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MATHEMATICS

GROUP III (Paper I) AND SUBSIDIARY SUBJECT (15a)

ARITHMETIC, ALGEBRA, AND TRIGONOMETRY

1918. 2\ 1/2 Hours

Not more than nine questions should be attempted by any Candidate. The easier Questions A, B, C, D, E should be attempted only by Candidates who offer Subsidiary Subject (15a), and must not be attempted by those who offer Group III as their Principal Subject.

1. Using the tables of logarithms calculate the values of
   \[(1) \ 10^{-\frac{3}{4}}; \quad (2) \ \frac{3}{4}^{10}.
   \]
   Which of the two results is the greater? Can you give any general reason for anticipating which is the greater?

2. If \(q^2\) is approximately equal to \(p\), so that \(p - q^2\) is small, verify that the percentage error in taking \(q\) to represent the square root of \(p\) is nearly equal to
   \[
   \frac{50}{p} (p - q^2) / p.
   \]
   Hence, or by direct calculation, estimate the percentage error in taking \(\sqrt{\frac{11}{9}}\) as the square root of 11.

3. A boy thinks of an odd number: he multiplies the number by 3 and divides by 2, finding that the quotient is even. He again multiplies the quotient by 3 and divides by 2; and states that his result is 175. Prove that he is wrong; and, assuming that his only error is in taking the final figure to be 5, find what was the original number. Test your result.

4. Verify that if \(ps = qr\), the value of the fraction \(\frac{p + q}{r + x}\) does not depend on the value of \(x\).

5. Show further that, if \(ad = bc\), the value of the fraction
   \[
   \frac{bed + cba + dab + abc + x(a + b)(c + d)}{(a + c)(b + d) + x(a + b + c + d)}
   \]
   is independent of the value of \(x\).

6. A motor-boat can travel at 12 miles an hour in still water. It makes a trip on a river of 15 miles downstream and returns against the current; show that, whatever may be the speed of the current, the trip takes longer than the boat would take over the same distance in still water.

   Find the speed of the current, if the extra time taken is 10 minutes.
6. Prove that
\[ 1.2 + 2.3 + 3.4 + \ldots + n (n+1) = \frac{1}{6} (n+1) (n^2+2) \]
and that
\[ 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3} (n^2 + n + 1) \cdot (2n+1) \cdot (2n+2) \cdot (2n+3) \cdot \frac{1}{6} \]
7. If \[ a = \frac{1}{3} (\sqrt[3]{13} - 1), \] prove that
\[ a^3 + a^2 = 3, \quad (2 - a)(3 + a) = 3 \]
and that
\[ 2 - a = 4 + a. \]
8. By means of the binomial theorem, write down the expanded form of \((1+x)^n\) and prove that its value for \(x = \frac{1}{2} \) is slightly less than \( \frac{18}{11} \).

Deduce or prove in any way that War Savings Certificates, which may be bought for 15s. 6d., and are worth £1 at the end of 5 years, give a better investment than compound interest at 5 per cent. per annum.

9. Write down the series for \( \log_{10} (1+x) \) in ascending powers of \( x \), stating the limits of \( x \) in order that the expansion may be valid.

Prove that, if \( n > 2 \),
\[ \frac{1}{2} \log_{10} \left( \frac{n+2}{n-2} \right) = \log_{10} \left( \frac{n+1}{n-1} \right) = \frac{1}{n} \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \right) \]
where \( z = \frac{1}{2} (n^2 - 3n) \) and \( \mu = \log_{10} e \).

Verify this result from the tables when \( n = 6 \), taking \( \mu \) to be 0.4343.

10. If \( t = \tan \theta \), verify that
\[ \sin 2\theta = \frac{2t}{1+t^2}, \quad \cos 2\theta = \frac{1-t^2}{1+t^2}. \]

Prove that if
\[ \sec 2\theta + \tan 2\theta = \nu, \]
then
\[ t = \frac{\nu-1}{\nu+1}, \]
and find expressions for \( \sin 2\theta \) and \( \cos 2\theta \) in terms of \( \nu \).

11. The relation between \( y \), the angle of refraction, and \( x \), the angle of incidence, of a ray of light on a block of glass, is given by
\[ \sin y = \frac{3}{5} \sin x. \]

Plot a graph of \( y \) as \( x \) varies from 0° to 90°, taking an inch to represent 10°.

Find from the graph the value of the angle \( x \), at which \( x - y = 30° \).

and show from the form of the graph that \( x - y \) is greatest when \( x = 90° \).
4. Express \( x^2 \) in the form \( A(x-1)(x-2)+B(x-1)+C \). Having given that \( \frac{4x^2+3}{x-1}(4x+3) \) can be expressed in the form \( \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{4x+3} \), where \( A, B, C \) are independent of \( x \), find \( A, B \) and \( C \).

5. The force \( (P) \) acting on a body is inversely proportional to the square of the distance \( (r) \) of that body from a fixed point. If to the square of the velocity \( (v) \) a certain fixed number is added, the result is inversely proportional to \( r \).

It is found that \( P = 2, v^2 = 7\frac{1}{2} \) when \( r = 3 \), and that

\[
P = 1\frac{1}{2}, v^2 = 4\frac{1}{3} \quad \text{when} \quad r = 4.
\]

Prove that \( P = 4\frac{3}{4}, v^2 = 13\frac{1}{2} \) when \( r = 2 \).

Also express \( P \) in terms of \( v \).

6. What is the "sum to infinity" of a geometrical progression, and what is the condition that such a sum should exist?

A man borrows \( £P \) at 5 per cent. per annum compound interest and agrees to discharge the debt by paying at the end of each year four-sevenths of the sum then due; prove that at the end of \( n \) years he will have paid \( £P \left( 1 - (0.05)\frac{n}{4} \right) \), and will still owe \( £(0.05)\frac{P}{4} \).

If \( P = 1000 \), find to the nearest shilling the amount paid and the amount still owing at the end of 10 years.

7. Expand \((1+x)^3\) by the Binomial Theorem as far as the term containing \( x^3 \).

Apply the Binomial Theorem to show that £1100 will amount to £1480 (to the nearest pound) in 10 years at 4 per cent. per annum compound interest.

8. Newton's Law of cooling states that if a body at a temperature of \( T \) degrees is allowed to cool, its temperature after \( t \) minutes will be \( T e^{-\alpha t} \) degrees, where \( \alpha \) is a constant. Apply the exponential theorem to determine correct to one place of decimals the percentage loss of temperature in 10 minutes of a body for which \( \alpha = 0.04 \).

9. Show that \( \cos \theta + b \sin \theta \) can be expressed in the form

\[
r \cos (\theta - \alpha),
\]

where \( r \) is positive and \( \alpha \) lies between \(-180^\circ\) and \(180^\circ\).

Find all the angles between \(0^\circ\) and \(360^\circ\) which satisfy the equation

\[
39 \cos \theta + 52 \sin \theta = 60.
\]

10. Prove that

\[
\cos \theta + \cos \phi = 2 \cos \frac{1}{2} (\theta + \phi) \cos \frac{1}{2} (\theta - \phi).
\]

Show that the value of

\[
\cos (a + \theta) + \sin (a + \theta) + \cos (a - \theta) + \sin (a - \theta),
\]

where \( a \) is a given positive acute angle, is not greater than \(2 (\cos a + \sin a)^2\) and not less than \(2 (\cos a - \sin a)^2\).

11. Two points \( A, B \) on the earth's surface are in the same latitude \( \beta \); the difference of their longitudes is \( \alpha \). Prove that the chord \( AB \) is \(2 \beta \cos \frac{1}{2} \alpha \), and that the arc \( AB \) measured along the parallel of latitude is \(\frac{\pi x}{180} \cos \beta \); where \( x \) is the radius of the earth, assumed to be a sphere.

Also find an expression for the angle which the chord \( AB \) subtends at the centre of the earth.

If \( A', B' \) are points on the equator such that the chord \( A'B' \) is equal to the chord \( AB \), show that, when \( r = 4000 \) miles, \( x = 69 \), \( x = 30 \), the difference between the area \( A'B' \) and \( AB \) is about 9 miles.

A. If a company earns a profit of 22% on its capital, and out of this profit pays a dividend of 9% on its capital, places £1300 to a reserve fund, and carries forward £1105 to next year's account, find the amount of capital.

B. Prove that the sum of the cubes of three consecutive numbers exceeds three times the product of the numbers by three times their sum.

C. Solve the equations:

\[
x^2 - 2y^2 + 4y = 0, \quad x + y = 2.
\]

If

\[
3x - 4y = 10, \quad 4x + y = 7,
\]

find the value of \(5x + 11y \).

D. The total area of the surface of a cone, whose slant side measures 11 in. and the radius of whose base is \( r \) in. is \( \pi r (r + 11) \). Plot the graph of \( r (r + 11) \), and use it to determine the radius of the base of such a cone whose total superficial area is 125 square inches.

Check your result by an algebraical solution. [Take \( \pi = 3.14 \).]

E. Two roads cross at right angles; a man travelling on one of these roads observes a house on the other and notes that its direction makes an angle of 52° with the direction in which he is going; he then walks on for 600 yards, passing the cross-road, and observes that the direction of the house makes an angle of 67° with the road along which he has come. Find the distance of the house from the crossing of the roads.

Check your result by a diagram to the scale of 1 inch to 150 yards.

[6]
1920 ARITHMETIC, ALGEBRA, TRIGONOMETRY GROUP III

1920. 2½ Hours

1. The volume of a piece of piping of length 10-5 in. is found to be 132 cu. in., and the external diameter is 5 in. Find the internal diameter, taking \( \pi = \frac{22}{7} \).

If the volume was really four per cent. less, find the true value of the internal diameter to two places of decimals.

2. (a) Simplify

\[
\frac{(2x-y)^2 + (x-2y)^2}{x^4 + 2x^2y^2 + y^4}.
\]

(b) It is given that \( x + 2 \) is a factor of

\[
x^4 - bx^3 - 11x^2 + 4 (b+1) x + a,
\]

which is also given to be a perfect square. Find \( a \) and \( b \).

3. An aeroplane travelling against the wind, the velocity of which is 20 ft. per sec., travels half as long again to travel a certain distance as another which is travelling with the wind. If there is no wind the first takes 46 min. and the second takes 44 min. Find the distance and the speeds of the aeroplanes.

4. Prove that the roots of the equation

\[ (a^2 + b^2) x^2 + 2 (a^2 + b^2 + ab) x + b^2 + c^2 = 0 \]

must be real, if \( a, b, c \) are real.

If \( \alpha, \beta \) are the roots of \( ax^2 + bx + c = 0 \), show that the roots of \( a^2x^2 + (2a^2 - b)b x + c = 0 \) are \( \alpha/\beta, \beta/\alpha \).

5. A rectangle one of whose sides is 2x in. is inscribed in a circle of radius 3 in. Draw a graph showing how the area of the rectangle varies with \( x \).

Show from your graph, or otherwise, that the greatest value of the area is 18 square inches, and find the rate of change of the area with \( x \) when \( x = 1 \) in.

6. Explain what is meant by Mathematical Induction.

Use this method, or any other, to prove (1) the Binomial Theorem for a positive integral index; (2) that the sum to \( n \) terms of the series whose \( n \)th term is \( n (n+1) (2n+1) \) is equal to \( \frac{3}{4} n (n+1)^2 (n+2) \).

7. Express \( \frac{9x}{(x+2)(2x-1)^2} \) in the form

\[
\frac{A}{x+2} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2}.
\]

Hence expand this expression in a series of ascending powers of \( x \) as far as \( x^3 \).

GROUP III ARITHMETIC, ALGEBRA, TRIGONOMETRY 1920

Calculate the value of \( (\frac{126}{4})^4 \) to five places of decimals by means of the Binomial Theorem.

8. Assuming that

\[
\log_e (1+x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \ldots,
\]

prove that

\[
\log_e \frac{P}{Q} = 2 \left( \frac{P-q}{P+q} + \frac{1}{3} \left( \frac{p-q}{p+q} \right)^3 + \frac{1}{5} \left( \frac{p-q}{p+q} \right)^5 + \ldots \right).
\]

Given that \( \log_e 3 = 0.4772 \) and \( \log_e 5 = 0.6990 \), calculate the values of \( \log_e 28 \) and \( \log_e 105 \) to four places of decimals.

9. Define the projection of a line \( PQ \) upon another line \( AB \), and express the length of the projection in terms of the length of \( PQ \) and the angle between the lines.

Prove that

\[
\cos \theta + \cos (\theta + 120^\circ) = 0.6665.
\]

Find by a graphical construction, or otherwise, the angles \( \theta \) and \( \phi \), where

\[
\sin \theta + 2 \sin \phi = 1.8, \quad \cos \theta + 2 \cos \phi = 1.9.
\]

10. Sketch roughly the graph of \( \sin \theta + \cos 2\theta \) between \( 0^\circ \) and \( 360^\circ \), and find approximately from your graph the values of \( \theta \) between these limits which satisfy

\[
\sin \theta + \cos 2\theta = 0.2.
\]

Find these values of \( \theta \) to the nearest 15 minutes by solving the equation.

11. Prove that if \( \sin \theta = \tan \theta \), then \( \theta = \frac{1}{2} \pi \), and that

\[
\sin \theta = \theta = \tan \theta,
\]

if \( \theta \) be small so that cubes and higher powers of \( \theta \) may be neglected.

Show that the distance to the horizon as seen from the top of a cliff \( h \) feet high is \( 1.23\sqrt{h} \) miles, approximately, where the radius of the earth is assumed to be 4000 miles, and that the dip of the horizon is \( 3250 \) radians.

A. A man’s capital on Jan. 1, 1918, was £4375, and on Jan. 1, 1920, was £4395. The interest per cent. during 1918 was equal to the decrease per cent. during 1919. What was his capital on Jan. 1, 1919?

B. Find the condition that the equations

\[
a x^2 + bx + c = 0, \quad c x^2 + b x + a = 0
\]

may have a common root, where \( a \) is not equal to \( c \).

Prove that this root must be +1 or -1.
1920 ARITHMETIC, ALGEBRA, TRIGONOMETRY GROUP III

C. (1) Solve the equations
\[ 2x - 3y = 1, \]
\[ x^2 - y^2 + 3xy + x = 2 = 0.\]

(2) Find the square root of 57 - 12√15.

D. Draw the graph of \[ y = x^2 \] between \( x = -3 \) and \( x = +3.\)

Use the graph to find
(1) the roots of the equation \( x^3 - 4x - 2 = 0 \), approximately.
(2) the rate of compound interest per cent. per annum at which \( a \) sum of money will treble itself in three years.

E. A blackboard slopes at an angle of 55° to the ground with two edges horizontal. A straight line drawn on the blackboard makes an angle of 30° with a horizontal edge. Find the angle which this line makes with the ground.

1921. 2\(\frac{1}{2}\) Hours

1. A hollow closed iron cylinder with sides and ends of the same thickness (t inches), is 20 in. long and 5 in. radius (outside measurements), and weighs 224 oz/lb. Prove that \( t \) is given by the equation
\[ 20t^2 + 125t - 80 = 0, \]
if 1 cubic in. of iron weighs 0.28 lb.

Calculate the values of the expression on the left-hand side of this equation for \( t = 0.7 \), \( t = 0.8 \); and by plotting these values on a graph, or otherwise, determine an approximate value of \( t \).

2. If \( y = ax + bx + cx^2 \) and corresponding values of \( x \) and \( y \) are given by
\[ x = -\frac{1}{2}, \quad y = \frac{1}{2}, \quad \frac{1}{2}, \]
find the values of \( a, b, c \) and the greatest positive value of \( y \).

3. Prove that \( t^2 - 1 + \frac{1}{2} \) is an even function of \( t \), i.e. one which does not change sign when \( -t \) is substituted for \( t \).

If this expression is denoted by \( \phi (t) \), prove that
\[ \phi (2t) = \phi (t) + t^2 \phi \left( \frac{t}{2} \right). \]

4. An express train, travelling at a certain rate, passes a certain point two minutes after its scheduled time. Forty miles farther on it is four minutes late. Its speed is then increased by twelve miles an hour, and it arrives punctually at a point twenty miles farther on. At what rate, in miles per hour, would it have travelled if it had been running to time the whole way?

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5. Find numbers \( A, B, C \) independent of \( n \) to satisfy the identity
\[ \frac{n^6}{A} = Ax^n + Bx^{n-1} + Cx^{n-2} \]
\[ + Anx^{n-3} + Bnx^{n-4} + Cnx^{n-5}. \]

Hence, or otherwise, when \( n \) is a positive integer, find an expression for
\[ n^6 = (n-1)^6 + (n-2)^6 + \ldots + (n-1)^6. \]

6. Show that in the expansion by the binomial theorem of \( (1+x)^{2n} \) when \( x = \frac{1}{2} \), there are two terms which are greater than all the rest; and find their value.

Prove that, when \( x \) is a small fraction,
\[ \left( \frac{1 + x}{1 - x} \right)^{1/2} = 1 + \frac{3}{2}x + \frac{3}{2}x^2 + \frac{4}{5}x^3 + \ldots \]

7. Prove that
\[ \log_a \frac{m}{n} = \frac{m-n}{m+n} \log \left( \frac{m-n}{m+n} \right) + \frac{m-n}{m+n} \log \left( \frac{m-n}{m+n} \right) + \ldots \]

Prove that \( \log_{10} \frac{10}{11} \approx 0.023706 \ldots \)

Noting that 128 is a power of 2, that 125 is a power of 5, and that \( \log_{10} 2 \approx 0.30103 \), prove that \( \log_{10} 20 = 2.30103 \) correct to three places of decimals.

8. Prove the formula \( \theta + \phi = \cos \theta \cos \phi - \sin \theta \sin \phi \), when \( \theta \) and \( \phi \) are positive acute angles and \( \theta + \phi \) is obtuse.

Solve for \( x \) and \( y \) the equations
\[ x \cos 2a + y \cos 3a = c \cos \beta, \quad x \sin 2a + y \sin 3a = c \sin \beta. \]

9. \( AOB \) is a triangle whose sides \( AO, AB, AC \) are of lengths 1 in. and 2 in. respectively; \( AOB \) is \( \theta \), \( ABO \) is \( \phi \), \( BO \) is \( z \). Prove that
\[ \sin \theta = \frac{1}{2} \sin x, \quad \theta = \cos x^2 + 2 \cos y. \]

Plot the graphs of \( y \) and \( z \) as \( x \) varies from 0 to 180.

[Take 0-4 in. to represent 1°, and the unit of length as 1 in.]

10. All that is known about an angle \( A \) is that its sine is \( s \); prove that \( \sin A \) has one of the four values
\[ \pm \frac{1}{2} (1 + s) \pm \frac{1}{2} (1 - s) \]

But if it is also known that the cosine of \( A \) is \( c \), prove that \( \sin A \) has one of the two values
\[ \pm \frac{1}{2} (1 + s - c) \pm \frac{1}{2} (1 + s + c). \]

Give a careful explanation of the reasons for these results.

[10]
GROUP III ARITHMETIC, ALGEBRA, TRIGONOMETRY 1922

Find $A, B, C$, and verify your answer by simplifying the expression thus obtained.

4. The force $(F)$ acting on a body is a quantity proportional to the distance $(s)$ the body has travelled diminished by a quantity proportional to the square of its speed $(v)$; it is also found that its speed is found by subtracting $e$ from a fixed number. To the values $3, 5, 10$ of $s$ correspond the values $202, 450, 1000$ of $F$.

Express $F$ in terms of $s$, and find its value when the body is at rest. Find also the value of $e$ when $F$ vanishes, assuming that $F, v$, and $e$ are all positive quantities.

5. Find the conditions that the roots of the equation

$$3x^2 - 6xy + 10y - 3 = 0,$$

considered as an equation in $x$, may be (i) real and different, (ii) coincident, (iii) imaginary.

Prove that $3$ is a minimum and $\frac{3}{4}$ a maximum value of $6x^2 - 10$, and show the apparent paradox.

Illustrate your answer by drawing a rough graph of the function.

6. If $E_n(z)$ denotes the series

$$1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!},$$

prove that the sum of the terms of degree $r$ in the product $E_n(x)E_r(y)$ is $\frac{(x+y)^r}{r!}$, provided that $r$ is not greater than $n$; and show that, when $x$ and $y$ are positive numbers, $E_n(x+y)$ greater than $E_n(x)+E_n(y)$ and less than $E_n(x+y).

Show that if $x$ is so small that its squares and higher powers may be neglected

$$(x^2 - 4x^2 + 6e^2 - 4e + 1) = 16(1 - x).$$

7. $OAB$ is a triangle, right-angled at $A$, in which $OA = a, AB = b, OB = k$; with centre $O$ is described a circle of radius $c(<k)$. $BM, BN$ are the tangents from $B$ to this circle. Prove that the angles $AOM, AON$ satisfy the equation $\cos \theta + \sin \theta - \epsilon = \cos \theta$, angles being reckoned positive when measured in the same sense as $OAB$.

Find all the angles between $-180^\circ$ and $180^\circ$ which satisfy the equation

$$5 \cos \theta - 3 \sin \theta = 4.$$ 

8. Prove the formulae for a plane triangle

(1) \hspace{1cm} a = b \cos C + c \cos B;

(2) \hspace{1cm} a \sin \frac{1}{2} (B + C) = (b - c) \cos \frac{1}{2} A.

$B$ is a point due east of $A$ and 200 yards from it; $D$ is a point 400 yards from $A$ bearing east $65^\circ$ north; $C$ is a point 300 yards
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from B bearing west 47° north. Find the bearing and the distance (to the nearest foot) of B from C.

9. Draw the graph of
   \[ y = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \]
   for values of \( x \) between 0 and 180.
   Find the values of \( x \) between 0 and 180 for which
   \[ x = 120 \sin x - 60 \sin 2x + 40 \sin 3x. \]
   [Take 2 inches as unit along the axis of \( y \); take 20° = 1 inch, and
   plot for \( x = 0, 30, 45, 60, 90, 120, 135, 150, 180, \).]

A. Three solid lead spheres, of radii 6 in., 8 in., 10 in., are melted together and recast as a solid sphere. Find the percentage diminution of the total surface exposed.

B. Find all the factors of
   \[ 4n^3 - (a^2 - b^2 + 1)^3. \]
   Simplify
   \[ \frac{b}{x+a} + \frac{a}{x+b} - \frac{a+b}{x+a+b} = \frac{x^2}{2} \]
   and find the value of the expression when \( b = -a \).

C. Solve the equations:
   \[ \begin{align*}
   (1) \quad & 4x^2 + 4xy - y^2 = 7, \\
   (2) \quad & 3y^2 - 2x = 7.
   \end{align*} \]

D. Prove that \( \sin^2 \theta + \cos^2 \theta = 1 \).
   Two roads cross at right angles; a man travelling on one of
   these roads observes a house on the other, and notes that its direction
   makes an angle of 50° with the direction in which he is going; he
   then walks on for 100 yards, passing the cross-road, and observes that
   the direction of the house makes an angle of 65° with the road
   along which he has come. Find the distance of the house from the
   crossing of the roads.

1923. 24 Hours

1. A man wishes to found an annual prize of value £5, to be given in twenty successive years; find the sum that he should pay now, in order to cover the annual payments, the first prize being awarded at the end of the first year, and interest being reckoned at 5 per cent.

   If he were to pay down £80 now, for how many years could the
   prize be given under the same conditions?

[14]

GROUP III ARITHMETIC, ALGEBRA, TRIGONOMETRY 1923

2. A, walking at 4 miles an hour, and B, cycling at 8 miles an hour, start at the same moment from one town X to another Y; C, motoring at 20 miles an hour, starts half an hour later from X to Y, and, on reaching Y, stays there for half an hour, and then returns to X at the same rate as before. If the interval between the instants when C passes A and B on the outward journey is half that between the instants when C passes B and A on the return journey, find the distance between X and Y.

3. If \( f(a) \) denote the sum of a number of terms containing integral powers of \( x \), and be divided by \( x - a \), show that the remainder is \( f(a) \). Also, if \( f(x) \) be divided by \( (x - a)(x - \beta) \), show that the remainder is \( px + q \), where
   \[ p = \frac{f(a) - f(\beta)}{a - \beta} \]
   and \( q = \frac{f(\beta) - f(a)}{\beta - a} \).

   Find the factors of
   \[ a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2). \]

4. Show that the roots of the quadratic equation
   \[ ax^2 + bx + c = 0 \]
   are
   \[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]
   and
   \[ \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]
   and hence show that the sum of the roots is \(-b/a \) and the product of the roots \(c/a \).

   Find three quadratic equations which are such that, in each,
   the sum of the squares of the roots is greater by 40 than the sum of
   the roots, and the sum of the cubes of the roots is greater by 20
   than the sum of the squares of the roots.

5. In the series
   \[ \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots, \]
   show that the remainder after \( n \) terms lies between
   \[ \left(1 + \frac{1}{n+1}\right) \frac{1}{n!} \text{ and } \left(1 + \frac{1}{n}\right) \frac{1}{n!}. \]
   and, taking \( n = 8 \), show that the sum of the series is 2.7183 correct to 4 places of decimals.

   Find the coefficients of \( x^n \) and \( x^n \) in the expansion of
   \[ e^x(1 + 2x - 4x^2). \]

6. If \( \alpha, \beta \) and \( \alpha + \beta \) be positive acute angles, find geometrically
   \( \sin(\alpha + \beta) \) and \( \cos(\alpha + \beta) \) in terms of \( \sin \alpha, \cos \alpha, \sin \beta \) and \( \cos \beta. \)

[15]
1923 ARITHMETIC, ALGEBRA, TRIGONOMETRY Group III

If \( \sin a + \sin b = a \) and \( \cos a + \cos b = b \), express \( \sin (a+b) \) and \( \cos (a+b) \) in terms of \( a \) and \( b \).

7. (1) If \( \sin a + \sin b = 0 \), prove that the values of \( \theta \) which satisfy this equation form two arithmetical progressions, and find their common differences.

(2) Solve the equation

\[
\sin 2\theta + \cos 2\theta = \sin \theta + \cos \theta.
\]

8. Draw the graphs of \( \sin^2 x \) and \( \frac{x}{300} + \cos x \) for values of \( x \) in degrees from 0 to 180, and find the value between 0 and 180 (to the nearest degree) for which

\[
\sin^2 x = \frac{x}{300} + \cos x,
\]

and test the solution by the tables. [Take one inch along the axis of \( x \) to represent 20°, and 3 inches along the axis of \( y \) to represent unity.]

9. An isosceles triangle, the sides of which are respectively 25, 25, and 14 feet in length, stands in a north-and-south vertical plane with its shortest side in contact with a horizontal plane; find (to the nearest square foot) the area of the shadow cast by the triangle on the horizontal plane when the sun is in the direction 20° north of east and at an altitude of 32°.

A. A publisher prints 5,000 copies of a book, the price of which is fixed at 7s. 6d. net; the cost of printing and paper is 1s. 3d. a copy, that of binding 3d. a copy, and that of carriage and advertising £55; he sells them to booksellers, charging 25 copies as 24 and 30 per cent, less than the published price, and upon the whole receipts takes 25 per cent, commission for himself; find the respective gains of the author, publisher, and booksellers on this edition.

B. (1) Find the factors of \( x^3 + y^3 - xy(x+y) \).

(2) Given that

\[
x^2 + 2ax^2 - 3x^2 - 8x - 4 = (x^2 + ax + 2)(x^2 + bx - 2),
\]

find \( a \) and \( b \); and hence find the four factors of the first expression.

C. Solve the equations:

(1) \( x^2 - 2x - x - a = a^2 - b^2 \)

(2) \( x^2 - x^2 - 2x + a^2 - x - a = a^2 - b^2 \)

D. A fine string passes tightly round part of a circular lamina of radius 5 inches and also round a point in the plane of the lamina and 10 inches from its centre; find the length of the string in inches to two places of decimals.

[16]
GROUP III ARITHMETIC, ALGEBRA, TRIGONOMETRY 1925

1925. 2½ Hours

1. Given that ten letters can be placed in ten addressed envelopes so that every letter is in a wrong envelope in $N_{10}$ different ways, where

$$N_{10} = 10! \left[ \frac{1}{2!} \frac{1}{3!} \frac{1}{4!} \frac{1}{5!} \cdots \frac{1}{10!} \right],$$

compute the value of $N_{10}$.

It takes a minute to place ten letters in ten envelopes and a minute to take them out again. If to-day you start putting the letters into the envelopes, every letter into a wrong envelope, taking them all out again and then reinserting them, find the date on which you will finish the task of inserting the letters into the envelopes in all the $N_{10}$ ways if you work at the rate of eight hours per day for seven days per week.

2. Write down the expansion of $(1+x)^n$ in ascending powers of $x$.

Prove that, when $x$ is small,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

is nearly equal to 1.

3. A slow train from London to Bristol travels the first 36 miles of its journey at 30 m.p.h., the next 42 miles at 20 m.p.h., and the remaining 42 miles at 30 m.p.h. A fast train leaves Bristol for London when the slow train has spent half of its time on the journey, and travels the 120 miles at 60 m.p.h. Find the distance of the trains from London when they meet.

[Full credit will be given for neat and accurate graphical solutions.]

4. Prove that

$$a^4 - a^3 b - a b^3 + b^4 = \frac{n(n+1)}{2} \left[ \frac{1}{2} \right] \left[ \frac{1}{4} \right] \left[ \frac{1}{6} \right] \cdots \left[ \frac{1}{n} \right];$$

$$a^2 = \frac{1}{2} \left( a + b \right)^2 = \frac{1}{4} \left( a + b \right)^2 = \frac{1}{6} \left( a + b \right)^2.$$

Deduce that, if $n$ is a positive integer,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)}{2} \left[ \frac{1}{2} \right] \left[ \frac{1}{4} \right] \left[ \frac{1}{6} \right] \cdots \left[ \frac{1}{n} \right].$$

5. If $a = \log_{10} 2$, $b = \log_{10} 3$, $c = \log_{10} 5$, prove that

$$\log_2 2 = 3a + b + c.$$

* July 14, 1925.
ARITHMETIC, ALGEBRA, TRIGONOMETRY GROUP III

Assuming that
\[ \log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, \]
compute the values of \( a, b, \) and \( c \) to four places of decimals, and
deduce the value of \( \log_2 3 \) to three places of decimals.

6. Prove the formulae
\[ \cos (A+B) = \cos A \cos B - \sin A \sin B, \]
\[ \sin (A+B) = \sin A \cos B + \cos A \sin B. \]

If \( A, B, C \) are three angles whose sum is two right angles,
prove that
\[ \tan A + \tan B + \tan C = \tan A \tan B \tan C. \]

7. Prove that the general solution of the equation
\[ \sin 4\theta = \sin 3\theta \]
is \( \theta = \frac{2n\pi}{7} \) or \( \frac{(2n+1)\pi}{7} \), where \( n \) is any integer.

Express
\[ \sin 3\theta \quad \text{and} \quad \sin 4\theta \]
in terms of \( \sin \theta \), and hence or otherwise deduce that the roots of the
equation
\[ 8x^3 - 4x^2 - 4x + 1 = 0 \]
are \( \cos \frac{\pi}{7}, \cos \frac{3\pi}{7}, \) and \( \cos \frac{5\pi}{7} \).

8. A man standing on the Llanberis Pass, 400 feet above sea-level,
observes the summit of Snowdon to be at a distance of 13,000 feet
and to be due South, and the summit of Glyder-fawr to be at a
distance of 10,000 feet and to be due East. If the heights of the sum-
mits of Snowdon and Glyder-fawr are 3,500 feet and 3,290 feet above
sea-level respectively; find the angle which the line joining their
summits subtends at his eye.

9. The sides and angles of an acute-angled triangle are \( a, b, c \) and
\( A, B, C \). A second triangle is constructed whose sides are \( a \cos A, \]
\( b \cos B, c \cos C \). Prove that the angles of the second triangle are
\( 180^\circ - 2A, 180^\circ - 2B, 180^\circ - 2C. \)

A. Factorize :
1. \( a^2 (b - c) + b^2 (c - a) + c^2 (a - b) \)
2. \( 2x^2 - 3x^2 - 3x + 2 \)
3. \( (x^2 - 5x)^2 + 10 (x^2 - 5x) + 24 \)
B. Solve the equations :
1. \( x^2 - (a - 1) x - a = 0 \)

GROUP III ARITHMETIC, ALGEBRA, TRIGONOMETRY 1925

(2)
\[ \frac{x}{x - a} = \frac{(a + b)(x - 2b)}{x - a - b} \]

C. In four years from now \( A \) will be two-thirds of \( B \)’s present age,
in eleven years from now \( B \) will be double \( C \)’s present age, and
14 years ago \( C \) was half of \( A \)’s present age. Find the present ages
of \( A, B, \) and \( C \).

D. The sides \( BC, CA, AB \) of a triangle are of lengths 5, 6, and
7 feet. Points \( D, E, F \) are taken on the sides respectively so that
\[ BD : DC = 3 : 2, CE : EA = 1 : 2, AF : FB = 4 : 3. \]
Prove that the sides of the triangle \( DEF \) are proportional to
\( \sqrt{10}, 3, \sqrt{7} \).

GROUP III (PAPER I)

1926. 2½ Hours

1. The population of England and Wales in 1801 was 8,893,000
and in 1901 was 32,528,000. Assuming that the population increases
according to the compound interest law, estimate the population
(1) in 1851, (2) in 1926.

2. A train travels from \( A \) to \( B \), forty miles off, in one hour, its
distance from \( A \) in miles, \( t \) minutes after starting, being given by
the formula
\[ t^2 (90 - t)/2700. \]

A cyclist leaves \( A \) half an hour before the train, travels with uniform
speed towards \( B \), and has travelled 20 miles when the train reaches
\( B \). Find, graphically, how far the cyclist is from \( A \) when the train
overtakes him.

3. Without assuming the formula
\[ b = \frac{\sqrt{b^2 - ac}}{a} \]
for the roots of the quadratic equation \( ax^2 + 2bx + c = 0 \), prove that
the sum of the roots is \( -2b/a \), and that the product of the roots is \( ac/a \).

If \( a, b \) are the roots of \( x^2 - 4x + 1 = 0 \), prove that
\[ a^3 + b^3 = 52, \]
and deduce that \( a + b = 2\sqrt{2} \).

4. Find rational numbers \( a \) and \( b \) such that
\[ 3 + \sqrt{2} = (a + b\sqrt{2}) (6 - \sqrt{2}) \).

[20]
1926 ARITHMETIC, ALGEBRA, TRIGONOMETRY GROUP III

Solve the equation
\[ \sqrt{x-1} + \sqrt{x+4} = \sqrt{3x+10}, \]
testing whether your answers actually satisfy the equation.

5. Write down the Binomial expansion for \( \sqrt{1+x} \) in powers of \( x \), giving the general term.

Prove that the first two terms of the expansion exceed \( \sqrt{1+x} \)
by exactly
\[ \frac{1}{2}x^2 \]
\[ 1 + \frac{1}{2}x + \sqrt{1+x} \]

By taking \( x = 0.024 \) prove that \( \sqrt{10} \) is approximately equal to
3.1625 ( = 1012.320), and estimate to how many places of decimals
this approximation is correct.

6. Prove the formula
\[ \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}. \]

Solve the equation
\[ \sin \theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0, \]
giving general solutions.

7. Prove that in any triangle
\[ \tan \frac{B-C}{2} = \frac{b-c}{b+c} \tan \frac{A}{2}. \]

The lengths of two sides of a triangle are 1,734 feet and 933 feet,
and the included angle is 49°. Solve the triangle without using
formulae unadapted for logarithmic computations.

8. A straight line \( AB \) is horizontal and 700 feet long; another
horizontal line \( DC \), 100 feet long, bisects \( AB \) at right angles at \( C \);
and \( D \) is the foot of a tower 520 feet high. Find the angle subtended
at the eye of an observer at the top of the tower by the line joining
\( A \) to the top of a building 90 feet vertically above \( B \).

9. If \( E_1, E_2, E_3 \) are the ex-centres and \( R \) is the circum-radius
of the triangle \( ABC \), prove that the radius of the ex-circles with centre
\( E_1 \) is \( 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \), and that the length of
\( E_2E_3 \) is \( 4R \cos \frac{A}{2} \).

Prove also that the area of the triangle \( E_1E_2E_3 \) is
\[ 8R^2 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. \]

GROUP III ARITHMETIC, ALGEBRA, TRIGONOMETRY 1927

1927. 2\frac{1}{2} Hours

1. An expression of the third degree in \( x \) is to have the values
\( A, B, C, D \) when \( x \) has the values \( a, b, c, d \) respectively. Prove that
is an expression which satisfies these conditions.

Is this expression the only one which satisfies all the conditions?
An expression of the third degree is denoted by \( f(x) \). If
\[ f(-1) = 6, f(0) = 9, f(3) = 19, f(3) = 11, \]

find the gradient of the graph of \( f(x) \) at \( x = 0 \).

2. Find the formula for the sum of \( n \) terms of a geometric progression.

Prove, or verify, that
\[ a + \{a + b\}r + \{a + 2b\}r^2 + \ldots + \{a + (n-1)b\}r^{n-1} \]
\[ = \frac{a(1-r)(1-r^n)}{(1-r)^2} + b\{r - nr^n + (n-1)r^{n-1}\}. \]

3. Solve the equations:

\[ \begin{align*}
1) & \quad x + y = 3, \quad x^2 + y^2 = 819; \\
2) & \quad \sqrt{x+1} + \sqrt{y+8} = \sqrt{6x+1}. 
\end{align*} \]

Test whether your answers to (2) actually satisfy the equation.
If any of them do not, explain why they do not and state the
surd-equation resembling (2) which they do satisfy.

4. Prove the formula for the number of combinations of \( n \) things
taken \( r \) at a time.

Sixteen oarsmen are to be selected to form two crews of eight
each. Four of the men can row only on bow side, and five can row
only on stroke side; the rest can row on either side. In how many
different ways can the crews be made up, regard being paid to the
order in which they row?

5. Write down the expansions of \( \log_e (1 + x) \) and
\[ \log_e \frac{1+x}{1-x} \]
in ascending powers of \( x \), when \( -1 < x < 1 \).

Prove that, if \( a, b, \) and \( y \) are positive,
\[ (a+b+2y) \log_e \frac{a+y}{b+y} \]
\[ = 2(a-b) \left[ 1 + \frac{1}{3} \left( \frac{a-b}{a+b+2y} \right)^2 + \frac{1}{5} \left( \frac{a-b}{a+b+2y} \right)^4 + \ldots \right]. \]
Deduce that, if \( a < b \), then the effect on the expression \( \left( \frac{a}{b} \right)^{a+b} \) of increasing \( a \) and \( b \) by the same amount is to increase its value.

