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HIGHER CERTIFICATE MATHEMATICS
1928-1932

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MATHEMATICS

Group III (Paper 1).

ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.

TUESDAY, JULY 17TH, 1928. 2½ HOURS.

1. A polynomial in \(x\) of the third degree vanishes when \(x = 2\) and \(x = 1\). Its gradient, when \(x = 2\), is 1, and its gradient, when \(x = 1\), is \(-\frac{1}{2}\). Find the third value of \(x\) for which the polynomial vanishes and the corresponding gradient.

2. (i) Solve the simultaneous equations

\[ ax^2 + xy - y^2 + 3x + y + 8 = 0, \quad x + 2y = 1. \]

(ii) If \(2x = \sqrt{(a+b)^2} - y \pm \sqrt{(a-b)^2} - y\), find \(y\) in terms of \(x\).

3. Write down the expansion of \(e^x\) as a series in ascending powers of \(x\).

If the product of \(e^x\) and the polynomial \(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\) is written as a series in ascending powers of \(x\), show that it contains no powers of \(x\) below the fourth, and find and simplify as much as possible the coefficient of \(x^n\) in this series, when \(n\) is not less than 4.

Deduce that for all real values of \(x\), \(e^{-x}\) is greater than \(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\).

4. Find the ratio of the seventh term to the sixth in the expansion of \((1 + 0.05)^{12}\) by means of the binomial theorem.

Show that the sum of all the terms after the sixth is less than 0.00008.

5. Write down the expansion of \(\log_{10}\left(\frac{1+x}{1-x}\right)\) in ascending powers of \(x\), stating the limits of \(x\) for which the expansion is valid.
Expand \( \log_a \left( 1 + \frac{1}{n} \right) \) in ascending powers of \( \frac{1}{2n+1} \), stating the necessary restrictions on the value of \( n \).

Prove that, if \( n \) has any positive value, \( \log_a \left( 1 + \frac{1}{n} \right) \) is greater than \( \frac{2}{2n+1} \) and less than \( \frac{2n+1}{2n} \cdot \frac{1}{(n+1)} \).

6. Draw the graph of \( \cos^3 x, x \) being the radian measure of an angle, between the values \(-\frac{1}{2} \pi\) and \( \pi \) of \( x \), taking 2 inches (or 4 centimetres) to represent \( \frac{1}{2} \pi \) on the \( x \) axis and 2 inches (or 4 centimetres) as the unit of \( y \).

Find as accurately as you can from your graph the solutions of the equation \( \pi \cos^3 x + x = 0 \) which are not outside the limits \(-\frac{1}{2} \pi, \pi \) of \( x \).

7. Prove the formula for a plane triangle
\[
a^2 = b^2 + c^2 - 2bc \cos A,
\]
considering all cases.

The tangents at the points \( B \) and \( C \) to the circle circumscribed about the triangle \( ABC \) meet at the point \( A' \). \( D \) is the middle point of \( BC \). Prove that \( AA' = AD \sin A \) and that the angle \( CAD \) is equal to the angle \( BAA' \).

8. Prove that
\[
\cos (A + B) = \cos A \cos B - \sin A \sin B,
\]
taking \( A, B, \) and \( A + B \) to be acute angles. How could you extend the theorem to angles of any magnitude, having proved it for this case?

Prove that, if \( \tan x \tan (A - x) = k \), then
\[
\cos (2x - A) = \frac{1 + k}{1 - k} \cos A.
\]

Solve completely the equation
\[
4 \tan x \tan (60 \^\circ - x) = 1.
\]

9. A ship \( A \) is steaming at 15 knots on a course 27\(^\circ\) west of north, and a second ship \( B \) is steaming at 12 knots on a course 25\(^\circ\) east of north. At a certain instant the distance and bearing of \( B \) from \( A \) are respectively 12 nautical miles and 22\(^\circ\) east of north. Find by calculation the distances (in nautical miles) of \( A \) south and west from \( B \) an hour later, and deduce the distance and bearing of \( B \) from \( A \).

Check your calculations by means of an accurately drawn figure.

