \(3:2\), \(-53:38\). 3. \(10b + k = -10\).
4. \(x(y_1 + y_3 - c) = 2(x_1 y_3 + c)(2y_3 - c)\).
5. \(ax^2 + hy^2 = 1\). 6. \(x^2 + bx = 0\); \(ax^2 + hy^2 = 0\). 7. \(a^2 + bx = 0\).
8. \((x(y_2 + x_2 - c)) = 2(x_1 y_3 + c)(2y_2 - c)\).
9. \(ax^2 + hy^2 = 1\). 10. \(x^2 + bx = 0\); \(ax^2 + hy^2 = 0\). 11. \(x^2 + bx = 0\);
12. \(x^2 + bx = 0\). 13. \(16x - 7y = 2\); \(5x - 6y = 2\);
14. \(x^2 + hy = 0\); \(ax^2 + hy^2 = 0\). 15. \(2x^2 + 3y^2 = 3\); \(2x^2 - 3y^2 = 3\).
16. \(x + y = 0\); \(2x - 3y = 0\).

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Exercise 54

2. \(x = 2a\) and \(y = 2b\). 3. \(a = 2b\).
4. \(x = 2a\) and \(y = 2b\). 5. \(x = 2a\) and \(y = 2b\).
6. \(x^2 + y^2 = 2\); \(x = 2a\) and \(y = 2b\). 7. \(x = 2a\) and \(y = 2b\).
8. \(x = 2a\) and \(y = 2b\). 9. \(x = 2a\) and \(y = 2b\).
10. \(x = 2a\) and \(y = 2b\). 11. \(x = 2a\) and \(y = 2b\).
12. \(x = 2a\) and \(y = 2b\). 13. \(x = 2a\) and \(y = 2b\).
14. \(x = 2a\) and \(y = 2b\). 15. \(x = 2a\) and \(y = 2b\). 16. \(x = 2a\) and \(y = 2b\).
17. \(x = 2a\) and \(y = 2b\). 18. \(x = 2a\) and \(y = 2b\). 19. \(x = 2a\) and \(y = 2b\). 20. \(x = 2a\) and \(y = 2b\). 21. \(x = 2a\) and \(y = 2b\). 22. \(x = 2a\) and \(y = 2b\). 23. \(x = 2a\) and \(y = 2b\). 24. \(x = 2a\) and \(y = 2b\). 25. \(x = 2a\) and \(y = 2b\). 26. \(x = 2a\) and \(y = 2b\). 27. \(x = 2a\) and \(y = 2b\). 28. \(x = 2a\) and \(y = 2b\). 29. \(x = 2a\) and \(y = 2b\). 30. \(x = 2a\) and \(y = 2b\). 31. \(x = 2a\) and \(y = 2b\). 32. \(x = 2a\) and \(y = 2b\). 33. \(x = 2a\) and \(y = 2b\). 34. \(x = 2a\) and \(y = 2b\). 35. \(x = 2a\) and \(y = 2b\). 36. \(x = 2a\) and \(y = 2b\). 37. \(x = 2a\) and \(y = 2b\). 38. \(x = 2a\) and \(y = 2b\). 39. \(x = 2a\) and \(y = 2b\). 40. \(x = 2a\) and \(y = 2b\). 41. \(x = 2a\) and \(y = 2b\).
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Exercise 63
9. \(3(x - 2y - 2) = 4(x + 3y + 12) + 6xy = 0.\)
10. \((2x - 3y - 5)(2x - 3y - 5) - 5(5x - 2y - 18)(5x - 2y - 7) = 0.\)
11. \(28x^2 + 110xy - 29y^2 = 60.\)
12. \(x^2 + 6xy - y^2 = 11x - 12y + 5.\)
13. \(x^2 + 3y^2 = (x - 2y)^2 + 25x - 29y + 24i = 0.\)
14. \((p + aq)(b - px - qy) = 2pq + px + py - q^2.\)
15. \((x - 2y)^2 = 2x + 4y - 1.\)
16. \((a + b)^2 + (a - b)(a + b) = 2a^2 - b^2.\)
17. \(0 = 180^\circ - \theta, \theta = (a - b)c = (a - b)c = (a - b)c.\)
18. \(a^2 = b^2, \theta = (a + b)c = (a + b)c = (a + b)c.\)
19. \(x^2 + y^2 = 10x - 10y.\)
20. \(2(x^2 + y^2) = (x^2 + y^2) + (x + y)^2 = 1.\)
21. \(x^2 + y^2 = 2., \theta = \frac{1}{2}(4x^2 + 4a^2 - 4c^2) = 1.\)
22. \(x^2 + y^2 = a^2(x^2 + y^2) = 4(a - c)(b - d)xy.\)