6. Draw in one diagram the graphs of \( \sin x \) and \( 1 - \frac{1}{2}x^2 \) from \( x = 0 \) to \( x = \pi \).

Estimate, as accurately as you can from your figure, the solution of the equation \( x^3 = 4 - 4 \sin x \) which lies between 0 and \( \pi \).

[Take 6 inches as \( \pi \) units for abscissae and 2 inches as 1 unit for ordinates.]

7. Prove the formula:

\[
\sin (A + B) = \sin A \cos B + \cos A \sin B.
\]

Solve the simultaneous equations

\[
\sin X + \sin Y = \frac{3}{4}, \quad \cos X + \cos Y = \frac{5}{4},
\]

giving all solutions which lie between \(-360^\circ\) and \(+360^\circ\).

8. A field is in the shape of a quadrilateral \( ABCD \), and the following observations are made of its dimensions:

\( AC \) is 100 yards. The bearings of \( B, C, D \) from \( A \) are respectively East, \( 7^\circ \) South, East, \( 43^\circ \) North, and North, \( 11^\circ \) West. The bearings of \( B \) and \( D \) from \( C \) are South, \( 10^\circ \) West, and West, \( 5^\circ \) South. Find the area of the field.

9. Prove that, with the usual notation for a triangle,

\( [ABCD] = \frac{abc}{4R} \).

The circumcircles of a triangle \( ABC \) is drawn, and the tangents at \( A, B, C \) to the circumcircle form a triangle \( DEF \). Prove that the area of the triangle \( DEF \) is

\[
R^2 \tan A \times \frac{1}{2} \tan B \times \frac{1}{2} \tan C.
\]

GROUP IV (PAPER I) AND SUBSIDIARY SUBJECT (15 a).

1926. 2\(\frac{1}{2}\) HOURS

1. The population of England and Wales in 1801 was 8,903,000 and in 1901 was 32,528,000. Assuming that the population increases according to the compound interest law, estimate the population

(i) in 1851,

(ii) in 1926.

2. A train travels from \( A \) to \( B \), forty miles off, in one hour, its distance from \( A \) in miles, \( t \) minutes after starting, being given by the formula

\[
20 - \frac{1}{2}t^2.
\]

A cyclist leaves \( A \) half an hour before the train, travels with uniform speed towards \( B \), and has travelled 20 miles when the train reaches \( B \). Find, graphically, how far the cyclist is from \( A \) when the train overtakes him.

GROUP IV ARITHMETIC, ALGEBRA, TRIGONOMETRY 1926

3. Without assuming the formula

\[
\frac{b + \sqrt{b^2 - 4ac}}{2a}
\]

for the roots of the quadratic equation \( ax^2 + bx + c = 0 \), prove that the sum of the roots is \(-\frac{b}{a}\), and that the product of the roots is \(\frac{c}{a}\).

If \( a, b \) are the roots of \( x^2 - 4x + 1 = 0 \), prove that

\[
a^2 + b^2 = 52,
\]

and deduce that \( a^2 + b^2 = 2702 \).

4. Obtain the formula for the sum of a terms of an arithmetical progression.

Find (1) the sum of all numbers represented by three digits, (2) the sum of all numbers represented by three digits, none of the digits being zero.

5. Write down the Binomial expansion for \( \sqrt{1 + x} \) in powers of \( x \), giving the general term.

Prove that the first two terms of the expansion exceed \( \sqrt{1 + x} \) by exactly \( \frac{1}{4}x^2 \).

By taking \( x = 0.024 \) prove that \( \sqrt{10} \) is approximately equal to 3.1622 (= 31622/1000).

6. A regular hexagon of side \( 2a \) is cut out of cardboard. Circles of radius \( a \) are drawn with their centres at the vertices of the hexagon, and the cardboard is cut along these circles so as to remove the vertices. Prove that the area remaining is about \( 0.395 \) of the original area of the hexagon.

7. Prove the formula

\[
\tan (A + B) = \frac{\tan B}{\cos A \cos (A + B)}.
\]

An observer in a boat measures the angles of elevation of the foot and top of a tower on a high cliff, and finds them to be \( 22^\circ \) and \( 26^\circ \). If the height of the tower is known to be 50 feet, find the height of the cliff and the distance of the foot of the tower from the observer.

8. Prove that in any triangle

\[
\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}.
\]

The lengths of two sides of a triangle are 1,734 feet and 953 feet, and the included angle is 49°. Solve the triangle without using formulae unadapted for logarithmic computations.
9. A straight line $AB$ is horizontal and 700 feet long; another horizontal line $DC$, 100 feet long, bisects $AB$ at right angles at $C$; and $D$ is the foot of a tower 320 feet high. Find the angle subtended at the eye of an observer at the top of the tower by the line joining $A$ to the top of a building 90 feet vertically above $B$.

1927. 2½ Hours

1. Solve the equations:
   
   (1) $x - y = 3, \quad x^2 - y^2 = 819$.
   
   (2) $6x - y = 41, \quad 6y - x = 0, \quad 6x - y = 0, \quad 6y - x = 41$.

2. An expression of the second degree in $x$ is to have the values $A, B, C$ when $x$ has the values $a, b, c$ respectively. Prove that

   $$A(x - b)(x - c) + B(x - c)(x - a) + C(x - a)(x - b)$$

   is an expression which satisfies these conditions.

   Is this expression the only one which satisfies all the conditions?

   An expression of the second degree is denoted by $f(x)$. If $f(1) = 7, f(2) = 22, f(3) = 17$, find the value of $f(4)$, and also find the gradient of the graph of $f(x)$ at $x = 2$.

3. Obtain the formulae (1) for the sum of $n$ terms of an arithmetic progression, (2) for the sum of $n$ terms of a geometric progression.

   Two sums of £100 start accumulating on January 1, 1927, the first at 10 per cent. per annum simple interest, the second at 5 per cent. per annum compound interest; the interest in both cases is paid annually on December 31. Find the date of that payment of interest which first makes the amount of the second sum exceed the amount of the first sum.

4. Prove the formula for the number of combinations of $n$ different things taken $r$ at a time.

   Sixteen oarsmen are to be selected to form two crews of eight each. Four of the men can row only on bow side, and five can row only on stroke side; the rest can row on either side. In how many different ways can the crews be made up, regard being paid to the order in which they row?

5. Write down the expansion of $\log_2 (1 + x)$ in ascending powers of $x$ when $-1 < x < 1$.

   Deduce the expansion of $\log_2 \frac{1 + x}{1 - x}$ in ascending powers of $x$.

   By taking $x = \frac{1}{2}$, compute $\log_2 \frac{1}{2}$ to seven places of decimals, given that $\log_2 2 = 0.301030972$.

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GROUP IV ARITHMETIC, ALGEBRA, TRIGONOMETRY 1927

6. The circumscribed and inscribed circles of a regular octagon are drawn. Find the ratio of the area which is inside the circumscribed and outside the octagon to the area which is inside the octagon and outside the incircle. Give your answer to three significant figures.

7. Draw in one figure the graphs of $\frac{x}{2} + \frac{1}{4} \sin x$ and of $2 \cos x$ from $x = 0$ to $x = 180$.

   Hence estimate the solution of the equation

   $$4 \cos x - \sin x = 1$$

   which lies between $0^\circ$ and $180^\circ$.

   [Take 1 inch to represent $30^\circ$ for absicssae and 2 inches as the unit of length for ordinates.]

8. Prove the formula

   $$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

   Solve the simultaneous equations

   $$\sin X + \sin Y = \frac{3}{5}, \quad \cos X + \cos Y = \frac{3}{5},$$

   giving all solutions between $-360^\circ$ and $360^\circ$.

9. A field is in the shape of a quadrilateral $ABCD$, and the following observations are made of its dimensions:

   $AC$ is 100 yards. The bearings of $B, C, D$ from $A$ are respectively $E. 7^\circ S., E. 45^\circ N.,$ and $N. 11^\circ W.$ The bearings of $B$ and $D$ from $C$ are $S. 10^\circ W.$ and $W. 5^\circ S.$ Find the area of the field.

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SPECIAL SUBSIDIARY PAPER (I).

ARITHMETIC, ALGEBRA, AND GEOMETRY

1918. 2½ Hours

Any nine questions may be attempted.

1. A man's salary is fixed at the rate of £15 a month; after deduction of income-tax, he receives £13 1s. 9d. At what rate in the pound does he pay income-tax?

2. Calculate to five places of decimals the values of $\sqrt{17}$ and $\sqrt[3]{7}$; and prove that the error in taking $\frac{17}{10}$ to represent $\sqrt{17}$ is about $\frac{1}{100}$th per cent.

3. Assuming that the area of a circle of radius $r$ is equal to $\pi r^2$, prove that the area of a circular path bounded by concentric circles of radii $r - \frac{1}{4}, r + \frac{1}{4},$ is equal to $2\pi r t$.

   If the width of the path is 10 feet and its area is 8,000 square feet, prove that the total area enclosed by the outer fence of the path is a little over 55,000 square feet, taking $\pi$ to be 3.1416.
4. Find the factors of
\[42x^2 + xy - 30y^2.\]
Verify that \(2x - a\) is a factor of
\[4x^3 - 3ax^2 + a^3,
\]and obtain all the factors.

5. Solve the equations:
\[
\begin{align*}
(1) & \quad \frac{2}{x} + \frac{y}{2} = \frac{1}{3} + \frac{y}{y} \\
(2) & \quad x(y - 1) = 8, \quad y(x - 1) = 9.
\end{align*}
\]

6. A motor-boat can travel at 12 miles an hour in still water. It makes a trip on a river of 15 miles down-stream and returns against the current. Show that, whatever may be the speed of the current, the trip takes longer than the boat would take over the same distance in still water.

Find the speed of the current, if the extra time taken is 10 minutes.

7. Using the tables of logarithms calculate the values of
\[
(1) 10^{-\frac{1}{2}}; \quad (2) \left(\frac{1}{2}\right)^{-10}.
\]

Which of the two results is the greater? Can you give any general reason for anticipating which is the greater?

8. Construct an isosceles triangle \(ABC\), such that \(AB = AC = 2\) inches, and with the angle \(B\) equal to 30°; and find a point \(D\) in the base \(BC\) such that \(AD = 1\frac{1}{2}\) inches.

If any other triangle \(PQR\) is constructed with \(PQ = 2\) inches, \(PR = 1\frac{1}{2}\) inches, and the angle \(Q\) equal to 30°, prove that the triangle \(PQR\) is equal in all respects to one or other of the two triangles \(ABD\) or \(ACD\).

9. Draw two circles of diameters 1\(\frac{1}{2}\) and 1\(\frac{1}{2}\) inches, which touch externally; and construct the centre of a third circle of diameter 4 inches, which touches both of these circles and encloses them both.

Construct the points of contact \(P, Q\) of the third circle with the other two and measure the distance \(PQ\).

10. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

If the quadrilateral is a parallelogram, prove that it must be a rectangle: and if in addition another circle can be inscribed in the quadrilateral, prove that the quadrilateral must be a square.

11. Prove that the square on a side of a triangle is greater or less than the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse or acute.
Draw a circle of radius 1½ in.; and draw two tangents to it which are inclined to one another at 30°.

Draw a circle to touch the two tangents and the given circle.

9. Show that, if two angles of a triangle are equal, the sides opposite to them are also equal.

In the annexed sketch (which is not drawn to scale), $AB$ is a horizontal line, 4 in. long; $BC$ is vertical and −3 in. The broken line $ACD$ is 6 in. long, and the parts $AP$, $BD$ are equally inclined to the vertical. Give a geometrical construction for drawing $AD$ and $CD$.

10. Prove that triangles on the same base and of the same altitude are equal in area. [Do not assume any formula for the area of a triangle.]

$ABCD$ is a quadrilateral. $E$ is the middle point of the diagonal $AC$; $DE$ produced meets in $H$ the line drawn through $B$ parallel to $AC$. Prove that the area of the quadrilateral is double that of the triangle $ABH$.

11. Define a tangent to a circle, and prove from your definition that the tangent at any point of a circle is perpendicular to the diameter through the point of contact.

Draw a circle of 2 inches radius, and draw a straight line at a distance of 3 in. from its centre. Find a point on the line such that one of the tangents from it to the circle may bisect the angle between the given line and the line joining the point to the centre of the circle.

12. Prove that, if two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.

$A$, $B$, $C$ are three points in order on a straight line; $AB = 2$ in., $BC = 4$ in.; with centre $B$ and radius 1 in. a circle is described. Any point $P$ on the circumference is joined to $A$ and $AP$ is pro-
8. Prove that the opposite sides of a parallelogram are equal and that the opposite angles are also equal.

A parallelogram \(ABCD\) is such that \(AB = 2BC\). If \(E\) is the middle point of \(AB\), show that \(CE\) and \(DE\) are perpendicular.

9. Construct a triangle \(ABC\) such that \(AB = 4\) in., \(BC = 5\) in., \(CA = 3\) in.

Take a point \(D\) in \(BC\), such that \(BD = 3\) in., and construct a quadrilateral \(ABDE\) equal in area to the triangle \(ABC\) and having \(DE = 4\) in.

10. Prove that the angle between a tangent to a circle and a chord drawn through the point of contact is equal to the angle in the alternate segment.

\(ABC\) is a given triangle. A circle is drawn touching \(AB\) at \(B\) and passing through \(C\), another circle is drawn touching \(BC\) at \(C\) and passing through \(A\). These two circles intersect at a second point \(O\). Find the angles \(BOC\), \(COA\) in terms of the angles of the triangle.

Hence show that the circle \(AOR\) touches \(AC\) at \(A\).

11. Given a straight line and two points \(P\), \(Q\), lying on the same side of it, show how to find a point \(R\) on the line such that \(PR + RQ\) shall be as small as possible.

\(ABC\) is an isosceles triangle right-angled at \(B\). Find points \(P\) in \(AB\) and \(Q\) in \(BC\) such that \(AP = BQ\), so that \(PQ\) shall be a minimum.

12. Show how to construct a fourth proportional to three lines of given lengths.

If \(a\), \(b\), \(c\) are the sides of a triangle in descending order of magnitude, show that it is possible to construct a triangle whose sides are the fourth proportions to \(a\), \(b\), \(c\); \(b\), \(c\), \(a\); \(c\), \(a\), \(b\); if \(1/a^2 < 1/b^2 + 1/c^2\).

1921. 2½ Hours

1. If a bankrupt's liabilities are £237581 17s. 6d. and his assets are £8967 16s. 8d., what can he pay in the pound (to the nearest penny)?

What will a creditor receive to whom he owes £209 17s. 5d. (to the nearest penny)?

2. Prove that \(log_a x - log_a y = log_a (x/y)\).

Find the value of

\[ (1) \quad (0.987)^{100} \quad (2) \quad 3.74 \times 43.2 \quad (3) \quad 784 \times 0.987 \]

3. Factorize \(ab(1 + c^3) - c(a^3 + b^2)\).

Find what the value of \(k\) must be if the expression \(9x^2 - 4x - k\) contains \(x + 1\) as a factor. What is the other factor?

4. If \(y = ax + bx + cx^2\) and corresponding values of \(x\) and \(y\) are given by

\[
\begin{array}{c|c|c}
\hline
x & y & \\
\hline
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\hline
\end{array}
\]

find the values of \(a\), \(b\), \(c\) and the greatest positive value of \(y\).

Draw the graph of \(y\).

5. Solve the equations

\[ 3z^2 + 5xy - y^2 = 1; \quad 2x + 3y = 1. \]

If \(y = \frac{3 - 2z}{4z - 8}\) and \(z = \frac{5 + 4x}{3x + 2}\), find \(y\) in terms of \(x\) and \(z\) in terms of \(y\).

6. An express train travelling at a certain rate passes a certain point two minutes after its scheduled time. Forty miles farther on it is four minutes late. Its speed is then increased by twelve miles per hour, and it arrives punctually at a point twenty miles farther on. At what rate, in miles per hour, would it have travelled if it had been running to time the whole way?

Expand \((1 + x)^5\) by the binomial theorem, calculating the actual values of the numerical coefficients.

Hence find to the nearest pound the amount after 12 years of £1000 at 5 per cent, per annum compound interest.

Compare your result with that obtained from four-figure logarithm tables.

8. Prove that if two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

On the side \(OP\) of a triangle \(OPQ\) an equilateral triangle \(OPR\) is described external to the triangle \(OPQ\); another equilateral triangle \(QRS\) is described on \(RQ\) and on the same side of \(RQ\) as \(O\). Prove that \(OS\) is equal to \(PQ\).

9. \(ABCD\) is a rectangle, \(O\) is any point in the diagonal \(AC\); \(MON\) parallel to \(AD\) meets \(AB\) in \(M\) and \(DC\) in \(N\); \(LOE\) parallel to \(AB\) meets \(AD\) in \(L\) and \(BC\) in \(E\). Prove that the rectangles \(LOND\), \(MORE\) are equal in area.

Construct a quadrilateral \(ABCD\), given \(A = 70°\), \(B = 60°\), \(C = 90°\), \(AB = 3\) in., \(CD = 1.5\) in.

Construct a rectangle of area equal to that of the quadrilateral on a base 1 in. long; not using multiplication.

10. Prove that the angle in a semicircle is a right angle.

\(ABC\) is a triangle inscribed in a circle \(BCD\). \(BL\) is a diameter of this circle; the perpendicular drawn from \(A\) to \(BC\) meets the
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perpendicular drawn from C to AB in H. Prove that HL is bisected by AC.
11. ABC is a triangle in which the vertices B and C are fixed and the angle BAC is given. BA is produced to D, so that AD = AC. Prove that the locus of D is a segment of a circle through B and C.
Construct a triangle ABC in which BC is 8 cm., the sum of AB and AC is 15 cm., and the angle BAC is 60°. Describe the various steps of your construction.
12. Define similar figures, and prove that two equiangular triangles are similar.

ABCD is a quadrilateral inscribed in a circle. O is any point on the circumference of this circle; OL, OM, ON, OR are perpendiculars from O on AB, BC, CD, DA respectively. Prove that the triangles ABO, BCO are equiangular, and that rect. OLOM = rect. OMAR.

1922. 2 $\frac{1}{2}$ Hours

1. Three solid lead spheres of radii 7 inches, 8 inches, 12 inches are melted together and recast as a solid sphere. Find the volume of the new sphere, and the percentage diminution of the total surface exposed.

2. Solve the equations:

(1) $\frac{z^2+ab}{x} - \frac{b}{a} = \frac{x+a}{a+b}$

(2) $4x^2+4xy-y^2 = 7, 3y-2x = 7$.

3. Draw on the same scale the graphs of $6-x^2$ and $\frac{3}{x}$.

By means of these graphs find as accurately as you can all the roots of the equation $x^2-6x+3 = 0$.

4. An oblong field is 100 yards long. If it were reduced to a square by cutting an oblong piece off one end, the ratio of the piece cut off to the remainder would be less by $\frac{1}{10}$ than the ratio of the remainder to the original field. Find the breadth of the field.

5. Prove that, if $x$ is numerically less than 1, $6(1-x)^4 = 1.2.3+2.3.4x+3.4.5x^2+4.5.6x^3+\ldots$.

A population increases every year by 3 per cent. of itself. If it is 1,000,000 now, what will it be 7 years hence? Obtain your answer (1) by the binomial theorem, (2) by use of logarithm tables.

6. Prove that in any triangle the square on a side opposite an acute angle is less than the sum of the squares on the sides containing that angle by twice the rectangle contained by one of these sides and the projection on it of the other.

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ABC is a triangle in which BC = 8 in., CA = 5 in., AB = 7 in. BM is taken on BC equal to BA. Prove that $7AM^2 = 3AB^2$.

7. Draw two circles of radii 1 in. and 2 $\frac{1}{2}$ in. respectively to touch externally at a point I. Construct one of the external common tangents AB.

Let the common tangent to both circles at I meet the tangent AB in M. Prove that $AM = MB - MI - 1\frac{1}{2}$ in.

Verify your results by measurement.

8. Prove that the sum of a pair of opposite angles of a cyclic quadrilateral is two right angles.

ABC is an equilateral triangle inscribed in a circle; P is a point on the arc BC which does not contain A; CP is produced to B' so that $PB = PB'$. Prove that $CB = AP$.

9. Prove that, if the sides of two triangles are proportional, their corresponding angles are equal.

ABCD is a plane quadrilateral. AY parallel to BC meets AD (produced if necessary) in Y; similarly BX parallel to AB meets AC in X. Prove that XY is parallel to CD.

SUBSIDIARY SUBJECT (15 c)

ARITHMETIC, ALGEBRA AND GEOMETRY 1923. 2 $\frac{1}{2}$ Hours

1. A publisher prints 5000 copies of a book, the price of which is fixed at 5s. 6d. net; the cost of printing and paper is 1s. 3d. a copy, that of binding 8d. a copy, and that of carriage and advertising 2s. 6d.; he sells them to bookstalls, charging 25 copies as 24 and 30 per cent. less than the publisher's price, and upon the whole receipt takes 25 per cent. commission for himself. Find the respective gains of the author, publisher, and booksellers on this edition.

2. Solve the equations:

(1) $\frac{x-2b}{x-b} - \frac{x-2a}{x-a} = \frac{a^2-b^2}{ab}$

(2) $x^2+xy = 2, xy+3y^2 = 1$.

3. Draw, on the same scale, the graphs of the expressions

$x^2-6x+8$ and $\frac{3}{x}$.

Hence, find approximately the roots of the equation

$x^2-6x+8x+3 = 0$,

and verify, algebraically, the values of the two irrational roots.

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4. \( A \), walking at 4 miles an hour, and \( B \), cycling at 8 miles an hour, start at the same moment from one town \( X \) to another \( Y \); \( C \), motoring at 20 miles an hour, starts half an hour later from \( X \) to \( Y \), and, on reaching \( Y \), stays there for half an hour, and then returns to \( X \) at the same rate as before. If the interval between the instants when \( C \) passes \( A \) and \( B \) on the outward journey is half that between the instants when \( C \) passes \( B \) and \( A \) on the return journey, find the distance between \( X \) and \( Y \).

5. Prove the binomial theorem when the index is a positive integer.

Show that
\[
(1 + 3x)^4 = 1 + 12x + 18x^2 + 18x^3 + 9x^4 + ....
\]

6. Describe a quadrilateral \( ABCD \) in which the sides \( AB \), \( BC \), \( CD \), \( DA \) are respectively \( 1 \frac{1}{2}, 2, 1, \) and \( 3 \) inches long and \( BC \) is parallel to \( AD \); and explain the construction. Measure the distance between the parallel sides and find the area of the quadrilateral.

7. Prove that, in a right-angled triangle, the square on the side opposite the right angle is equal to the sum of the squares on the sides containing it.

\( A \), \( B \), \( C \) are points in order on a straight line; \( AB \), \( BC \), and on the same side of \( AC \), squares \( ABDE \), \( BCFG \) are described; prove that the sum of the squares on \( BC \), \( DF \) is equal to three times the sum of the squares on \( AB \), \( BC \).

8. If the line joining two points subtends equal angles at two other points on the same side of it, prove that the four points lie on a circle.

In the triangle \( ABC \), the side \( AB \) is greater than the side \( AC \), and \( D \) is a point in \( AB \) such that \( AD \) is equal to \( AC \); the internal bisectors of the angles \( B \), \( C \) meet in \( I \); show that the four points \( B, D, I, C \) lie on a circle. Show also that this circle cuts \( AC \) produced in a point \( E \) such that \( AE = AB \).

9. Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

The angle \( A \) of the triangle \( ABC \) is bisected by a line which meets \( BC \) in \( D \); through \( D \) a line \( FE \) is drawn perpendicular to \( AD \) meeting \( AB, AC \), produced if necessary, in \( F, E \), respectively; prove that \( BF:CE = BD:DC \).

1924. 2½ Hours

1. A train, 88 yards long, overtakes and passes \( A \), who is walking in the same direction at 4 miles an hour, in 6 seconds; it also passes \( B \), who is riding, in 4 seconds. At what rate, and in which direction, is \( B \) riding?
9. Prove that, if two triangles are equiangular, their corresponding sides are proportional.

1 is the incentre of the triangle \(ABC\), and \(ID\) is drawn perpendicular to \(IC\) to meet \(BC\) in \(D\); show that \(BI\) is a mean proportional between \(BD\) and \(BA\).

1925. 2½ Hours

1. In four years from now \(A\) will be two-thirds of \(B\)'s present age, in eleven years from now \(B\) will be double of \(C\)'s present age, and 14 years ago \(C\) was half of \(A\)'s present age. Find the present ages of \(A\), \(B\), and \(C\).

2. A slow train from London to Bristol travels the first 36 miles of its journey at 30 m.p.h., the next 42 miles at 20 m.p.h., and the remaining 42 miles at 30 m.p.h. A fast train leaves Bristol for London when the slow train has spent half of its time on the journey, and travels the 120 miles at 60 m.p.h. Find the distance of the trains from London when they meet.

[Full credit will be given for neat and accurate graphical solutions.]

3. Given that ten letters can be placed in ten addressed envelopes so that every letter is in a wrong envelope in \(N_{10}\) different ways, where

\[
N_{10} = 10! \left[ \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + \frac{1}{10!} \right],
\]

compute the value of \(N_{10}\).

It takes a minute to place ten letters in ten envelopes and a minute to take them out again. If to-day you start putting the letters into the envelopes, every letter into a wrong envelope, taking them all out again and then reinserting them, find the date on which you will finish the task of inserting the letters into the envelopes in all the \(N_{10}\) ways if you work at the rate of eight hours per day for seven days per week.

4. If \(a\) and \(b\) are the roots of the quadratic equation

\[ax^2 + 2bx + c = 0,\]

express \(a + b\) and \(ab\) in terms of \(a\), \(b\), and \(c\).

Prove that the quadratic equation whose roots are

\[\frac{1}{a} \quad \text{and} \quad \frac{1}{b}\]

is

\[ax^2 + 2b(a + c)x + (a + c)^2 = 0,\]

5. Draw the graph of \(x^3 - 2x^2\) from \(x = -1\) to \(x = 2\), taking two inches as your unit of length.

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3. Draw in the same diagram the graphs of $x^2 + 3x + 1 = 0$ and of $-1/x$.
   Hence find the roots of the equation $x^2 + 3x + 1 = 0$ as accurately as your diagram permits.

4. Without assuming the formula
   \[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
   for the roots of the quadratic equation $ax^2 + 2bx + c = 0$, prove that
   the sum of the roots is $-2b/a$, and that the product of the roots is $c/a$.
   If $a, b$ are the roots of $x^2 - 4x + 1 = 0$, prove that
   \[ a^2 + b^2 = 2702 \]
   and deduce that $a^2 + b^2 = 2702$.

5. The population of England and Wales in 1801 was 8,809,000 and in 1901 was 32,528,000. Assuming
   that the population increases according to the compound-interest law, estimate the population
   in 1861, (2) in 1926.

6. Prove that, in an obtuse-angled triangle, the square on the side opposite the obtuse angle exceeds
   the sum of the squares on the other two sides by twice the rectangle contained by one of
   these sides and the projection on it of the other.
   A point $D$ is taken on the base $BC$ of a triangle $ABC$ so that $BD : DC = m : n$. Prove that
   \[ mAC^2 + nAB^2 = (m+n) AD^2 + mDC^2 + nDB^2. \]

7. Prove that the sum of the opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.

8. Prove that the diagonals of a quadrilateral $ABCD$ are at right angles.

9. Prove that, if two triangles are equiangular, their corresponding sides are proportional.
   In the triangle $ABC$, $AD$ is the perpendicular from $A$ to $BC$; $O$ is the circumcentre, $AO$ meets the
circumcircle again at $P$, and $PN$ is the perpendicular from $P$ to $BC$. Prove that
   \[ PN \cdot AD = BD \cdot CD. \]

1927. 2$\frac{1}{2}$ Hours

1. A train travels from London to Birmingham (110 miles) in $2$ hours.
   At the instant the train starts an aeroplane, which travels

2. Solve the equations:
   \begin{align*}
   (1) & \quad \frac{1}{x-1} - \frac{1}{x-5} = \frac{1}{x-7} - \frac{1}{x-3} \\
   (2) & \quad x - y = 3, \quad x^2 - y^2 = 819.
   \end{align*}

3. Find the formulae (1) for the sum of $n$ terms of an arithmetic progression, (2) for the sum of $n$ terms of a geometric progression.

Two sums of £1000 start accumulating on January 1, 1927, the first at 10% per cent. per annum simple interest, the second at 5% per cent. per annum compound interest; the interest in both cases is paid annually on December 31. Find the ratio of the amount of the second sum to that of the first on January 1, 1967. [Give your answer to four significant figures.]

4. Draw in the same diagram the graphs of $\frac{1}{2}x^2$ and of $4 - \frac{1}{x}$
   from $x = -2$ to $x = 2$, taking 1 inch as the unit of length.
   Hence find the roots of the equation
   \[ x^2 - 8x + 2 = 0 \]
   as accurately as your diagram permits.

5. Find the number of combinations of $n$ different things taken $r$ at a time.

A boat's crew is to be composed of eight oarsmen of whom three can row only on bow side and two can row only on stroke side; the other three can row on either side. Find the number of possible ways of arranging the crew.

6. Prove that, if two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

Two equal circles intersect at $A$ and $B$. A point $P$ is taken on the major arc of one circle, so that $PA$ and $PB$ meet the minor and major arcs of the other circle respectively at $Q$ and $R$. Prove that the distance of $Q$ from $P$ is less than the distance of $Q$ from $R$.

7. Prove that the diagonals of a parallelogram bisect one another.

$AOA', BOB', COC'$ are three diameters of a circle; $AB$ and $AC'$ meet at $X$, $AC$ and $AB'$ meet at $Y$. Prove that $XY$ passes through the centre of the circle.

8. From a point $O$ outside a circle two straight lines $OP, OQ, OR, OS$ are drawn to cut the circle at $P, Q, R, S$. Prove that the rectangle contained by $OP, OQ$ is equal to the rectangle contained by $OR, OS$.
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Two circles intersect at $P$ and $Q$. From a point $O$ on $PQ$ produced a chord $OB$ of one circle is drawn, and a tangent $OT$ is drawn to the other circle. Prove that a circle can be described to pass through $R$ and $S$ and to touch $OT$ at $T$.

9. Prove that, if two triangles are equiangular, their corresponding sides are proportional. $AC', BC'$ are two chords of a circle intersecting at a point $C$. The circumcircles of the triangles $ABC, A'B'C'$ are drawn, and they intersect at $C$ and $D$. Prove that the triangles $DAB, D'A'B'$ are similar.

GROUP III (Paper 2) AND SUBSIDIARY SUBJECT (15 b)

PURE GEOMETRY AND TRIGONOMETRY

1918. 2½ Hours

Not more than nine questions are to be attempted by any Candidate. Questions A, B, C, D, E, F are to be attempted only by Candidates taking Mathematics as a Subsidiary Subject.

1. Prove (without assuming any formula for the area of a triangle or a parallelogram) that triangles that are on the same base and of the same altitude are equal in area.

$ABCD$ is a parallelogram, $EF$ is a fixed straight line parallel to $AD$ and $BC$, meeting $AB$ in $E$ and $CD$ in $F$; $Y$ is any point on $EF$, and $XY$ parallel to $DC$ cuts $BC$ in $X$; $AX$ and $DC$ produced if necessary meet in $Z$. Prove that the area of the triangle $AYZ$ is constant, and equal to half the area of the parallelogram $EBCF$.

2. Prove that in an obtuse-angled triangle the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing that angle by twice the rectangle contained by one of these sides and the projection of the other.

$ABC$ is a triangle in which $AB = 5$ in., $AC = 3$ in., $BC = 7$ in.; $M$ and $N$ are the feet of the perpendiculars from $B$ and $C$ on $CA$ and $BA$ respectively. Prove that

$$MN = \frac{1}{2} BC.$$

3. Prove that, if the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.

A triangle $AOP$ is rotated in its own plane into the position $BOQ$. Prove that, if $AB$ and $PQ$ intersect in $M$, the angles $AOP$ and $AMP$ are equal or supplementary; and that, if $OAP$ is a right angle, $PQ$ is bisected at $M$.

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4. Prove that a straight line which is parallel to one side of a triangle divides the other sides proportionally; and, conversely, a straight line which divides two sides of a triangle proportionally is parallel to the third side.

$CD$, $CL$ are the perpendiculars on the internal and external bisectors of an angle $BAC$ from a point $C$ in the line $AC$. Prove that $KL$ bisects $CA$ and $CB$.

5. $TP$, $TQ$ are tangents to a circle of centre $O$ and radius $OA$; $OT$, $OP$ meet in $M$. Prove that $OM\cdot OT = OA^2$.

If $TO$ is produced to $N$ so that $ON = MO$, prove that the tangents from $T$ to the circle, whose centre is $N$ and radius $2OA$, are equal to $TM$.

6. Prove that, if a transversal cuts the sides $BC, CA, AB$ of a triangle in $X, Y, Z$ respectively,

$$BX \cdot CY \cdot AZ = CX \cdot AY \cdot BZ.$$

Of the escribed circles of the triangle $ABC$ that is inscribed to $BC$ touches $AB$ in $Z$ and $AC$ in $Y$; that which is escribed to $AC$ touches $AB$ in $Y$; that which is escribed to $AB$ touches $AB$ in $W$. Prove that $YW, VW, and BC$ are concurrent.

7. How is the angle between two non-intersecting (non-coplanar) straight lines measured?

$A, B$ are points on a diameter of a sphere, of radius $a$, at equal distances $b$ from the centre. A straight line through $A$, perpendicular to $AB$, meets the sphere in $P$; a straight line through $B$, perpendicular to $AB$, meets the sphere in $Q$. Prove that, if the angle between $AP$ and $BQ$ is $\theta$,

$$PC^2 = 4a^2 \sin \theta - 4b^2 \cos \theta.$$

8. Prove that any plane section of a sphere is a circle.

A straight line cuts a sphere in $AB$, and a concentric sphere in $CD$; prove that $AC = DB$.

9. Prove the formula

$$\frac{2 \sin \frac{A}{2}}{2} = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}.$$

Explain geometrically why there are four possible values of $\sin \frac{A}{2}$ when $\sin A$ is given.

What signs are to be used when $A$ lies between 180° and 270°?

10. Prove that in a triangle $ABC$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{(s-a)}}.$$

Find the least angle, and the radius of the circumcircle, of the triangle whose sides are 17, 21, and 24 inches.
11. In a triangle $ABC$, $AD$ is the perpendicular from $A$ on $BC$, and $M$ is the middle point of $BC$. Prove that 
\[ \cot B + \cot C = BC/AD \] and \[ \cot B - \cot C = 2 \cot AMC. \]

A steamer, pursuing a straight course at 14 miles per hour, observes a lighthouse at 9 o'clock, at 9.30, and at 10. At 9 the lighthouse bears $28^\circ 43' \text{N}$, of the steamer's course, at 9.30 it bears $80^\circ 4' \text{N}$ of the steamer's course; the course being reckoned always in the same direction. Find the two possible directions of the steamer's course, and show that she is 12 miles from the lighthouse at 9.30 o'clock.

12. Prove that $\sin 2A - \sin 2B = 2 \sin (A - B) \cos (A + B)$.

The sides $BC, CA, AB$ of a triangle make angles $\theta, \phi, \psi$ (measured in the same sense) with a given straight line. Express $\phi$ and $\psi$ in terms of $\theta$, and show that

- $\sin A \sin \theta + \sin B \sin \phi + \sin C \sin \psi = 0$;
- $\sin A \cos A \sin^2 \theta + \sin B \cos B \sin^2 \phi + \sin C \cos C \sin^2 \psi = \sin A \sin B \sin C$.

A. Prove that, if all the sides of a quadrilateral are equal, the opposite sides are parallel and the diagonals bisect each other at right angles.

B. Prove that, in equal circles, if chords are equal they cut off equal arcs.

Two equal circles cut in $A$ and $B$; a circle, whose centre is $A$, cuts them in $P$ and $Q$. $PQ$ and $OQ$ being on the same side of $AB$. Prove that $BPO$ is a straight line.

C. Draw two circles of radii 1 in. and 2 ½ in. to touch internally; and draw a circle of radius 0 8 in. to touch one of these circles externally and the other internally. Explain your construction.

D. Define the sine of an angle in such a way as to be true for all values of the angle.

Prove that $\cos \theta = \pm \sqrt{(1 - \sin^2 \theta)}$. Given that $\sin 11^\circ 51' = 0.2$, calculate the value of $\cos \theta$, and check your results by reference to tables.

E. Prove that \[ \cos (A + B) = \cos A \cos B - \sin A \sin B. \]

Prove that
\[ \sin (3\theta + \phi) + \sin (\theta + 3\phi) = 2 \sin (\theta + \phi) \cos 2\theta + \cos 2\phi. \]

F. Prove that in any triangle
\[ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \]

Prove also that
\[ a \cos 2B + 2 b \cos A \cos B = c \cos B - b \cos C. \]

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straight lines, prove that the orthogonal projection of PQ on either line is equal to \(2\sin \frac{1}{2} \theta\), where \(\theta\) is the angle between the lines XP, YQ; and that \(PQ^2 = XY^2 + 4y^2 \sin^2 \frac{1}{2} \theta\).

8. A sphere of radius \(r\) is cut by a plane at a distance \(x\) from the centre of the sphere: prove that the section is circular and find a formula for its radius.

A sphere rests in a horizontal circular hole of radius \(a\), with the lowest point of the sphere at distance \(c\) below the plane of the hole: prove that the radius of the sphere is equal to \(\frac{1}{2} \left( a^2 + c^2 \right) / c \).

9. Prove that \(\cos \theta - \cos 2\theta = \frac{1}{2} \left( 1 - \cos \frac{1}{2} \theta \right) \cos \frac{1}{2} \theta \).

Solve the equation in \(\theta\):

\[\cos \theta - \cos 2\theta = \frac{1}{2},\]

finding the values of \(\theta\) between 0 and \(\pi\); and verify the solutions from the tables.

10. In the diagram BC is horizontal, AB, CD make angles of 30° and 40° with the horizontal, and \(AB = 120, BC = 140, CD = 160\), as indicated; AK, DL are vertical. Calculate the distances AK, KL, LD; and hence determine the distance BP, where P is the point in which AD cuts BC.