(A knot is a speed of 1 nautical mile per hour.)

**Group III (Paper 2) and Subsidiary Subject (15 d).**

**PURE AND ANALYTICAL GEOMETRY.**

**Wednesday, July 18th, 1928. 2\(\frac{1}{2}\) Hours.**

1. The middle points of the sides of a triangle \( ABC \) are \( D, E, F \). Prove that the medians \( AD, BE, CF \) meet at a point \( G \) which divides each of them in the ratio 2 : 1.

Show how to construct a triangle, given (i) the lengths of two sides and of the median which bisects the third side, (ii) the lengths of the three medians.

2. Prove that the areas of similar triangles are proportional to the squares on corresponding sides.

An equilateral triangle \( ABC \) is inscribed in a circle and \( P \) is any point on the arc \( BC \). By using properties of similar triangles or otherwise, prove that
\[
PA = PB + PC.
\]

3. Two given circles have their centres at \( A \) and \( B \), and \( AP, BQ \) are any two parallel radii; prove that \( PQ \) passes through one or other of two fixed points.

If the second circle passes through \( A \) and if the two real common tangents to the circles touch the second circle at \( C \) and \( D \), prove that the straight line \( CD \) touches the first circle.

4. A straight line \( OA \) is perpendicular to each of two straight lines \( OB, OC \). Prove that \( OA \) is perpendicular to every straight line through \( O \) lying in the plane \( BOC \).

Two points \( P, Q \) are taken on \( BC \), and the circles \( OAP, OAQ, APQ \) are drawn. Prove that there is a sphere on which all three circles lie.
5. If \(a_1 x + b_1 y + c_1 = 0, \ a_2 x + b_2 y + c_2 = 0\) are two given straight lines, prove that any straight line through their point of intersection may be represented by an equation of the form 
\[a_3 x + b_3 y + c_3 + \lambda (a_2 x + b_2 y + c_2) = 0.\]
A mirror perpendicular to the plane \(xOy\) meets it in the line 
\[ax + by + c = 0;\] the path of a ray of light in the plane \(xOy\) is the line 
\[px + qy + r = 0.\]
Prove that the path of the reflected ray is the line 
\[(a^2 + b^2) (px + qy + r) = 2 (ap + bq) (ax + by + c).\]

6. Prove that the equation 
\[ax^2 + 2hxy + by^2 = 0\]
represents a pair of straight lines through the origin and that the sine of the angle between them is equal to 
\[\frac{2 \sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}.\]
Prove that the length intercepted by these lines on the line \(lx + my + n = 0\) is 
\[\frac{2n \sqrt{(l^2 + m^2) (h^2 - ab)}}{am^2 - 2bhm + bk^2}.\]

7. Find the equation of the tangent at \((x_1, y_1)\) to the circle 
\[x^2 + y^2 + 2gx + 2fy + c = 0.\]
A triangle has two of its sides along the coordinate axes, and its third side is any tangent to the circle 
\[x^2 + y^2 - 2ax - 2ay + a^2 = 0.\]
Prove that the locus of the circumcentre of the triangle is 
\[a^2 (x^2 + y^2) = (a (x + y) - 2xy)^2.\]

8. The coordinates of any point \(P\) on the parabola 
\[y^2 = 4ax\] are given in terms of a parameter by the expressions \((at^2, 2at)\). Write down the equation of the normal at the point \(P\), and prove (do not merely verify) that the parameter of the point \(Q\) at which the normal meets the parabola again is 
\[t^2 + \frac{2}{t}.\]

The line \(PQ\) is divided at \(R\) so that \(PR : RQ = 1 : 3\); prove that the locus of \(R\) is the parabola 
\[y^2 = ax - 3a^2.\]

9. Obtain the equation of the chord joining the points with eccentric angles \(\alpha\) and \(\beta\) on the ellipse 
\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\] in the form 
\[\frac{x}{a \cos \frac{\alpha + \beta}{2}} + \frac{y}{b \sin \frac{\alpha + \beta}{2}} = \cos \frac{\alpha - \beta}{2}.\]
A perpendicular \(CY\) is drawn from the centre of the ellipse to any tangent. Prove that the locus of \(Y\) is 
\[(x^2 + y^2)^2 = b^2 y^2 + a^2 x^2.\]

10. Prove that, as \(t\) varies, the point with coordinates 
\[\frac{a}{2} \left( i + \frac{1}{t} \right), \quad \frac{a}{2} \left( i - \frac{1}{t} \right)\] traces out the whole of a rectangular hyperbola.