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Exercise 64
3. \(\frac{1}{2}(b + a)(b - a), b(a + b)(b - a).\)
7. 1, 2, \(\frac{-i}{2}, \theta = y = 0, x = -2y = 1.x = 0, x + 2y = 2; x = 6y, 3x + 4y = 2.\)
8. 1, \(\frac{-2}{2}, 23, x = -2y, x - y = 1, y = 2, 4x + 3y = -10, 5x - 8y = -7.\)
9. 1, \(\sqrt{2}, x = 3x - 8y = 0, \theta = \frac{1}{2}(x^2 + 3x + x + 2y).\)
14. Conic touching \(AB, AC\) at \(C, D.\)
15. Circumcircle \(\triangle ABC: 2L_1 + L_2 = 2L_1 + L_2 = 0.\)
16. Conic through \(B, C, L_2 + kL_1 = 0, L_2 + kL_1 = 0, L_2 + L_2 = kL_1 = 0.\)

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Exercise 65
1. Touching joins of \(1, -2, 2, 1\) to each of \(3, 1, (1, -1).\)
2. Touching joins of \(1, -2, 1, (1, 0)\) to each of \(0, 1, 0, 0, 0, 0/0.\)
3. \(A = 1, B(0), B(0, 1), C(1, 1),\) touching \(CA, CB, AB,\) A, B.
4. \(A(2, 0), B(3, 0), C(2, 4),\) touching \(CA, CB,\) with asymptote \(y = 0.\)
5. \(9(x - 2m + 2n)(2m - 5n - 7m)(3m + n - 5m) = 0.\)
6. \((3m + 2n)(2m - 7n + n)(3m + 5n + 5m) = 0.\)

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Exercise 66
12. \(L_2 + L_4 = 0, kL_1 - L_1 = 0.\)
17. \((b + c)(x + y + a)(x + y - a) = 2b - (a-b)xy = 2(a-b) = 0.\)
18. \(a = b; 2a(x^2 + y^2) + 2ax + 2by = 0.\)

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Exercise 67
2. \(-3(x + 4m + 2n)(m + n, m = 2m + 3m + 1; 4m).\)
4. \((x - 4a)(y - 2b) = k^2.\)
10. \(3x^2 + y^2 = x(a + b + c) - y(a - b - c) = 0.\)
13. \(2x^2 + y^2 = 2x + 2y.\)
14. Conical, radical axis \(x = 0.\) If \(k = 4,\) circle touches \(y = 4ax\) at \(L, L'.\)
19. \(t = -2; f = 2a_1x, g = -a(2 + 3t_1^2), c = -3a_1^2t_1^2; (2a + 3a_1x, -2a_1t_1^2) = 0.\)

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Exercise 69
3. \(x = (a + b - 1, y = (a + b - 1, 4a + 5b = xz.\)
6. \(L_2 + L_4 + L_5 = 0.\)
12. \(b + c = 0.\)
19. \(e - 3x + 2y = 0, 3. 2x - y = 0, 2x + 2y + 4z = 0.\)
20. \(1, 2,\) double point, \(2x + x = 0.\)

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Exercise 70
Q.R. 29. \(1. h^2 = ab; 2. \pm \sqrt{2} \pm \sqrt{2}(r - p) \pm p - q.\)
4. \((0, 0).\)
5. \(x^2 + x^2 + p^2 = 0.\)
6. \((0, 0).

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Exercise 70
Q.R. 29. \(1. c = -\frac{1}{2}; 10x + 23y = 0, c = -\frac{1}{2}, 12x + 29y = 0.\)
2. \(c = -5x + 4y = 0; (0, 4y = 0).\)
3. \(a - h^2(p^2 + q^2) = a + h.\)
5. \(2x - y = 1, 32 + 3y = 2, x \pm 2 \pm 2 = \pm 2 \pm 2 = \pm 2 \pm 2.\)
6. \(10x + 12y = 0; (0, -5).\)
Q.R. 33. \(1. p + q + r = 0, (p + q + r); 2. (p + q) + n = 0, (a + b) + (a + b) = 0.\)
3. \(a + b = 0, 2 + 5 = 0, 2 + 5 = 0.\)
4. \(a + b = 0, 2x + 2y + 2z = 0.\)
6. \(h^2 = 1 - 2x - y, k = 0, h^2 = 0, h^2 = 1 - 2x - y.\)
Q.R. 34. \(2. 4x + 27y - 4z + 2z = 0, 4z + 0 = 0.\)
6. \(p + q = 0, 1 \pm p \pm q, 1 \pm p \pm q.\)
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Q.R. 33. 1. (i) Intercepts a, b; (ii) perpendicular form; (iii) rect. hyperbola. 2. 15: -4: 3. (x-2)^2 - 4y^2 = 9. 4. \( \frac{1}{2} \left( \frac{x}{a} - \frac{y}{b} \right) \). 5. \( k = \frac{1}{2} \). 6. \( a^2 - b^2 = a^2 \). 