11. Prove that in a triangle \(ABC\),

\[c \sin \frac{1}{2} (A - B) = (a - b) \cos \frac{1}{2} C,\]

\[c \cos \frac{1}{2} (A - B) = (a + b) \sin \frac{1}{2} C.\]

Given \(a = 325, b = 248, C = 70° 12'\), calculate the above expressions and hence or otherwise solve the triangle.
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1. If $T_A$, $T_B$ are tangents to a circle at $A$ and $B$; $AC$, $BC$ are chords joining $A$ and $B$ to any point $C$ on the circumference of the circle; $TPQ$ parallel to $AC$ meets the circle in $P$ and $Q$ in $B$. Prove that $T_A$, $T_B$, $T_C$ are concurrent, and that $C$ is the midpoint of $PQ$.

4. Prove that two triangles whose sides are proportional are also equiangular.

$AB$ is a chord of a circle; $PQ$ is another chord parallel to the tangent to the circle at $B$; a straight line $QR$, parallel to $PQ$, meets $AQ$ in $R$ and $AB$ in $U$. Prove that the rectangle $AR.AP = $ the rectangle $AU.AB$.

5. Prove that the perpendiculars from the vertices of a triangle on the opposite sides meet in a point (the orthocentre).

If $H$ is the orthocentre of the triangle $ABC$ and $a$, $b$, $c$ are the points of the circumcircle diametrically opposite to $A$, $B$, $C$, prove that $HA$, $HB$, $HC$ bisect $BC$, $CA$, $AB$ respectively.

6. $AP$, $BQ$ are parallel radii of two fixed circles whose centres are $A$ and $B$; prove that $PQ$ meets $AB$ in one or two fixed points.

Prove also, when the circles cut one another, that their common points lie on the circle which has the above fixed points as ends of a diameter.

7. Draw a perpendicular to a given plane from a point outside it.

$OA$, $OB$ are two straight lines; $OC$ is a third straight line bisecting the angle $AOB$. Prove that any straight line which is perpendicular to $OC$ makes with $OA$ and $OB$ angles which are equal or supplementary.

8. $OA$ is a fixed radius of a sphere, $OP$ a variable radius such that the angle $AOP$ is constant; prove that the locus of $P$ is a circle.

Prove that the difference between the area of the curved surface of a segment of a sphere and the area of its plane face is equal to the area of a circle whose radius is the height of the segment.

9. Prove that

$$\frac{1}{2} \left( \tan (x+h) + \tan (x-h) \right) - \tan x = \frac{\tan x \sin^2 h}{\cos^2 x - \sin^2 h}.$$

Given that $x - h = 80^\circ 24'$, $x + h = 80^\circ 30'$, and taking $3'$ as 0.00087, apply this formula to find $\tan 80^\circ 27'$; and show that the use of the rule of proportions to find $\tan 80^\circ 27'$ involves an error of about 2 in the fourth place of decimals.

[Notice that in the 4 or 5 figure tables generally used the difference column for tangents of angles greater than $80^\circ$ is blank.]

10. Draw the graph of $4 \sin 3\theta - 3 \sin 4\theta$ for values of $\theta$ between $0^\circ$ and $180^\circ$.

1921. 2½ Hours

1. In a triangle $ABC$ which has a right angle at $C$, prove that $AB^2 = AC^2 + BC^2$. 

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If $BA$ is produced to a point $D$ such that $BA = AD$, prove that $CD = 4\cos^2 BC$.

2. Show how to construct the common tangents to two given circles.

A common tangent is intersected by the three other common tangents in $P$, $P'$, $T$, where $T$ is on the line which contains the centres $OC'$ of the circles. Prove that

$$TP \cdot TP' = TC \cdot TC'.\]

3. Prove that if two triangles are equiangular their corresponding sides are proportional.

$AB$, $CD$ are segments of parallel lines with the same sense, and $E'$ is the mid-point of $CD$. The lines $AE$, $BE'$ meet in $F$ and the lines $BE$, $CD$ meet in $G$. Prove that $FG$ is parallel to the segments, and that

$$\frac{1}{FG} = \frac{1}{AB} + \frac{1}{ED}.\]

4. The feet of the perpendiculars from the vertices $A$, $B$, $C$ of a triangle to the opposite sides are $D$, $E$, $F$ respectively; and the mid-points of the sides $BC$, $CA$, $AB$ are $A'$, $B'$, $C'$ respectively. Prove that the six points $A'$, $B'$, $C'$, $D$, $E$, $F$ lie on a circle.

If the circle with $CD$ as diameter cuts $DE$, $DF$ respectively in $L$, $M$, prove that $LM$ is perpendicular to $AB$.

5. Prove that the difference of the squares of the tangents drawn from a point to two circles is proportional to the distance of the point from the radical axis of the circles.

Prove that the locus of a point, such that its distance from a fixed line is proportional to the square of its distance from a fixed point, is a circle.

6. Prove that if a line is perpendicular to each of two intersecting lines $OA$, $OB$, it is perpendicular to every line in the plane $OAB$.

Each of three concurrent straight lines $OA$, $OB$, $OC$ is perpendicular to a fourth line. Prove that the lines $OA$, $OB$, $OC$ are in the same plane.

7. Solve the equations:

(i) $4 \sin \theta - 2 \cos \theta = 3$;
(ii) $4 \sin \theta - 2 \cos 2\theta = 3$;

giving the values of $\theta$ between 0 and 180°.

8. Prove that in any triangle $ABC$

$$\tan \frac{1}{2} (A - B) = \frac{a - b}{a + b} \tan \frac{1}{2} (A + B).$$

A flagstaff on a hill is observed from two stations $A$, $B$ in the same horizontal plane. The elevation and compass bearing of the staff from $A$ are found to be 15° and 60° respectively, and from $B$ the corresponding angles are found to be 12° 30' and 110° respectively. Find the compass bearing of $A$ from $B$. [N.B. The compass bearings are all measured from the northerly direction in the clockwise sense.]

9. A rope $ABC$ is stretched symmetrically over a drum in the shape of a circular cylinder with its axis horizontal and has its ends $A$, $B$ at the level of the axis. Having given that the length of the rope is 12 ft., and that the radius of the drum is 2 ft., prove that the straight parts of the rope make with the horizontal an angle $\theta$ which satisfies the equation

$$\cot \theta + \theta = 3.$$ 

By use of graphical methods, or otherwise, solve the equation approximately, and hence find the distance $AB$.

10. A linear segment $AB$ is inclined at an angle $a$ to a line in space. Prove that the projection of $AB$ on the line is $\cos \alpha$.

In a regular tetrahedron $ABCD$ prove that the common perpendicular to two opposite edges is inclined at an angle of 45° to each of the other four edges.

A. Prove that the sum of the angles of a triangle is equal to two right angles.

Prove that the bisectors of the interior angles of a parallelogram, which is not a rhombus, form a rectangle.

B. Construct a triangle $ABC$ in which the side $BC$ is 8 cm., the perpendicular from $B$ on $AC$ is 2 cm., and the angle $BAC$ is 50°.

Measure the length of the perpendicular from $C$ on $AB$.

C. Draw the graph of $\sin x + 2 \cos 2x$ for values of $x$ between 60° and 180°.

What is the maximum value of the function?

D. In any triangle $ABC$ prove that

$$\cos^2 A = \frac{a^2 + b^2 - 2ab \cos C}{4}.$$ 

An equilateral triangle $ABC$ is described on the side of $AB$ remote from $C$. Prove that

$$CC'^2 = \frac{1}{2} (a^2 + b^2 + c^2) + 2 \sqrt{3} c.$$ 

where $d$ is the area of the triangle $ABC$.\]

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1. Prove that the sum of either pair of opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.

\[ ABCD \text{ is a quadrilateral inscribed in a circle; the bisectors of the angles } A, B, C, D \text{ meet the circumference in } P, Q, R, S \text{ respectively; show that } PQRS \text{ is a rectangle.} \]

2. Show that the sum of the rectangles contained by pairs of opposite sides of a cyclic quadrilateral is equal to the rectangle contained by the diagonals.

\[ ABCD \text{ is a cyclic quadrilateral and the bisector of the angle } ABD \text{ meets } AD \text{ in } E; \text{ show that the angles } ACE, DCE \text{ are unequal.} \]

3. Draw an external common tangent to two given circles.

\[ \text{Show that the sum of an external and an internal common tangent to any two excircles of a triangle is equal to the perimeter of the triangle.} \]

4. Show that a point which moves so that the tangents drawn from it to two given circles are equal lies on a straight line perpendicular to the line joining the centres of the circles.

\[ \text{From a point } A \text{ tangents } AP, AQ \text{ are drawn to a circle with centre } O, \text{ and the straight line joining the mid-points of } AP, AQ \text{ cuts } OA \text{ in } D; \text{ show that the point } D \text{ lies outside the circle.} \]

5. Two straight lines are parallel to one another and one of them is perpendicular to a plane; show that the other is perpendicular to the same plane.

\[ OA, OB, OC \text{ are straight lines mutually at right angles, } OD \text{ is perpendicular to } BC \text{ and } OE \text{ to } AD; \text{ show that } OE \text{ is perpendicular to the plane } ABC. \]

6. Assuming the formulae for \( \sin (A + B) \) and \( \cos (A + B) \), find \( \tan (A + B) \) in terms of \( \tan A \) and \( \tan B \).

\[ ABC \text{ is a triangle right-angled at } A; \text{ the lengths of } CA, AB \text{ are } 5 \text{ and } 9 \text{ inches respectively; } AB \text{ is divided at } E \text{ and } F \text{ so that } AE = EF = FB; \text{ find the tangents of the angles } ACE, ECF, \text{ and } FCB. \]

7. (i) Solve the equation \( \sin \theta + \cos \theta = 1/\sqrt{2}. \)

(ii) Solve the equation \( 2 \cos \theta = 1/\sqrt{2} \) and show that it has three roots between 0 and \( 2\pi \). Show also that the equation \( \cos \theta = \sin \theta \) has only one root between 0 and \( 2\pi \) if \( c < 2 \). Illustrate these results graphically.

8. Find the cosine of any angle of a triangle in terms of the sides.

\[ \text{Two circles, of radii } 5 \text{ and } 3 \text{ inches, have their centres } 6 \text{ inches apart; find the angle at which they cut (that is, the angle between the tangents to the circles at either point of intersection).} \]

9. A hill-side is a plane surface inclined to the west at an angle \( \alpha \); a road is made up the hill-side so that the horizontal projection of the road is in the direction of S. of E.; find the inclination of the road to the horizon.

\[ \text{From the ends of a horizontal base-line } 100 \text{ yards long, the directions in which a tower standing on the horizontal plane through the base-line is seen are at right angles to one another, and the elevations of the top of the tower are } 28^\circ 30' \text{ and } 17^\circ 15'; \text{ find the point in the base-line at which the elevation of the top of the tower is greatest and the amount of the elevation at that point.} \]

\[ \text{A. If one side of a triangle be produced, show that the exterior angle so formed is equal to the sum of the interior opposite angles.} \]

\[ D \text{ is any point in the side } BC \text{ of a triangle } ABC; \ AB \text{ is produced to } F \text{ so that } BF = BD, \text{ and } AC \text{ to } E \text{ so that } CE = CD; \text{ show that the angle } PDE \text{ is obtuse.} \]

\[ \text{B. Construct a quadrilateral } ABCD \text{ in which the side } DC \text{ is parallel to } AB, \text{ the angle } A \text{ is } 55^\circ, \text{ and the lengths of } AB, BC, CD \text{ are respectively } 3,5, 7, 2, \text{ and } 2.8 \text{ inches long. Measure the length of the fourth side.} \]

\[ \text{C. Draw the graph of } \sin x + \cos x \text{ for values of } x \text{ ranging from } 0^\circ \text{ to } 180^\circ. \]

\[ \text{Find, from the graph or otherwise, the value or values of } x \text{ between } 0^\circ \text{ and } 180^\circ \text{ for which (i) } \sin x + \cos x = \sqrt{2}, \text{ (ii) } \sin x + \cos x = 1, \text{ and (iii) } \sin x + \cos x = 0. \]

\[ \text{D. A man walks a certain distance due east, then } 7 \text{ miles in the direction } 12^\circ \text{ North of East, and lastly } 10 \text{ miles in the direction } 20^\circ \text{ North of East; if the last point reached be } 15^\circ \text{ North of East of the starting-point, find the distance he walks due east.} \]

\[ \text{GROUP III (PAPER 2) AND SUBSIDIARY SUBJECT (15 b) \text{ \textit{PURE AND ANALYTICAL GEOMETRY}}} \]

\[ \text{1923. } 2^{1/2} \text{ Hours} \]

\[ \text{Not more than nine questions are to be attempted by any Candidate. The easier questions A, B, C, D are to be attempted only by Candidates taking Mathematics as a Subsidiary Subject; they must not be attempted by Candidates offering Group III as their principal subject.} \]

\[ \text{1. Show that the areas of similar polygons are in the same ratio to one another as the squares of corresponding sides.} \]

\[ \text{Prove that the areas of the rectangles formed by the external and internal bisectors, respectively, of the angles of a parallelogram} \]
are in the ratio \( (a+b)^2 : (a-b)^2 \), where \( a, b \) are the lengths of the sides of the parallelogram.

2. Describe a circle so as to touch one side of a triangle and the other sides produced.

3. If two straight lines are drawn from a point outside a plane perpendicular one to the plane and the other to a straight line in the plane, prove that the line joining the feet of the perpendiculars is perpendicular to the line in the plane.

Three non-planar lines \( OA, OB, OC \) of any lengths meet in \( O \); \( X, Y, Z \) are the centres of the circles described about the triangles \( BOC, COA, AOB \), respectively; lines are drawn through \( X, Y, Z \) perpendicular to the corresponding planes; show that these perpendiculars pass through one point.

4. \( P \) is any point on the circle described about the triangle \( ABC \); show that the feet of the perpendiculars from \( P \) on the sides, or the sides produced, of the triangle lie along a straight line (the \textit{pedal line} of \( P \) with respect to the triangle \( ABC \)).

5. Show that the equation \( ax^2 + 2bxy + by^2 = 0 \) represents a pair of straight lines through the origin.

6. If \( f, g, \) and \( c \) be constants, show that the equation \( x^2 + y^2 + 2gx + 2fy + c = 0 \) represents a circle, and find its centre and radius.

7. If \( r \) be the radius of the circle which passes through the points \( (a, 0), (-a, 0), \) and \( (b, 0) \); and \( r' \) that of the circle which passes through the points \( (0, b), (0, -b), \) and \( (0, a) \); show that \( r : r' = a : b \).

8. Find the equation of the normal at any point of the parabola \( y^2 = 4ax \) in the form \( y = mx - 2am - am^2 \).

9. Two normals to the parabola \( y^2 = 4ax \) make angles \( \tan^{-1} m \) and \( \tan^{-2} m \) with the axis of \( x \); show that the locus of the point of intersection of the normals is the given parabola, and that the locus of the point of intersection of the corresponding tangents is a straight line parallel to the directrix.

10. Find the equation of the polar of the point \( (x', y') \) with respect to the ellipse \( x^2/a^2 + y^2/b^2 = 1 \). Show also that, if the point \( Q \) lies on the polar of \( P \), then \( P \) lies on the polar of \( Q \).

11. Given that the equation of a rectangular hyperbola referred to its principal axes is \( x^2 - y^2 = a^2 \), deduce the equation of the curve referred to its asymptotes as axes.

12. If \( c \tan \theta, -c \cot \theta \) and \( c \tan \theta', -c \cot \theta' \) are two points on the rectangular hyperbola \( xy = c^2 \) where \( \theta + \theta' = \text{constant} \), show that the chord joining the points passes through a fixed point on the conjugate axis of the hyperbola.

A. Show that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

B. ABCD is a quadrilateral inscribed in a circle; \( AB, DC \) produced meet in \( E \); and \( BC, AD \) produced meet in \( F \); if the angles \( A, C \) are equal, show that \( AC \) is a diameter of the circle.

C. Show that the sum of the squares on two sides of a triangle is equal to twice the sum of the squares on half the third side and on the median corresponding to that side.

D. Find the point of intersection of, and the angle between, the lines \( 2x^2 + 3xy - 2y^2 - 15x - 5y + 25 = 0 \).

E. Show that the line \( y = mx + c \) cuts the parabola \( y^2 = 4ax \) in two points, and find the condition that the points may be real and different.

F. Find the value of \( c \) if the parabola \( y^2 = 4ax \) intercepts a length \( 4a \) on the straight line \( y = x + c \).

1924. \( 2 \frac{1}{2} \) Hours

I. Show that the locus of a point such that the tangents from it to two non-intersecting circles are equal is a straight line.

From any point on the radical axis of two circles tangents are drawn to the circles; show that the four points of contact lie on a circle, and deduce that the chords of contact intersect on the radical axis.
2. A quadrilateral is inscribed in a circle; prove that the rectangle contained by the diagonals is equal to the sum of the rectangles contained by pairs of the opposite sides.

The diagonals of a quadrilateral inscribed in a circle intersect at right angles; prove that the tangents at the vertices form a quadrilateral inscribable in a circle.

3. Prove that, if two triangles have one angle of one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.

In a triangle $ABC$ the perpendiculars from the vertices of the opposite sides meet those sides in $D$, $E$, and $F$, and intersect in $P$. Prove, by the use of similar triangles, that

\[ AD / PD = DE / DF. \]

4. Show that the equation of any straight line can be written in the form $y = mx + b$.

If $A$, $B$ are two points on the axes of reference at distances $a$ and $b$ from the origin, obtain the coordinates of any point $C$ dividing $AB$ in the ratio $m : n$.

Show that, if $C$, $D$ are two points on $AB$ such that $CD$ subtends a right angle at the origin $O$, then

\[ BC \cdot BD = AC \cdot AD \cdot \frac{OB^2}{OA^2}. \]

5. Find the equations of the tangents to the two circles

\[ x^2 + y^2 = 2ax, \quad x^2 + y^2 = 2by \]

at their points of intersection, and verify that they cut at right angles at both points.

6. Find the equation of the chord joining the two points $(am, b)$ and $(b, am)$ on the parabola $y^2 = 4ax$.

Prove that, if the chord $PQ$ passes through the point of intersection of the axis and the directrix, and the tangents at $P$ and $Q$ meet in $T$, and the normals at $P$ and $Q$ meet in $G$, $TG$ is bisected by the axis of the parabola.

7. Prove that the line $x \cos a + y \sin a = p$ is a tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$, if $p^2 = a^2 \cos^2 a + b^2 \sin^2 a$.

If two points are taken on the minor axis of an ellipse at the same distance from the centre as the focus, prove that the sum of the squares of the perpendiculars from these points on any tangent to the ellipse is constant.

8. Define the eccentric angle of any point on an ellipse, and show that the equation of the chord joining two points of eccentric angles $(\alpha \pm \beta)$ is

\[ x \cos \alpha + y \sin \alpha = \cos \beta. \]
and $AB$ as radius describe a circle ($C_1$); with $B$ as centre and $BA$ as radius describe a circle ($C_2$). Let these circles meet at $C$. Commencing with $C$ step off on the part of the circumference of ($C_2$) remote from $A$ two areas $CD, DE$ with $CA$ as radius. With centre $E$ and radius $EA$ describe a third circle to meet ($C_1$) at $F$. Then the circle with $F$ as centre and $FA$ as radius will pass through the middle point of $AB$.

3. Prove that the feet of the perpendicular drawn to the sides of a triangle $ABC$ from a point $P$ on the circle circumscribed about the triangle $ABC$ are collinear.

A straight line meets the sides $BC, CA, AB$ of a triangle $ABC$ in the points $P, Q, R$ respectively. The circles circumscribed about the triangles $ABC$ and $PQC$ meet again at the point $O$. By means of the converse of the foregoing theorem, or from any other considerations, prove that

1. the circles circumscribed about the triangles $AQR$ and $BRP$ also pass through $O$;
2. the centres of these four circles and the point $O$ all lie on a circle.

4. Prove that a sphere can be drawn to pass through any four points not in one plane.

Prove that if two circles in different planes have two common points, then a sphere can be described on which both circles lie.

5. If the equation of a straight line is written in the form

$$\frac{x}{a} + \frac{y}{b} = 1,$$

interpret $a$ and $b$ geometrically.

Through the point $(x', y')$ two straight lines are drawn forming the equal sides of an isosceles triangle of which the base is the axis of $x$, the base angles of the triangle each being $\alpha$. One of these equal sides meets the axis of $x$ at $P$ and the axis of $y$ at $Q$. The other meets the axis of $x$ at $P'$ and the axis of $y$ at $Q'$. Find the coordinates of the point of intersection of $PQ$ and $P'Q'$.

6. Prove that the length of the perpendicular drawn from the point $(h, k)$ to the straight line, whose equation is

$$x \cos \alpha + y \sin \alpha = a - p$$

is $p - h \cos \alpha - k \sin \alpha$, the upper or lower sign being taken according as the point and the origin are on the same or opposite sides of the line.

Two circles have their centres at the points $(a, 0)$ and $(-a, 0)$, and the equations of their two external common tangents are

$$x \cos \beta + y \sin \beta = a \cos \beta.$$
meets $BC$ at $Q$. Prove that the area of the quadrilateral $ABQP$ is half that of the quadrilateral $ABCD$.

If the quadrilateral were cut out of cardboard and the point $P$ marked on $AD$, show by using the properties of similar figures, how $Q$ could be determined by lines drawn entirely on the cardboard, it being assumed that the position of $P$ is such that $Q$ lies between $E$ and $C$.

2. Prove that the only point $P$ in the plane of a triangle $ABC$, such that $PA^2 - BC^2 = PB^2 + CA^2 - PC^2 + AB^2$, is the orthocentre of the triangle.

Find the locus of $P$, if $P$ is not restricted to lie in the plane of the triangle.

3. $A$ and $B$ are two fixed points on a circle and $CD$ is a variable chord of the circle of constant length $AD$ and $BC$ meet at $E$, $AC$ and $BD$ meet at $F$. Prove that the loci of $E$ and $F$ are circles passing through $A$ and $B$.

Prove also that the tangents at $E$ and $F$ to their respective loci are parallel to $CD$.

4. Prove that the locus of a point from which the tangents drawn to two circles are of equal length is a straight line perpendicular to the line joining the centres of the circles.

$ABC$ is a triangle inscribed in a circle and the internal bisector of the angle $A$ of the triangle meets the circle again at $D$. Prove that the foot of the perpendicular from $D$ to $BC$ bisects the distance between the points of contact of $BC$ with the inscribed circle of the triangle and the circle escribed to the side $BC$, and hence or otherwise that the radical axis of these two circles is the straight line joining the feet of the perpendiculars drawn from $D$ to the sides of the triangle $ABC$.

5. Find the coordinates of the point which divides the distance between the two points $(x_1, y_1)$ and $(x_2, y_2)$ internally in the ratio $p : q$.

A variable straight line passing through the fixed point $(h, k)$ meets the axis of $x$ at $L$ and the axis of $y$ at $M$. Find the equation to the locus of the point which divides $LM$ internally in the ratio $p : q$.

6. Prove that the equation of a circle, which touches the straight lines $x^2\sin^2\alpha - y^2\cos^2\alpha = 0$ and lies in the angle $2\alpha$ between them, may be written in the form

$$y^2\cos^2\alpha - x^2\sin^2\alpha + (x - a)^2 = 0,$$

and that the equation of a circle, which touches these two straight lines and lies in the other angle between them, may be written in the form

$$x^2\sin^2\alpha - y^2\cos^2\alpha + (y - b)^2 = 0,$$

where $a$ and $b$ may have any values.

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the points $P$ and $Q$. $A$ is any point on the first circle and the length of the tangent from $A$ to the second circle is $t$. Prove that

$$AP_A Q = R^2 \sqrt{O'}.$$

2. Prove that the bisector of an angle of a triangle divides the opposite side into two parts, which are in the ratio of the sides containing the angle.

The bisector of the angle $A$ of the triangle $ABC$ meets the circumcircle of the triangle again at $D$. The bisector of the angle $ABC$ meets $AC$ at $E$ and the bisector of the angle $ADB$ meets $AB$ at $F$. Prove that $EF$ is parallel to $BC$ and passes through the centre of the inscribed circle of the triangle.

3. The middle points of the edges $BC, CA, AB, DA, DB, DC$ of a tetrahedron $ABCD$ are respectively $P, Q, R, P', Q', R'$. Prove that $PP', QQ', RR'$ meet at a point and bisect each other.

Prove that if $BC$ is perpendicular to $DA$ and $CA$ is perpendicular to $DB$, then $AB$ is perpendicular to $DC$.

4. Prove that all squares inscribed in a given circle have the same area, which is greater than that of any quadrilateral, not a square, which can be inscribed in the circle.

Deduce from this a property of the ellipse by orthogonal projection.

5. A straight line makes intercepts $x$ and $y$ on the axes (rectangular) of coordinates. The coordinates of a point $P$ on the line are $(x', y')$. Prove that the straight line, whose intercepts on the axes are $E^2 / x'$ and $y^2 / y'$, passes through the point whose coordinates are $(x, y)$, and prove also that the straight line joining the origin to the intersection of these two straight lines and the straight line joining the origin to $P$ are equally inclined to each of the axes of coordinates.

6. Find the equation of the circle which passes through the points $A, B, C$, whose coordinates are respectively

$$(a, 0) \quad (-a, 0) \quad (x', y').$$

Find the coordinates of the other extremity of the diameter of this circle drawn through $C'$; and prove that, if $A$ and $B$ are fixed and $C$ varies so that the area of the triangle $ABC$ is fixed, the locus of this other extremity is a parabola.

7. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(a^2, 2at)$. Prove that, if two or more pairs of tangents to a parabola intercept equal lengths on one other tangent to the parabola, they will intercept equal lengths on every other tangent.
3. Find the condition that the circles
\[ x^2 + y^2 + 2px + 2fy + c = 0 \] and \[ x^2 + y^2 + 2g'x + 2f'y + c' = 0 \]
may cut at right angles.

Prove that each pair of the circles
\[ x^2 + y^2 - 8x - 36 = 0, \quad x^2 + y^2 + 18x - 36 = 0, \]
\[ x^2 + y^2 - 20y + 36 = 0 \]
cut one another at right angles, and that the common chord of the first pair passes through the centre of the remaining circle.

4. Define the centres of similitude of the two circles.

Find the coordinates of the centres of similitude of the circles \( x^2 + y^2 - 8x - 36 = 0, \) \( x^2 + y^2 - 18x - 36 = 0 \); and show that the centres of similitude lie on the circle which is described on the common chord of the two circles as diameter.

5. Prove that the distance of any point on the parabola \( y^2 = 4ax \)
from the point \((a, 0)\) is equal to its distance from the straight line \( x + a = 0. \)

\( P, P' \) are points on a parabola, whose focus is \( S; \) the tangents at \( P \) and \( P' \) meet in \( T; \) \( PK, P'K' \) are perpendiculars on the directrix.

Prove that \( TK^2 = TK'^2 = TS^2 = PS \times PS'. \)

6. Find the equation of the normal at \((x', y')\) to the conic whose equation is \( ax^2 + by^2 = 1. \)

\( P \) is the point \((6, 4), Q \) the point \((-8, 3)\) on the ellipse \( x^2 + 4y^2 = 100. \) The tangents at \( P \) and \( Q \) meet in \( T, \) the normals meet in \( G. \) Find the coordinates of \( T \) and \( G, \) and show that the diameter through \( G \) is perpendicular to \( PQ. \)

7. Find the equation of the chord of contact of the tangents from the point \((k, k)\) to the conic \( x^2 + y^2 + 2x + 2y = 1. \)

\( SZ, S'Z', CY, PK \) are the perpendiculars from the foci, the centre, and the point \( P \) on the polar of \( P \) with regard to the above conic. Prove that (regard being had to sign) \( SQ, S'Q' - CY, PK = 0. \)

8. Obtain the equation of the tangent at \( x', y' \) to the rectangular hyperbola \( xy = c^2. \)

The normal at \( P(3a, 4a) \) on the rectangular hyperbola \( xy = 12a^2 \) meets the curve again in \( Q. \) Show that \( PQ \) is equal to \( 12a/12. \)

A. Obtain the equation of a straight line in terms of the intercepts which it makes on the axes of coordinates.

\( A \) is the point \((5, 0), B \) is \((0, 7), \) \( C \) is \((10, -7). \) A variable straight line through \( C \) meets the axes of \( x \) and \( y \) in \( M \) and \( N \) re-
Two perpendicular lines are drawn through the origin so as to form an isosceles right-angled triangle with the line

\[ lx + my + n = 0 \]

show that their equations are given by

\[ (l-m)x + (l+m)y = 0, \]
\[ (l+m)x + (m-l)y = 0. \]

3. Find the locus of a point \( P \) which moves so that \( BP = k \cdot AP \), where \( A, B \) are the points \((c, 0), (-c, 0)\), respectively.

Show that the locus is cut orthogonally by any circle passing through \( A \) and \( B \).

4. Find the equation to the tangent at any point \((x', y')\) on the circle \( x^2 + y^2 + 2px + 2qy + e = 0 \).

Show that the line \( lx + my + n = 0 \) is a tangent to the circle provided that \( (y + mf - n)^2 = (f^2 + g^2 - e) \).

5. Defining a conic section as the locus of \( P \), which moves so that \( SP = e \cdot PM \), where \( S \) is a focus and \( PM \) is the perpendicular on the corresponding directrix, show how the equation to the conic can be reduced to the form

\[ (1 - e^2)x^2 + y^2 = P/(1 - e^2) \]

where \( l/e \) is the perpendicular from \( S \) to the directrix.

Show that it is also possible to write the equation in the form

\[ y^2 = 2lx - x^2(1 - e^2). \]

6. Find the equation to the normal at the point \((ap^2, 2ap)\) on the parabola \( y^2 = 4ax \).

Prove that the normal cuts the curve again in the point given by \((at^2, 2at)\) where \( t = -(p + 2/p) \);

and that the length of the chord intercepted is equal to

\[ 4a(p^2 + 1) \sqrt{p^2}. \]

7. Explain how an ellipse may be regarded as the orthogonal projection of a circle, and prove that conjugate diameters of an ellipse correspond to perpendicular diameters of the circle.

The tangent to an ellipse at \( P \) cuts the tangents at \( K, K' \) (opposite ends of a diameter) in \( T, T' \), respectively. Prove that

\[ TK, TK' = CL, \quad TP, PT' = OC. \]

where \( CL, CQ \) are semidiameters conjugate to \( CK, CP \) respectively.

8. Define the asymptotes of the hyperbola

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

and establish the result which justifies the use of this name for these lines.

Show that if the tangent at \( P \) to the hyperbola cuts the asymptotes in \( Q, Q' \) then \( QQ' \) is bisected at \( P \); and that the area of the triangle \( QQ'P \) is equal to \( ab \).

A. Find an expression for the area of a triangle in terms of the coordinates of the three vertices \((x, y), (x, y), (x, y)\).

Determine the locus of \( P \) if the area of the triangle \( OAP \) is equal to \( \frac{1}{2}ab \), where \( A \) is the point \((c, 0)\) and \( O \) is the origin.

B. Find the equation to the circle with the points \((2, 3), (-3, 2)\) as ends of a diameter.

Verify that the point \((-1, 5)\) and the origin are at opposite ends of a diameter.

C. Prove that the equation to the tangent at \((ap^2, 2ap)\) to the parabola \( y^2 = 4ax \) is \( x + py + ap^2 = 0 \).

Show that the perpendicular to this tangent from the focus \((a, 0)\) meets the tangent in the point \((0, ap)\).

D. Express the coordinates of any point on the ellipse

\[ x^2/a^2 + y^2/b^2 = 1 \]

in terms of the eccentric angle \( \theta \).

Show that the distance of the point from the focus \((ae, 0)\) is equal to \( a(1 - e \cos \theta) \).

1920. 1½ Hours

1. Find the coordinates of the orthocentres of the three triangles whose vertices are the points \((a, 0), (-a, 0), (b, k)\).

\( P \) is the point \((5, 2)\) on the circle \( x^2 + y^2 - 10y - 9 = 0 \), \( A \) and \( B \) are the points in which the circle meets the axis of \( x \). \( H \) is the orthocentre of the triangle \( APB \), and \( PH \) meets the circle again in \( K \); prove that \( HK \) is bisected by \( AB \).

2. Prove that the equation

\[ 8x^2 + 38xy - 33y^2 + 26x + 68y + 21 = 0 \]

represents two straight lines.

Show also that the angle between them is \( \tan^{-1} 2 \).

3. Find the equation of the tangent to the circle,

\[ x^2 + y^2 - 2kx + 3 = 0 \]

at the point \((x', y')\); and deduce the equation of the polar of the point \((p, q)\) with regard to the circle.

The pole with regard to the circle \( A \) of the radical axis of two
circles $A$ and $B$ is the centre of $B$; prove that the centre of $B$ is the pole of the radical axis with regard to $A$.

4. Obtain the equation of the pair of lines from the origin to the points of intersection of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

and the straight line $lx + my + n = 0$.

Prove that the locus of the middle points of chords of the circle

$$x^2 + y^2 - 2ax + 2by = 0$$

which subtend a right angle at the origin is the circle

$$x^2 + y^2 - ax + by = 0$$

5. Find the equation of the normal to the parabola $y = 4ax$, at the point $P (am^2, 2am)$.

$S$ being the focus of the parabola, the perpendicular to $SP$ at $S$ meets the normal at $P$ in $Q$; $NP, MQ$ are the ordinates of $P$ and $Q$; $X$ is the point of intersection of the axis and the directrix of the parabola. Prove that $XM = 2XN$.

6. Find the equation of the chord joining the points on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

whose eccentric angles are $\alpha + \beta$, $\alpha - \beta$; and prove that the locus of the middle points of all chords of the ellipse which are parallel to the above chord is a diameter of the ellipse.

Prove that, if $CP$ and $CQ$ are conjugate semidiameters of the ellipse,

$$CP^2 + CQ^2 = a^2 + b^2.$$

7. Prove that, if chords of the conic $ax^2 + by^2 = 1$ are drawn through the fixed point $(h, k)$, the locus of the point of intersection of pairs of tangents at the extremities of these chords is a straight line.

From each point of the line $x = 4$ a perpendicular is drawn to its polar with regard to the conic $3x^2 + 4y^2 = 24$. Prove that the locus of the feet of these perpendiculars is a circle.

8. Obtain the equation of the tangent to the rectangular hyperbola, $xy = a^2$, at the point $x', y'$; and show that it may be put into the form

$$x'y' + y'x' = 2.$$

Show that the hyperbolas $x^2 - y^2 = 20, xy = 24$ cut at right angles at all their common points.

A. Lay off the points $(4, 3), (1, 2), (7, -3), (-1, -3)$ on a diagram; and calculate the area of the quadrilateral of which they are the corners.

B. Obtain the equation of a straight line, given $p$ the length of the perpendicular on it from the origin, and $\theta$ the angle which this perpendicular makes with the axis of $x$.

Find the complete locus of the point whose distance from the straight line $3x - 4y + 2 = 0$ is $\sqrt{2}$ times its distance from the straight line $7x - y + 10 = 0$.

C. Find the equation of the circle through the points $(0, 0), (4, 0), (0, 2)$; and show that the circle is touched by the line $2x + y = 10$.

Prove analytically that the line equal to the point of contact to the origin is a diameter of the circle.

D. Find the equation of the normal to the ellipse

$$x^2 + 4y^2 = 100$$

at the point $P (8, 3)$.

If the normal at $P$ meets the major axis in $G$, and $CY$ is the perpendicular from the centre $C$ to the tangent at $P$, prove that $PG.CY$ is equal to the square on the minor semi-axis.

1921. 1½ Hours

1. Find from first principles the condition that the lines $y = mx, y = mx$ should be perpendicular.

The coordinates of two points $P_1, P_2$ referred to axes $Ox, Oz$ respectively, $Q_1, Q_2$ are the feet of the perpendiculars from the points to $Ox$. Find the equations of the lines drawn from $Q_1$ perpendicular to $OP_1$, and from $Q_2$ perpendicular to $OP_2$; and prove that the lines meet on the perpendicular drawn from $O$ to $P_1P_2$.

2. Prove that the perpendicular distance of the point $(h, k)$ from the line $ax + by + c = 0$ is $\frac{\pm (ah + bk + c)}{\sqrt{a^2 + b^2}}$.

Find the centre of the circle inscribed in the triangle which has its sides along the lines $3x - 4y = 2, 3x + 4y = 10, y = 0$.

3. Prove that the equation $x^2 + y^2 + 2px + 2fy + c = 0$ represents a circle, and give the coordinates of the centre and the radius.

Show that the common chord of $x^2 + y^2 - 4x + 2y = 0$ and $x^2 + y^2 - 10x = 0$ is a diameter of the first circle, and find the angle at which the two circles cut.

4. Find the equation of the tangent and normal at the point $(a\theta, 2a\theta)$ on the parabola $y^2 = 4ax$.

The normal at a point $P$ on a parabola whose focus is $S$ meets the axis in $G$. Prove that there are two positions of $P$ such that the triangle $SPG$ is equilateral, and that the sides then have length $4a$. 

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5. Find the equation of the chord joining points on the ellipse 
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] 
which have eccentric angles \( \alpha, \beta \) respectively.

Through the point whose eccentric angle is \( \alpha \) a chord is drawn which makes with the major axis an acute angle equal to that made by the tangent at the point. Prove that the eccentric angle of the other end of the chord is \( 2\pi - 3\alpha \).

6. Find the locus of the mid-points of chords of the ellipse 
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] 
which are parallel to \( y = mx \).

A given circle has its centre on the major axis of the ellipse, and a line \( PQP' \) is drawn in a given direction to meet the ellipse in \( P, P' \) and the circle in \( Q, Q' \). Prove that there is one and only one position of the line in which \( PQ = P'Q' \), and that in this position the distance of the mid-point of \( PP' \) from the minor axis is independent of the given direction.

7. Define the polar of a point with respect to a conic, and find the equation of the polar of \( (x', y') \) with respect to the hyperbola
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \]
Prove that the pole with respect to the hyperbola of the tangent
to the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at a point \( P \) is a point \( Q \) on the ellipse; and that the pole of the tangent at \( Q \) is \( P \).

8. Show that the equation of a rectangular hyperbola, referred to its asymptotes as coordinate axes, is \( xy = c^2 \).

Find the coordinates of the foci and the equations of the directrices of the hyperbola referred to the above axes.