The normal to a rectangular hyperbola at \(P\) meets the axes of the hyperbola at \(G\) and \(H\), and the asymptotes at \(Q\) and \(R\). Prove that \(PG\) and \(PH\) are both equal to the geometric mean of \(PQ\) and \(PR\).

**Group III (Paper 3).**

**DIFFERENTIAL AND INTEGRAL CALCULUS.**

**THURSDAY, JULY 26TH, 1928. 2 1/2 Hours.**

1. Find the differential coefficient of the function 
\[6x^3 + 9x (1 + 4 \cos x) - 9 \sin x (4 + \cos x),\]
and simplify your result as much as possible.

Prove that this differential coefficient is positive.

What inferences about the original function can you draw from this result (i) when \(x\) is positive, (ii) when \(x\) is negative?
2. State and prove Leibniz's theorem for the nth differential coefficient of a product.

Prove that the result of differentiating the equation

\[(1-x^2)^m \frac{dy}{dx} + 2mxy = 0\]

n + 1 times with respect to x is

\[(1-x^2) \frac{d^{n+2}y}{dx^{n+2}} + 2 (m-n+1) x \frac{d^{n+1}y}{dx^{n+1}} + (n+1) (2m-n) \frac{d^ny}{dx^n} = 0.\]

Hence verify that, if

\[z = \frac{d^m (1-x^2)^m}{dx^m},\]

then z is a solution of the equation

\[(1-x^2) \frac{d^2z}{dx^2} + 2 (m-n-1) x \frac{dz}{dx} + (n+1) (2m-n) z = 0.\]

3. Show how to find values of x which make a given function of x a maximum or minimum; and state a test which discriminates between maxima and minima.

A circle of radius a and centre O has two fixed tangents TA and TB drawn to it, the angle ATB being 2\(\beta\). A point X is taken on the minor arc AB, so that \(\angle AOX = \theta\), and the tangent at X meets the fixed tangents at P and Q. Find the area of the triangle TPQ, and prove that its only stationary value for values of \(\theta\) between 0 and \(\pi - 2\beta\) is a maximum when \(\theta = \frac{1}{2} \pi - \beta\).

4. State Maclaurin's theorem.

By using Maclaurin's theorem, or otherwise, obtain the expansion of \(\log (1 + \cos 2x)\) in powers of x as far as the term in \(x^4\).

5. The coordinates of a point on a curve are given in terms of a parameter by the equations

\[x = a \cos^2 t, \quad y = a \sin^3 t.\]

Prove that the equation of the tangent at the point with parameter \(t\) is

\[x \sin t + y \cos t = a \sin t \cos t.\]

Prove that this tangent is also a normal to the curve if \(\tan 2t = \pm 2\),

and that the parameter of the foot of this normal also satisfies the equation \(\tan 2t = \pm 2\).

Prove that the distance between the foot of the normal and its point of contact is \(2a / \sqrt{5}\).

6. Evaluate the following integrals:

\[\int \frac{dx}{(x-1)(x-3)}, \quad \int \frac{x^2 dx}{(x-1)^2}, \quad \int \sqrt{(x-1)(3-x)} dx, \quad \int \tan^2 x dx, \quad \int \cos^3 x dx.\]

7. Give a rough sketch of the curve

\[y^2 = (2a-x)(x-a)^2,\]

and prove that the area of its loop is

\[\frac{1}{2}a^2 (4-\pi).\]

8. Write down a formula for the volume of a surface of revolution.

A lamina in the shape of the parabola \(y^2 = 4ax\), bounded by the chord \(x = a\), is rotated (i) about the axis of y, (ii) about the line \(x = a\). Prove that the two volumes thus generated are in the ratio 3:2.