Q.R. 35. 1. \( x + y \cos \alpha = 2 + 3 \cos \omega \); \( x + y \cos \alpha = \frac{2 \pi}{3} \). 2. 2x = y - 2. 3. \( \frac{2}{3} \). 4. \( \cos \alpha + \cos \beta = 1 \). 5. \( \frac{\pi}{4} \). 6. \( \cos \alpha + \cos \beta = 1 \). 

Q.R. 37. 1. \( \{ f(x, y, z, a, b) \} \). 2. \( x^2 + y^2 + z^2 + 1 = 0 \). 3. \( 3, 1 \frac{1}{2} \). 

Q.R. 38. 1. \( 12xy - 4k^2y^2 - 9x^2 = 2 \). 2. \( t = - \cos \omega \); vertex. 3. \( 3x^2 + y^2 = 3 \). 4. \( \frac{5}{3} \). 5. \( \cos \omega = 0 \). 6. \( \sin \omega = 0 \). 

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Q.R. 38. 1. \( \sin \omega(x_1y_2 - x_2y_1) \). 2. \( \sin \omega(x_1y_2 - x_2y_1) \). 3. \( \cos \omega(x_1y_2 - x_2y_1) \). 4. \( \cos \omega(x_1y_2 - x_2y_1) \). 5. \( \cos \omega(x_1y_2 - x_2y_1) \). 6. \( \cos \omega(x_1y_2 - x_2y_1) \). 

Q.R. 39. 1. \( \sin \omega(x_1y_2 - x_2y_1) \). 2. \( \sin \omega(x_1y_2 - x_2y_1) \). 3. \( \cos \omega(x_1y_2 - x_2y_1) \). 4. \( \cos \omega(x_1y_2 - x_2y_1) \). 5. \( \cos \omega(x_1y_2 - x_2y_1) \). 6. \( \cos \omega(x_1y_2 - x_2y_1) \). 

Q.R. 40. 1. \( 9 - 36 \). 2. \( \frac{1}{2} \). 3. \( \cos \omega(x_1y_2 - x_2y_1) \). 4. \( \cos \omega(x_1y_2 - x_2y_1) \). 5. \( \cos \omega(x_1y_2 - x_2y_1) \). 6. \( \cos \omega(x_1y_2 - x_2y_1) \). 

Q.R. 41. 1. \( 22 . \quad 22 \). 2. \( h = a(2 - \Sigma t_i / t_i) \). 3. \( a = 2 + t_i / t_i \). 4. \( -2 + t_i / t_i \). 5. \( a = 2 + t_i / t_i \). 6. \( a = 2 + t_i / t_i \). 

Q.R. 42. 1. \( \sin (a - b) \). 2. \( \sin (a - b) \). 3. \( \sin (a - b) \). 4. \( \sin (a - b) \). 5. \( \sin (a - b) \). 6. \( \sin (a - b) \). 

Q.R. 43. 1. \( \sin (a - b) \). 2. \( \sin (a - b) \). 3. \( \sin (a - b) \). 4. \( \sin (a - b) \). 5. \( \sin (a - b) \). 6. \( \sin (a - b) \). 

Q.R. 44. 1. Line; rect. hyperbola. 2. \( \sin (a - b) \). 3. \( \sin (a - b) \). 4. \( \sin (a - b) \). 5. \( \sin (a - b) \). 6. \( \sin (a - b) \). 

Q.R. 45. 1. \( \sin (a - b) \). 2. \( \sin (a - b) \). 3. \( \sin (a - b) \). 4. \( \sin (a - b) \). 5. \( \sin (a - b) \). 6. \( \sin (a - b) \). 

Q.R. 46. 1. \( \sin (a - b) \). 2. \( \sin (a - b) \). 3. \( \sin (a - b) \). 4. \( \sin (a - b) \). 5. \( \sin (a - b) \). 6. \( \sin (a - b) \).