A. The sides of a parallelogram are in pairs parallel to \( x+y=0 \) and \( x-2y=0 \); and the points \((-1, 3), (0, 1) \) are ends of one diagonal. Find the equation of the second diagonal.

B. Find the equation of the circle which passes through the points \((-2, 0), (1, 0), (2, -1) \).

C. Find the equation of the tangent to the circle at \((-2, 0) \).

D. Find the coordinates of the mid-point of the chord of the locus which lies along the line \( 3y = 2x + 4 \).

D. Find also the eccentricity of the ellipse \( 4x^2 + 9y^2 = 36 \).

Find the equations of two tangents to the ellipse which are parallel to \( 2y = x \).
B. Find the equation of the circle which has the same centre as the circle \( x^2 + y^2 - 4x + 6y - 3 = 0 \) and which passes through the point (3, -6).

Show that the origin lies inside both circles.

C. Find the coordinates of the vertex and focus, and the equation of the directrix, of the parabola \( y^2 = 8x + 4y + 20 \).

This curve is cut by a straight line passing through the origin and the vertex; find the coordinates of the other point of intersection.

D. A focus of an ellipse is at the point (3, 0), the equation of the corresponding directrix is \( x - 1 = 0 \), and the eccentricity is \( \frac{1}{2} \); find the equation of the ellipse.

Find the length of the major axis of this ellipse.

GROUP III (PAPER 4) AND SUBSIDIARY SUBJECT (150)

DIFFERENTIAL AND INTEGRAL CALCULUS

1918. 1 1/2 Hours

Not more than seven questions should be attempted by any Candidate. The easier Questions A, B, C, D should be attempted only by Candidates who offer Subsidiary Subject (150); they must not be attempted by Candidates who offer Group III as their principal Subject.

1. Prove that as \( x \) tends to 1, the limit of

\[
\frac{\xi^x - 1}{\xi^2 - 1}
\]

is equal to \( p/q \), when \( p \), \( q \) are positive integers; and show that the result remains valid when \( p \) is a negative integer.

By writing \( \xi = 1 + h/x \), \( p/q = n \), prove that, as \( h \) tends to zero,

\[
\lim_{h \to 0} \frac{(x + h)^n = x^n}{(x + h)^2 - x^2} = n x^{n-1}
\]

and state this result in the notation of differentiation.

[The Binomial series for fractional and negative indices must not be used.]

2. State the rules for finding the maximum and minimum values of a function by means of the differential calculus.

Apply the method to the function

\[
y = \sin x + \frac{1}{3} \sin 3x
\]

between the limits \( x = 0 \) and \( x = 2\pi \).

Draw a rough graph to illustrate your conclusions.

3. Show how to find the equation to the tangent at any point of a curve on which the coordinates are given as functions of a variable \( \theta \).
A. State and prove the formula for differentiating a quotient \( \frac{u}{v} \).

Differentiate \( \frac{x^2}{x+1} \) and \( \frac{x}{x^2+1} \).

B. Prove that the maximum and minimum values of the function \( \frac{1}{2} (5x^2 - 2x) \) are \( \frac{1}{\sqrt{5}} \) and \( -\frac{1}{\sqrt{5}} \) respectively.

Show that as \( x \) varies from \(-1\) to \(+1\), the greatest and least values of the function are \( 1 \) and \(-1 \) respectively. Draw a rough sketch to explain these conclusions.

C. State and prove the rule for integration by parts.

Evaluate the integrals:

\[
\int x \sin x \, dx, \quad \int \log x \, dx.
\]

What is the value of the former integral between the limits \( 0 \) and \( \frac{1}{2} \pi \)?

D. Explain why the integral \( \int_0^\pi f(x) \, dx \) represents a certain area connected with the curve \( y = f(x) \), defining carefully the area in question.

The parabola \( y^2 = 2x \) is cut at \( PP' \) by the line \( x = a \); prove that the area between \( PP' \) and the curve is two-thirds of the area of the rectangle bounded by \( PP' \), the axis of \( y \), and the two lines drawn through \( P \), \( P' \) parallel to the axis of \( x \).

**Differential and Integral Calculus**

**Group III (Paper 4) and Subsidiary Subject (15 d)**

1. If \( x = e^{-3at} \sin 4at \), find \( \frac{dx}{dt} \) and \( \frac{d^2x}{dt^2} \); and prove that

\[
\frac{d^2x}{dt^2} + 6n \frac{dx}{dt} + 25a^2x = 0.
\]

If \( x \) is the distance at time \( t \) of a point, which is moving in a straight line, from a fixed point in that line, and if \( u = \pi/180 \) and \( t \) is measured in seconds, prove that the point will be instantaneously at rest about \( 13 \frac{1}{2} \) seconds after it has passed through the fixed point.

2. Show how to find and how to discriminate between maximum and minimum values of a function of a single variable.

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**Group III**

**Integral Calculus**

A tin canister, in the form of a right circular cylinder with flat ends, contains 150 cu. inches; it is to be made as cheaply as possible, i.e. to require as little material as possible. Prove that its length and its diameter are both \( 4 \) in. (Neglect any overlap.)

3. Find the equations of the tangent and the normal at the point \( t \) of the curve \( x = f(t), \ y = g(t) \).

Prove that the normals at the points \( t \) of the curves

\[
x = a \cos^2 t, \quad y = a \sin^2 t,
\]

and

\[
x = a \cos^2 t + b \cos t, \quad y = a \sin^2 t + b \sin t,
\]

meet at the point \( a \cos 3t, -a \sin 3t \).

4. Prove the formulae for a plane curve

\[
p = s \frac{d\theta}{ds} = x \frac{dy}{ds} - y \frac{dx}{ds}.
\]

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**Group III**

**Integral Calculus**

5. Integrate \( \frac{x}{a^2 + x^2} \) and \( x^2 (1 + x) \). Let \( a^2 = e \).

Show that

\[
\int_0^\pi x \sin^2 x \, dx = \frac{\pi^3}{4}.
\]

6. If \( x = \frac{1}{2} \left( u + \frac{1}{u} \right) \), prove that

\[
\int \frac{dx}{\sqrt{u^2 - 1}} = \frac{1}{u}.
\]

By applying this substitution, or otherwise, show that

\[
\int \frac{dx}{\sqrt{(2x^2 - 1)^2 + (x^4 - 1)^2}} = \frac{\pi}{3 \sqrt{3}}.
\]
7. The equation to a parabola being
\[ 2ahy = x^2 - (c - 2a)x + ax (c - a) + 2bh, \]
find the values of \( y \) when \( x = -a, 0, h \) respectively.

Prove that the area included between the curve, the axis of
\( x \) and the ordinates \( x = -a, x = h \) is
\[ \frac{4}{3}h (a + 4b + c). \]

Apply this result to obtain an approximate value of
\[ \int_{-\pi/3}^{\pi/3} \sin x \, dx, \]
by assuming that \( y = \sin x \) is approximately a parabola between
the values \( x = -\frac{\pi}{3}, x = \frac{\pi}{2} \).

Compare your result with the actual value of the integral.

8. Prove that the area of the curved surface of a frustum of a cone
is \( \pi (r + r') l \), where \( r, r' \) are the radii of the ends of the frustum and
\( l \) is the length of its slant side.

Deduce the expression \( 2\pi \int y \, ds \) for the area of a surface of
revolution.

Find the area of the surface of the parabolic cup formed by
the revolution about its axis of the portion of the parabola \( y^2 = 4ax \)
cut off by the line \( x = 3a \).

A. Differentiate
\[ \frac{dx}{1 + x^2}, \quad \frac{\sin^2 x}{\cos^2 x}. \]

If \( u = x^n y^m \), prove that
\[ \frac{\Delta u}{\Delta x} = m \frac{\Delta x}{x} + n \frac{\Delta y}{y}. \]

The radius of a right circular cylinder is measured as 13 cm.,
but is believed to be 2 per cent. in excess of the true measure; the
length, measured as 46 cm., is believed to be 3 per cent. in defect.
By how much per cent. is the calculated volume wrong, and is the
total error in excess or defect ?

B. Plot the curve \( y = 4x - x^2 \).

Find the equation of the tangent at the point in which the
curves cut the positive part of the axis of \( x \). Prove that this tangent
meets the curve again in the point \((-4, 48)\).

Find the maximum and minimum values of \( y \). Are these the
greatest and least values of \( y \)?

At what point is the slope of the curve a maximum?
Differential and integral calculus

5. Evaluate
   \[ \int \frac{dx}{x(a^2-x^2)} \]
   \[ \int \frac{dc}{(a^2-x^2)^{3/2}} \]
   \[ \int \tan^{-1}x \, dx \]

6. Prove that
   \[ \int \frac{dx}{(b-x)(b-x)^{1/2}} \]
   \[ = \frac{\pi}{3} \]

7. Show how to find the mean value of \( f(x) \) between \( x = x_1 \) and \( x = x_2 \), taken for equidistant values of \( x \).

8. Prove that the area bounded by the curve \( r = f(\theta) \) and the radii vectors \( \theta = \theta_1 \), \( \theta = \theta_2 \) is
   \[ \int_{\theta_1}^{\theta_2} \frac{r^2}{2} \, d\theta. \]

A. Show how to find \( \frac{dy}{dx} \) from the graph of \( y = f(x) \).

B. Differentiate \( x/(1+x^2) \), \( x \cos x \).

C. Prove that \( \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx \). Evaluate
   \[ \int_1^x \log x \, dx \]
   \[ \int_0^x \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \, dt, \]

D. Show that the length of the arc of a plane curve is equal to
   \[ \int_0^a \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \, dt, \]
   where \( x, y \), the coordinates of any point on the curve, are expressed as functions of \( t \).

1. Prove that, when \( n \) is a positive integer, that \( \frac{dx^n}{dx} = nx^{n-1} \); and deduce that, when \( p \) and \( q \) are any integers, positive or negative,
   \[ \frac{d}{dx} \frac{r^p}{q} = \frac{r^{p-1}}{q} \]

2. Apply Maclaurin's theorem to prove that
   \[ \tan y = 1 + \frac{1}{2} y^2 + \frac{1}{2} y^4 + \frac{1}{2} y^6 + \ldots \]
   \[ 3. \] State and prove rules for determining the maxima and minima values of \( f(x) \).
Differential and Integral Calculus

Show that the function \((x-3)^2(x+2)^3\) has a maximum value when \(x=1\) and a minimum when \(x=3\), but that these are not the greatest and least values.

What is the nature of the value when \(x=-2\)?

4. The coordinates of any point \(P\) on the curve \((a-x)y^2 = x^3\) may be written \(\frac{a^3}{1+m^2}, \frac{am^3}{1+m^2}\). Find in terms of \(m\) the equation of the tangent at \(P\).

Find the coordinates of the point in which the tangent at \(\frac{1}{4}a, \frac{1}{2}a\) meets the curve again; and prove that the intercepts which the tangents at these two points make on the axis of \(y\) are in the ratio 8:1.

5. State and prove the rule for integration by parts.

Prove that \(\int e^{-x}(x^3-1)dx = 0\).

Integrate \(x^{-3}, \cos^8 x \sin^3 x, \frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x-1}}\).

6. Evaluate \(\int \frac{(1+x^2)dx}{1+x^2}\) by means of the substitution \(u = \frac{1}{x}-x\).

Prove that \(\int_0^{\frac{1}{2}} \frac{(1+x^2)^2}{1+x^2}dx = \frac{\sqrt{2}+1}{4} \pi\).

7. Apply the integral calculus to find the volume of a pyramid of height \(h\) on a base of area \(A\), and to find the distance of the centre of gravity of the pyramid from the base.

Find the position of the centre of gravity of a circular plate, of small but variable thickness, the thickness at any point being proportional to the distance of that point from a fixed point on the edge of the plate.

8. Show how to obtain by means of the integral calculus the volume of a solid of revolution.

The curve \(cy = (a-x)(x-b)\) cuts the axis of \(x\) in the points \(A, B\); find (1) the area included between the curve and the chord \(AB\), (2) the volume of the solid generated by the revolution of this area about the axis of \(x\).

A. Differentiate \(\frac{x^2-1}{x}, \sec^2 x \tan x, \left(\frac{x+1}{x-1}\right)^{\frac{1}{3}}\).

If \(y = x^8 \log x\), show that \(x^2 \left(\frac{dy}{dx^2} - 3\right) = 2y\).

[80]
5. Integrate \((x^2+1)f(x, x \cos x, x\sqrt{1+x^2})\), and evaluate
\[
\int_0^\frac{\pi}{2} \sin^3 x \, dx \quad \text{and} \quad \int_0^\frac{\pi}{2} \frac{dx}{1 - \cos ax \sin x}.
\]

6. Explain how to find the length of the arc of a curve given by the equations \(x = \phi(\theta), y = \psi(\theta)\) between two points corresponding to given values of \(\theta\).

Find the length of the arc of the curve,
\[
x = a (\theta - \sin \theta), \quad y = a (1 - \cos \theta)
\]
between the points for which \(\theta = 0\) and \(\theta = \pi\).

Find the area included between the curve, the axis of \(x\), and the ordinates of the points for which \(\theta = 0\) and \(\theta = \pi\).

7. The portion of a parabola bounded by the latus rectum revolves about the tangent at the vertex; find (1) the volume, (2) the curved surface, of the solid so formed.

A. Find, from the definition, the differential coefficient of \(x^4\) with respect to \(x\).

Differentiate,

\[
\frac{dy}{dx} = x^3, \quad \log \frac{x}{1+x}, \quad \text{and} \quad \cos^{-1} \sqrt{1-x^2}.
\]

B. If \(\frac{dy}{dx}\) be positive, show that \(y\) increases as \(x\) increases; and, if \(\frac{dy}{dx}\) be negative, show that \(y\) decreases as \(x\) increases.

Find the values of \(x\) for which the expression

\[2x^3 - 21x^2 + 60x + 20\]

is increasing, and also those for which it is decreasing.

C. Integrate

\[
x + 2, \quad x + \tan x, \quad \text{and} \quad \frac{x}{(x-1)(x-2)}.
\]

Find the area between the curve \(y = a \sin x\) and the axis of \(x\) from \(x = 0\) to \(x = \pi\).

D. Find the volume of the smaller segment of a sphere of radius \(a\) cut off by a plane at distance \(a - c\) from the centre.

A cylindrical hole is bored through a hemisphere, the axis of the hole passing through its centre and being at right angles to its plane surface; the radius of the hole is three-fifths of that of the sphere; find the volume of the portion removed.

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[82]
8. Obtain a formula of reduction for \( \int \sin^n x \, dx \).

Prove that \( \int_0^\pi (2 \sin^5 x - 3 \sin^3 x) \cos^2 x \, dx = 0 \).

Verify the formula of reduction
\[
(m + n)(m + n - 2) \int \sin^n x \cos^m x \, dx = -(m - 1)(n - 1) \int \sin^{n-2} x \cos^{m-2} x \, dx
\]
\(- (m - 1) \sin^{n-1} x \cos^{m-1} x = (n - 1) \sin^{n-1} x \cos^{m-1} x \).

9. Give formulae for the area of the curve \( r = f(\theta) \), and for the coordinates of the centre of gravity of the area bounded by the curve and two radii vectors.

Show that the area of the cardioid \( r = a (1 + \cos \theta) \) between the radii \( \theta = 0 \) and \( \theta = \pi \) is \( \frac{3}{2} \pi a^2 \); and show that the ordinate of the centre of gravity of the area is \( 16a^3/9\pi \).

A. Define the differential coefficient of a function of \( x \), and find the differential coefficient of \( \log x \).

Prove that \( x - \log x \) increases steadily when \( x > 1 \), and is always greater than unity.

Differentiate \( \cos x \cos 3x \), \( \tan^{-1} \sqrt{x} \), \( (\log x)^3 \).

B. Prove that the turning points of the function \( \frac{5x - 13}{x^2 - 1} \) are \( x = 3 \) and \( x = \frac{1}{2} \); and show that \( \frac{1}{2} \) is a maximum value and \( 12 \frac{1}{2} \) a minimum value of the function.

C. Obtain the integrals of \( \sin^2 x \), \( e^{+x} \), \( 1/(e^x + e^{-x}) \), \( 1/(\sqrt{x^2 - x^2}) \).

Prove that the area of the curve \( y = \frac{1}{2} (e^x - e^{-x}) \), and show that the area between the curve, the axis of \( x \), and any two ordinates is equal to the arc between those points multiplied by the unit of length.

D. The segment of the parabola \( y^2 = 4ax \) cut off by the line \( x = h \), revolves about the axis of \( x \). Prove that the volume generated is \( \frac{2}{3} \pi \) (volume of the enveloping cone generated by the tangents to the parabola on \( x = h \)).

The segment revolves about the ordinate; prove that the volume of the spindle generated is \( \frac{2}{5} \pi \) (volume of the enveloping cylinder generated by the tangent at the vertex).

[84]

GROUP III

INTEGRAL CALCULUS

1924. 2\(\frac{1}{2}\) Hours

1. Find the differential coefficients with respect to \( x \) of \( \sin x \) and \( \sin^{-1} x \).

Find the differential coefficients with respect to \( x \) of:

\[
\begin{align*}
(1) \quad &\log(x + \sqrt{1 + x^2}); \\
(2) \quad &\frac{\sqrt{x^2 + a^2} - \sqrt{x^2 - a^2}}{\sqrt{x^2 + a^2} + \sqrt{x^2 - a^2}}; \\
(3) \quad &\tan^{-1} \frac{3x^2 - x}{1 - 3x^4}.
\end{align*}
\]

2. If \( x \) and \( y \) are connected by the relation \( \Phi(x, y) = 0 \), where \( \Phi(x, y) \) is the sum of a number of terms of the form \( Ax^n y^m \), and \( n \) and \( m \) being zero or positive integers, show that

\[
\frac{dy}{dx} = -\frac{\partial \Phi/\partial y}{\partial \Phi/\partial x}.
\]

If \( (x+y)^n (x-y)^m = (x-y)^n y^m \), show that \( \frac{dy}{dx} = \frac{y}{x} \).

3. (1) Find the minimum value of \( a^2 \sec^2 \theta + b^2 \cos^2 \theta \).

(2) Find the values of \( \theta \) that give the maximum and minimum values of \( \tan \theta + 3 \log \cos \theta + \theta \),

where \( \theta \) lies between 0 and \( \frac{\pi}{2} \).

4. Find the equation of the tangent at any point of the semi-cubical parabola \( ay^2 = x^3 \).

The tangent at any point \( Q \) of the semi-cubical parabola \( ay^2 = x^3 \) meets the curve again in \( R \), and \( QR \) is divided at \( P \) in the ratio \( k : 1 \); show that the locus of \( P \) is also a semi-cubical parabola, and find the value of \( k \) when the locus becomes the axis of \( x \).

5. Evaluate the integrals:

\[
\begin{align*}
(1) \quad &\int \frac{x}{(x-1)(x-2)} \, dx; \\
(2) \quad &\int \frac{dx}{\sqrt{x-x^2}}; \\
(3) \quad &\int_0^\pi \tan^2 \theta \, d\theta; \\
(4) \quad &\int_0^{\pi/2} x^2 \sqrt{1-x^2} \, dx.
\end{align*}
\]

6. Show that \( \int_0^{\pi/2} \sin^2 \theta \, d\theta = \int_0^{\pi/2} \cos^2 \theta \, d\theta \), and find the value of each integral.

[85]
Show that \( \int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta \ d\theta = \int_0^{\frac{\pi}{2}} \cos^m \theta \sin^n \theta \ d\theta \), and hence, or otherwise, find the value of
\[
\int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^3 \theta \ d\theta.
\]

7. Find the area of the parabola \( r = 2a(1 + \cos \theta) \) between the radii \( \theta = \frac{\pi}{4} \) and \( \theta = \frac{3\pi}{4} \).

Find the distance of the centre of gravity of the same area from the axis.

8. A cycloid, given by the equations
\[ x = a(1 - \cos \theta), \quad y = a(\theta - \sin \theta), \]
revolves about its base \( x = 2a \); find the area of the surface, and the volume of the solid of revolution so formed.

A. A point moves along a straight line and, at the end of \( t \) seconds, its distance (feet) from a fixed point in the line is given by the equation
\[ s = \theta - \theta^2 + 24t - 18.\]

Show that the velocity vanishes for two values of \( t \) and the acceleration for one value of \( t \).

Find also the value of the velocity when the acceleration vanishes, and the values of the acceleration when the velocity vanishes.

B. Find the equation of the tangent to the curve
\[ 2y = 2x^3 - 4x^2 - 7x + 3 \]
at the point in which the curve cuts the axis of \( y \).

Find the points on this curve at which the tangent is parallel to the line \( x = 2y \).

C. Evaluate
\[
\int \frac{x^2 + 1}{x} \, dx, \quad \int \cos ax \, dx, \quad \text{and} \quad \int \sin 3x \cos x \, dx.
\]

Find the area included between the curve \( y = \sin x + \sin 3x \) and the axis of \( x \) from the origin to the first point in which the curve meets the axis on the positive side.

D. From any point on the curve \( ay^2 = x^3 \), perpendiculars are drawn to the axes; show that the curve divides the rectangle so formed into parts the areas of which are as \( 3 : 2 \).

Find the abscissa of the point if the volumes of revolution of each part about the corresponding axis are equal.

[86]
6. Evaluate the integrals

(1) \( \int \frac{dx}{\sqrt{x^2 - 4x + 3}} \);

(2) \( \int \frac{dx}{x(x^2 - 4x + 3)} \);

(3) \( \int \sin^3 x \, dx \).

By using the substitution \( x + 1 = 1/y \), or otherwise, prove that

\[ \int \frac{dx}{(x + 1)\sqrt{x^2 - 4x + 3}} = \frac{x}{2} \left( \sqrt{x^2 - 4x + 3} \right) + C. \]

7. A uniform wire is bent into the shape of that part of the curve 8ay^2 = x^8 which lies between the points (2a, -a) and (8a, 8a). Find the length of the wire and the area contained between the wire and the chord joining its ends.

8. An arc of a circle of radius \( c \) subtending an angle \( 2 \alpha \) at the centre is rotated about its chord. Prove that the area of the surface of revolution so formed is \( 4\pi c^2 (\sin \alpha - \alpha \cos \alpha) \), and that its volume is \( \frac{3}{2} \pi c^3 (2\sin \alpha + \sin \alpha \cos \alpha - 3\alpha \cos \alpha) \).

A particle moves along a straight line in such a way that at the end of \( t \) seconds its distance (in feet) from a fixed point on the line is given by the equation \( s = 1 + 2t \cos 3t \); find its acceleration at any instant.

Prove that the velocity of the particle never changes in direction, but that its acceleration vanishes whenever its velocity is equal to 13 feet per second or to 1 foot per second.

B. Find the slope of the tangent at any point of the curve

\[ y = x^4 + 4x^3 - 18x^2 - 44x + 57, \]

and prove that the tangent at \((-1, 80)\) is parallel to the axis of \( x \).

Prove that the only inflexions of the curve are at two of the points where it crosses the axis of \( x \).

C. Evaluate

(1) \( \int \frac{x \, dx}{\sqrt{x^2 + 1}} \);

(2) \( \int \frac{x \, dx}{\sqrt{x^2 + 1}} \);

(3) \( \int_0^\pi e^\sin x \, dx \).

Find the area between the parabola \( y^2 = 4ax \) and the straight line \( x = 2a \).

D. A sphere of radius \( a \) is cut into two portions by a plane at distance \( c \) from its centre. Prove that the volumes of the two portions are \( \frac{\pi}{3} (a - c)^2 (2a + c) \) and \( \frac{\pi}{3} (a + c)^2 (2a - c) \).
The tangent at a point \( t \) on one span of a cycloid is the normal to the cycloid at the point where it meets the cycloid at the next span on the right. Prove that \( t \) satisfies the equation \( \cos^2 t = \frac{\pi}{2} \).

What is the corresponding equation when the words ‘next span’ are changed to ‘next span but one’?

6. Integrate
\[
\int \frac{dx}{(x-1)(x-2)} = \int \frac{dx}{(x-1)^2(x-2)} = \int \cos^2 x \, dx, \quad \int \frac{dx}{\sqrt{1+x^2}}.
\]
Deduce that \( \int_0^1 \frac{dx}{\sqrt{1+x^2}} = 0.8813 \).

7. Interpret the expressions
\[
\int y \, dx, \quad \frac{1}{2} \int r^2 \, d\theta,
\]
in terms of areas of curves.

Prove that the area common to the parabolas
\[
y^2 = 2ax, \quad y^2 = 4ax
\]
is \( 2a^2/3 \).

8. The circle \( x^2 + y^2 = 2ay \) is rotated about the axis of \( x \). Prove that the volume generated is \( 2\pi a^3 \).

9. Find (1) the coordinates of the centre of gravity, (2) the moments of inertia about the axes of a uniform lamina of mass \( M \) whose edges are the axis of \( x \), the line \( x = a \) and the arc of the parabola \( y^2 = 4ax \) joining \((0, 0)\) to \((a, 2a)\).

1927. 2½ Hours

1. Prove that the 2nth differential coefficients of
\[
\frac{1}{1-x^2} \quad \text{and} \quad e^{\alpha x} \sin \alpha x
\]
are respectively
\[
\frac{(2n)!}{2} \left[ \frac{1}{1-x^{2n+1}} + \frac{1}{1+x^{2n+1}} \right]
\]
and
\[
2^n a^{2n} \alpha^n \sin (\alpha x + \frac{1}{2} n\pi)
\]
2. If \( y = \sqrt{\frac{1-x^2}{1+x^2}} \), prove that \( \frac{dy}{dx} = \frac{1-y^4}{1+y^4} \).

By differentiation, or otherwise, prove that
\[
\int \frac{dx}{\sqrt{\cos 2x - \cos 4x}} = -\frac{1}{\sqrt{6}} \sinh^{-1} \frac{\sqrt{3}x}{\sin^2 x}
\]
[90]

Group III

3. Prove that the only maximum value of the function
\[
\log \frac{x}{a}
\]
ocurs when \( x = e \).

If \( a \) is a given positive number less than \( 1/e \), prove that the equation \( \frac{\log x}{x} = a \) has two and only two real solutions.

If \( a \) is chosen so that one of these solutions is \( x = 1.5 \), verify that the other solution lies between 7.40 and 7.41.

4. Obtain the expansion of \( \sin (a + x) \) in powers of \( x \) and \( x^2 \), as far as the term in \( x^2 \).

Given that \( \sin 45^\circ = 1.1442136 \), obtain the value of \( \sin 45^\circ \).

5. Write down the values of
\[
\int \frac{x \, dx}{1+x^2}, \quad \int \frac{x \, dx}{\sqrt{1+x^2}}, \quad \int \frac{1+x^2 \, dx}{x}
\]
Prove that
\[
\frac{d}{dx} \tan^{-1} x = (a-1) (\tan^{-2} x + \tan^{-1} x),
\]
and deduce, or prove otherwise, that
\[
\int_{\tan^{-1} \sqrt{x}}^{\sqrt{x}} \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{76}{105}
\]

6. Prove that the point with coordinates
\[
\left( \frac{2a \alpha^2}{1+\alpha^2}, \frac{2a \alpha^2}{1+\alpha^2}, \frac{1+\alpha^2}{1+\alpha^2} \right)
\]
always lies on the curve whose equation is \( y^2 (2a - x) = x^2 \).

Prove that the tangent and normal to this curve at the point with parameter \( t \) have for their equations
\[
x (3t^2 - 2y) - 2y = 2a \alpha^2, \quad 2a \alpha^2 y (3t^2 - 2y) - 2a^2 (2t^2 - 2) \]
Find the parameter of the point at which this tangent meets the curve again.

7. A triangular lamina of variable density has two of its sides along the coordinate axes, and its third side is \( 8x + 12y = 60 \). The density of the lamina at any point is proportional to the distance of the point from the axis of \( y \). Prove that the coordinates of the centre of gravity of the lamina are \((6, 5/4)\).

8. Show that the volume generated is half the volume of a sphere of diameter \( a \).
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If the volume generated is composed of material of uniform density, prove that its moment of inertia about the axis of $x$ is $\frac{1}{2}$ of the moment of inertia about a diameter of a sphere of diameter $a$, composed of the same material.

9. A particle of mass $M$ starts at time $t = 0$ to move along the axis of $x$ from the origin, under the action of a variable force of such magnitude that the distance travelled in time $t$ is $2a(2t - \sin 2t)$, where $a$ is a constant. Prove that—

(1) the magnitude of the force is $8Ma\sin 2t$;

(2) the work done by the force from starting until time $t$ is $32Ma^3\sin^3 t$;

(3) the power exerted by the force at time $t$ is $128Ma^3\sin^2 t \cos t$.

GROUP IV (PAPER 2) AND SUBSIDIARY SUBJECT (16 b)

ANALYTICAL GEOMETRY AND DIFFERENTIAL AND INTEGRAL CALCULUS

1926. 2½ HOURS

No credit will be given for attempts to solve the questions on Analytical Geometry by measurements in carelessly drawn figures.

1. If the equation of a straight line is $\frac{x}{a} + \frac{y}{b} = 1$, interpret geometrically $a$ and $b$. Do not restrict your interpretation to the case in which $a$ and $b$ are both positive.

The two straight lines drawn through the point $(2, 3)$ inclined at angles of $45^\circ$ to the axes of coordinates meet the axis of $x$ at the points $A$ and $A'$ and the axis of $y$ at the points $B$ and $B'$ respectively. Find the equations of the straight lines $AB'$ and $A'B$ and the coordinates of their point of intersection.

2. Find the coordinates of the point which divides the distance between the points $(x_1, y_1)$ and $(x_2, y_2)$ internally in the ratio $l:m$. (Full credit will not be given for a proof which assumes that both points lie in the first quadrant.)

A straight line drawn through the point $(1, 1)$ meets the axes of coordinates at the points $P$ and $Q$. Show that the equations of the loe of the points of trisection of $PQ$ are

$3zx = x + 2y$ and $3zy = 2x + y$.

3. Find the angles between the straight lines

$y = mx + c$, $y = mx' + c'$.

Find the equations of the perpendiculars of the triangle, whose

vertices are the points $(2, 1)$, $(1, 3)$, $(-1, -1)$, and also find the coordinates of its orthocentre.

4. Show that the equation of any circle can be written in the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

Find the length of the radius and the coordinates of the centre of the circle which passes through the origin and the points $(3, 4)$, $(-4, 3)$.

5. Prove that the equation of the circle on the line joining the points $(x_1, y_1)$, $(x_2, y_2)$ as diameter is

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$.

Prove the accuracy of the following geometrical method of solving any quadratic equation, $x^2 + bx + c = 0$. Take any two numbers $p$ and $q$, whose product is equal to $c$. Plot the points $A$ and $B$, whose coordinates are $(0, p)$ and $(2a, q)$. Draw the circle on $AB$ as diameter. If the quadratic has real roots, their values are those of the $x$ coordinates of the points in which the circle meets the axis of $x$. If the quadratic has imaginary roots, their values are $a \pm t\sqrt{-1}$, where $t$ is the length of either tangent drawn to the circle from the point $(a, 0)$.

6. Find the first differential coefficient of $x^2 \sin x$ and the second differential coefficient of $\sqrt{a^2 + x^2}$.

Write down the values of

$$\int_1^{x} \frac{dx}{1+x^2}, \int_0^\infty \frac{1}{x^2 + 1} dx,$$

and evaluate

$$\int_0^{\infty} \frac{x}{(a^2 - x^2)} dx.$$

6 Show that, if $y$ is a function of $x$, a value of $x$, which makes $\frac{dy}{dx}$ vanish, in general makes $y$ take a maximum or minimum value.

If $x$ is restricted to lie between $a$ and $b$, will the greatest and least values assumed by $y$, as $x$ increases from $a$ to $b$, necessarily correspond to values of $x$ which make $\frac{dy}{dx}$ vanish, and how are they to be found if there is no such value of $x$ between $a$ and $b$?

The height of a cone is $h$ and the radius of its base is $a$. A cylinder is inscribed in the cone. Find an expression for the total surface of such a cylinder, that is the sum of the area of its curved surface and the areas of its two flat ends, in terms of $h$, $a$, and $x$, the distance of the top of the cylinder from the vertex of the cone, and discuss the greatest and least values of this total surface (b) when $h = 3a$, (ii) when $h = 3a/2$. [93]
8. The hypotenuse and one side of a right-angled triangle are measured and found to be 15 inches and 8 inches respectively. Each measurement is liable to an error of 1 per cent. in either direction. Prove that the length of the remaining side, as calculated from these measurements, is liable to an error of as much as very nearly 1.3 per cent.

9. Use the calculus to find the equations of the tangent and normal to the curve \( ay^2 = x^2 \) at the point \((4a, 8a)\).

10. Prove by integration that

- (1) the volume of a segment of a sphere of radius \( R \) is \( \frac{1}{2} \pi h^2 (3R - h) \), where \( h \) is the height of the segment;
- (2) the moment of inertia of a uniform thin circular wire, of radius \( a \) and mass \( M \), about any diameter is \( \frac{1}{2} M a^2 \).

1927. 2\(\frac{1}{2}\) Hours

1. Find the angles between the straight lines

\[ y = mx + c, \quad y = mx' + c' \]

The coordinates of the vertices \( A, B, C \) of a triangle are \((1, 1), (4, 3), (-2, 2)\) respectively. Find the angle \( BAC \) to the nearest minute.

2. Obtain an expression for the area of a triangle in terms of the coordinates of its angular points.

The coordinates of the vertices of a triangle are \((3, -1), (-1, 4), \) and \((-2, -2)\); the triangle is cut into four pieces by the coordinate axes. Find the area of each piece.

3. Prove that the perpendicular distance of the point \((x', y')\) from the line \( ax + by + c = 0 \) is

\[ \pm \frac{ax' + by' + c}{\sqrt{a^2 + b^2}} \]

How do you ascertain whether two given points are on the same side, or on opposite sides, of the line ?

Without drawing a figure, determine whether the point \((7, 11)\) is inside or outside the triangle formed by the lines whose equations are

\[ 4x + 3y = 60, \quad y = x + 8, \quad y = 2x - 20. \]

4. Find the radius and the coordinates of the centre of the circle whose equation is

\[ x^2 + y^2 + 2gx + 2fy + c = 0. \]

Prove that the square of the tangent from \((x_1, y_1)\) to this circle is equal to

\[ x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \]

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5. A uniform cubical block, of side $2a$ and weight $W$, rests on a rough horizontal plane. A man pushes it at right angles to one of its faces. Prove that the least push that will move it is the smaller of the two $\frac{\mu W}{h}$, $\mu h$, where $h$ is the height of the man's hands above the ground, and $\mu$ is the coefficient of friction between the block and the ground.

6. The diagram represents a framework of nine smoothly jointed light rods, supported by vertical reactions at $A$ and $D$, and loaded with 60 cwt. at $F$ and 90 cwt. at $E$; the angles in the figure are all 45° or 90°. Determine graphically, or otherwise, the stresses in the rods, marking with a double line those which are under thrust.

7. Given the velocities of the points $A$ and $B$, find the velocity of $A$ relative to $B$.

8. A destroyer, steaming N. 30° E. at 30 knots, observes at noon a steamer, which is steaming due N. at 12 knots, and overtakes the steamer at 12.45 p.m. Find the distance and bearing of the steamer from the destroyer at noon.

9. The angle of projection of a shot, fired with given velocity, that its range on the horizontal plane through the point of projection may be the greatest possible.

The greatest range of a gun is 16 miles: find the muzzle velocity of the shot, and prove that, when the shot has travelled 4 miles horizontally, it has risen 3 miles.

Are these results true in practice?
At what rate (measured in H.P.) is the engine working (1) just before, (2) just after, beginning the ascent?

10. State and explain the third law of motion.

A shot of mass \( m \) is fired from a gun of mass \( M \), placed on a smooth horizontal plane and inclined at an angle \( \alpha \) to the horizontal. If \( v \) is the velocity of the gun's recoil at the instant when the shot leaves it, prove that the horizontal component of the impulsive pressure on the shot is \( Mv \cos \alpha \) and that the component at right angles to the gun's length is \( mv \sin \alpha \). (It is assumed in each case that the impulsive pressure is resolved into two components at right angles.)

Prove that the initial direction of the shot's motion is inclined at \( \tan^{-1} \left( \frac{v \sin \alpha}{M \cos \alpha} \right) \) to the horizontal.

11. State and explain the principle of conservation of energy.

A fly-wheel, 2 ft. in diameter and weighing 20 lb., is keyed on to a shaft, of 6 in. diameter, which can turn freely in smooth horizontal bearings; a long fine chain is attached to and wrapped round the axle and carries at its other end a mass of 16 lb.

The wheel is turned till it acquires a speed of 480 revolutions per minute, and is then left running. Prove that it will come to rest after about 33 more revolutions. (Neglect the masses of the axe and chain, and assume the mass of the wheel to be concentrated in and uniformly distributed round its rim.)

12. Prove that the acceleration of a point describing a circle of radius \( r \) with uniform speed \( v \) is \( v^2/r \), and is directed towards the centre of the circle.

A bicyclist is describing a curve of 60 ft. radius at a speed of 10 miles per hour; find the inclination to the vertical of the plane of the bicycle.

What is the least coefficient of friction between the bicycle and the road, that the bicycle may not side-slip?

[Assume the rider and his machine to be in one plane.]

A. Find the magnitude and line of action of the resultant of two like parallel forces.

A light horizontal rod, 12 in. long, is supported by two vertical props, each 3 in. from an end of the rod, and is loaded with 16 lb. at each end. What weights hung from the ends will produce in one prop a pressure double and in the other prop a pressure half of that produced by the 16 lb. weights?

B. A man wishes to pull a smooth lawn-roller of diameter 20 in. and weight 200 lb. over a kerb-stone 4 in. high. Find the direction in which he should pull, in any position of the roller, so as to raise the roller with the least effort; and show that the greatest force he need exert is 100 lb. weight.

C. A uniform bar \( AB \), 3 ft. long and weighing 4 lb., has a cord 5 ft. long attached to its ends. The cord passes through a smooth ring \( O \) fixed in a smooth vertical wall; and the rod is placed in a vertical plane perpendicular to the wall with the end \( A \) against the wall and vertically below \( O \). Prove that the rod will be in equilibrium if \( OA \) is 2 ft.; and show that the tension of the string is 3 lb. weight.