9. A particle moves in a straight line under the action of a variable force so that its velocity \(v\) when it has travelled a distance \(x\) is given by the formula

\[v = uw^2/w,\]

where \(w\) is the initial velocity and \(a\) is a constant. Prove that the force at any instant is proportional to \(v^2\), and that the power exerted at that instant is proportional to \(v^3\).

**Group III (Paper 4).**

**STATICS AND DYNAMICS.**

**Monday, July 16th, 1928. 3 Hours.**

[Not more than nine questions are to be attempted by any Candidate.]

In numerical calculations take \(g = 32\) foot-second units.]

1. Prove that, if the lines of action of two forces intersect, the algebraic sum of the moments of the forces about any
point $O$ in their plane is equal to the moment about $O$ of their resultant.

Forces $P, 4P, 2P, 6P$ act along the sides $AB, BC, CD, DA$ of a square $ABCD$ of side $a$. Find the magnitude of their resultant, and prove that the equation of its line of action, referred to $AB$ and $AD$ as coordinate axes, is

$$2x - y + 6a = 0.$$

2. A uniform wire of length $6a$ and weight $W$ is bent so as to form three sides $AB, BC, CD$ of a square. It is freely suspended from $A$; prove that, in equilibrium, $AB$ makes an angle $\tan^{-1} \frac{3}{4}$ with the vertical.

If a horizontal force is now applied at $C$ so as to maintain the wire in equilibrium with $BC$ horizontal and below $A$, find the magnitude of this force, and the magnitude and direction of the pressure on the support at $A$.

3. The light jointed framework shown in the diagram has all its angles equal to $45^\circ$ or $90^\circ$; it is supported at its ends with five of its bars horizontal, and it carries loads of 3 tons and 9 tons. Find graphically the stress in each bar, indicating which are in tension and which are in compression.


A uniform rod rests with its ends upon two fixed equally rough planes which intersect along a horizontal line and which are each inclined at an angle $\alpha$ to the horizontal; the vertical plane through the rod is perpendicular to the line of intersection of the two planes. If $\mu$ is the coefficient of friction and $\mu < \tan \alpha < 1$, prove that the greatest angle which the rod can make with the horizontal is

$$\tan^{-1} \left( \frac{\mu}{\sin^2 \alpha - \mu^2 \cos^2 \alpha} \right).$$

5. A uniform wire is bent so as to form a triangle. Prove that its centre of gravity is at the centre of the inscribed circle of the triangle whose vertices are the middle points of the triangle formed by the wire.

A uniform square lamina $ABCD$ of side $2a$ is folded along two straight lines, the first fold bringing $A$ to $O$, the centre of the square, and the second fold bringing $B$ to $O$. Find the distance from $O$ of the centre of gravity of the folded lamina.

6. A train travels for the first $nt$ part of its journey with uniform acceleration, the last $n$th part with uniform retardation, and the intermediate part with uniform speed $V$. Prove that its average speed is $\frac{nV}{n + 2}$.

If the train were to travel for the first $n$th part of the time with uniform acceleration, for the last $n$th part of the time with uniform retardation, and for the intermediate time with uniform speed $V$, prove that its average speed would be $\frac{n - 1}{n} V$.

7. A particle is projected under gravity with velocity $V$ from a point on a plane inclined at an angle $\alpha$ to the horizontal. Find its maximum range (i) up the plane, (ii) down the plane.

If $r_1$ and $r_2$ are these maximum ranges, prove that

$$\frac{1}{r_1} + \frac{1}{r_2}$$

is independent of the inclination of the plane.

8. Define Horse Power.

A motor-car weighing 15 cwt. travels up a hill a mile long and 1000 feet high at a uniform speed of 20 miles per hour. Find the H.P. exerted by its engine in overcoming the force of gravity.
If the engine is actually working at 30 H.P., find the total resistance (in pounds) due to friction, air resistance, &c.

9. A particle is describing a circle of radius \( a \) with uniform speed \( v \). Prove that the acceleration of the particle is \( v^2/a \).