D. Prove the formula \( s = ut + \frac{1}{2}at^2 \) for uniformly accelerated or retarded motion.

A point, moving in a straight line, describes 16 ft. in the 2nd second of its motion, 25 ft. in the 5th second, 52 ft. in the 11th second. Prove that these distances are consistent with the supposition that the motion of the point is uniformly accelerated; also find the whole distance described in 10 seconds from the beginning of motion.

E. Define (1) a foot-pound, (2) kinetic energy.

A force equal to the weight of 5 lb. acts on a mass of 30 lb., originally at rest, for 10 seconds. Find, in foot-pounds, the distance travelled by the mass, and in foot-pounds the kinetic energy generated in it.

F. A ball weighing 4 oz. and moving at the rate of 20 ft. per sec. is struck by a bat and rebounds with a velocity of 44 ft. per sec. Find in foot-pounds the work done on the ball, and in lb., the average pressure on the bat, assuming the bat and the ball to be in contact for 0.1 sec.

1919. 2½ Hours

1. State the theorems known as the 'Parallelogram of Forces' and the 'Triangle of Forces'; and prove that they are equivalent.

The resultant of two forces \( P \), \( Q \) is equal to \( P \) in magnitude; and that of two forces \( 2P \), \( Q \) (acting in the same directions as before) is also equal to \( P \). Find the magnitude of \( Q \) and prove that the direction of \( Q \) makes an angle of 150° with that of \( P \).

2. Explain how to obtain the resultant of a number of forces acting at a point in one plane, by resolving the forces in two perpendicular directions.

Four horizontal wires are attached to a telephone post and exert the following tensions on it: 20 lb. north, 30 lb. east, 40 lb. south-west, and 50 lb. south-east. Calculate the resultant pull on the post and find its direction.

3. Prove that if three forces acting in the same plane are in equilibrium their lines of action must either meet in a point or be parallel.
A pole rests with its lower end $P$ in a socket and is supported by a rope joining a point $Q$ of the pole to a point $R$ vertically above the socket. Prove that if the vertical through the centre of gravity of the pole cuts $QR$ in $S$, the triangle $PQS$ forms a triangle of forces for the weight, the tension in the rope and the reaction of the socket.

4. Explain how to find the position of the centre of gravity of a composite body, when the masses and centres of gravity of the parts of the body are known.

A pile in the shape of a flight of stairs is formed of $n$ equal uniform cubical blocks, one edge of each being displaced a small distance $c$ horizontally with reference to the block below. Determine the position of the centre of gravity of the pile and deduce that the pile must topple over if $(n-1)c > a$.

5. The accompanying diagram represents a framework of pin-jointed light rods resting on two supports at $A$, $D$ and carrying loads of 50 tons at $B$ and 20 tons at $C$. Calculate the pressures on the supports and draw a stress-diagram to show the stresses in all the members. Mark those under thrust with a double line.

[All the angles in the diagram are either 45° or 90°.]

6. Draw a velocity-time graph for the case of a body moving in a straight line, at first with uniform acceleration, secondly with uniform velocity and finally with uniform retardation. A train of 180 tons starts from rest with an engine-pull of 3 tons and makes a run of one mile from one station to rest at the next. At the instant of maximum speed, the steam is shut off and the brakes are applied, producing an effective coefficient of friction 0.1. Prove that the time occupied is about 172 seconds and that the maximum speed is about 42 miles per hour.

(Frictional resistances, other than those due to the brakes, are neglected.)

7. If a particle is projected with a velocity whose horizontal and vertical components are $u$, $v$ respectively, find expressions for its coordinates at time $t$ referred to horizontal and vertical axes through $A$, the point of projection.

The particle is viewed from a point $B$ on its path, against the vertical through $A$: show that the particle appears to rise along this vertical at a uniform rate 1/2$gt_0$, where $t_0$ is the time taken by the particle to reach $B$.

8. A shot weighing 18 lb. is fired horizontally from a gun weighing 9 cwt. If the muzzle velocity of the shot is 1680 feet per second calculate that of the gun.

Calculate the total kinetic energy produced (in the shot and gun) in foot-pounds; and if the distance travelled along the bore of the gun is 7 feet, prove that the average force applied to the shot is a little over 51 tons. How far will the gun have moved when the shot leaves the muzzle?

9. Prove that if a particle describes a circle of radius $r$ with uniform angular velocity $\omega$, its acceleration is equal to $ur^2$ and is directed towards the centre of the circle.

A particle of mass 4 lb. is whirled round at the end of a string 10 inches long, so as to describe a horizontal circle, making 60 revolutions per minute; calculate the tension in the string (in pounds weight) and prove that the fixed end of the string is a little less than 10 inches above the centre of the circle.

10. A wheel has a diameter of 2 feet and a mass of 50 pounds which may be regarded as distributed uniformly round the rim. Calculate the number of foot-pounds of energy stored in the wheel when it is making 600 revolutions per minute.

If the wheel is to be stopped in 50 seconds by a brake pressing on the rim, calculate the pressure required, assuming that the coefficient of friction is 0.1 between the brake-block and the rim.

A. Find the resultant of two parallel forces $P$, $Q$ acting in opposite directions. Prove that the sum of their moments about any point is equal to the moment of the resultant about the same point.

B. A pulley carries a weight of 30 lb. and can slide freely up and down a smooth vertical groove. It is held up by a string passing round the pulley so that the two parts of the string make angles of 30° and 60° with the horizontal; show that the tension in the string is slightly under 22 lb.

C. Draw a velocity-time graph for a particle falling freely from rest, and deduce the formula $s = \frac{1}{2}gt^2$.

If a stone falls past a window, 8 feet high, in half a second, find the height from which the stone started.

D. State the theorem known as the triangle of velocities.

Find the true course and the true speed of a steamer travelling through the water at 12 knots and steering due north by the compass through a current of 3 knots which sets south-east. Find also the direction in which the steamer should steer in order to make its true course due north, and the true speed on this course.
1920

STATICS AND DYNAMICS

GROUP III

1920. 2 1/2 Hours

In numerical calculations take \( g = 32 \) foot-second units or 981 centi-
metre second units.

1. Forces \( P \) and \( Q \) act at a point at an inclination \( \alpha \); find the
magnitude of their resultant and its inclination to the direction of \( P \).

Find, graphically or otherwise, the magnitude and the direction
of the resultant of the following forces acting at a point: 20 lb. E.,
42 lb. N.W., 60 lb. W. 30° S., 15 lb. S.

2. Show that the algebraical sum of the moments about any point
of two forces whose lines of action are concurrent is equal to the
moment of their resultant about that point.

A square piece of thin wood \( ABCD \), 8 in. by 8 in., resting on
a smooth table is acted on by forces of 3 lb. wt. along \( AB \), 2 lb. wt.
along \( BC \), 5 lb. wt. along \( DC \), and 4 lb. wt. along \( DA \); but the square
does not move as it is pinned to the table by a smooth pin through a
down the middle of \( AB \) and \( DC \). Find the position of the pin and the magnitude of the pressure exerted on it
by the board.

3. Find the distance from a straight line of the centre of gravity
of masses \( m, m', m'', \ldots \) which lie in a plane through the line, whose
respective distances from that line are \( x, x', x'' \).

\( ABCD \) is a vertical face of a rectangular block, the horizontal
face which has \( BC \) for an edge being in contact with the ground; \( BC \)
is 40 in., \( CD \) is 25 in.; \( E \) is a point in \( BC \) 15 in. from \( B \). Find the
position of a point \( F \) in \( CD \), such that the block will lie on the point
of toppling over when a prism is cut from it by cutting it along \( EF \)
at right angles to the face \( ABCD \).

4. The diagram represents a framework of pin-jointed light rods, attached
to a wall at \( A \) and \( E \), and loaded with 10 cwt. at \( D \) and 20 cwt. at \( C \).

Draw a force diagram to represent the stresses in all the rods and the
reaction of the pin at \( A \), and mark the
rods under thrust with a double line.

Find the values of the stresses, assuming
that there is no stress in the rod \( AE \). [The angles are either 90° or
45°.]

5. A car with four equal wheels has a wheel-base \( a \) (i.e. \( a \) is the
distance between the points of contact with the ground of the front
and the hind wheels); its centre of gravity is equidistant from all

four wheels and at a height \( h \) above the ground. Find the pressures
on the wheels when the car rests with the hind wheels locked on
a slope of inclination \( \alpha \) facing (1) uphill, (2) downhill.

Show that, if the hill is slippery, the car may be able to rest in
position (1) but not in position (2).

If \( a = 10 \) ft. and \( h = 2 \) ft., find the least coefficient of friction
between wheels and road that the car may be able to rest in either
position on a slope of 10°.

6. A bullet is fired from a point \( O \) with a velocity whose horizontal
and vertical components are \( u \) and \( v \) respectively; find the direction
in which it is moving after a time \( t \).

If \( u \) is 96 f.p.s., \( v = -288 \) f.p.s., prove that at two points the
direction of the bullet's motion is at right angles to the line joining
the bullet to \( O \) and the positions of these points.

7. Masses of 100 grams and 60 grams are attached to the end of
a fine string which passes over a smooth fixed pulley. Find the
accelerations of the masses and prove that the tension of the string
is equal to the weight of 75 grams.

The pulley, whose mass is 50 grams, is now detached from its
fastening and attached by means of another fine string to a mass of
100 grams, which lies on a smooth table over whose edge the string
passes. Prove that the pulley moves as if the original weights were
removed and its own mass were increased by 150 grams.

8. The tension (or thrust) of an elastic spring varies as its exten-
sion (or compression). Deduce by the aid of a force-space graph, or
otherwise, that the work done in stretching (or compressing) such
a spring from its natural length is measured by half the product of
the final tension (or thrust) into the extension (or compression).

A weight of 4 kilos. will compress a spring through 2.5 cm. A
model truck, weighing 230 grams, runs into the spring, used as a
buffer, with a velocity of 90 cm. per sec. How far will the spring be
compressed before the truck is brought to rest?

9. Prove that the acceleration of a point, which is describing a
circle of radius \( r \) with uniform speed \( v \), is \( r^2/v^2 \) and is directed towards
the centre of the circle.

A ball of mass 6 oz. is attached to one end of a thread, 27 in.
long, which can just support a weight of 18 oz. without breaking;
the other end of the thread is fixed. The ball is held with the string
untwisted and at a height of 13 1/2 in. above the level of the fixed point,
and is started at right angles to the string with a velocity of 6 feet per
second. Show that the string will break when the ball is directly
below the starting-point.

10. A wheel, with a diameter of 3 feet and a mass of 70 lb., which
may be regarded as distributed uniformly round the rim, is rolling along a horizontal road at a speed of 10 miles per hour. Calculate the number of foot-pounds of energy stored in the wheel.

If it comes to a hill rising 1 in 5 (i.e. 1 foot rise to 5 feet) along the road, how far will it go before it stops?

[Note.—In a rolling motion no work is done against friction.]

A. What is force and how is it measured? If a kilogram is suspended by a string, express the tension of the string in dynes.

A bullet weighing 30 grams is fired into a fixed block of wood with a velocity of 294 metres per second and is brought to rest in \(\frac{1}{30}\) sec. Find in dynes, and in grams weight, the resistance exerted by the wood, supposing it to be uniform.

B. Prove the formula, \(v^2 = u^2 + 2as\), for uniformly accelerated motion.

A ball is thrown vertically upwards with a velocity of 56 ft. per sec.; find its height when it is moving at the rate of 40 ft. per sec., and find the time between the instants at which it is at this height.

C. Show how to find the resultant of forces acting at a point by resolving in two directions at right angles.

A uniform rod, 3 ft. long, is suspended by a light string of length 5 ft. passing over a smooth peg, and rests horizontally, the string being attached at the ends of the rod. If the rod weighs 7 lb., find the tension of the string in pounds weight.

D. Define the moment of a force about a point and explain its physical meaning.

Eight feet of a plank, 24 ft. long and weighing 200 lb., project over the side of a quay. What weight must be placed on the end of the plank that a man weighing 150 lb. may be able to walk to the other end without the plank tipping over?

1921. 2 ½ Hours

1. A body which weighs 20 lb. is suspended from a fixed point by a string, and is in equilibrium with the string inclined at 20° to the vertical under the action of a force in a direction making an angle of 60° with the downward vertical. Find graphically, or otherwise, the magnitude of the force and the tension of the string.

Assuming that the force remains constant in magnitude but varies in direction, find the greatest possible inclination of the string to the vertical.

2. Define the moment of a force about a point, and prove that the sum of the moments of two like parallel forces about any point in their plane is equal to the moment of their resultant about the point.

A horizontal beam \(ABC\) rests on two supports at \(B, C\), where

\[ AB = BC = CD. \]

It is found that the beam will just tilt when a weight of \(p\) lb. is hung from \(A\) or when a weight of \(q\) lb. is hung from \(D\). Find the weight of the beam, and prove that its centre of gravity divides \(AD\) in the ratio \(2p + q : p + 2q\).

3. A uniform wire \(ABCD\) is bent at right angles at \(B\) and \(C\) in such a way that \(BA\) and \(CD\) are in the same sense, and its lengths of the parts \(AB, BC, CD\) are 6, 4, 2 inches respectively. Find the distances of the centre of gravity of the wire from \(AB\) and \(BC\).

Show that the wire can be suspended with each part equally inclined to the vertical by a string attached at a point \(P\), and give the length \(BP\).

4. State the laws of statical friction.

A wheel situated in a vertical plane is free to turn about its centre \(C\). A uniform rod \(AB\), weight \(W\), is smoothly hinged at \(A\), which is at the same level as \(C\), and rests in contact with the wheel at \(D\). Prove that to turn the wheel a couple of moment greater than \(\mu W AB CD/2\) is required, \(\mu\) being the coefficient of friction between the rod and the wheel.

Prove also that, when the wheel rotates in the direction shown, the reaction at \(A\) will be vertical, if the inclination of the rod to the horizontal is \(\tan^{-1}\mu\).

5. In the symmetrical framework of pin-jointed rods shown, the triangles \(ABC, ADC\) are equilateral and the angle \(AEC\) is 30°. The framework carries loads of 20 lb. at each of the joints \(A, B, C, E\) and is supported at \(D\). Find the stresses in the rods \(AE, AD, AB, AC\).
6. Find the horizontal range of a projectile in terms of its initial velocity.

A shell is observed to explode at the level of the gun from which it is fired after an interval of 10 seconds; and the sound of the explosion reaches the gun after a further interval of 3 seconds. Find the elevation of the gun and the speed with which the shell is fired. [Assume the velocity of sound to be 1,100 ft. per sec.]

7. A ball, mass 4 oz., is released when at rest from a position 6 ft. above the floor of a lift, which is descending with a uniform acceleration of 4 ft. per second. Prove that, if the ball rises to the same position after striking the floor, the impulse on the floor at impact is nearly 9\frac{1}{2} feet-pound-second units, and find the times of descent and ascent of the ball relative to the lift.

8. Find the acceleration of a particle which describes a circle with uniform speed.

A mass of 8 oz. is attached to two fixed points $A$, $B$, which are 18 inches apart and in the same horizontal line, by means of two strings each 15 inches in length. The mass is held with the strings taut in the horizontal plane through $A$ and $B$, and is then released. Find the tension of a string, when the mass is in the vertical plane through $AB$.

9. Find the horse-power required to pump 500 gallons of water per minute from a depth of 100 ft., the water being delivered through a circular pipe 3 inches in diameter. [Assume that 1 cu. ft. of water is 6\frac{1}{2} gallons and that 1 gallon of water weighs 10 lb., and neglect friction.]

10. State the principle of the conservation of linear momentum.

Three small bodies of masses 4, 5, 6 ounces respectively lie in order in a straight line on a large smooth table, the distance between consecutive bodies being 6 inches. Two slack strings, each 2 ft. in length, connect the first with the second and the second with the third. The third body is projected with a speed of 15 ft. per second directly away from the other two. Find the time which elapses before the first begins to move and the speed with which it starts. Find also the loss in energy.

A. Explain how the acceleration and the space covered may be obtained from the velocity-time diagram of a particle moving in a straight line.

The velocity-time diagram consists of two straight lines $AB$, $BC$ where the coordinates of $A$, $B$, $C$ are $(0, 10)$, $(10, 10)$, $(20, 25)$; the first coordinate in each case being the time in seconds and the second coordinate the velocity in feet per second. Describe the motion of the particle, and find the total distance covered.

B. State the theorem known as the ‘Triangle of Velocities’.

1922. 2½ Hours

1. Forces represented by $p$.OP and $q$.OQ act along OP and OQ respectively. Show that their resultant is represented by $(p+q)OG$, where $G$ is the mean centre of the points $P$ and $Q$ for multiples $p$ and $q$.

Forces represented by $2BC$, $CA$, $BA$ act along the sides of a triangle $ABC$. Show that their resultant is $6DE$, where $D$ bisects $BC$ and $E$ is a point on $CA$ such that $CE = \frac{1}{2} CA$.

2. Define the term ‘Centre of Parallel Forces’ and the property which justifies the use of the term.

Three equal parallel forces act at the corners of a triangle $ABC$; find their centre (1) when the forces are alike, (2) when two are alike and the third unlike.

3. Five equal weights are attached to a light string which hangs from two points $P$ and $Q$ in the same horizontal. In equilibrium the horizontal projections of the six intervals of the string are all equal to $a$, and the depth below $PQ$ of the lowest weight is $3 \frac{1}{2} a$. Show that the inclinations to the horizontal of the parts of the string are $\tan^{-1}(1/3)$, $\pi/4$, and $\tan^{-1}(5/3)$. Prove that the external angles of the polygon are $\tan^{-1}(1/4)$, $\tan^{-1}(1/5)$, and $\tan^{-1}(3/4)$.

4. The angles in the framework of light rods freely jointed are 30° or 60° or 90° or 120°. It is fixed by a joint at $A$ and tied at $B$; a weight 30 lb. is suspended from $C$ and 10 lb. from $D$. Find the stresses in the rods, and prove that two rods are tied and three are struts.
5. A circular cylinder of weight \( W \), with its axis horizontal, is supported in contact with a rough vertical wall by a string wrapped partly round it and attached to a point of the wall, and making an angle \( \alpha \) with the wall. Show that the coefficient of friction must not be less than \( \cos \alpha \), and that the normal pressure on the wall is \( W \tan \frac{\alpha}{2} \).

6. Prove the formula for motion in a straight line with uniform acceleration \( f, a = u^2 + \frac{1}{2} f t^2 \) and \( v^2 = u^2 + 2ft \).

A train starts from \( A \) with uniform acceleration \( \frac{1}{2} \) ft. per second per second. After 2 minutes the train attains full speed and moves uniformly for 11 minutes. It is then brought to rest at \( B \) by the brakes producing a constant retardation \( 5 \) ft. per second. Find the distance \( AB \).

7. Express the horizontal and vertical components of the velocity of a projectile and its coordinates in terms of the time and the initial components of velocity.

The projectile is to pass through a point whose angular elevation is \( \gamma \), and at that point to impinge perpendicularly on an inclined plane of slope \( \beta \) to the horizontal. Show that the angle of elevation, \( \alpha \), of the gun is given by

\[
\tan \alpha = \cot \beta + 2 \tan \gamma.
\]

8. Two masses 1 oz. and 3 oz. are connected by a fine string which passes over a small pulley at the vertex of an isosceles right-angled wedge, whose base is horizontal. Show that the string slips over the pulley with an acceleration \( 11.3 \) ft. per second approximately.

The wedge is of mass 1 lb. and rests on a rough horizontal table; find the vertical pressure on the table, assuming that the wedge does not slide, and show that the condition that the wedge shall not slide is that the coefficient of friction between it and the table exceeds \( 2/39 \).

9. A particle suspended by a fine string from a fixed point describes a circle uniformly in a horizontal plane. If it makes 3 complete revolutions every 2 seconds, show that the vertical depth below the fixed point is 4-3 inches approximately. (Take \( \pi = 22/7 \).)

10. A particle is projected inside a smooth straight tube of length \( a \), which lies at rest on a smooth horizontal table, and is closed at the ends. The mass of the tube is twice that of the particle; the coefficient of restitution is \( \frac{1}{2} \). Show that between the first and second impacts and between the second and third the distances traversed by the tube are both \( a \), and that five-eighths of the kinetic energy is destroyed by the first two impacts.
A in a smooth wall and is supported in a vertical plane perpendicular to the wall, with $AB$, $CD$ horizontal and $AD$ below $AB$, by the pressure of the wall against $D$. Find the reactions of the hinge at $A$ and of the wall at $D$.

Show that the stresses in the rod at $B$ and $C$ each consist of a force and a couple; find the reactions of the parts $AB$, $DC$ on the part $BC$, and verify the equilibrium of $BC$.

3. A portion, of mass $m$, of a body of mass $M$ is displaced so that its centre of gravity moves through a distance $c'$; prove that the centre of gravity of the whole body is displaced a distance $mc'/M$ in a direction parallel to the displacement $c'$.

A circular plate, centre $O$ and radius $a$, is pierced with four circular holes, each of radius $c$. The centres of these holes are at $A$, $B$, $C$, and $D$, where $ABC$ is an equilateral triangle in a circle, centre $O$ and radius $b$ ($> 2c$), and $D$ is the other end of the diameter of this circle through $A$. Find the centre of gravity of the plate.

4. State the laws of statical friction.

A sphere, of weight $W$ lb., is placed on a rough plane, of inclination $45^\circ$ to the horizon, and is kept in equilibrium, if possible, by a horizontal force $P$ applied at the highest point of the sphere:

1. Show that equilibrium is impossible if $\mu$, the coefficient of friction between the plane and the sphere, is less than $\sqrt{2} - 1$.

2. Show that equilibrium is possible if $\mu$ is equal to or greater than $\sqrt{2} - 1$; find the value of $P$, and determine whether equilibrium is limiting or not when $\mu$ is greater than $\sqrt{2} - 1$.

5. The framework in the figure consists of five light rods, smoothly jointed at $A$, $B$, $C$, $D$, hinged to a fixed support at $A$ and further supported by a vertical reaction at $B$. Find the magnitude and direction of the reaction of the support at $A$ and the stresses in the rods when a weight of 80 lb. is suspended from $D$. (Mark with a double line rods which are under thrust.)
least force parallel to the plane which will just drag the load up the plane.

B. Find the line of action of the resultant of two unlike parallel forces.

ABCD is a quadrilateral in which AB = BC, CD = DA, A and C are right angles, B is 60°, D is 120°. Equal forces $\sqrt{3}P$ act along AD and BC; equal forces $P$ act along CB and BA. Find the magnitude of their resultant, and in which it cuts BD produced.

C. Prove the formula $s = ut + \frac{1}{2}at^2$ for uniformly accelerated motion.

A train approaching a station does two successive quarters of a mile in 16 and 20 seconds respectively. Assuming the retardation to be uniform, draw a graph to show the variation of the velocity with the time during this interval of 36 seconds.

Prove that the train runs 1,761 ft. 10 in. further before stopping, provided that the same constant retardation is maintained.

D. Define—work, power, one horse-power.

Find the H.P. of an engine which can fill a cistern, 200 feet above the level of a river, with 30,000 gallons of water in 24 hours; assuming that a gallon of water weighs 10 lb., and that only two-thirds of the work actually done by the engine is available for raising the water.

1924. 3 HOURS

1. Define the term ‘Centre of Parallel Forces’, and prove the property which justifies the use of the term.

Parallel forces $2P$, $P$, and $-P$ (two like and one unlike) act at the vertices $A$, $B$, and $C$ of a triangle; show that their centre is $G$, where $BDAG$ is a parallelogram, $AD$ being the median of the triangle through $A$.

2. Prove that the moment of the resultant of two forces, which intersect, about any point, is equal to the algebraic sum of the moments of the forces about that point.

ABCD is a square; four forces, whose algebraic magnitudes form an arithmetical progression, act along the sides taken in order. Show that, if their resultant passes through a corner of the square, the progression is a diminishing one, in which, if the common difference is $2P$, the greatest force is $5P$ or $3P$.

3. The total mass, $M$, of a body and the position of its centre of gravity are given; find the position of the centre of gravity of the part left when a portion of mass $m$, whose centre of gravity is known, has been removed.

A rigid framework ABCDE of four equal rods, forming part of

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Group III

1924. 3 HOURS

a regular hexagon, is suspended from $A$. Show that the angle made by $AB$ with the vertical is $\tan^{-1}\left(\frac{a\sqrt{3}}{7}\right)$. Prove that this angle falls short of 45° by less than 20°.

4. A thin hemispherical shell rests with its curved surface in contact with a rough horizontal plane, whose coefficient of friction is $\mu$, and with a rough vertical plane, whose coefficient is $\mu'$. If the shell is on the point of slipping when the plane of its rim makes 30° with the horizontal, find the relation connecting $\mu$ and $\mu'$; and prove that, if $\mu' < \frac{1}{3}$, $\mu$ must lie between 1/5 and 1/4.

The centre of gravity of a thin hemispherical shell bisects the radius.

5. The framework consists of seven light, smoothly jointed rods, the horizontal and vertical rods being equal. It is loaded at $C$ and $D$ with equal weights 10 lb., and a horizontal force 10 lb. acts at $E$. The framework is hinged at $B$ to a fixed support and anchored at $A$; find the stresses in the rods, and mark those rods which are ties.

6. A ship leaves a certain port and steams N.E. at 15 knots; five hours later another ship leaves the same port and steams due W. at 20 knots. Their wireless instruments can maintain communication up to 250 nautical miles; find to the nearest nautical mile the distances of the ships from the port when communication ceases.

[A knot = a nautical mile per hour.]

7. Prove the formula for motion in a straight line with uniform acceleration: $s = ut + \frac{1}{2}at^2$, $v^2 = u^2 + 2as$.

A body moving in a straight line traverses distances $AB$, $BC$, $CD$ of 163 ft., 215 ft., and 217 ft., respectively, in successive intervals of 3, 5, and 7 secs. Show that these facts indicate a uniform retardation, and find the speed and the distance traversed when the body comes to rest.

8. A projectile is fired from a point on a cliff to hit a mark 200 ft. horizontally from the point and 200 ft. vertically below it. The velocity of projection is that due to falling freely under gravity through 100 ft. from rest. Show that the two possible directions of projection are at right angles, and that the times of flight are approximately 2-7 and 6-5 secs.

9. What do you understand by the equation $\mathbf{M}f = \mathbf{P}$?

A light string $ABCD$ has one end fixed at $A$, and passing under a movable pulley of mass $M$ at $B$ and over a fixed pulley at $C$,
10. Define angular velocity, and state how it is measured.

Prove that a point moving in a circle of radius $r$ with uniform angular velocity $\omega$ must have normal acceleration $\omega^2 r$.

A particle, attached to a fixed point by a string one yard long, describes a circle in a horizontal plane. The string can only support a tension equal to fifteen times the weight of the particle; show that the greatest possible number of revolutions per second is just over two. ($\omega = \frac{22\pi}{7}$).

A. Prove that, if three forces acting at a point are represented in direction and magnitude by the sides of a triangle taken in order, they are in equilibrium, and conversely.

Determine the effect of the forces, if the sides of the triangle are their actual lines of action.

B. Find the condition of equilibrium in the third system of pulleys, assuming that there are three movable pulleys, each of weight $W$, a power $P$, and that the four strings are attached to a block of weight $W$.

If the radius of each pulley, including the fixed pulley, is $a$, show that for the block to remain horizontal and the strings vertical the horizontal distance of its C.G. from the point of attachment of the longest string must be equal to $\frac{11P + 5a}{W}$.

C. Prove that the path of a projectile under gravity is a parabola. If $u$, $v$ are the horizontal and vertical components of the initial velocity, prove that the greatest height attained above the point of projection is $u^2/2g$, and the range on the horizontal plane $2uv/g$.

D. Enumerate the principles which determine the velocities after impact of two elastic balls of given masses impinging directly with given velocities.

Three balls $A$, $B$, $C$, of masses $3m$, $2m$, $2m$, and of equal radii, lie on a smooth table with their centres in a straight line. Their coefficient of restitution is $1/4$; show that, if $A$ is projected with velocity $V$ to strike $B$, there are three impacts, and that the final velocities are $(50, 57, 60)$.

1925. 3 HOURS

1. State and explain the theorems known as 'the triangle of forces' and 'the polygon of forces'.

$ABCD$ is a plane non-re-entrant quadrilateral, in which the

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last 20 seconds of its motion and moves at a uniform speed during the rest of its motion. Find the acceleration and retardation.

7. Explain the term foot-pound and show how to estimate kinetic energy in foot-pounds.

An engine in 7 seconds has raised a load of 1 ton through a height of 3 feet and has communicated to it a speed of 10 feet per second. At what average horse-power has it been working?

8. Show how to estimate the difference of potential energy in a system, acted on by gravity only, in two different configurations.

A particle of mass $M$ is attracted to two particles, each of mass $m$, by means of two light inextensible strings passing over two small smooth fixed pegs, distant 2c apart at the same level. Show that, when the mass $M$ falls from a position, in which the strings are each inclined to the vertical at an angle $\theta$, to the position, in which they are each inclined to the vertical at an angle $\phi$, there is a loss of potential energy of amount

$$Mgc (\cot \phi - \cot \theta) - 2 Mgc (\cosec \phi - \cosec \theta).$$

Deduce that, if there is equilibrium when the strings are each inclined to the vertical at an angle $\phi$, and the system is released from a position in which the strings are each inclined to the vertical at an angle $\theta$, it will next come to instantaneous rest when the inclination is $\phi$, where

$$\tan^2 \frac{\phi}{2} = \tan \frac{\phi}{2} \tan \frac{\phi}{2}.$$

9. A particle is projected with a given velocity so that the direction of projection makes an angle $\alpha$ with a plane inclined to the horizontal at an angle $\beta$, the motion taking place in a vertical plane through a line of greatest slope of the inclined plane. Find the range on the plane.

If the direction of projection is such that $\alpha$ is a maximum, show that $R$, this maximum range, and $T$, the time of flight, are connected by the relation $R = \frac{1}{2} g T^2$.

10. Find the direction and magnitude of the acceleration of a particle describing a circle of radius $a$ ft. with a uniform speed $v$ ft. per sec.

A plane horizontal circular disk is constrained to rotate uniformly about its centre, describing two complete revolutions per second. Show that the greatest distance from the centre of the disk at which a small object can be placed so as to stay on the disk is very approximately $2.45 \mu$ inches, where $\mu$ is the coefficient of friction between the object and the disk.

A. Determine the magnitude and line of action of the resultant of two like parallel forces.

GROUP III (Paper 4)

STATICS AND DYNAMICS

1926. 3 Hours

Not more than nine questions are to be attempted by any Candidate. In numerical calculations take $g = 32$ foot-second units.

1. Define the moment of a force about a point, and prove that the algebraic sum of the moments of two forces, whose lines of action intersect, about any point in their plane is equal to the moment of their resultant about the same point.

The moments about the points $A$, $B$, $C$ of a force, whose line of action lies in the plane of the triangle $ABC$, are $p$, $q$, $r$ respectively. If the force is resolved into three forces, one along each of the sides of the triangle, determine the magnitudes and senses of these forces.
2. Investigate the conditions that a rigid body may be in equilibrium under the action of a system of coplanar forces.\[AB\] and \[BC\] are two uniform rods of weights \(W\) and \(W'\) respectively. They are freely hinged together at \(B\) and the end \(A\) is freely pivoted to a fixed point \(A\), while the end \(C\) is constrained to move on a fixed horizontal wire, passing through \(A\), by means of a small smooth ring of negligible mass. Show that the horizontal force which must be applied at \(C\) to keep the rods in the position in which the angles \(\angle CAB\) and \(\angle ACE\) are \(\theta\) and \(\phi\) and \(B\) is below \(AC\), is \[
\frac{1}{2} (W + W') \cos \phi \cos \theta \cos (\theta + \phi).
\]

3. A framework consists of seven light rods \(AB, BC, CA, BD, CD, DE, CE\), smoothly jointed together so that \(ABC, BCD, CDE\) are right-angled isosceles triangles, the right angles being at \(B, C, D\) respectively, and \(ACE\) are in one straight line, which is parallel to \(BD\). It is pivoted to a fixed point at \(A\) and rests with \(AB\) vertical and \(B\) above \(A\) and the joint \(E\) against a support, which exerts a pressure at right angles to \(AE\). It carries loads of 10 oz. at each of the joints \(B, C, D\). Show that there is no stress in \(CD\), and find the stresses in the other rods, indicating which are tensions and which are compressions.

4. Define limiting friction, and give some examples from everyday life of equilibrium maintained by friction, which is not limiting.

A uniform cube, whose edges are each 4 ft, stands on a rough horizontal plane. A gradually increasing force is applied to one of its vertical faces at a height \(a\) vertically above the centre of that face. Determine how equilibrium will be broken

1. when the coefficient of friction between the plane and the cube is 0.5;
2. when this coefficient is 0.7.

5. Define momentum, and explain Newton's second law of motion.

Waves are striking against a vertical sea-wall with a speed of 50 ft. per se. Taking a cubic foot of sea-water to weigh 64 lb., show that the pressure on the wall, due to the destruction of the momentum of the waves, is very approximately 34-7 lb. weight to the square inch.

6. A cyclist and his machine together weigh 200 lb. Riding along a certain road he observes that, when he is free-wheeling down a slope of 1 in 40, his speed, when it has become uniform, is 20 miles an hour, and is 30 miles an hour when he is free-wheeling down a slope of 1 in 20. If the air-resistance varies as the square of the speed, and other resistances remain constant, find in horse-power the rate at which he must work to maintain a speed of 15 miles an hour on the level.

7. State the principles which are employed and the experimental

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results which are used to determine the motion after impact of two small spheres moving in the same straight line.

\(A, B, C\) are three exactly similar small spheres at rest in a smooth horizontal straight tube. \(A\) is set in motion and impinges on \(B\). Show that \(A\) will impinge on \(B\) again after \(B\) has impinged on \(C\), and show that there will be no more impacts, if \(e\) the coefficient of restitution between the spheres, is not less than \(3 - \sqrt{8}\).

8. A small object, projected towards a vertical rectangular screen perpendicular to the vertical plane of the path of the object, just clears it at the highest point of its path. The screen is \(a\) feet in front of the point of projection and the top of the screen is \(b\) feet higher than the point of projection. Find the vertical and horizontal components of the velocity of projection.

If the object is projected with the requisite velocity, but the vertical plane through the direction of projection, instead of being at right angles to the screen, makes an angle \(\theta\) with the screen, prove that it will fail to clear the screen by \(b\) (\(\cos \theta - 1\)) feet.

9. Prove (do not merely state) that the direction of the acceleration of a particle, describing a circle with uniform speed, is at any instant towards the centre of the circle, and find an expression for its magnitude.

Two small weights, of 2 oz. and 1 oz. respectively, are connected by a light inextensible string, a foot long, which passes through a smooth fixed ring. The 1 oz. weight hangs at a distance 9 inches below the ring, while the 1 oz. weight describes a horizontal circle. Show that the plane of this circle is 3\(\frac{1}{2}\) inches below the ring, and show also that the 1 oz. weight makes very approximately 153 revolutions per minute.

10. Explain what is meant by the 'equivalent simple pendulum' in the case of a rigid body which can rotate freely about a fixed horizontal axis, and show how to find its length.

A uniform circular disk of radius \(a\) has a particle of mass equal to that of the disk fixed to a point of its circumference. The disk can turn freely about a fixed horizontal axis through its centre at right-angles to its plane. Assuming that the radius of gyration of the disk about this axis is \(a\sqrt{2}\), show that the length of the equivalent simple pendulum for small oscillations of the system about its position of stable equilibrium is \(3a\sqrt{2}\).

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1927. 3 Hours

1. Prove that the lines of action of three forces in equilibrium meet at a point.

A uniform sphere of radius \(a\) is to be kept at rest on a smooth
plane inclined to the horizontal at an angle $\alpha$ by means of a string attached to a point of the plane and to a point on the surface of the sphere, and the tension in the string is not to exceed the weight of the sphere. Prove that the length of the string must not be less than $\alpha (\sec \alpha - 1)$.

2. Find the conditions that a system of forces acting in one plane should be in equilibrium.

Two smooth planes, each inclined to the horizontal at an angle $\alpha$, have a common horizontal line. A uniform cylinder, of weight $W$, rests in the space between them touching each plane along a generator. A second uniform cylinder, of weight $W'$, is placed between the first cylinder and one of the planes, and its radius is such that, in this position, the plane through the axes of the two cylinders is horizontal. If the system remains in equilibrium, find the reactions between the planes and the two cylinders and the reaction between the two cylinders, and show that equilibrium cannot exist if $W'$ is greater than $W$.

3. The above figure, in which each of the triangles $ABC, BCD, DBE, EDF$ is right-angled and isosceles, represents a framework of nine light rods, freely pivoted to a fixed point at $A$, resting against a vertical wall at $F$ with $AB, BE, CD$, and $FD$ horizontal, and carrying loads of 10 cwt. at each of the joints $B, C, D, E$. Determine the stresses in each of the rods, indicating which are thrusts and which are tensions.

4. $ABCD$ is a uniform plane quadrilateral lamina, in which $AB$ is parallel to $DC$. The length of $AB$ is $a$ and the length of $DC$ is $b$. Prove that the centre of gravity of the lamina coincides with that of four particles of masses proportional to $a, a + b, a - b, a + b$, placed at $A, B, C, D$ respectively.

If $AD = BC = c$ and $b > a$, find the coordinates of the centre of gravity referred to rectangular axes, of which one is $DA$ and the other is the perpendicular to $DA$ drawn through $D$, and show that a uniform prism, which has such a quadrilateral for its cross-section, cannot rest on a horizontal plane with the faces corresponding to $W'$ as its upper face.

5. A uniform rod $AB$ lies on a rough horizontal ground. A cord attached to the end $B$, passes over a small pulley fixed at a point $D$, vertically over a point $C$ lying in the line $AB$ produced beyond $B$. Prove that, if the cord is pulled with a gradually increasing force, the rod will begin to slide along the ground, if $\mu$, the coefficient of friction between the rod and the ground, is less than cot $DBC$; but that, otherwise, it will begin to turn about the extremity $A$, and the extremity $A$ will not slip if $\mu (\tan \beta - \tan \alpha) = 1$, where $\alpha$ and $\beta$ are the angles which $AB$ and $BD$ respectively then make with the horizontal.

6. A train of total weight 400 tons is travelling on the level at 60 miles an hour, the engine working at 800 H.P. If the resistances, apart from air resistance, are 10 lb. weight per ton, find in lb. weight the magnitude of the air resistance.