What is the direction of this acceleration? Give a proof of your statement.

A motor-car is travelling with uniform speed, and \( P \) is a point on the circumference of one of its wheels. Prove that, if \( \theta \) is the angle between the direction of the velocity of \( P \) and the direction of the acceleration of \( P \), then \( \theta \) is equal to 60° once in each revolution of the wheel, and \( \theta \) is equal to 120° once in each revolution; and that, at each of these two instants, the direction of the velocity \( P \) is inclined at 60° to the vertical.

10. Explain what is meant by the ‘simple equivalent pendulum’ of a rigid body which can turn freely about a fixed horizontal axis, and show how to obtain its length in terms of lengths associated with the body.

The pendulum of a clock consists of a thin uniform rod 2 feet long attached to the circumference of a uniform circular disk of diameter 1 foot, the rod lying along a radius of the disk produced; the point of suspension is the end of the rod remote from the disk, and motion takes place in the plane of the disk. If the weights of rod and disk are 1 lb. and 4 lb., find the length of the simple equivalent pendulum correct to a hundredth of an inch.

[The moment of inertia of a circular disk about a line through its centre perpendicular to its plane is \( \frac{1}{2} M a^2 \).]

Group III (Paper 5).

MATHEMATICAL DISTINCTION PAPER.

MONDAY, JULY 23RD, 1928. 3 HOURS.

1. Evaluate as a product of factors the determinant

\[
\begin{vmatrix}
2a, & 1, & (x+1)^3 \\
2b, & 1, & (y+1)^3 \\
2c, & 1, & (z+1)^3 \\
\end{vmatrix}
\]

2. If \( (A_1, B_1, C_1), (A_2, B_2, C_2), (A_3, B_3, C_3) \) are three essentially different sets of values of \( A, B, C \) satisfying the equation \( A + B + C = 0 \), prove that the vanishing of the determinant

\[
\begin{vmatrix}
A_1^3, & B_1^3, & C_1^3 \\
A_2^3, & B_2^3, & C_2^3 \\
A_3^3, & B_3^3, & C_3^3 \\
\end{vmatrix}
\]

is equivalent to the relation

\[A_1 A_2 A_3 + B_1 B_2 B_3 + C_1 C_2 C_3 = 0.\]

3. Show that, if the roots, \( \alpha \) and \( \beta \), of the quadratic

\[ax^2 + (b+c)x + d = 0\]

are unequal, the equation

\[a u_n u_{n+1} + b v_n + c v_{n+1} + d = 0\]

can be put into the form

\[
\frac{u_{n+1} - \alpha}{u_n - \alpha} = k \frac{u_{n+1} - \beta}{u_n - \beta},
\]

where \( k \) is given by the equation

\[k = \frac{b - c}{a}.\]

Show that \( u_{n+r} = u_n \), if \( b^2 - c^2 + 2bc \cos \theta = 4ad \cos \theta \), where \( \theta \) is an integral multiple of \( \pi/r \).

4. Show that any convergent to the continued fraction

\[a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \ldots}}}\]

where \( a_1, a_2, a_3, a_4, \ldots \) are positive integers, is nearer to the value of the fraction than any preceding convergent.

If \( p_{n+1}/q_{n+1} \) and \( p_n/q_n \) are the \((n+1)\)th and \( n \)th convergents to the fraction and \( a \) is a positive integer such that \( 2a < a_{n+1} \), prove that \( (p_{n+1} - a q_n)/(q_{n+1} - a q_{n}) \) is a closer approximation to the value of the fraction than \( p_n/q_n \).

5. By means of the identity

\[(e^x - e^{-x}) (e^x + e^3 x + e^5 x + \ldots + e^{2n-1} x) = e^{2nx} - 1,\]

or by any other method, prove that \( s_n \), the sum of the \( r \)th
powers of the first \( n \) odd positive integers, is a polynomial in \( n \) of degree \( r + 1 \).

Show that the polynomial contains only odd powers of \( n \) if \( r \) is even, and contains only even powers of \( n \) if \( r \) is odd, and that in both cases the polynomial vanishes when \( n \) vanishes.