If air resistance varies as the square of the speed, find the rate at which the engine is working when drawing the same train up a gradient of 1 in 200 at a steady rate of 30 miles an hour; and find the acceleration which the train would have on this gradient at this speed if the engine were working at 800 H.P.

7. Two weights $W$ and $W'$ ($W > W'$) are connected by a light inextensible string which passes over a small smooth fixed pulley. $W$ hangs freely and $W$ is on a smooth plane, inclined to the horizon at an angle $\alpha$. Prove that, in the position indicated by the figure, in which the string is supposed to be taut and in the vertical plane containing the line of greatest slope through the point at which $W$ is in contact with the plane, the potential energy of the system differs by a constant from

$$h (W' \cos \theta - W \sin \alpha \cot \theta) \cos \alpha,$$

where $h$ is the height of the pulley above the plane.

If this is not the equilibrium position and the system is held in
this position and then released, show that in the subsequent motion the system will first come to instantaneous rest when \( \theta \) has changed to \( \theta' \), where \( \tan \frac{1}{2} \theta' \tan \frac{1}{2} \theta = \tan^2 \frac{1}{2} \beta \), \( \beta \) being the value of \( \theta \) in the equilibrium position.

8. \( A \) and \( B \) are two points, such that the coordinates of \( B \) referred to \( A \) as origin, the axis of \( x \) being horizontal in the vertical plane through \( AB \) and the axis of \( y \) being the upward-drawn vertical, are \( a \) and \( b \). It is required to project a particle from \( A \) to pass through \( B \), the magnitude of the velocity of projection being \( V \). Show that this is impossible if \( V^2 \) is less than \( g \left\{ b + \frac{\sqrt{a^2 + b^2}}{a} \right\} \), but that, if \( V^2 > g \left\{ b + \frac{\sqrt{a^2 + b^2}}{a} \right\} \), there are two possible directions of projection.

Show that the ratio of the difference between the greatest heights attained during these two paths and the difference between the ranges of these two paths on the horizontal plane through \( A \) is of \( \frac{a}{b} \).

9. A particle of mass \( m \), attached to a fixed point by a light inextensible string of length \( l \), describes a horizontal circle with uniform angular velocity \( \omega \). Find the depth of the plane of the circle below the fixed point, and find also the tension in the string.

Two equal particles \( A \) and \( B \) are connected by a light inextensible string of length \( a \), and \( A \) is attached to a fixed point by a light inextensible string of length \( b \). The particles describe horizontal circles with the same uniform angular velocity, the string joining \( A \) to the fixed point making an angle of 45\( ^\circ \) with the vertical, and the string joining \( A \) to \( B \) making an angle of 60\( ^\circ \) with the vertical. Find the ratio of \( a \) to \( b \).

10. A rigid body is free to turn about a smooth fixed horizontal axis. Show how to find the period of its small oscillations about its position of stable equilibrium.

If the body consists of a uniform rod, 6 feet long, with a particle of weight equal to that of the rod attached to the middle point of the rod, and the axis passes through an extremity of the rod, calculate this period.

GROUP IV (PAPER 3) AND SUBSIDIARY SUBJECT (15 d)

STATICS AND DYNAMICS

1926. 3 HOURS

Not more than nine questions are to be attempted by any Candidate.
In numerical calculations take \( g = 32 \) feet-second units.

1. State and prove the theorem which determines the magnitude and position of the resultant of two unlike parallel forces.

A uniform beam rests in a horizontal position supported at a point distant 2 feet from one end and carrying a weight of 10 lb. suspended from this end. The pressure on the support is 30 lb. weight. Determine the weight and length of the beam.

2. State the conditions of equilibrium of a body acted on by a system of forces in one plane.

\( AB \) and \( BC \) are two uniform exactly similar rods, each of weight \( W \), freely hinged together at \( B \), and carrying small rings of negligible weight which enable the ends \( A \) and \( C \) to move without friction on a fixed horizontal wire. The rods are placed so as to include a right angle, with the joint \( B \) below the wire, and are prevented from closing up by means of a rigid stay of negligible weight joining the middle points of the rods. Find the stress in this stay and the reactions at \( A \), \( B \), and \( C \).

3. \( ABCD \) is a plane quadrilateral in which the angles \( A \) and \( B \) are each right angles, the angle \( ADC \) is 60\( ^\circ \), and the sides \( AD \) and \( BC \) are equal. A force \( P \) acts along \( BC \) in the direction from \( B \) to \( C \). Find graphically or otherwise the magnitudes and directions of three forces along \( CD \), \( DA \), \( AB \) respectively, which will be in equilibrium with \( P \).

4. Seven light rods \( AB \), \( BC \), \( CA \), \( BD \), \( CD \), \( DE \), \( CE \) are jointed together to form a framework in which \( ABCD \), \( BCD \), \( CDE \) are equilateral triangles. The framework rests on smooth vertical supports at \( A \) and \( E \) with \( AE \) and \( BD \) horizontal and \( BD \) above \( AE \). It carries loads of 5 cwt. at each of the joints \( B \), \( C \), \( D \). Determine the stresses in the various rods, indicating which are thrusts and which are tensions.

5. Explain clearly what is meant by limiting friction, and give some examples from everyday life of equilibrium maintained by friction which is not limiting.

A uniform cube, whose edges are each 8 feet long, stands on a rough horizontal plane. A gradually increasing horizontal force is applied to one of its vertical faces at a height of 2 feet above the centre of that face. Determine how equilibrium will be broken (1) when the coefficient of friction between the plane and the cube is 0·5; (2) when this coefficient is 0·7.

6. Explain what is meant by a velocity-time graph, and give the interpretation of the area under such a graph. Employ such a graph to prove the formula, \( s = ut + \frac{1}{2}at^2 \), for uniformly accelerated motion.

Find in lb. weight per ton the force exerted by the brakes of a train travelling at 60 miles an hour, which will bring it to rest in half a mile, and find the time during which the brakes act.
7. A particle held at rest on a smooth table is attached by a light inextensible string to a second particle, of the same mass as the first, which hangs over the edge of the table, the string being taut and at right angles to the edge of the table. If the particle on the table is released, find the acceleration with which it will begin to move.

If the string connecting the particles is 5 feet long and both particles are initially on the table, one 4 feet from the edge and the other at the nearest point of the edge from the first, and the particle at the edge is dropped gently over the edge, find the time that elapses before the other particle reaches the edge.

8. Explain the term horse-power, making it clear that it is neither a unit of force nor a unit of work.

An engine is raising water from a depth of 55 feet and discharging 16 gallons a second with a velocity of 44 feet per second. Taking the weight of a gallon of water to be 10 pounds, find separately in foot-pounds the potential energy and the kinetic energy of the water discharged per second, and find the horse-power at which the engine is working.

9. A small sphere of mass \( m \), moving with velocity \( u \), impinges directly on another small sphere of mass \( m' \), moving in the same straight line with velocity \( u' \). Write down, without proof, equations which determine their subsequent motion.

Three small exactly similar spheres \( A, B, C \) are at rest in a smooth straight horizontal tube. The coefficient of restitution between any two of the spheres is \( 0.5 \). \( A \) is projected towards \( B \) with a velocity \( u \). Determine the velocities of the three spheres after \( B \) has impinged on \( C \) and \( A \) has impinged a second time on \( B \), and show that there will be no more impacts.

10. A particle attached by a light inextensible string to a fixed point describes a horizontal circle with uniform angular velocity \( \omega \), the plane of the circle being at a distance \( h \) below the fixed point. Prove that \( g = h \omega^2 \).

11. If the length of the string is 3 inches, and it is inclined at an angle of 60° to the vertical, and the weight of the particle is 1 oz., find the tension in the string and the number of revolutions per minute that the particle is making.

1927. 3 Hours

1. Show how to find the magnitude and position of the resultant of two like parallel forces.

A uniform beam, 6 feet long, which weighs 40 lb, is supported in a horizontal position by two vertical strings, each connecting an end of the beam with a fixed point above the beam. Neither string can support a tension of more than 30 lb. Find the greatest weight which will not cause either of the strings to break, wherever it is placed on the beam, and determine the portion of the beam on which a weight of 15 lb. can be placed without causing either of the strings to break.

2. A uniform equilateral triangular lamina \( ABC \), of weight \( W \), has the vertex \( A \) pinned to a fixed point, about which it can turn freely in a vertical plane, and rests with \( AB \) vertical, \( B \) being above \( A \), and the vertex \( C \) in contact with a smooth vertical wall. Find the reaction between the lamina and the wall, and the magnitude and direction of the reaction at \( A \).

3. Find the position of the centre of gravity of a uniform triangular lamina, and show that it coincides with that of three equal masses placed one at each vertex of the lamina.

\( ABCD \) is a uniform quadrilateral lamina, in which \( BC \) is parallel to \( AD \). The length of \( BC \) is \( a \) and the length of \( AD \) is \( b \). Prove that the centre of gravity of the lamina coincides with that of masses placed at \( D, A, B, C \), proportional to \( b, a, a + b \) respectively.

4. Determine the stresses in the rods composing the framework of light rods represented by the figure, which rests on vertical supports at \( A \) and \( F \) with \( ABEF \) horizontal, and carries loads of 10 cwt. at each of the joints \( C \) and \( E \). Each of the triangles \( ABC, BCD, DBE, DEF \), is a right-angled isosceles triangle.

Show which stresses are thrusts and which are tensions.

5. A uniform solid cube stands on a rough horizontal plane and an exactly similar cube is placed on it so that the faces coincide. The coefficient of friction between the two cubes is \( \mu \) (-1) and the coefficient of friction between the lower cube and the plane is \( \mu \). A gradually increasing horizontal force is applied to the upper cube at right angles to one of its faces at the centre of that face. Prove that, when equilibrium is broken, the upper cube will slide on the lower, while the lower remains at rest, or both cubes will move together as a single rigid body according as \( 2\mu \) is greater or less than \( \mu \).

6. A light inextensible string has one end attached to the underside of the edge of a smooth horizontal table. It passes through a small smooth ring of weight \( W \), and has its other end attached to
a weight $W'$ resting on the table. If the system is held so that the string is taut and is then released, prove that $W'$ will move with an acceleration

$$2Wg/(W + 4W').$$

If the weight on the table has a velocity $V$ when it reaches the edge of the table, find its horizontal and vertical velocities just after it has gone over the edge.

7. A train of total weight 400 tons is travelling on the level at 60 miles an hour, the engine working at 800 H.P. If the resistances, apart from air resistance, are 10 lb. weight per ton, find in lb. weight the magnitude of the air resistance.

If air resistance varies as the square of the speed, find the rate at which the engine is working when drawing up a gradient of 1 in 200 at a steady rate of 30 miles an hour; and find the acceleration which the train would have on this gradient at this speed if the engine were working at 800 H.P.

8. A particle projected from a point meets the horizontal plane through the point of projection after traversing a horizontal distance $a$, and in the course of its trajectory attains a greatest height $b$ above the point of projection. Find the horizontal and vertical components of the velocity of projection in terms of $a$ and $b$.

Show that when it has described a horizontal distance $x$, it has attained a height $4bx(a-x)/a^2$.

9. State the law of conservation of linear momentum.

A truck, weighing 5 tons, is moving on a set of level rails at the rate of 5 feet per second, and impinges on a second truck, weighing 10 tons, which is standing at rest on the same rails. If after the impact the second truck moves on at the rate of 2 feet per second, find the rate at which the first truck moves after the impact, and calculate in foot-pounds the amount of kinetic energy destroyed by the impact.

10. The shape of a cycle track at a corner is that of an arc of a circle whose radius is 100 yards. Find the angle at which the track should be inclined to the horizontal, that a rider can take the corner at 30 miles an hour without any lateral reaction between his bicycle and the track.

If a motor-cyclist can take the corner safely at 60 miles an hour, find the least possible value of the coefficient of friction between the track and his tyres.

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1. Determine the values of $A$, $B$, $C$ so that

$$1 + Ax + Bx^2 + Cx^3$$

may be divisible by $1 + x$, the quotient being a perfect square.

Verify that then the equation

$$1 - (Ax + Bx^2 + Cx^3)(1 - x^2)(1 - 12x^2 + 16x^4)$$

is identically true.

[It is supposed that the coefficients $A$, $B$, $C$ are different from zero.]

2. Write down in terms of $\lambda$ the condition that the quadratic

$$S + \lambda S' = 0$$

should have equal roots in $x$, where

$$S = px^2 + 2qx + r, \quad S' = x^2 + c.$$

Show that if the values of $\lambda$ which satisfy this condition are equal, and if all the coefficients are real, then

1. if $c > 0$, $S = pS'$;

2. if $c < 0$, $S$ and $S'$ have a factor in common.

What is the conclusion when $c = 0$?

3. If $x$, $y$ are both positive and $x^n + y^n = \text{const.}$, prove that $x^n + y^n$ decreases as $x$ increases if $n > m$ and $x < y$, but increases if $x < m$ and $x < y$.

Show that if $x$, $y$, $z$ are positive and

$$x^n + y^n + z^n = 3c^n,$$

then $x^n + y^n + z^n \geq 3c^n$, as $n \rightarrow m$.

[Both $n$ and $m$ are supposed positive.]

4. By considering the expansion of

$$\log \{(1 - at)(1 - bt)(1 - ct)\}$$

in ascending powers of $t$, or otherwise, prove that if $a + b + c = 0$ and $x^n + y^n + z^n = 3c^n$, then

$$s_1 = \frac{1}{3} s_2, \quad s_3 = \frac{1}{3} s_2, \quad s_5 = \frac{1}{3} s_4.$$

5. From the vertices of a triangle $ABC$ are drawn perpendiculars $p$, $q$, $r$ to a straight line which is in the plane of $ABC$ but passes entirely outside the triangle. Prove that

$$a^2(p - q)(p - r) + b^2(q - p)(q - r) + c^2(r - p)(r - q) = 4 \Delta^2,$$

where $\Delta$ denotes the area of the triangle.

6. Prove that, if $n$ is an integer, $\sin nx/\sin x$ can be expressed as a polynomial of degree $(n - 1)$ in cos $x$, and that the coefficient of the term of highest degree is $2^{n-1}$.
Prove that, if \( n \) is odd and equal to \( 2p + 1 \),
\[
\sin n\theta = n\prod_{r=1}^{p} \left[ 1 - \frac{\sin^2 \theta}{\sin^2 (r\pi/n)} \right]
\]
and that
\[
\prod_{r=1}^{p} \{2 \sin (r\pi/n)\} = \sqrt{n}.
\]

[Note. The remaining coefficients in the original polynomial need not be determined explicitly.]

7. Two perpendicular lines \( C A, C B \) each of length \( l \) are drawn on level ground; borings are taken at each of the points \( A, B, C \), and rock is found at depths \( a, b, c \) respectively. Prove that, if the face of the rock is a plane, the inclination \( \theta \) of this plane to the horizontal is given by
\[
P \tan \theta = (a - b)^2 + (a - c)^2.
\]

Prove also that the volume of a triangular prism with vertical faces, bounded at the top and bottom by the plane \( ABC \) and the face of the rock, is
\[
\frac{1}{2} \cdot l \cdot (a + b + c).
\]

8. Explain how to solve an equation of the form
\[
a x^4 + b x^3 + c x^2 + b x + a = 0
\]
by writing \( y = x + 1/x \).

Illustrate by means of the equation
\[
x^4 + x^3 + x^2 + x + 1 = 0.
\]

Hence or otherwise prove that if \( d \) is the side of the decagon (regular polygon of ten sides) and \( p \) that of the pentagon (regular polygon of five sides) inscribed in a circle of radius \( r \), then
\[
d^2 + rd = r^2, \quad p^2 = r^2 + d^2.
\]

9. Draw a rough graph of the curve
\[
y = x + 1 - \frac{1}{x} + \frac{1}{1-x},
\]
and verify that it consists of three parts, along each of which \( y \) steadily increases with \( x \).

Verify that, for any real value of \( c \), the equation in \( x \),
\[
x + 1 - \frac{1}{x} + \frac{1}{1-x} = c
\]
has three real roots, one negative, one between 0 and 1, and the third greater than 1.

Prove also that if one root is \( a \), the others are
\[
\beta = 1 - \frac{1}{a}, \quad \gamma = \frac{1}{1-a}.
\]

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1. Prove that if
\[
x + 1 = y + 1 = z + 1 = \lambda,
\]
then either (i) \( x = y = z \), or (ii) \( x^2 + y^2 + z^2 = \lambda \).

2. Show that, if \( (p^2 + q^2)(z + 1) \) is written for \( x \) in the expression
\[
y = \frac{2x^2 - 14x + 11}{2x^2 - 14x + 5},
\]
values of \( p \) and \( q \) can be found such that the expression, when simplified, takes the form
\[
ax^2 + b
\]
\[
ax^2 + c
\]

Hence, or otherwise, prove that the value of \( y \) lies between 3 and -1 for all real values of \( x \).

3. Sum the following series:
   (1) \( x + 3x^2 + 7x^3 + 15x^4 + 31x^5 + \ldots \) to \( n \) terms.
   What does this sum become when \( x = \frac{1}{2} \)?

   (2) \[
   \frac{2}{3} - \frac{3}{5} + \frac{4}{7} - \frac{5}{9} - \frac{6}{11} + \ldots
   \]
to \( n \) terms and to infinity.

4. Express \( (x + a_1)(x + a_2)\ldots(x + a_n) \)
   \( (x - a_1)(x - a_2)\ldots(x - a_n) \)
in the form
\[
P + \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n},
\]
where \( P, A_1, A_2, \ldots, A_n \) are independent of \( x \).

By putting \( x = 0 \), or otherwise, prove that
\[
\sum \frac{(a_1 + a_2)(a_1 + a_3)\ldots(a_1 + a_n)}{(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n)}
\]
is zero if \( n \) is even and 1 if \( n \) is odd, the summation being taken over the letters \( a_1, a_2, \ldots, a_n \).

5. Prove that the determinant
\[
\begin{vmatrix}
0, & -k, & y, & x \\
k, & 0, & -f, & y \\
-y, & f, & 0, & z \\
x', & -y', & -z', & w
\end{vmatrix}
\]
is the product of two linear factors, and find the factors.
6. \( O \) is the circumcentre, \( H \) is the orthocentre of a triangle \( ABC \), the radius of whose circumcircle is \( R \); prove that
\[
(1) \quad OH = \frac{R^2 - 8R^2 \cos A \cos B \cos C}{2R}.
\]
\[
(2) \quad \tan AOH = \frac{\sin 2B \cos 2C}{1 + \cos 2B \cos 2C}.
\]
7. Prove that, if \( x, y, u, v \) are all real and if
\[
\tan (x+iy) = u + iv,
\]
then
\[
u/v = \sin 2x/e^{2y}
\]
and
\[
\frac{1 - u^2 + e^{2y}}{1 + u^2 + e^{2y}} = \cos 2x/e^{2y}.
\]
8. Prove that, if \( x < 1 \),
\[
\frac{1 - x}{1 - 2x \cos 2\theta + x^2} = \sum_{n=0}^{\infty} x^n \cos (2n+1)\theta, \quad \sec \theta.
\]
Hence, or otherwise, show that
\[
\cos (2n+1)\theta, \quad \sec \theta = \sum_{r=0}^{n} (n+r)! \left(-4\sin^2 \theta\right)^r.
\]
Solve the equation \( 2x^3 + 6x^2 + 2x = 0 \).
Find also the product of the squares of the differences of its roots.

1920. 2 Hours

1. Find the conditions that
\( ax^2 + 2bx + c \),
\( ax^2 + 2kxy + 2qy^2 + 2px + 2py + c \),
may be positive for all real values of the variables \( x \) and \( y \).
Show that
\[
ax^2 + 2bx + c
\]
is capable of taking all real values for real values of \( x \) if \( b^2 > a'c' \), and
\( 4(a' b - a'b') (b' c - b c') > (a' c - a c')^2 \).
2. Prove that
\[
1 + 2x + 3x^2 + \ldots + nx^n - 1 = \frac{1 + x}{(1 - x)^2}, \quad -1 < x < 1.
\]
Hence show that
\[
1 - 2^n C_1 + 3^n C_2 - \ldots = 0,
\]
if \( n \) is a positive integer which is greater than \( 2 \).
Prove that
\[
1 - 2^n C_1 + 3^n C_2 - \ldots = 0,
\]
if \( n \) and \( r \) are positive integers such that \( n \) is greater than \( r \).

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Show that, \( n \) being a positive integer,
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 2 \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right).
\]

8. Prove that
\[
\frac{1}{2} \log \left( 1 + 2r \cos \theta + r^2 \right) = r \cos \theta - \frac{1}{2} r^2 \cos 2\theta + \frac{1}{4} r^4 \cos 4\theta + \ldots
\]

Use this expansion to express \( \cos \sigma \theta \) as a series of descending powers of \( \cos \theta \), and write down the general term in the expression.

9. Form the equation whose roots exceed by 3 the roots of the equation \( x^2 - 10x^2 + x - 9 = 0 \).

Describe Horner’s method, or any other, of approximating to a real root of a rational equation with numerical coefficients.

Prove that all the roots of the equation
\[
x^4 - 61x^2 + 168x - 115 = 0
\]
are real, and obtain the root which is approximately equal to unity correct to two places of decimals.

1921. 2 Hours

1. (1) If
\[
\frac{1}{a-b} - \frac{1}{c-a} = \frac{1}{b-c}
\]
prove that
\[
b = \frac{1}{c-a}.
\]

(2) If \( x(x-c) = yz \), \( y^2 = xz \), \( z(z-c) = xy \), and \( x, y, z \) are not zero, prove that
\[
(x+y+z) = bc + ca + ab.
\]

2. Show that, \( n \) being a positive integer,
\[
(2n+1)^{(n+1)} - (2n-1)^n = (2n+1) \times 2^{2n} \times (2n+1)^{n-1},
\]
contains only even powers of \( n \), and that the coefficient of \( x^{n-2} \) is
\[
2(2r-2p+1) \times (2p+1) \times (r+2p) \times (r-2p),
\]
where \( p < r/2 \).

If \( S_n = \sum_{k=1}^{n} k^2 \), prove that
\[
(1) \quad S_n = \frac{n^2(n+1)(2n+1)}{6}
\]
\[
(2) \quad S_n = \frac{n(n+1)^2}{6} \times \frac{n^2}{6}.
\]

State, without further calculation, a generalization of these results.

3. If \( a, b, c \ldots \) are all positive and less than unity, prove that
\[
(1-a)(1-b)(1-c) \ldots \text{ lies between 1 and } 1-a-b-c-\ldots.
\]

If \( n \) is a positive integer and
\[
u_n \equiv 1 + x + x^2 + \ldots + x^{n-1},
\]
prove, when \( x \) is positive, that \( \left( 1 + \frac{x}{n} \right)^n \) lies between \( u_n \) and
\[
u_n = \frac{x^n}{2n} u_{n-2}.
\]

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GROUP III MATHEMATICAL DISTINCTION PAPERS

If \( n \) is so large that squares and higher powers of \( \frac{1}{n} \) may be neglected, prove that
\[
\left( 1 + \frac{x}{n} \right)^n = e^x \left( 1 - \frac{x^2}{2n} \right).
\]

4. Find the limiting value, when \( n \) is indefinitely increased, of
\[
u_n = \log \left( \frac{1}{n} + \frac{1}{\sqrt{n}} \right) - \frac{1}{x + \sqrt{n}}.
\]

Hence, or otherwise, prove that the series
\[
u_1 + \nu_2 + \nu_3 + \ldots
\]
is divergent, unless \( x = \frac{1}{2} \).

5. Evaluate the determinant
\[
a + b, a + 2b, a + 3b, a + 4b
\]
\[
a + 2b, a + 3b, a + 4b, a + b
\]
\[
a + 3b, a + 4b, a + b, a + 2b
\]
\[
a + 4b, a + b, a + 2b, a + 3b
\]

6. The escribed circles of a triangle \( ABC \) touch \( BC, CA, AB \) internally at \( D, E, F \) respectively; prove that
\[
(1) \quad EF^2 = a^2 (1 - \sin B \sin C);
\]
\[
(2) \quad \text{area } DEF; \quad \text{area } ABC = r \times 2R.
\]

7. Prove that
\[
x \sin \theta + x^2 \sin 2\theta + \ldots + x^n \sin n\theta
\]
\[
= \frac{(x \sin \theta - x^2 \sin (n+1) \theta + x^{n+1} \sin n\theta)}{(1 - x \cos \theta \times x^2)}.
\]

Multiply the infinite series
\[
(1 + x^2) \cos \theta - 2x / (1 + x^2 - 2x \cos \theta).
\]

by \( 1 - 2x \cos \theta + x^2 \), and show that the product is
\[
\left( 1 + x^2 \right) \cos \theta - 2x / (1 + x^2 - 2x \cos \theta).
\]

8. Express \( x^n \) as the product of \( n \) real quadratic factors.

Prove that
\[
(1 + x^n)(1 - x^n) = \frac{x^n + 1}{4} \times \left( 2 - x^n \right) \times \left( 2 - \frac{x^n}{4} \right);
\]
and deduce, or prove otherwise, that
\[
S_n = \frac{\pi}{2}, \quad S_n = \frac{\pi}{4}, \quad S_n = \frac{\pi}{4}, \quad S_n = \frac{\pi}{4}.
\]

9. Explain some method of solving a biquadratic equation, and apply it to solve
\[
x^2 + 3x^2 - 3x^2 - 2x + 2 = 0.
\]

If \( a, b, c \) are the roots of this equation, find
\[
(B \gamma + a \beta) (\gamma a + \beta b) (a \beta + \gamma b)
\]
in its simplest form.
1. Prove that if \( xy + x + y = 0 \), then \( \frac{1}{x+2} + \frac{1}{y+2} = 1 \).

Show that \( X \) and \( Y \) satisfy the same relation as \( x \) and \( y \) where

\[
X = \frac{x^3 + 2x}{2x^2 + 3x + 4}, \quad Y = \frac{y^2 + 2y}{2y^2 + 3y + 4}.
\]

2. Determine independently the number \( C_n \) of combinations of \( m \) things taken \( r \) at a time,

and the number \( C_r \) of permutations of \( m \) different things taken \( r \) at a time.

3. Show that the Binomial Series \( 1 + m \alpha + m \beta + \ldots \) is convergent if \( |\alpha| > 1 \), and that, if \( |\alpha| = 1 \), it is convergent if \( m > 0 \).

By considering the product of the infinite series for the sum and difference of \((1-\alpha)^{\frac{1}{2}}\) and \((1+\alpha)^{\frac{1}{2}}\), show that

\[
\sum \binom{r}{p+1} (2p+1)(2p+3) \ldots (4p-1) (4q+1) (4q-1) \ldots (2q+3) = 2^n |m|,
\]

where the summation extends to all positive integral values of \( p \) and \( q \), including zero, such that \( p+q = n \).

4. Evaluate \( \log \frac{n^2}{q} \) by the transformation \( x = \frac{a}{b} \).

Show that the remainder after \( n \) terms of the series in the bracket lies between

\[
\frac{1}{2^n+1} \left( \frac{p-q}{p+q} \right)^{2^n+1} \quad \text{and} \quad \frac{1}{2^n+1} \left( \frac{p-q}{p+q} \right)^{2^n+1} \left( \frac{p+q}{p+q} \right)^{2^n+1}.
\]

Hence, taking \( n = 4 \), show that

\[
\log 3 - \log 2 = 0.40546510.
\]

5. Prove that the roots of the equation

\[
\cos \theta \cos (\theta - \alpha) \cos (\theta - \beta) \cos (\theta - \gamma) = \cos \alpha \cos \beta \cos \gamma
\]

are given by \( \theta = n \pi \) and

\[
2 \tan \theta = \tan \alpha + \tan \beta + \tan \gamma + \tan \alpha \tan \beta \tan \gamma.
\]

6. Prove that the radius of the circumscribed circle of a triangle \( ABC \) is equal to \( 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \).

[134]

GROUP III MATHEMATICAL DISTINCTION PAPERS 1922

Show that the centre of the circumscribing circle is the mean centre of the centres of the inscribed and escribed circles.

7. Prove that with a suitable choice of \( a \), the roots of the equation

\[ z^n = 1 \]

are \( 1, a, a^2, \ldots, a^{n-1} \).

Show that the fifteen roots of unity, other than cube and fifth roots, are the roots of the quotient of \( z^3 + z + 1 \) on division by \( z^2 + z + 1 \).

8. Evaluate \( \int \frac{dx}{(x-a)(x-b)} \) by the transformation \( x = \frac{a}{b} \).

Show that \( \int \frac{dx}{(x-a)(x-b)} \) is an algebraic function of \( x \).

Obtain the value of \( \int \frac{\sin \theta d\theta}{3 \cos \theta + 4 \sin \theta} \) in the form

\[
\frac{1}{2 \alpha} \log (3 \cos \theta + 4 \sin \theta).
\]

[135]
3. Find the sum of $n$ terms of the series:

(1) $6 \div \frac{12}{2 \cdot 5 \cdot 8 + 5 \cdot 8 \cdot 11 + 8 \cdot 11 \cdot 14 + \ldots}$

(2) $a(a+b)-(a+b)(a+2b)+(a+2b)(a+3b)-\ldots$

4. If $p_n/q_n$ is the $m$th convergent to the continued fraction $a_1, b_1/b_2, b_3/b_4, \ldots$.

prove that $p_n = a_n p_{n-1} - b_n p_{n-2}$.

If $p_1/q_1$ is the $m$th convergent to the recurring continued fraction

$a^2 - b^2 = a^2 + b^2 - a^2 - a^2 - b^2 - \ldots$,

prove that $p_{2n+1} - ap_{2n+2} = b(p_{2n+2} - aq_{2n+1}) = ab(p_{2n+2} - aq_{2n+1})$.

and

$p_{2n+2} = a^2 q_{2n+1}$.

[Note. These convergents are not reduced to their lowest terms.]

5. Find $\sin 15^\circ$.

Show that

$1 + \cos 2A - 2 \cos 4A = 16 \cos (A - 27^\circ) \cos (A + 9^\circ) \sin (A + 27^\circ) \sin (A - 9^\circ)$.

6. $O, I, H, I_1$ are the circumcentre, the incentre, the orthocentre, and the excentre opposite $A$ of a triangle $ABC$; $\angle A$, $\angle B$, $\angle C$ produced meet the circumcircle again in $D'$, $E'$, $F'$. Prove that

(1) $OH = \sqrt{9R^2 - 4R^2 \cos A \cos B \cos C}$.

(2) $H$ is the incentre of the triangle $D'E'F'$ if the triangle $ABC$ is acute-angled and the excentre opposite $D'$ if the angle $A$ is obtuse.

Deduce that $OH^2 = R^2 - 2Rr$ and $O_1^2 = R^2 - 2Rr.

7. State de Moivre's theorem and prove it for a rational positive exponent.

Prove that

$m + 2(n - 1) \cos \theta + 2(n - 2) \cos 2\theta + 2(n - 3) \cos 3\theta + \ldots \div 2 + \cos (n - 1) \theta = (1 - \cos \theta)/(1 - \cos \theta),$

and

$(n - 1) \sin \theta + (n - 2) \sin 2\theta + \ldots \div \sin (n - 1) \theta = \frac{1}{2} [n \sin \theta - \sin n \theta]/(1 - \cos \theta)$.

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8. If $y = (x^2 + 2x + 7)^{1/2}$, prove that, if $y$, denotes $\frac{dy}{dx}$,

$\left( \frac{y_1}{y_2} \right)^2 - \frac{y_2}{y_1}$ and $\frac{y_2}{y_1}$ are all of the form $A/B$.

where $A$ and $B$ are independent of $x$, and that

$\frac{y_2}{y_1} + 4 \frac{(y_1)^2}{y_2} = 0$.

9. Prove that $\int (1 - x^2) dx = \int (1 + x^2) dx$ may be evaluated by means of the respective substitutions $u = x + x^{-1}$, $v = x - x^{-1}$.

Hence, or otherwise, evaluate $\int \frac{dx}{x^4 + x^2}$.

[1924. 3 Hours]

1. Show that the necessary and sufficient condition that the relation between $x$ and $y$, obtained from the relations

$a_1 x + b_1 (x + y) + c_1 = 0$, $a_2 y + b_2 (y + z) + c_2 = 0$,

$a_3 w + b_3 (w + z) + c_3 = 0$ should be of the form $A(x + B(x + y) + C = 0$ is that the determinant

$\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix}$

should vanish.

2. Find a polynomial $f(x)$ such that

$f(n) - f(n - 1) = (2n - 1)^2$.

Find the sum of the series

$(2n - 1)^2 - (2n - 3)^2 + (2n - 5)^2 - \ldots \div (1)^{2} - (-1)^{2} - 1$.

3. Resolve $(x + 1)^{3}$ into partial fractions.

Assuming that the sum of the infinite series, whose $n^{th}$ term is $n^{-2}$, is $\frac{1}{3}x^2$, prove that the sum of the infinite series, whose $n^{th}$ term is $(x^{-2} + 1)^{-2}$, is $10 - n^2$.

4. If $x$ is positive and greater than unity, prove that

$\frac{1}{n} > \log_e \left( \frac{1 + \frac{1}{n}}{n} \right) > \frac{1}{n + 1}$.

Prove that the value of the product

$\left( 1 + \frac{1}{m^2} \right) \left( 1 + \frac{2}{n^2} \right) \ldots \left( 1 + \frac{n}{m^2} \right)$

lies between $e^k$ and $e^{k + 2}$.
5. Prove that the value of the determinant
\[
\begin{vmatrix}
1 & a_1 & a_2 & a_3 & \cdots & a_n \\
a_1 & 1+a_1 & a_2 & a_3 & \cdots & a_n \\
a_2 & a_1 & 1+a_2 & a_3 & \cdots & a_n \\
a_3 & a_2 & a_1 & 1+a_3 & \cdots & a_n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_n & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1+a_n
\end{vmatrix}
\]
is \(1+a_1+a_2+a_3+\cdots+a_n\).

Deduce, or prove in any other manner, that the value of the determinant
\[
\begin{vmatrix}
x_1 & a_2 & a_3 & \cdots & a_n \\
a_2 & x_2 & a_3 & \cdots & a_n \\
a_3 & a_2 & x_3 & \cdots & a_n \\
a_n & a_{n-1} & a_{n-2} & \cdots & x_n
\end{vmatrix}
\]
is \(\left(1+\sum_{r=1}^{n} \frac{a_r}{x_r-a_r}\right) \times \prod_{r=1}^{n} (x_r-a_r)\).

6. The escribed circles of a triangle \(ABC\) touch the sides \(BC, CA, AB\) to which they are escribed at the points \(D, E, F\). Prove that the ratio of the areas of the triangles \(DEF\) and \(ABC\) is \(r:2R\).

Prove also that the squares on the sides of the triangle \(DEF\) are \(a^2-bc \sin^2 A, b^2-ca \sin^2 B, c^2-ab \sin^2 C\).

7. Show that \(\left(2x^{2n-1} - \frac{1}{x^{2n+1}}\right) \left(\frac{x^2-1}{x}\right)\) can be expressed as a polynomial in \(x^{-\frac{1}{x}}\) of degree \(2n\), containing only even powers of \(x^{-\frac{1}{x}}\), and determine the coefficient of the highest term and the term free from \(x^{-\frac{1}{x}}\).

Prove that
\[
\sin (2n+1) \theta = 2^{2n} \sin \theta \prod_{r=1}^{n} (\sin^2 \frac{\pi}{2n+1} - \sin^2 r \theta).
\]

8. If \(y = (1-x^2)^{n-3}\), prove that
\[
(1-x^2) \frac{dy}{dx} + (2n-1)xy = 0,
\]
and that
\[
(1-x^2) \frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{1}{2} y = 0,
\]
where \(x\) denotes \(\left(\frac{d}{dx}\right)^{n-1} y\).

Transform the last equation by the substitution \(x = \cos \theta\).

9. Prove that
\[
\int_{0}^{\pi} \frac{d\theta}{a+b \cos \theta} = \pi \left(\frac{a^2-b^2}{a^2}\right)^{-\frac{1}{2}},
\]
if \(a\) is positive, and find its value when \(a\) is negative.

Find also the value of the integral when the upper limit is \(\pi\), \(\pi\) being any positive integer.

1925. 3 HOURS

1. Eliminate \(y\) and \(z\) from the equations
\[
y = \frac{x+a}{by+1}, \quad z = \frac{y+a}{bz+1}, \quad x = \frac{z+a}{bz+1}.
\]

Deduce that \(x, y,\) and \(z\) are all equal and that their common value is either
\[
\sqrt[+]{\frac{a}{b}} \text{ or } \sqrt[+]{\frac{a}{b}},
\]
unless \(ab+3 = 0\).

Investigate what happens when \(ab+3\) is equal to 0.

2. Determine the differential coefficient of the function
\[
\sin x (2+\cos 2x) = x (1+2 \cos 2x),
\]
and prove that this differential coefficient is of the same sign as \(\sin x\) when \(x\) is positive, and that it is of the opposite sign to \(\sin x\) when \(x\) is negative.

What inferences concerning the function itself can you draw from these results?

Prove that, when \(x\) is positive and less than half a right angle,
\[
\sin x (2+\cos 2x) = x (1+2 \cos 2x)
\]
is positive and less than 0.0181.

3. Prove that
\[
a^2-bc \quad b^2-ca \quad c^2-ab = \begin{vmatrix}
\begin{array}{ccc}
a & b & c \\
c & a & b \\
b & c & a
\end{array}
\end{vmatrix}
\]

Deduce, or prove otherwise, that, if
\[
ax+by+cz = 1,
\]
\[
ax+by+cz = 0,
\]
then
\[
\begin{vmatrix}
x & y & z \\
x & a & b \\
z & a & b
\end{vmatrix} = 1.
\]

4. If \(O\) is the circumcentre and \(P\) is the orthocentre of the triangle \(ABC\), prove that
\[
OP^2 = R^2 (1-8 \cos A \cos B \cos C).
\]
Deduce that \( \cos A \cos B \cos C \) cannot exceed 1/8, and that it is equal to 1/8 only if the triangle is equilateral.

5. Prove that the roots of the equation

\[ x^3 + x^2 - x + 1 = 0 \]

are the values of

\[ 2 \pi k \]

\[ \frac{2 \pi n}{6} + i \sin \frac{2 \pi n}{6} \]

where \( r = 1, 2, \ldots, n-1 \).