Determine these polynomials explicitly, when \( r = 4 \) and \( r = 5 \).

5. Show graphically or otherwise that the equation
\[
es^x = ax + a,
\]
where \( a \) is real, has two real roots, if \( a \) is greater than unity and has no real roots, if \( a \) is less than unity.

Show also that the equation has no root of the form \( iv \), where \( v \) is real, and that, if \( u + iv \) is a complex root, \( u \) is positive.

6. Give any method of finding the sum of the \( r \)th powers of the roots of the equation
\[
x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,
\]
proving its accuracy.

If \( a + b + c + d = 0 \) and \( a^5 + b^5 + c^5 + d^5 = 0 \) and \( a, b, c, d \) are all real, show that
\[
(a + b)(a + c)(a + d) = 0.
\]

7. \( A'B'C' \) is a triangle whose sides are parallel to those of the triangle \( ABC \). \( ABC \) lies entirely inside \( A'B'C' \) and the point of concurrence of \( AA', BB', CC' \) lies inside \( ABC \).

If the distances between \( B'C' \) and \( BC \), \( C'A' \) and \( CA \), \( A'B' \) and \( AB \) are \( x, y, z \) respectively, with a suitable convention with regard to signs, prove that the ratio of a side of \( A'B'C' \) to the corresponding side of \( ABC \) is
\[
\frac{1 + ax + by + cz}{2 \Delta},
\]
where \( \Delta \) is the area of the triangle \( ABC \), and \( a, b, c \) are the lengths of its sides.

Prove that the sides of the triangle formed by the common tangents of the three escribed circles of the triangle \( ABC \), distinct from the sides of the triangle \( ABC \), are parallel to the sides of the triangle formed by the tangents at

\( A, B, C \) to the circle circumscribed to \( ABC \), and that the ratio of the sides of the first triangle to the sides of the second triangle is \( 1 + \cos A + \cos B + \cos C \).

8. Show that \( \sin (2n+1) \theta \) can be expressed as a polynomial in \( \sin \theta \) of the \( (2n+1) \)th degree and hence show that
\[
\sin (2n+1) \theta = (2n+1) \sin \theta \prod_{r=1}^{n} \left( 1 - \cos^2 \frac{r \pi}{2n+1} \sin^2 \theta \right).
\]

From the theory of partial fractions, or otherwise, justify the representation of \( (2n+1) \) \( \cos (2n+1) \theta \) in the form
\[
A \cos \theta + \sum_{r=1}^{n} B_r \left\{ \cos \left( \theta + \frac{r \pi}{2n+1} \right) + \cos \left( \theta - \frac{r \pi}{2n+1} \right) \right\}
\]
where \( A \) and \( B_r \) are independent of \( \theta \).

Prove that \( A = 1 \) and \( B_r = (-1)^r \).

9. By means of the substitution \( x = ab/y \), prove that
\[
\int_b^a \frac{(a^2 - x^2)(x^2 - b^2)^n}{x^{2n+2}} \, dx = ab \int_b^a \frac{\{a^2 - x^2\}(x^2 - b^2)^n}{x^{2n+2}} \, dx,
\]
a and \( b \) both being positive.

By integration by parts, combined with the preceding result, prove that
\[
\int_b^a \frac{(a^2 - x^2)^n (x^2 - b^2)^m}{x^{2n+2}} \, dx = (a-b)^2 \int_b^a \frac{(a^2 - x^2)^{n-1} (x^2 - b^2)^{m-1}}{x^{2n}} \, dx.
\]

**Group III (Paper B)**

**MATHEMATICAL DISTINCTION PAPER.**

**TUESDAY, JULY 24TH, 1928. 3 HOURS.**

**[Not more than eight questions to be attempted.]**

1. A straight line meets the sides \( BC, CA, AB \) of a triangle \( ABC \) at \( P, Q, R \) respectively. Prove that
\[
\frac{BP}{CQ} \cdot \frac{CQ}{AR} \cdot \frac{AR}{QA} \cdot \frac{QA}{RB} = -1,
\]
explaining why the minus sign is inserted.
Points $L, M$ are taken on the sides $AC, AB$ of a triangle, and $BL$ meets $CM$ at $O$. Prove that, if

$$AL : LC = BM : MA = 1 : 2,$$

then $BO : OL = 3 : 4$.

2. A system of coaxial circles is such that its members do not meet at real points. Prove that any circle through the limiting points of the system meets every circle of the system orthogonally.

Two points $A, B$ are inverses with respect to a circle $S$; the inverses of $A, B, S$ with respect to a circle $T$ are $A', B', S'$. Prove that the points $A', B'$ are inverses with respect to $S'$.

3. A tetrahedron $ABCD$ is such that $AD = BC$, $BD = CA$, $CD = AB$. By considering the circumscribing parallelepiped of the tetrahedron, or otherwise, prove that the sum of the squares on the six edges of the tetrahedron is equal to sixteen times the square of the radius of its circumscribing sphere.

Prove also that the centres of the inscribed sphere and the circumscribing sphere of this tetrahedron coincide.

4. Prove that the middle points of the three diagonals of a complete quadrilateral lie on a straight line.

Generalize this theorem by projection.

State the theorem which is the polar reciprocal of the generalized theorem.

5. Prove that the locus of intersections of corresponding rays of two homographic pencils is, in general, a conic.

Assuming that the feet of the perpendiculars from any point $O$ on the circumcircle of a triangle $ABC$ to the sides of the triangle lie on a straight line (the pedal line of $O$), prove that, through any point $P$ in the plane of the triangle, three pedal lines can, in general, be drawn; but that the number is reduced to two when $P$ lies on any side of the triangle.

6. Tangents are drawn from the point $(h, k)$ to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Prove that the area of the triangle formed by the tangents and their chord of contact is

$$ab\left(\frac{k}{a^2} + \frac{k}{b^2} - 1\right)^{\frac{3}{2}} \left(\frac{k^2}{a^2} + \frac{k^2}{b^2}\right).$$

Prove also that the area of the triangle whose base is the chord of contact and whose vertex is the centre of the ellipse is

$$ab\left(\frac{k^2}{a^2} + \frac{k^2}{b^2} - 1\right)^{\frac{3}{2}} \left(\frac{k^2}{a^2} + \frac{k^2}{b^2}\right).$$

7. Prove that four normals can be drawn to the rectangular hyperbola $xy = a^2$ from any point $H(k, b)$.

If $K, L, M, N$ are the feet of these normals and if the circle through $K, L, M$ has its centre at $U$ and meets the hyperbola again at $N'$, prove that the origin $O$ bisects $NN'$ and that the middle point of $ON$ divides $OH$ in the ratio $1:3$.

8. Verify that the equation of the chord joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

$$y(t_1 + t_2) = 2x + 2at_1t_2.$$  

A variable triangle $PQR$ is inscribed in the parabola. The side $PQ$ passes through the fixed point $(x_1, y_1)$ and the side $PR$ passes through the fixed point $(x_2, y_2)$. If $P$ has coordinates $(at^2, 2at)$, prove that the equation of $QR$ is

$$y = \frac{x}{y_1 + y_2}y_1 - \frac{y_1y_2 - 2ax_1 - 2ax_2 + 2ay_1^2 + ay_1t + ay_2t^2}{a(2x_1 - y_1t)(2x_2 - y_2t)}.$$  

If $y_1 = y_2$, verify that $QR$ envelopes a parabola.

9. Prove Brianchon's theorem that the diagonals of a hexagon circumscribing a conic are concurrent.

[If you prove Brianchon's theorem by reciprocating Pascal's theorem, you must first give a proof of Pascal's theorem.]
Three tangents \( AB, BC, CD \) to a parabola and the
direction of its axis are given; prove the following con-
struction for the tangent at the vertex:

Through \( B \) draw a line parallel to the axis and let \( N \) be
the foot of the perpendicular from \( C \) to this line. Through
\( N \) draw \( NM \) parallel to \( CD \) to meet \( AB \) at \( M \); then the line
through \( M \) parallel to \( CN \) is the required tangent.