If \( a = \cos \frac{2 \pi}{15} + i \sin \frac{2 \pi}{15} \), prove that

\[ a + a^2 + a^4 + a^6 + a^8 + a^{10} = 1 \]

and

\[ a^2 + a^4 + a^6 + a^8 + a^{10} = -1 \]

are the roots of the quadratic

\[ x^2 + x - 3 = 0. \]

6. If \( a + b + c = 3 \), \( \beta + \gamma + \alpha = a \), \( \beta \gamma + \alpha \beta = b \), and

\[ a^2 + \beta^2 + \gamma^2 = s_n \]

prove that \( s_n \) is equal to \( n \) times the coefficient of \( x^n \) in the expansion of

\[ (1 - px + qx^2 - rx^3) \]

in ascending powers of \( x \).

If \( \beta \gamma + \alpha \beta = 0 \), prove that

\[ 4(s_n - s_1) = (s_n - s_2)^2. \]

7. If \( a \) denotes any one of the roots of the equation

\[ x^3 = 2x + 1, \]

and if \( \beta = a^2 - a - 2 \), prove that \( \beta \) is another root of this cubic equation.

Prove further that \( \beta^2 - \beta - 2 \) is the third root.

8. Sketch the curve whose equation is

\[ x^2y^2 - 2axy(a^2 - x^2) + (a^2 - x^2)^2 = 0. \]

Write down a formula for the area contained between the axis of \( x \) and the lower branch of the curve as an integral, and by using the substitution \( x = a \sin \phi \), prove that this area is equal to

\[ 2a^2 \int_0^{\pi/2} (1 - \cos \phi) \, d\phi. \]

Prove that this expression is equal to

\[ (3 \pi - 8) a^2 / 2. \]

9. By using the substitution

\[ x = \frac{4 \sin \theta}{2 - \sin \theta}, \]

evaluate the integral

\[ \int_0^{\pi/2} (x - 1)^2 \sqrt{(14x - 3x^2 - 15)} \, dx \]

10. Solve the simultaneous equations

\[ x + y + z = -2, \]
\[ x^2 + y^2 + z^2 = 50, \]
\[ (y + 2) (x + cables) (x + y) = 90. \]

2. If \( a, b, c, d, e \) are constants and

\[ U_0 = a, \quad U_1 = ax + b, \quad U_2 = ax^2 + 2bx + c, \]
\[ U_3 = ax^3 + 3bx^2 + 3cx + d, \]
\[ U_4 = ax^4 + 4bx^3 + 6cx^2 + 4dx + e, \]

prove that

\[ U_0 U_2 U_4 + 2U_1 U_2 U_3 - U_0 U_3^2 - U_1 U_4 - U_2^2 \]

is independent of \( x \).

3. Prove that the arithmetic mean of \( n \) positive numbers is not less than their geometric mean.

If \( a, b, c \) are positive, prove that

\[ \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \]

4. Prove that the \( n \)th convergent of the continued fraction

\[ b \quad b \quad b \]
\[ a + a + a + \ldots \]

is equal to

\[ 2b \left( \frac{(a+c)^n - (a-c)^n}{(a+c)^{n+1} - (a-c)^{n+1}} \right) \]

where \( c = \sqrt{a^2 + 4b} \).

5. Explain briefly how complex numbers may be represented in a diagram.

If \( z = z^{-1} \) and if \( z \) describes the circle of unit radius whose centre is the origin, prove that the locus of \( Z \) is also a circle, and find its centre and its radius.

6. If

\[ a = \cos \frac{2 \pi}{7} + i \sin \frac{2 \pi}{7}, \]

prove that the quadratic whose roots are

\[ a + \alpha^2 + \alpha^4, \quad \alpha^3 + \alpha^5 + \alpha^6 \]

is

\[ x^2 + x + 2 = 0. \]

Deduce or prove otherwise that

\[ \sin \frac{\pi}{7} \sin \frac{2 \pi}{7} \sin \frac{3 \pi}{7} = \frac{\sqrt{7}}{8}. \]
7. If \( a, b, c \) are the roots of the cubic

\[ x^3 + 3Hx + G = 0, \]

prove that \( A a^2 + B a + C \) is expressible in the form

\[ \frac{L a + M}{N a + P}, \]

where \( L: M: N: P = 3HA^2 + B - AC \cdot GA^2 + BC : - A : B \).

If \( a, b, c \) are the roots of the cubic \( x^3 + 3x - 3 = 0 \), construct the cubic whose roots are

\[ a^3 + a + 4, \quad \beta^2 + \beta + 4, \quad \gamma^2 + \gamma + 4. \]

8. Prove that the differential coefficient of

\[ \text{sinh}^{-1} \frac{apx + b(x + p) + c}{(x - p) \sqrt{(ax + b^2)}} \]

is

\[ \frac{(x - p) \sqrt{(ax^2 + 2bx + c)}}{(x - p) \sqrt{(ax^2 + 2bx + c)}}. \]

9. If the coordinates of a point on a curve are given as functions of a parameter \( t \), prove that the sectoral area of the curve is

\[ \frac{1}{2} \left( \frac{dy}{dx} - y \frac{dy}{dx} \right) dt. \]

Prove that the complete area of the curve traced out by the point \( (2a \cos t + a \cos 2t, 2a \sin t - a \sin 2t) \) is \( 2 \pi a^2 \).

1927. 3 HOURS

1. Simplify the equation

\[ \frac{y + z}{1 - yz} + \frac{z + x}{1 - zx} + \frac{x + y}{1 - xy} = \frac{y + z}{1 - yz} + \frac{z + x}{1 - zx} + \frac{x + y}{1 - xy}. \]

giving your result in a symmetrical form.

Deduce that, if the equation is satisfied, then either

\[ x + y + z = xyz \] or else \( yz + zx + xy = 1. \)

2. Prove that the cubic equation

\[ x^3 + 3Hx + G = 0 \]

is reducible to the form

\[ (x + \mu)^2 - \nu (x + \mu)^3 = 0 \]

if \( \mu \) and \( \nu \) are taken to be the roots of the quadratic

\[ H \mu^2 - G \mu - H^3 = 0. \]

If the roots of the quadratic are real and unequal, prove that the cubic has only one real root.

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7. Write down the expansion of
\[ \sin x - x \cos x \frac{x}{\sqrt{3}}. \]
arranged in ascending powers of \( x \) as far as the term in \( x^3 \), and give the general term of the expansion.

Prove that, when \( 0 < x < 5 \), every term in the expansion is numerically less than the preceding term and of the opposite sign to it.

Deduce that, for this range of values of \( x \),
\[ 0 < \sin x - x \cos x \frac{x}{\sqrt{3}} < \frac{x^5}{270}. \]

8. Prove that \( x^2 - 2x \cos \theta + 1 \) is expressible in quadratic factors in the form
\[ \prod_{r=0}^{n-1} \left[ x^2 - 2x \cos \left( \theta + \frac{2r\pi}{n} \right) + 1 \right]. \]

Deduce, or prove otherwise, that
\[ \tan^2 \frac{\theta}{2} \cdot \cos \phi = \prod_{r=0}^{n-1} \left[ \frac{\cos \phi - \cos \left( \frac{2r\pi}{n} \right)}{\cos \phi + \cos \left( \frac{2r\pi}{n} \right)} \right]. \]

9. With the help of the substitution
\[ x = \frac{y - 1}{y + 1}, \]
or otherwise, prove that
\[ \int_{-1}^{1} \frac{(x+1) \, dx}{(3x^2 - 2x + 3) \sqrt{(3x^2 + 2x + 3)}} = \frac{\pi}{6\sqrt{3}}. \]

GROUP III (PAPER 7)

MATHEMATICAL DISTINCTION PAPER

1918. 2 Hours

1. A, B, C are three collinear points; \( O \) is any other point. Prove that
\[ BC \cdot OA^2 + CA \cdot OB^2 + AB \cdot OC^2 = -BC \cdot CA \cdot AB. \]

2. \( PAC \) is a triangle; \( O \) is a point outside the plane \( PAC \) such that \( OP \) is perpendicular to \( PA \) and \( PC \). Prove that it is also perpendicular to \( PB \), where \( B \) is any point in \( AC \).

3. \( H \) is the orthocentre of a triangle \( ABC \); \( P, Q, R \) are its reflections in the sides of the triangle. Prove that \( P, Q, R \) lie on the circumcircle.

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8. Prove that the equation of the director circle of the conic
\[ u \equiv ax^2 + 2hxy + by^2 + c = 0 \]
is the axes being rectangular.

Prove that \( \omega - \lambda a = 0 \), where \( \lambda \) is a root of the equation
\[ \lambda^2 - \lambda (a+b) + ab - b^2 = 0, \]
is the equation of a pair of directrices of the conic.

9. Find the coordinates of the reflection in BC, a side of the triangle of reference, of the point whose trilinear coordinates are \( \alpha, \beta, \gamma \).

Prove that the equation of the reflection in BC of the line
\[ l_{a+m+b+n} = 0 \]
is
\[ l_{a+m+b+n} - 2a(1-m)\cos C - n\cos \beta = 0. \]

1919. 2 Hours

1. Triangles \( PAB, ABQ \) are drawn on the same side of \( AB \) so as to be directly similar; and a line \( PR \) is drawn so as to be equal and parallel to \( AB \) (and in the same sense). Prove that \( RAP \) is directly similar to the first two triangles.

2. Describe the method of inversion; and prove that if \( A', B' \) are the points inverse to \( A, B \) with respect to a circle of radius \( k \) and centre \( O \), then
\[ \frac{A'B'}{AB} = \frac{OA' \cdot OB'}{OA \cdot OB} = \frac{k^2}{k^2}. \]

Invert the theorem that in general
\[ \frac{AB + BC + AC}{BC + AC} \]
with respect to (i) a circle in the plane \( ABC \), (ii) a sphere in space.

3. Show that if a point \( P \) moves so that the tangents from \( P \) to two given circles are in a constant ratio, then the locus of \( P \) is a circle coaxial with the given circles.

Two circles \( S, S' \) are external to each other, and the points \( A, B \) are their centres of similitude. Show that the circle on \( AB \) as diameter is coaxial with \( S, S' \); and that \( S \) and \( S' \) subtend equal angles at any point on this circle.

4. State and prove Pascal's theorem on the intersections of opposite sides of a hexagon 123456 inscribed in a conic.

Consider the special case in which 1, 2 coincide and also 4, 5; and show that the theorem enables us to construct any number of points lying on a conic which touches two given lines at given points and passes through another given point.

5. Find the equation to a parabola referred to a point \( P \) on the curve as origin, the (oblique) axes being the diameter through \( P \) and the tangent at \( P \).

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(2) $X$, $Y$, $Z$ are points on the sides $BC$, $CA$, $AB$ of a triangle respectively. Prove that the value of the expression

$$BX \cdot CY \cdot AZ + ZX \cdot YA \cdot ZB$$

is unaltered by projection.

3. $PV$ is a fixed diameter of a parabola; $QVQ'$ is a variable ordinate of this diameter. Prove that the circle $QQ'$ meets the parabola in a second fixed point, and meets $PV$ again in a point whose distance from $V$ is fixed.

Prove that, if $QQ'$ is also a focal chord, the diameter of the circle is $2SP \left(9SP + 16AS\right)/AS^2$, where $S$ is the focus and $A$ the vertex of the parabola.

4. Prove that the reciprocal of a conic with regard to a circle whose centre is at a focus of the conic is a circle.

The orthocentre of a triangle which circumscribes a parabola lies on the directrix of the parabola. Prove this theorem by reciprocation, or otherwise.

5. The points $(f, g), (-f, -g)$ are extremities of a diameter of a conic; the points $(f', g'), (-f', -g')$ are extremities of the conjugate diameter. Prove that the equation of the conic is

$$(ax - fy)^2 + (g'y - f'g)^2 = (f'g - f'g)^2.$$  

Show that of the expressions

$$(f + g')^2 + (g - f')^2, \quad (f' - g)^2 + (g' + f)^2$$

one is the square of the sum and the other the square of the difference of the semi-axes of the conic.

6. $P$ is a point on the major axis of an ellipse, $P'$ is the point in which the polar of $P$ with regard to the ellipse cuts the major axis. Prove that, in general, there are two straight lines $(l, l')$ through $P'$, equally inclined to the axes, such that the square of the distance of any point on the ellipse from $P$ is in a fixed ratio to the product of its distances from $l, l'$.

Consider in particular the cases when $P$ is (i) a focus, (ii) an extremity of the major axis.

7. $PP'$ are the extremities of a double ordinate of an ellipse; $Q, Q'$ are the corresponding points on the auxiliary circle. Prove that $CO, CO'$ are the asymptotes of the confocal hyperbola through $P$ and $P'$, $C$ being the centre of the ellipse.

Prove that, as $P$ and $P'$ vary, the tangents at the ends of the latera recta of the confocal hyperbolas touch one or other of two fixed parabolas.

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8. Prove that the locus of mid-points of chords of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

which are parallel to the straight line $mx - ny = 0$ is

$$IX + mY = 0,$$

where

$$X \equiv ax + hy + g, \quad Y \equiv hx + by + f.$$

Show that each of the diameters $IX + mY = 0, I'X + m'Y = 0$ bisects chords parallel to the other, if

$$c + m = \alpha \left(mx + m'y + bmn'\right) = 0;$$

and show that, when this condition is satisfied, they will be perpendicular if $I' + m'm' = 0$.

Are these results true if the conic is a parabola?

9. Find the extent of the perpendicular from the point $a', b', c'$ on the straight line $lax + mb + ncy = 0$.

Find the condition that the above line may touch the inscribed circle of the triangle of reference; and deduce, or obtain otherwise, the trilinear equation of this circle.

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1921. 2 Hours

1. Prove that the locus of a point at which the equal sides of an isosceles triangle subtend equal angles consists of a line, a line with a segment excluded, and an arc of a circle.

2. If a transversal meets the sides $BC$, $CA$, $AB$ of a triangle at $D, E, F$ respectively, prove that

$$BD \cdot CE \cdot AF = - DC \cdot EA \cdot FB.$$  

The inscribed circle touches the sides $BC$, $CA$, $AB$ of a triangle at $X, Y, Z$ respectively, and the escribed circles touch the same sides at $X_1, Y_1, Z_1$ respectively. If $YZ_1$ meets $BC$ in $P, ZX_1$ meets $CA$ in $Q$, and $XY_1$ meets $AB$ in $R$, prove that $P, Q, R$ are collinear.

3. Prove that the polar reciprocal of a circle $S$ with respect to a circle $S'$ is a conic which has the centre of $S'$ as a focus. What is the reciprocals of a pair of points which are inverse with respect to $S$?

Reciprocate the theorem: The inverse of a circle $S$ with respect to a point $O$ is a straight line, or a circle, according as $S$ does, or does not, pass through $O$.

4. State and prove Pascal's Theorem on the intersections of opposite sides of a hexagon inscribed in a conic.

Points $E, F$ are taken on the sides $CD, DA$ of a parallelogram $ABCD$. Prove that the tangent at $B$ to the conic through $A, B, C, E, F$ is parallel to $EF$; and hence, or otherwise, prove that the line
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joining $B$ to the mid-point of $EF$ passes through the point of intersection of the join of the mid-points of $A'F'$ and $BC$ with the join of the mid-points of $AB$ and $CE$.

5. Two points $P, P'$ are taken on the axis of $x$ on the same side of the origin $O$, and two points $Q, Q'$ are taken on the axis of $y$ on opposite sides of the origin where

$$OP, OP' = OQ, OQ'. $$

Prove that, if $P, P'$ have any fixed positions whilst $Q, Q'$ vary, the locus of the point of intersection of $PQ$ and $P'Q'$ is a circle; and that if $Q, Q'$ have any fixed positions whilst $P, P'$ vary, the locus is a circle orthogonal to the first.

6. Find the equation of the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$.

Show that the locus of a point from which the tangents to $x^2/a^2 + y^2/b^2 = 1$ make equal angles with $y = mx$ is

$$x^2 - y^2 + xy (m - 1/m) = a^2 - b^2. $$

7. Show that four normals can be drawn from a point to the ellipse $x^2/a^2 + y^2/b^2 = 1$, and that, if $lx + my = 1$ is the chord joining the feet of two normals, then the chord joining the feet of the other two is $x/f^2 + y/g^2 = 1 = 0$.

Prove that, if one chord passes through a fixed point, the other chord envelops a parabola.

8. Find the lengths of the axes of the conic

$$ax^2 + 2hxy + by^2 = 1, $$

the coordinate axes being rectangular.

An ellipse of given area has a given centre and touches a given line: prove that it passes through two fixed points.

9. Find the condition that

$$xyz + mxz + nxy = 0 $$

should be a parabola, the coordinates being real.

Prove that there are two real parabolae which circumscribe the triangle of reference and pass through the point

$$(p, q, r) (p + q + r = 1) ,$$

provided that the product $pqr$ is negative. Interpret the condition geometrically, and illustrate by a figure.

1922. 3 Hours

1. Prove that the radical axes of three circles, taken two and two together, meet in a point.

$D, E, F$ are the feet of the perpendiculars from the vertices $A, B, C$ of a triangle on the respectively opposite sides. If $EP, EP'$

are drawn perpendicular to $BC$; $FQ, DQ'$ to $CA$; and $DR, ER'$ to $AB$; then the six points $P, P', Q, Q', R, R'$ lie on a circle.

2. Prove that any circle through a pair of points inverse with regard to a circle cuts that circle orthogonally.

$J, J'$ are a pair of points inverse with regard to a circle; $P$ is a point equidistant from $J$ and $J'$; a line through $P$ meets the circle in $Q, R$. Prove that, when $Q$ and $R$ lie on the same side of the diameter $IJ$, the sum of the angles $IQJ, IRJ$ is equal to the angle $IPJ$.

3. Prove that the inverse of a circle with regard to a circle in its plane is either a circle or a straight line.

The straight lines $u, v$ are the bisectors of the angles between two given straight lines $p, q$. These lines are inverted with regard to any point in their plane into circles $U, V, P, Q$ respectively. Prove that $P$ is the inverse of $Q$ with regard to either $U$ or $V$.

4. Define conjugate lines with respect to a conic, and prove that conjugate lines through a focus are at right angles.

Prove that any diameter of a conic, the perpendicular from a focus on a tangent at an extremity of this diameter, and the directrix corresponding to the focus are concurrent.

5. Find the coordinates of the foot of the perpendicular from the point $(p, q)$ to the straight line $lx + my + n = 0$.

The feet of the perpendiculars drawn from the point $(p, q)$ to the straight lines $ax^2 + 2hxy + by^2 = 0$ from the points $P$ whose coordinates are $p, q$, are $R$ and $S$. Show that the area of the triangle $PRS$ is

$$(ap^2 + 2hpq + bq^2) (a^2 - ab) / [(a - b)^2 + 4h^2]. $$

6. Find the coordinates of the pole of the chord joining the points $P, Q$ on the conic $ax^2 + by^2 + c = 0$ whose eccentric angles are $\alpha - \beta, \alpha + \beta$.

If $T$ is this pole and $S$ is a focus of the conic, prove that

$$ST^2 = sc/\beta, SP, SQ. $$

7. Find the equation of the tangents from the origin to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. $$

Prove that the origin will be a focus of this conic if

$$bc - f^2 = ca - g^2 $$

and show that the coordinates of the other focus are then

$$2 (hf - bg) / (ab - h^2), 2 (gh - af) / (ab - h^2). $$

8. Find separately the equations of the axes of the rectangular hyperbola

$$2x^2 + 3xy - 2y^2 - 11x + 13y - 5 = 0, $$

and its equation referred to them.
Show that the equations of the tangents to it at its real vertices are $x^2 - 3y^2 + 2 = 0$, and $x - 3y + 18 = 0$.

Give a rough drawing of the curve.

9. Find the condition that the straight lines whose trilinear equations are $a + m \beta + n \gamma = 0$, $a + m' \beta + n' \gamma = 0$, may be perpendicular.

Prove that the equation of the circle described on the base $BO$ of the triangle of reference as diameter is $\beta \gamma - a^2 \cos \theta + a \beta \cos \theta + a \gamma \cos \theta = 0$.

GROUP III (PAPER 6)

MATHEMATICAL DISTINCTION PAPER

1923. 3 HOURS

Not more than eight questions should be attempted by any Candidate.

1. A circle touches the arc of a given semicircle internally and also touches two ordinates to the diameter; prove that the lines joining the point of contact of the circle with the semicircle to the extremities of the diameter pass through the points of contact of the circle with the ordinates.

2. Give a geometrical construction for the circle passing through a given point and centred with two given circles, which do not meet in real points, (1) when the given point is on, (2) not on the common diameter of the two circles.

$U$ and $F$ are two fixed circles, $P$ a given point; $Q$ is the inverse of $P$ with respect to $U$, $R$ is the inverse of $Q$ with respect to $V$, $S$ the inverse of $R$ with respect to $U$, and so on. Prove that all these points are concyclic.

3. Prove that the anharmonic ratio of the pencil subtended at a variable point on a conic by four fixed points on the conic is constant. What relation holds between the four fixed points when the pencil is harmonic?

$PR$ and $QS$ are parallel chords of a conic, $T$ is the pole of $PO$. Prove that $PS$ and $QK$ intersect on the chord through $T$ parallel to the given chords, and that this chord is bisected at the point of intersection.

4. $P$ is a point on a central conic, $QQ'$ is a chord parallel to the tangent at $P$; a chord through $P$ meets the conic again in $U$.

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the circle $Q'Q''$ in $R$, and the chord $QQ'$ in $V$; also $DCD'$, $LCL'$ are the diameters parallel to $QQ''$ and $PU$ respectively. Prove that $FR \cdot FU = CD^2 \cdot CP^2$.

Deduce that the chord of curvature at $P$ in any direction is $2CD \cdot PE$, where $PE$ is the intercept made by $DCD'$ on the chord.

5. Find the equation of the lines bisecting the angles between the lines given by the equation $ax^2 + 2hxy + by^2 = 0$.

The latter lines are rotated about the origin positively through an angle $\alpha$; prove that their equation in the new position is

\[ (ax^2 + 2hxy + by^2) \cos^2 \alpha + (hx^2 + 2hxy + oy^2) \sin^2 \alpha + 2(hs - b)(x^2 - y^2) \sin \cos \alpha \cdot a - 0. \]

6. $AB$, $CD$ are given finite portions of the same straight line; prove that the locus of a point at which $AB$ and $CD$ subtend equal (finite) angles is a circle.

Prove that the circle is coaxial with the circles on $AB$ and $CD$ as diameters, and that it is always real, unless one portion lie entirely within the other.

7. Find the equation of the normal to the ellipse $ax^2 + ay^2/b^2 = 1$ at the point of eccentric angle $\phi$.

From a point $P$ on the ellipse, whose coordinates are $h, k$, the three normals (other than the normal at the point) are drawn. Prove that the conic $x(x+h)/a^4 + y(y+k)/b^4 = 0$ passes through their feet and the other end of the diameter through $P$; and show that the general equation of a conic through the feet of the three normals is

\[ \lambda (by^2 + ax^2 - a^2 h^2) + \mu (by(hx + a^2 y + b^2 k)) + \nu (a^2 - b^2) xy - a^2 bh y + b^2 kx = 0, \]

where $\lambda, \mu, \nu$ are arbitrary constants.

8. Prove that through any point two conics confocal with $ax^2 + ay^2/b^2 = 1$ can be drawn; and that confocal conics cut at right angles.

Show that the polar of a given point $P$ with respect to a family of confocals envelop a parabola; and that the line joining $P$ to the centre is the directrix of the parabola.

9. Prove that in trilinear (or areal) coordinates the equation of the first degree represents a straight line.

$D, E, F$ are points on the sides of a triangle $ABC$ such that $AD, BE$ and $CF$ are concurrent. Any line cuts the sides of the triangle
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DEF in points P, Q, R: AP cuts BC in P', BQ cuts CA in Q', CR cuts AB in R'. Prove that P', Q', R' are collinear.

10. Prove that the reciprocal of a circle with regard to any point is a conic, and find the eccentricity and the lengths of the axes of this conic.

'A chord of a circle which touches a concentric circle is bisected at the point of contact.' Reciprocate this proposition with regard to any point.

1924. 3 Hours

1. Prove that the locus of a point, such that the lengths of the tangents drawn from it to two fixed co-planar circles are in a constant ratio, is a circle coaxal with the two circles.

Hence show that the feet of the perpendiculars drawn to the four common tangents to two circles from any point on the straight line joining their centres lie on a circle coaxal with the two circles.

2. If each edge of a tetrahedron is perpendicular to the opposite edge, prove that the perpendiculars drawn from the vertices to the opposite faces meet at a point.

Prove also that this is not true for any other kind of tetrahedron.

3. Define a cross-ratio of a range of four points and of a pencil of four straight lines, and show how to express all the cross-ratios in terms of any one of them.

A straight line OX meets the sides BC, CA, AB of a triangle ABC in the points A', B', C' respectively. Show that corresponding cross-ratios of the pencil O (A, B, C, X) and the range (A', B', C', O) are equal.

4. If A, B, C, D are four points on a conic and BC meets AD at L, CA meets BD at M, and AB meets CD at N, prove that the lines AN, BM, CL are concurrent.

5. Prove that the tangents at the ends of a focal chord of a conic meet the corresponding directrix, and that, if the conic is a parabola, they are at right angles.

A parabola cuts the parabola \( y^2 = 4ax \) orthogonally at each end of a focal chord. Prove that the chord is a focal chord of the first parabola, that the two parabolas have equal latus-recta, and

6. Prove that the straight line \( lx + my + n = 0 \) is a tangent to the circle \((x - a)^2 + (y - b)^2 = r^2\), if \((al + bm + n)^2 = a^2(b^2 + m^2)\).

Two circles with their centres at the points \((a, 0)\) \((-a, 0)\) cut orthogonally. Prove that the envelope of their common tangents is the ellipse \(2x^2 + y^2 = 2a^2\).

7. Find the coordinates of the point of intersection of the tangents to the ellipse \(b^2x^2 + a^2y^2 = a^2b^2\) at the points whose eccentric angles are \(a\) and \(b\).

8. A chord of this ellipse is a tangent to the confocal conic \(x^2/(a^2 - a) + y^2/(b^2 - b) = 1\). Prove that the intersection of the tangents at the extremities of the chord with the diameter parallel to it is the circle \(a(x^2 + y^2) = a^2b^2\).

9. The point \((a, c/f)\) on the rectangular hyperbola \(xy = c^2\) being called the point \(t\), find the condition that the points \(t_1, t_2, t_3, t_4\) should be the vertices of a triangle and its orthocentre.

Prove that, when this condition is satisfied, the three circles, which have as extremities of a diameter the three pairs of opposite vertices of the quadrilateral formed by the tangents at these points, touch each other at the centre of the hyperbola.

10. Prove that the middle points of chords of the conic

\[ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\]

which are parallel to the straight line \(x = y/\tan \alpha\), lie on the straight line \(lx + my = 0\), where \(X\) and \(Y\) denote respectively

\[ax + hy - g, \quad hx + by - f.\]

Prove that there are in general two pairs of conjugate diameters of the conic, which include an angle \(\alpha\), and that the equations of these two pairs are

\[(lx^2 - 2axy + aY^2) + \{b(x^2 - Y^2) - (a - b)XY\} \tan \alpha = 0,\]

and that the corresponding tangential equation is

\[fWX + gYN - kLM = 0,\]

A parabola touches the sides \(BC, CA, AB\) of the triangle of reference at the points \(P, Q, R\) respectively. Prove that the sides of the triangle \(PQR\) pass each through a fixed point, and that \(AP, BQ, CR\) meet at a point on the ellipse, whose equation in the original coordinates is \(y^2 + ax + xy = 0\).
1. Prove the accuracy of the following method of constructing a triangle \(ABC\) with given angles, such that the distances of its vertices from an internal point \(O\) are given.

Take \(A\) at the required distance from \(O\). On \(OA\) describe a triangle \(OAX\) similar to \(CAB\), the points \(O, A, X\) corresponding to \(C, A, B\) respectively. On \(OX\), on the side remote from \(A\), describe a triangle \(OBX\) such that \(OB\) is of the required length and \(BX : AX : OC : OA\). On \(AB\) on the side remote from \(X\) describe a triangle of the given shape. This will be the triangle required.

2. Prove that the inverse of a sphere is a sphere or a plane, according as the origin of inversion is not or is on the first sphere.

Prove that any plane cuts a system of spheres, which touch at the same point, in a system of coaxial circles, the radical axis being the line of intersection of the plane and the common tangent-plane of the spheres; and deduce by inversion with respect to the point or in any other manner that, if the sections of a sphere by a system of parallel planes be projected from any point of the sphere on to a plane parallel to the tangent-plane at the point, the projections will be a system of coaxial circles.

3. Prove that the reciprocal of a circle with respect to a second circle is a conic with a focus at the centre of the second circle, and determine the eccentricity of the conic.

A conic has a focus and one tangent given, and also its eccentricity. Prove that the envelope of the directrix corresponding to the given focus is a conic with a focus at this given focus.

4. Prove that the pencil formed by joining a variable point on a conic to four fixed points on the conic has constant cross-ratios.

\(A, B, C, D\) are four points on a conic. A straight line drawn through \(D\) meets \(BC, CA, AB\) at \(P, Q, R\) respectively and meets the conic again at \(S\). Prove that the range \(P, Q, R, S\) is on the line and the range \(A, B, C, D\) on the conic are projective.

5. Show that the equations of circles of a coaxal system with real intersections may be taken to be

\[x^2 + y^2 + 2\lambda x - a^2 = 0,\]

where \(\lambda\) is a variable parameter.

Prove that two circles of the system may be drawn to touch any straight line, which does not meet their radical axis between the common points, and that these two circles cut at an angle

\[2 \tan^{-1}\left(\frac{p_1 p_2}{a^2}\right),\]

where \(p_1\) and \(p_2\) are the lengths of the perpendiculars drawn to the straight line from the common points.

6. Find the length of that chord of the parabola \(y^2 = 4ax\) which

is drawn through the point \((k, k)\) so as to make an angle \(\theta\) with the axis of \(x\).

Two chords of a parabola are drawn in given directions so that their lengths have a given ratio. Show that the locus of their point of intersection is a straight line.

7. Prove that the equation of the pair of tangents to the ellipse \(x^2/a^2 + y^2/b^2 = 1\) at its intersections with the straight line \(lx + my + n = 0\) is

\[
\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)
\left(\frac{x^2}{l^2} + \frac{y^2}{l^2} - 1\right) = (lx + my + n)^2.
\]

Prove that any tangent to this ellipse meets each of the two conics, which are confocal with it and pass through the point \((a, b)\), in a pair of points, at each of which the tangents are perpendicular.

8. A variable tangent to the hyperbola \(xy = c^2\) meets the asymptotes at the points \(P\) and \(Q\). Find the coordinates of the point of intersection of the tangents at \(P\) and \(Q\) to the circle, which passes through \(P\) and \(Q\) and the centre of the hyperbola, and prove that the locus of the point is the hyperbola \(xy = c^2 \sec^2 \omega\), where \(\omega\) is the angle between the asymptotes.

9. Prove that, if the straight line \(lx + my + n = 0\) is a tangent to the conic \(ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\),

\[AP^2 + BP^2 + CP^2 + DP^2 = 2(lm + nh) + 2hP_1 + 2hP_2 + 2H_1h = 0,
\]

where

\[A \equiv bc - f^2, \quad P \equiv gh - af, \quad &c.\]

A parabola passes through a fixed point \(O\), and has double contact with a fixed conic \(S\). Prove that the envelope of the chord of contact is the conic, similar and similarly situated to \(S\), which has its centre at \(O\) and has the polar of \(O\) with respect to \(S\) for its finite common chord with \(S\).

10. Prove that the equation of the conic, which has the triangle of reference as a self-conjugate triangle and touches the conic \(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0\) at the point \((x', y', z')\), is

\[x' y' z' x + y' z' z + z' x' x + x' y' y = x'y' z' + y' z' x' + z' x' y' = 0,\]

where \(X', Y', Z\) denote respectively

\[ax' + by' + cz', \quad bx' + cy' + az', \quad cx' + ay' + bz'.\]
\( \sqrt{(AB \cdot AC)} \), is the fourth common tangent of the inscribed circle of the triangle and the circle escribed to the side \( BC \).

A circle touches \( AB \) and \( AC \), the chord of contact passing through the centre of the inscribed circle of the triangle \( ABC \). Prove that it touches the circle circumscribed about the triangle \( ABC \).

3. Prove that two straight lines, whether coplanar or not, are divided in the same proportion by a system of parallel planes.

A variable parallelogram \( A_1A_2A_3A_4 \) has its vertices on each of four straight lines \( l_1, l_2, l_3, l_4 \) no two of which are coplanar. Prove that the locus of the point of intersection of its diagonals is a straight line; and show how the parallelogram may be constructed when a point on the locus is taken to be the point of intersection of the diagonals.

4. Prove that a variable tangent to a conic determines two homographic ranges on two fixed tangents to the conic.

A triangle is formed by two fixed tangents to a conic and a third variable tangent. Prove that the locus of its orthocentre is a hyperbola with its asymptotes perpendicular to the two fixed tangents, when the conic is a central conic; and is a straight line, when the conic is a parabola.

5. Prove that:

(1) If the coordinates of the vertices \( A, B, C \) of a triangle \( ABC \) are respectively \( (x_1, y_1), (x_2, y_2), (x_3, y_3) \), the angle \( A \) is acute, right, or obtuse according as

\[
(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3)
\]

is positive, zero, or negative.

(2) If the equations of the sides \( BC, CA, AB \) are respectively

\[
l_1x + m_1y + n_1 = 0, \quad l_2x + m_2y + n_2 = 0, \quad l_3x + m_3y + n_3 = 0,
\]

the angle \( A \) is acute, right, or obtuse according as

\[
(l_1l_2 + m_1m_2)(l_2m_3 - l_3m_2) - (l_2m_1 - l_1m_2)(l_1m_3 - l_3m_1)
\]

is positive, zero, or negative.

6. The tangent at a point \( P \) on the ellipse, \( x^2a^2 + y^2b^2 = 1 \), meets the tangents at \( Q \) and \( Q' \), the extremities of a diameter of this ellipse, at the points \( X \) and \( Y \). Prove that the equation of the circle on \( XY \) as diameter, referred to \( P \) as origin and axes parallel to those of the ellipse, is

\[
x^2 + y^2 - x^2 \sin^2 \alpha - b^2 \cos^2 \alpha + 2(\alpha x \sin \alpha - by \cos \alpha) \cot (\alpha - \theta) = 0,
\]

where \( \alpha \) is the eccentric angle of \( P \) and \( \theta \) is the eccentric angle of \( Q \) or \( Q' \).

If \( P \) remains fixed while the diameter varies, show that the common points of the system of coaxial circles so formed lie, one on the

**GROUP III: MATHEMATICAL DISTINCTION PAPERS**

3. Prove that the centroid of a plane triangle is that point of trisection of the distance between its orthocentre and the centre of its circumcircle which is nearer to the centre of its circumcircle.

If \( P \) is the orthocentre of a face of a tetrahedron and \( Q \) is the foot of the perpendicular drawn to this face from the opposite vertex, prove that the perpendicular drawn to this face from the middle point of \( PQ \) intersects the line joining the centroid of the tetrahedron and the centre of the sphere which can be circumscribed about the tetrahedron.
2. A, B, C, D are four points, not necessarily coplanar. Prove by inversion with respect to one of them, or by any other method, that the sum of any two of the rectangles $DA, BC, DB, CA, DC, AB$ is greater than the third, unless the four points are either collinear or concyclic, and that then the sum of two of them is equal to the third.

$ABC$ is a triangle, in which the angle $A$ is greater than $120^\circ$. Let $BA$ be produced beyond $A$ to $D$, so that $AB$ is equal to $AC$. Let $P$ be any point other than $A$.

By applying the first part of the question to the four points $A, D, C, B$, we find that $PA + PB + PC$ is greater than $AB + AC$.

3. $S$ is the external centre of similitude of two fixed circles. A variable circle $(C)$, whose centre is $X$, touches each of the two fixed circles, one at the point $P$, the other at the point $Q$, the contacts being both internal or both external. Prove that

(1) $P, Q,$ and $S$ are collinear and $(C)$ cuts orthogonally a fixed circle $(C_1)$, which has its centre at $S$;

(2) the locus of $X$ is a hyperbola, which has its foci at the centres of the two fixed circles, and the locus of the harmonic conjugate of $S$ with respect to $P$ and $Q$ is the reciprocal of the locus of $X$ with respect to the circle $(C_1)$.

4. Prove that a range of four collinear points and the pencil formed by their polars with respect to any conic have the same cross-ratios (or are projective).

Assuming that a conic can be found with respect to which any two given conics are reciprocals, prove that the two pairs of tangents drawn to two orthogonal circles from a point on the chord of contact of either with their common tangents are pairs of harmonic conjugates.

5. The coordinates of the vertices $A, B, C$ of a triangle $ABC$ are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ respectively. By changing the origin to one of the vertices, or in any other manner, prove that, according to the sense of description of the triangle in the order $ABC$ is positive or negative (with the usual convention),

(1) the area of the triangle is given positively or negatively by the value of

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(2) that value of $\tan^{-1} \frac{y_2 - y_1}{x_2 - x_1} - \tan^{-1} \frac{y_3 - y_1}{x_3 - x_1}$ which lies between $0$ and $\pi$ is the value of the angle $A$ of the triangle or of the supplement of this angle.

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6. Obtain the area of a triangle inscribed in the parabola $y^2 = 4ax$, in terms of the ordinates of its vertices.

Prove that the latus-rectum of a parabola circumscribed about a triangle is equal to $2R \sin \alpha \sin \beta \sin \gamma$, where $R$ is the radius of the circle circumscribed about the triangle and $\alpha, \beta, \gamma$ are the angles which the sides of the triangle make with the axis of the parabola.

7. Obtain in its simplest form the equation of the chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$, which joins the points whose eccentric angles are $\alpha \pm \beta$.

A parallelogram is inscribed in this ellipse and one of its sides touches the ellipse $x^2/a^2 + y^2/b^2 = 1$. Prove that its other three sides also touch this ellipse and that the parallelogram has a constant perimeter $4 \sqrt{a^2 + b^2}$.

8. Interpret the locus given by the equation $MN = L^2$, where $L = 0, M = 0, N = 0$ are the equations of the sides of a triangle.

Two tangents to an ellipse meet on its minor axis. Prove that the two circles, which touch these tangents at the points of contact, where they are met by the tangents at the extremities of the minor axis, pass through the foci of the ellipse; and prove also that they cut orthogonally, if $a^2 = 2b^2$, where $a$ and $b$ are the semi-axes of the ellipse.

9. Prove that, if $l, m, n$ are connected by a relation of the form $al + bm + cn + 2fmn + 2glm + 2kmn + 2glm = 0$, the straight line $lx + my + nz = 0$ is a tangent to a fixed conic, and the coordinates of its point of contact are

$$\frac{a}{l} + \frac{b}{m} + \frac{c}{n}$$

$g^2 + fmn + on^2 = 0 + fmn + on^2$.

A conic passes through the fixed point $A$, has a fixed straight line $OX$ as an asymptote, and touches the fixed straight line $OY$.

Prove that the envelope of the other asymptote is a conic, which touches $OA$ at $A$, has $OX$ for one asymptote, and has its other asymptote parallel to $OX$.

10. Prove that, when the harmonic triangle of the quadrangle formed by four points is taken to be the triangle of reference, the coordinates of the four points can be taken to be $X, \pm Y, \pm Z$.

Show that two conics can be described to pass through these four points and touch the straight line $lx + my + nz = 0$,

which does not pass through any one of the points, and that the points of contact of the line and the two conics lie on the conic $LX^2 + mY^2 + nZ^2 + XYZ = 0$.

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Mathematical Distinction Papers Group III

1918. 2 Hours

1. Show that if \( \tan y = k \tan x \), where \( k > 1 \), the maximum value of \( (y-x) \) is

\[
\tan^{-1} \left( \frac{k-1}{\sqrt{k^2-1}} \right)
\]

all the angles being supposed to be acute.

Prove also that then \( x+y = \frac{\pi}{2} + x \).

2. If \( \phi \) denotes the angle between the tangent at \((x, y)\) to a curve and the axis of \(x\), and if

\[
p = x \sin \phi - y \cos \phi, \quad q = x \cos \phi + y \sin \phi,
\]

verify the differential relations

\[
dp = q \, d\phi, \quad dq = ds - pd\phi,
\]

and deduce that the radius of curvature \( \rho \) is given by

\[
\rho = \frac{p^2 + q^2}{dp/dx}.
\]

Show that, for the ellipse \( x^2/a^2 + y^2/b^2 = 1 \), the relation between \( p \) and \( \phi \) is given by

\[
p^2 = \frac{a^2 \sin^2 \phi + b^2 \cos^2 \phi}{a^2 \cos^2 \phi - b^2 
\]

and deduce that

\[
p^2 + q^2 = \frac{a^2 - b^2}{a^2 b^2} p^2.
\]

Prove that

\[
\rho = \frac{a^2 b^2}{p^2}.
\]

3. Given that the expression

\[
u = A x^4 + 4 B x^3 y + 6 C x^2 y^2 + 4 D x y^2 + E y^4
\]

satisfies the relation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,
\]

prove that \( C, D, E \) can be determined in terms of \( A \) and \( B \).

Prove that \( u \) can then be expressed in the form

\[
u = \frac{1}{2} \left( (d - i R) (x + iy)^4 + (A + i E) (x - iy)^4 \right),
\]

where \( i = \sqrt{1} \).

4. The chord \( AB \) of a circular arc \( ACB \) is of length \( 2c \), and the height \( MC \) is \( b \), where \( M \) is the mid point of the chord \( AB \) and \( MC \) is perpendicular to \( AB \). Show that, referred to \( AB \) and \( MC \) as axes, the equation to the circle may be written in the form

\[
y = \frac{1-c^2}{c^2} \left( b - y \right);
\]

and that, when \( b/c \) is small, this equation is approximately equivalent to

\[
y \approx \left( 1 - \frac{c^2}{c^2} \right) \left( 1 + \frac{b x^2}{c^4} \right)
\]

for points on the arc \( ABE \).

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Group III Mathematical Distinction Papers 1918

Deduce that the area between the arc and the chord is approximately equal to

\[
\frac{4}{3} \sqrt{a} \left( 1 + \frac{b^2}{5c^2} \right).
\]

5. For a certain curve \( y = a \tan \phi \), where \( \phi \) is the angle between the tangent and the axis of \(x\). Deduce the values of \( x \) and \( \phi \) as functions of \( \phi \), and show that

\[
y = A e^{i \phi},
\]

where \( A \) is an arbitrary constant.

6. Explain the reasons for using the formula \( 2 \pi \int y \, ds \) to calculate the area of a surface of revolution.

Apply this formula to show that the area of the surface obtained by revolving the curve

\[
r = a \left( 1 + \cos \theta \right)
\]

about the line \( \theta = 0 \) is equal to \( \frac{3}{2} \pi a^2 \).

7. Prove that the work done by a couple of moment \( L \) in turning a body through a small angle \( d\theta \) (in the plane of the couple) is equal to \( L \theta \) (neglecting \( d\theta^2 \)).

A uniform ladder of length \( l \) and weight \( W \) is held against a smooth vertical wall, and with its lower end on a smooth horizontal surface; a man of weight \( W' \) stands on the ladder a distance \( l' \) from its lower end. Show that if the ladder is kept from slipping by means of a couple, the moment of the couple is equal to

\[
\frac{1}{2} \left( W + W' \right) l' \sin \theta,
\]

where \( \theta \) is the inclination of the ladder to the vertical.

Draw a diagram to illustrate the sense of the couple.

8. When a particle moves in a straight line, express the velocity \( (v) \) and acceleration \( (f) \) as differential coefficients of the space \( (s) \) with respect to the time \( (t) \).

If \( 2t = a s^2 + 2b s + c \), verify that

\[
f = -a v^2;
\]

and that the average velocity, taken with respect to the time, over any part \( PQ \) of the path is equal to the velocity at the middle point of \( PQ \).

9. Prove that the moment of inertia of a uniform cylindrical tube of mass \( M \), about its axis, is equal to

\[
\frac{1}{2} M (a^2 + b^2),
\]

where \( a, b \) are the internal and external radii of the tube.

The tube stands from rest and rolls, with its axis horizontal, down an inclined plane (making an angle \( \alpha \) with the horizontal).
By applying the principle of energy, or otherwise, show that \( T \), the time occupied in travelling a distance \( l \) along the plane, is given by
\[
l \left( 3 + \frac{a^2}{b^2} \right) = y T \sin \alpha.
\]

1919. 2 Hours

1. Prove that, if \( x^2 = a^2 \cos \theta + b^2 \sin^2 \theta \),
\[
\frac{dx}{dt} + x = a^2 \frac{dt}{dx}.
\]

Differentiate \( n \) times the equation
\[
\frac{d^2 x}{dx^2} - 2x \frac{dx}{du} + 2n u = 0;
\]
and prove that the resulting equation is satisfied if
\[
\frac{d^m x}{dx^m} = \frac{e^n}{x^n}.
\]

2. Express \( \frac{dx}{du} \) in terms of \( \frac{dy}{dx} \) and \( \frac{du}{dx} \).

If the variables \( z \), \( y \) are connected with the variables \( x \), \( y \) by the relations
\[
x \frac{dy}{dx} - y = \eta^{-2}, \quad y \frac{dx}{dy} - x = \xi^{-2},
\]
prove that
\[
x \frac{dy}{dx} + y = \eta^{-2}, \quad y \frac{dx}{dy} + x = \xi^{-2},
\]
and
\[
x \frac{dy}{dx} + y = \eta^{2}, \quad y \frac{dx}{dy} + x = \xi^{2}.
\]

3. Show that the radius of curvature of a plane curve is
\[
\left( 1 + \left( \frac{dy}{dx} \right)^2 \right) \frac{dx}{dy},
\]
and that the coordinates of the centre of curvature may be written in either of the forms
\[
(i) \quad x = y \left( 1 + p^2 \right), \quad y = 1 + p^2,
\]
\[
(ii) \quad x = \frac{dy}{dx}, \quad y = \frac{dx}{dy},
\]
where \( p \) denotes \( \frac{dy}{dx} \), \( q \) denotes \( \frac{dx}{dy} \), and \( \phi \) is the inclination of the tangent to the axis of \( x \).

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8. A billiard ball of mass \( m \) strikes another ball of mass \( M \) at rest, in a direction making an angle \( \theta \) with the line of centres. Prove that the tangent of the angle of deviation of the first ball is

\[
\frac{1 + \epsilon}{m - \epsilon M + (m + M) \tan^2 \theta}.
\]

When \( m \) is greater than \( \epsilon M \), determine the angle \( \theta \) for which the deviation is a maximum, and show that this deviation is \( 90^\circ - 2\theta \).

9. Find the moment of inertia of a uniform circular disc about its axis.

Two equal solid fly-wheels, each of mass \( m \) and radius \( a \), are in the same vertical plane and free to move in that plane about their centres, which are fixed. A connecting rod, of mass \( M \) and length equal to the distance between the centres of the wheels, is smoothly jointed to each wheel at a point on its rim, so that as the wheels revolve it is always parallel to the line joining their centres. Show that, as the system moves under gravity, the angular motion of the wheels is the same as that of a simple pendulum of length \( a (1 + \epsilon m/M) \).

1920. 2 Hours

1. If \( y = xe^{x} \cos \beta \), prove that

\[
\frac{\partial^2 y}{\partial x^2} - 2x(\alpha x^2 + 1) \frac{dy}{dx} + \{(x^2 + \beta^2) x^2 + 2\alpha x + 2\} y = 0.
\]

Prove by Induction that

\[
\frac{\partial^n}{\partial x^n} (e^x) = e^x \left[ (2n)! x^n + n(n - 1)(2n)! \right].
\]

2. Trace the curves

\[
y^2 = 4(a - x)(x + 1)(x - 4),
\]

where (1) \( \alpha = 1 \), (2) \( \alpha = 3 \).

Show that (1) consists of an oval curve whose least radius vector is \( 1 \) and whose maximum radius vector is approximately 1.072, measured from the origin.

3. Explain how to find the envelope of the family of curves \( f(x, y, c) = 0 \), where \( c \) is a variable parameter.

Circles are described having their centres on the circle \( x^2 + y^2 = a^2 \), each passing through the point \((a, 0)\). Show that the circle whose centre is at \((a \cos \theta, a \sin \theta)\) is intersected by the consecutive circle at the point \((2a \cos \theta - a \cos 2\theta, 2a \sin \theta - a \sin 2\theta)\).

Show that the envelope of the circles is

\[
8a^4 x = 3a^4 + 6a^2 (x^2 + y^2) - (x^2 + y^2)^2.
\]
9. Prove that the moment of inertia of a uniform rod of length 2a about an axis intersecting the rod at right angles at a distance b from its centre is \( M \left( \frac{a^2}{3} + b^2 \right) \), where \( M \) is the mass of the rod.

A wheel of radius \( a \) is formed of a thin uniform rim of mass \( M \) and \( a \) uniform spokes of length \( a - b \), each of mass \( m \), which are fastened to the rim and to an axe of radius \( b \) and mass \( m' \). The wheel rolls down an inclined plane of inclination \( \alpha \). Find the acceleration of its centre.

1921. 2 Hours

1. Prove that, if a function \( f(x) \) is continuous in the interval from \( x = a \) to \( x = b \), and if it possesses a definite differential coefficient at each point in the interval, there is a number \( c \) lying between \( a \) and \( b \) such that \( f(b) - f(a) = (b - a) f'(c) \).

2. Discuss this result for the functions \( x^{-1} \), \( x^{-2} \), when \( b = 3 \), \( a = -2 \). Show that, when \( f(x) = x^{-1} \), no such value of \( c \) exists, and that, when \( f(x) = x^{-2} \), there is a value of \( c \), but that it does not lie between 3 and -2.

3. How do these results illustrate the above theorem?

4. Obtain the \( n^{th} \) differential coefficients of \( e^x \), \( e^{-x} \) when \( x \) is a function of \( u \).

If \( \cosh x = \frac{1}{2} (e^x + e^{-x}) \), \( \sinh x = \frac{1}{2} (e^x - e^{-x}) \), then \( \cosh x \) and \( \sinh x \) are functions of \( x \).

5. Prove that \( \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = -\sinh x \frac{d^2z}{dx^2} + 4 \frac{d^3y}{dx^3} + 12 \frac{d^4y}{dx^4} = \cosh x \frac{d^2z}{dx^2} \).

3. If \( z = r \cos \theta \), \( y = r \sin \theta \), find in terms of \( r \) and \( \theta \) the value of \( \frac{\partial^2 \theta}{\partial x^2} \) (1) when \( y \) is constant, (2) when \( r \) is constant.

4. If \( u = x + y \), \( v = xy - 1 \), where \( y \) is a function of \( x \), prove that \( (1 + x^2) \frac{dy}{dx} + (1 + y^2) = \frac{w}{dx} \left( \frac{w}{dx} \right) \).

5. Find the asymptote of the curve \( x^2 - y^2 = ay(x + y) \).

6. There are two other tangents to this curve which are parallel to the asymptote. Find their equations and the coordinates of their points of contact.

Trace the curve.

5. (1) Integrate \( \int_0^\frac{\pi}{4} \frac{a^2b^2d\theta}{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)d\phi} \) by the substitution \( b \tan \theta = a \tan \phi \), or otherwise.
2. Prove that \((\sqrt{1+x^2+1})^\frac{1}{2}\) and \((\sqrt{1+x^2-1})^\frac{1}{2}\) are solutions of
\[
(1+x^2)\frac{dy}{dx^2} + x\frac{dy}{dx} - \frac{1}{2} y = 0.
\]

Hence, by the use of Leibnitz’s and Maclaurin’s theorems, obtain the general term in the expansion of
\[
(\sqrt{1+x^2+1})^\frac{1}{2}
\]
in ascending powers of \(x\).

3. Prove that the curve \((x^2+9y^2) = 24y\) has a point of flexion at the origin, and give a rough drawing of the curve.

Show that the radius vector through the origin, which is inclined at \(30^\circ\) to the axis of \(x\), is normal to the curve at its ends.

4. A solid is generated by the revolution of the conic
\[
y^2 = ax^2 + 2bx + c
\]
about the axis of \(x\). Prove that the volume of the segment cut out by two planes perpendicular to the axis of \(x\) at distance \(h\) apart is \(\frac{1}{2} h (A_1 + 4A_2 + A_3)\), and that the centroid of the segment is at distance
\[
\frac{1}{2} h A_2 - A_1
\]
from its middle point, where \(A_1\) and \(A_2\) are the areas of the ends, \(A_3\) the area of the section midway between them.

5. A rigid square framework of rods is hung over two smooth pegs in the same horizontal line. Prove that if the distance between the pegs is greater than \(\frac{1}{2}\) (diagonal) and less than \(\frac{1}{2}\) (side) of the square, an unsymmetrical position of equilibrium exists, and show that it is stable.

6. A step-ladder with no cross-tie whose legs are of equal lengths but unequal weights stands on a rough horizontal plane. Find the reactions at the feet, and show that as the angle between the legs is increased the lighter leg slips first.

7. A lift contains an Atwood’s machine in which the moving masses are \(m_1\) and \(m_2\). The system is initially at rest; the force acting on the lift gives at first a constant upwards vertical acceleration \(g'\), then zero acceleration, and lastly a constant vertical retardation \(g''\). Show that, when the lift comes to rest, each moving mass has velocity
\[
m_1\sqrt{m_2 + m_2} g T^2,
\]
and has described relatively to the lift a distance
\[
\frac{1}{2} m_1 m_2 g \left( T^2 + \frac{T^2 - T^2}{n + n} \right).
\]

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Group III Mathematical Distinction Papers

2. Prove that \(T\) is the whole time of motion, and \(\theta\) the interval during which the lift moves uniformly.

8. Define simple harmonic motion, and prove that the period of an oscillation is independent of its amplitude.

A particle moving with acceleration \(-\mu x\) has coordinates \(x_1\) and \(x_2\) and velocities \(v_1\) and \(v_2\) at any two moments. At the moment midway in time between them its coordinate and velocity are \(\bar{x}\) and \(\bar{v}\); show that
\[
x_{1} - x_{2} = 0
\]
\[
v_{2} - v_{1} = \frac{\mu}{2}\]
and that
\[
x_{1} + x_{2} = \frac{\bar{x}}{2}
\]
\[
v_{1} + v_{2} = \frac{\bar{v}}{2}.
\]

9. A fine circular hoop of weight \(W\) is free to move about a fixed horizontal tangent. It falls over from the position in which it is vertical so that its centre describes a circle in a vertical plane perpendicular to the tangent. Show that in the positions, when the hoop is vertical, the stress on the support is \(\frac{11}{3}\) or \(W\).

Group III (Paper 7)

Mathematical Distinction Paper

1923. 3 Hours

Not more than eight questions should be attempted by any Candidate.

1. Prove that, if \(f(x)\) is a finite and continuous function of \(x\) for \(a \leq x \leq b\),
\[
\frac{f(a) - f(b)}{a - b} = f\left(\frac{a + b}{2}\right) + \theta \frac{f'(a - b)}{2},
\]
where \(1 > \theta \geq 0\).

If \(f(x) \equiv a\), show that, if \(a\) and \(b\) have the same sign, there is a unique value of \(\theta\); but if \(2a + b = 0\), the equation is satisfied by \(\theta = -\frac{1}{2}\) and \(\theta = 1\). Illustrate graphically.

2. The inverse of the rectangular hyperbola \(r^2 \cos \theta = 2a^2\) with respect to the origin, the radius of inversion being \(\sqrt{2a}\), is the lemniscate \(r^2 = 2a^2 \cos 2\theta\). Deduce, or prove otherwise, that, if \(S, S'\) are the points \((a, 0), (-a, 0)\), the equation of the lemniscate can be written as \(SP \cdot S'P = a^2\), or as \(SP - S'P = \sqrt{2}, OP\).

3. The coordinates of any point on a parabola are \(a \sin^2 \alpha, 2a \sin \alpha\). Express the arc between any two points in terms of the parameters of these points.

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The tangents at two points $P$ and $Q$ intersect in $T$; show that $TP + TQ = PQ$ is constant, if the difference of the parameters of $P$ and $Q$ is constant.

4. Two forces $P$, $Q$ act at the origin in directions making angles $\alpha$ and $\beta$ respectively with the axes $Ox$ and $Oy$; two forces $P'$, $Q'$ act at the point $(h, k)$ in directions respectively perpendicular to $P$ and $Q$.

Show that for equilibrium

$$P'(h \cos \beta - k \sin \beta) = P'(h \cos \alpha - k \sin \alpha) = Q'(h \cos \beta + k \sin \beta) = Q'(h \cos \alpha + k \sin \alpha).$$

5. A uniform plank, length $l$, standing on rough ground is to be lowered from the vertical position by paying out from a fixed point $Q$ a rope tied to the plank at $P$. The upper end of the plank initially coincides with $Q$; $\mu$ is the coefficient of friction with the ground. Show that, if $P$ is above the centre of the plank, it will not upset; and this condition being satisfied, that if $\mu < 1$, the plank must slip; that if $\mu > 1$, the plank will slip unless the distance of $P$ from the free end $< \frac{l}{2\mu - 1}$. 

6. The resistance to the motion of a train is 160 lb. weight per ton mass at all speeds. It is moving on the level with uniform speed $V$ and comes to an incline of 1 in 70. The engine continues to work at the same rate as before; prove that if $v$ is the velocity, and $x$ the distance described up the incline in time $t$, then

$$v^2 = V^2 - \frac{1}{3} g (6x - 5Vt).$$

7. Form the equations of motion of a particle moving in a straight line against a resistance varying as the square of the velocity; and show that the distance described in any time is the same as if the particle moved uniformly with its velocity at the mid-point of the distance.

The particle passes three points at distances $d_1$, $d_2$ apart at equal intervals of time; show that its velocities at these points are inversely in the ratios $d_2^2 - 2d_1^2 : d_1^2 : d_2^2 + 2d_1^2$, $d_2 = d_1 + d_2$.

8. The ends of a light string of length $2a$ are attached to two points in the same vertical at distance $2c$ apart, and the string passes through a smooth ring of weight $w$. Show that if the ring revolves in a horizontal circle at the level of the lower point of attachment its velocity is $\left(\frac{a^2 - c^2}{c}\right)^{\frac{1}{2}}$ and the tension of the string is $\frac{1}{2} w \left(\frac{a + c}{c}\right)$. 

9. A light spiral spring is fixed in a vertical position, and a given weight resting on it produces a compression $\delta$. Show that if the weight is let fall on the spring from a height $\frac{3}{2} \delta$ above it, the compression of the spring in the ensuing motion is $3\delta$.

10. A circular hoop of radius $a$ rolling on a rough horizontal plane impinges on a rough peg of height $\frac{1}{2} a$ fixed in the plane. Find the angular velocity with which the hoop begins to turn about the peg.

If $V$ be the velocity of the centre before the impact, prove that if $81 V^2 < 80 a$, the hoop leaves the peg immediately.

1924. 3 HOURS

1. If $V = (x^2 + y^2 + z^2)^{\frac{3}{2}}$ satisfies the relation

$$\frac{\partial}{\partial x} \left( \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial V}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial V}{\partial z} \right) = 0,$$

where $u$ is a function of $z$ only, prove that $u$ is of the form $Ax^{-\alpha + 1}$, where $A$ is a constant.

2. Prove the formula $p = \frac{rdr}{dp}$ for a plane curve.

3. The origin, $P$ is a point on a plane curve, $C$ is the corresponding centre of curvature. Prove that

$$\cot POC = \frac{2d\phi}{dr},$$

where $\phi$ is the angle between $OP$ and the tangent at $P$. 

4. Trace the curves $r = a \cos \theta$, $r = a \cos \frac{1}{2} \theta$.

Show that their complete lengths are equal to each other and to that of an ellipse of axes $a$ and $3a$.

5. Verify the following:

$$\int \frac{dx}{\{(a-x)^2 + (x-b)^2\}^{\frac{1}{2}}} = \frac{\sqrt{2} \tan^{-1} \frac{z}{\sqrt{2}} - \tan^{-1} \frac{z}{\sqrt{2}}}{(a-x)^2 - (x-b)^2}.$$

where $z = \frac{2(a-x)(x-b)}{(a-x)^2 - (x-b)^2}$.

6. A uniform rod $AB$, of weight $W$ and length $a$, is free to turn about its end $A$, which is fixed. A fine string is attached to $B$ and, passing over a smooth pulley vertically above $A$ and at distance $h$ from $A$, carries a weight $w$. Show that if

$$\frac{h + a}{h} > \frac{2w}{W}, \quad \frac{h - a}{h},$$

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the two positions in which $AB$ is vertical are stable positions of equilibrium.

6. Obtain the equations for a common catenary,

$$y = c \cosh (x/c), \quad s = c \sinh (x/c).$$

A chain 30 ft. long is hung from two points in the same horizontal, and the sag in the middle is 10 ft. Given log 5 = 1.6094, show that the span is 20-12 ft., and that the tension at the points of support is the weight of 10-25 ft. of chain.

7. A shot of mass $m$ is fired from a gun, which is mounted on a track free to move on smooth horizontal rails parallel to the vertical plane through the gun. Assuming that the energy set free by the explosion is always the same, show that with angle of elevation $\alpha$, the greatest height attained is $\frac{(M + m)h \sin^2 \alpha}{M + m \sin^2 \alpha}$, and the range on the horizontal plane through the gun is $\frac{2Mh \sin 2\alpha}{M + m \sin^2 \alpha}$, where $h$ is the vertical height to which the gun can send the shot, and $M$ is the mass of the gun and truck.

8. A particle, suspended from a fixed point by a string of length $a$, is projected horizontally so as to describe part of a circle in a vertical plane; show that if the parabolic path of the particle after the string becomes slack passes through the original point of projection, the velocity of projection is $\left(\frac{\rho}{2} \sqrt{g}a\right)$.

9. A fine elastic string $OAB$, whose modulus of elasticity is $\lambda$ and unstretched length $a$, has one end fixed at $O$, and passes over a small smooth pulley fixed at $A$, where $OA = a$. A particle of mass $m$ hangs in equilibrium at $B$. Show that if a horizontal impulse $I$ is applied to the particle, it will move in a horizontal line with simple harmonic motion of amplitude $I \left(\frac{a}{\lambda m}\right)^{\frac{1}{2}}$.

10. A thin uniform rod of mass $m$ and length $2a$ can turn freely about one end, which is fixed. A uniform bar, whose mass is $\frac{1}{2} m$ and length $3a$, can be clamped to the rod so that its centre occupies any position on the rod. Show that the length of the simple equivalent pendulum for oscillations in which the bar and the rod remain in a vertical plane lies between $\frac{1}{2} a$ and $2a$.

GROUP III MATHEMATICAL DISTINCTION PAPERS 1925

1. If $y = \frac{(1+k)x}{1+kex}$, prove that

$$\frac{dy}{dx} = \frac{(1+e^2)(1-ke^2x^2)}{(1-y^2)(1-ke^2y^2)}^{\frac{1}{2}},$$

where $k' = 2ek'(1+e)$.

2. $P$ is a point on a plane curve and the straight line joining $P$ to a fixed point $O$ in its plane makes an angle $\phi$ with $PT$, the tangent at $P$. A point $Q$ is taken on $PT$ such that $Q$ and $T$ are on opposite sides of $P$ and $PQ = OP$. Prove that the tangent at $Q$ to the locus of $Q$ makes with $PQ$ an angle $\cot^{-1} \left(\frac{\rho}{r} (1+\cos \phi)\right)$, where $\rho$ is the radius of curvature at $P$ and $r$ is the length of $OP$.

3. Show how to find the equations of the linear asymptotes of a curve, whose equation is given in Cartesian coordinates, and show that, if a curve of the $n$th degree has $n$ distinct linear asymptotes, their finite intersections with the curve lie on a curve of degree $n-2$.

A cubic curve has the straight lines $z+a=0$, $y+b=0$ as asymptotes and has a node at the origin. Prove that the finite intersections of the cubic with its third asymptote lies on the straight line $bx+cy=0$.

4. A plane area $(A)$ revolves about an axis in its plane, which does not intersect the boundary of the area. Prove that the volume of the solid traced out is $2\pi yA$, where $y$ is the distance of the centroid of the area from the axis.

A solid is formed by the revolution of the curve $2\rho^2 = (a-x)^2 (2a-x)$ about its asymptote. Prove that the volume traced out by the area of the loop and the volume traced out by the area between the curve and the asymptote are each equal to the volume of a sphere of radius $a$.

5. Two equal uniform rods $AB$, $BC$, each of weight $W$, are freely hinged together at $B$ and rest in a vertical plane with the angle $ABC$ a right angle, and the ends $A$ and $C$ on a rough horizontal plane. A string is attached to the middle point of $BC$ and pulled parallel to $AC$ away from $AB$ with a gradually increasing horizontal force. Prove that, if the coefficient of friction between the rods and the plane is $\frac{1}{2}$, equilibrium will be broken by $C$ slipping while $A$ remains at rest, as soon as the force exceeds $2W$, but that, if the coefficient of friction is equal to $\frac{1}{2}$, equilibrium will be broken by $A$ slipping, while $C$ remains at rest, as soon as the force exceeds $3\frac{1}{2}W$. 

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6. From mechanical considerations prove the equations
\[ s = c \tan \phi, \quad y^2 = x^2 + c^2, \quad T = mgy. \]
for the common catenary.

A uniform string of length \( 64a \) rests symmetrically over two smooth pegs at the same level, the lowest point of the curved portion of the string between the pegs being at a depth \( a \) below the level of the pegs. Find the lengths of the portions of the string which hang vertically and in the form of a catenary respectively, and find the distance between the pegs. Prove also that the pressure on either peg is equal to the weight of a length \( 40a \) of the string.

7. Five equal uniform rods, \( AB, BC, CD, DE, EA \), each of weight \( W \), are freely hinged together to form a pentagon \( ABCDE \). The pentagon is suspended from the joint \( A \) and rests in the form of a regular pentagon, the configuration being maintained by a light horizontal stay joining the joints \( B \) and \( E \). Prove that the stress in this stay is \( W \cot 18^\circ \).

8. A smooth straight open tube is held so as to be inclined to the vertical and contains a number of particles held so as not to be in contact with each other. If the particles are simultaneously released, show that at any subsequent instant all the particles which have left the tube lie on a parabola.

9. A particle is attached by a light inextensible string of length \( a \) to a fixed point and is projected horizontally from its position of equilibrium with a velocity \( u \). Show that, if \( 5ag > u^2 > 2ag \), the string will go slack in the subsequent motion.

If the particle is held with the string stretched and making an acute angle \( \alpha \) with the upward drawn vertical and is then released, show that after the string tightens it will never go slack again and the particle will not rise to the level of the fixed end of the string if \( \alpha < \frac{\pi}{2} \); but that, if \( \alpha > \frac{\pi}{2} \), the particle will rise above this level and the string will go slack again, when the particle is at a height \( -\frac{a}{2} \cos a \cos 2a \) above this level.

10. The ends of a uniform rod of weight \( W \) are attached by means of two equal rigid stays of negligible weight to a fixed point, about which the system can turn freely in a vertical plane, the stays being of such a length that they form with the rod a right-angled isosceles triangle. If the system performs finite oscillations about the fixed point, the extreme positions being those in which the rod is vertical, prove that the stress in either stay is as great as possible, when the rod is inclined to the horizontal at an angle \( \tan^{-1} (0.1) \).

1926. 3 Hours

1. If two variables \( \theta, \phi \) are connected by the equation
\[
(a^2 + b^2 - 2ab \cos \theta)(a^2 + b^2 - 2ab \cos \phi) = (a^2 - b^2)^2,
\]
prove that
\[
\int \frac{\sin^m \theta d\theta}{(a^2 + b^2 - 2ab \cos \theta)^n} = \frac{1}{(a^2 - b^2)^{2n-1}} \int \frac{\sin^m \phi d\phi}{(a^2 + b^2 - 2ab \cos \phi)^{2n-1}}.
\]

Deduce that
\[
\int_0^{3\pi} \frac{\sin^4 \theta d\theta}{\cos \theta} = \frac{3\pi}{8} (a^2 - b^2)^4.
\]

2. The coordinates of any point on an epieycloïd are given by the equations
\[
x = (a + b) \cos t - b \cos \frac{a}{b} t, \quad y = (a + b) \sin t - b \sin \frac{a}{b} t,
\]
where \( t \) is a parameter. Prove that the equation of the tangent at any point to the epieycloïd is
\[
x \sin \frac{a + 2b}{2b} t - y \cos \frac{a + 2b}{2b} t = (a + 2b) \sin \frac{a}{2b} t.
\]

If \( a = b \), prove that the parameters of the remaining points at which this tangent meets the curve satisfy the equation
\[
\frac{\sin x}{\cos a - 2} = \frac{\sin t}{1 + \cos t}.
\]

3. A parabola, the length of whose latus rectum is \( 4a \), touches the axis of \( y \) at the origin. Prove that the locus of its focus is \( x^2 = a^2 (x^2 + y^2) \), and sketch this curve.

How do you account for the presence of the curve of the isolated point at the origin?

4. The curve traced out by the point
\[
x = a \log (\sec t + \tan t) - a \sin t, \quad y = a \cos t,
\]
as \( t \) increases from \( -\frac{\pi}{2} \) to \( \frac{\pi}{2} \), is rotated about the axis of \( x \). Prove that the whole surface generated is equal to the surface of a sphere of radius \( a \), and that the whole volume generated is half the volume of a sphere of radius \( a \).

5. A number of equal uniform rectangular blocks, each of length \( 2a \), are placed in a pile with each block projecting over the end of the block below it, the lowest block resting on a fixed horizontal plane. If the number of blocks is \( n \), and the distance which each projects over the one below it is so adjusted that the total overhang
(the horizontal distance between the vertical ends of the highest and lowest blocks) is a maximum, prove that this overhang is

\[ a \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right). \]

If, however, the distances which each block projects over the one below it are all equal, prove that the greatest overhang is

\[ 2a \left( n - 1 \right) \]

and verify that this never exceeds the maximum overhang.

6. Three equal uniform smooth spheres of radius \( a \) are placed in contact with each other in a fixed spherical bowl of radius \( b \). A fourth equal sphere is placed gently on top of them. Prove that the three lower spheres do not separate if \( b \) is less than \( a(1 + 2\sqrt{11}) \).

7. Seven equal uniform heavy rods, each of weight \( W \), freely joined together at their ends, form a heptagon \( ABCDEFG \). The heptagon is suspended from the vertex \( A \) and is maintained in the shape of a regular heptagon by a light strut \( BG \) and two light strings \( BE \) and \( GD \). Prove that the tension in each string is

\[ 2W \sin \frac{\pi}{7}. \]

8. A projectile is aimed with velocity \( u \) at a vertical wall whose distance from the point of projection is \( a \). Prove that the greatest height above the level of the point of projection at which the projectile can hit the wall is

\[ \frac{2}{2g} \left( u^2 - g^2a^2 \right). \]

9. Two particles each of mass \( M \) are attached to the ends of a taut string of length \( 2a \) which lies at rest on a smooth horizontal table, passing through a small ring attached to the table; the ring is at the middle point of the string. An impulse \( I \) is applied to one of the particles in a direction perpendicular to the string. Prove that, when the other particle reaches the ring, the velocities of the particles are in the ratio \( \sqrt{5} : \sqrt{3} \).

10. A uniform rod of length \( 2a \) is suspended from two fixed points at the same level by two vertical strings, each of length \( b \), attached to the ends of the rod. A fixed vertical wire passes through a small hole at the centre of the rod and acts as a smooth guide. The rod is twisted through an angle about the wire, the rod remaining horizontal. Prove that the period of small oscillations is

\[ \frac{2\pi}{\sqrt{3g}}. \]

1. The figure represents a meridian section of a uniform solid of revolution with one flat face, having a cylindrical hole bored through it, the axis of the cylinder coinciding with that of the solid. If the curved portions \( AB \) and \( DE \) are arcs of a parabola of latus rectum \( 4a \), whose axis coincides with that of the solid, and the length of the hole, \( AC \) or \( DF \), is \( l \), prove that the volume of the solid is

\[ 2\pi a^2 l, \]

and that the distance of its centroid from the plane face is \( \frac{1}{3} l \).

2. If \( y = A(\sqrt{x^2 + 1} + x)^n + B(\sqrt{x^2 + 1} - x)^n \), where \( A \) and \( B \) are any constants, prove that

\[ x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - ny^2 = 0. \]

By putting \( y = x^2 u \) and changing the independent variable from \( x \) to \( \xi \), where \( x^2 \xi = 1 \), or by any other method, prove that, if

\[ u = A(\sqrt{1 + \xi} + 1)^n + B(\sqrt{1 + \xi} - 1)^n, \]

\[ 4(1 + \xi) \frac{d^2 u}{d\xi^2} - 4(n - 4 + (4n - 6)\xi) \frac{du}{d\xi} + n(n - 1) u = 0. \]

3. The normal to a plane curve at a point \( P \) meets the axis of \( x \) at \( G \), and \( N \) is the foot of the ordinate at \( P \). \( OQ \) is drawn perpendicular to the axis of \( x \), so that \( OQ \) is equal to \( GP \), and \( G \) and \( P \) are on the same side of the axis of \( x \). The normal at \( Q \) to the locus of \( Q \) meets the axis of \( x \) at \( G' \). Prove that \( G \) is the middle point of \( NG' \).

4. Draw a rough sketch to indicate the shape of the curve whose points have coordinates expressed in terms of a parameter \( t \) in the form

\[ x = \frac{at}{1 + t^2}, \quad y = \frac{at^2}{1 + t^2}. \]

Find the relation connecting the parameters of three collinear points, and show that the tangents at two points, such that the lines joining them to the origin are equally inclined to each of the axes, intersect on the curve.

If \( O \) is the origin and \( P \) and \( P' \) are two points, such that the tangents at \( P \) and \( P' \) intersect on the curve, prove that the area between the chord \( PP' \) and the arc of the curve from \( P \) to \( P' \) is four-thirds of the area of the triangle \( POP' \).

5. Three equal rods, each of weight \( W \), are freely joined together at one extremity of each to form a tripod and rest with their other extremities on a smooth horizontal plane, each rod inclined at an angle \( \theta \) to the vertical, equilibrium being maintained by three equal
light strings, each joining two of these extremities. Prove, by means
of the principle of virtual work, or in any other manner, that the
tension in each string is \( W \tan \theta / 2 \sqrt{3} \).

6. Find the equation of the common catenary.

A uniform flexible chain of length \( 2a(1 + k) \) has its extremities
attached to two points at the same level, distant \( 2a \) apart. Prove
that, if \( k \) is so small that powers of \( k \) above the second may be
neglected, the sag of the chain is

\[
\frac{1}{2} a \sqrt{\frac{6k}{1 + \frac{7k}{20}}}.
\]

7. A particle is projected with a velocity of magnitude \( V \) from
a point of a plane, inclined to the horizontal at an angle \( \alpha \), in the
vertical plane through the line of greatest slope through the point
of projection. The direction of projection is up the plane and makes
an angle \( \beta \) with the plane. The coefficient of restitution between
the plane and the particle is \( e \). Prove that the range of the particle on
the plane at the moment of its second impact with the plane is
greatest when \( \cot 2\beta = (1 + e) \tan \alpha \), and that this greatest range is
\( V^2 (1 + e) \tan \beta/g \cos \alpha \), where \( \beta \) has the value given by the first
equation.

8. Two particles, moving in straight lines, the first acted on by
a constant force, the second acted on by a force doing work at a
constant rate, each have their velocities increased from \( V \) to \( 2V \) after
traversing a distance \( a \). Show that the time taken by the second is
\( 27/28 \) of that taken by the first.

Show also that the velocity acquired by the second after
traversing a distance \( x \), less than \( a \), is greater than that acquired by
the first after traversing the same distance.

9. Define simple harmonic motion and the period and amplitude
of such a motion, and prove that the period is independent of the
amplitude.

A system, held in equilibrium, consists of two equal particles,
each of mass \( M \), connected by a taut light inextensible string passing
over a smooth fixed pulley, and a particle of mass \( m \) suspended from
one of them by a light stretched elastic string. If the system is
released, prove that the motion of \( m \), relative to the particle \( M \) to
which it is attached, consists of a simple harmonic motion of amplitudex_0/(m + 2M), where \( x_0 \) is the extension of the elastic string
when the system is held in equilibrium, provided that \( 2M > m \).

Explain without detailed calculation what happens if \( 2M < m \).

10. A plane is inclined to the horizontal at an angle greater than
\( \tan^{-1} \sqrt{2} \). The coefficient of friction between this plane and a solid