10. A conic is inscribed in the triangle of reference of
areal coordinates. Prove that its equation may be written
in the form \( \sqrt{(p \alpha)} + \sqrt{(q \gamma)} + \sqrt{(r \zeta)} = 0 \),
and that the coordinates of its centre are proportional to
\( (q+n, r+p, p+q) \).

**Group III (Paper 7).**

**MATHEMATICAL DISTINCTION PAPER.**

**WEDNESDAY, JULY 25TH, 1928. 3 HOURS.**

[Not more than eight questions to be attempted.]

1. Find the area of the surface of the spheroid formed by
rotating an ellipse of eccentricity \( e \) and semi-axis major \( a \)
about its major axis.

If \( e \) is so small that its fourth and higher powers may
be neglected, show that this area is equal to that of the
surface of the sphere, whose volume is equal to that of the
spheroid.

2. Prove that
\[
(x+1) \left( \frac{d}{dx} \right)^n \{(x+1)^n (x-1)^n \}
= (n+1) \left( \frac{d}{dx} \right)^n \{(x+1)^n (x-1)^n \}.
\]

Prove also that \( \left( \frac{d}{dx} \right)^n \{(x+1)^n (x-1)^n \} \) satisfies
the equation
\[
(1-w^2) \frac{d^2 y}{dx^2} - (1+w) \frac{dy}{dx} + (n+1)^2 y = 0.
\]

3. A straight line is drawn through a point \( P \) of a given
plane curve in the plane of the curve so as to make a con-
stant angle with the tangent at \( P \). Prove that the normal
to the envelope of the line passes through the centre of
curvature of the given curve at \( P \).

Prove also that the centre of curvature of the envelope
lies on the circle which has the centre of curvature at \( P \) and
the centre of curvature of the evolute of the curve at the
centre of curvature at \( P \) as extremities of a diameter.

4. The coordinates of a point on a curve are expressed in
terms of a parameter \( t \). Determine what area is represented
by an integral of the form
\[
\frac{1}{2} \int \left( \frac{dy}{dt} - y \frac{dx}{dt} \right) dt.
\]

Give a rough sketch of the curve
\[
x = a (2t^3 - 3t), \quad y = a (2t^4 - 4t^2 + 1)\frac{1}{2}.
\]

The branches of this curve which pass from the origin
to infinity are obliterated, leaving a single loop. Prove that
the area of this loop is \( \frac{24a^2}{7} \sqrt{\frac{3}{2}} \).

5. Prove from mechanical considerations the formulae
\( y^2 = x^2 + a^2, \quad T = m \gamma \) for the common catenary.

If the tangents at the points \( P \) and \( Q \) of a catenary are
at right angles, prove that the tension at the middle point
of the arc \( PQ \) is equal to the weight of a length of the string
equal to half the arc \( PQ \).

6. A framework is formed of \( n \) equal and similar rods
\( A_0, A_1, A_2, A_3, \ldots, A_{n-1}, A_n \), each of weight \( W \), freely
jointed at \( A_1, A_2, \ldots, A_{n-1} \), and hanging in the form of \( n \)
sides of a regular polygon of 2\( n \) sides with the extremities
\( A_0 \) and \( A_n \) attached to two points at the same level,
the configuration being maintained by light inextensible strings
connecting the joints \( A_0, A_2, \ldots, A_{n-2} \) to the middle point.
of \( A_0 A_n \). Prove, by means of a stress diagram or otherwise, that the vertical component of the reaction on \( A_0 A_1 \) at \( A_0 \) is \( 2W \cos^2 \frac{\pi}{2n} \).

Prove that, when \( n = 6 \), the tensions in the strings are \( W (\sqrt{3} - 1) \), \( W (\sqrt{3} - 1) \), \( W \).

7. Two exactly similar light elastic strings each have one end attached to a heavy particle and have their other ends attached, one to a fixed point \( A \) and the other to a fixed point \( B \), vertically below \( A \). The particle rests in equilibrium at a distance \( b \) below the middle point of \( AB \) and between \( A \) and \( B \) with both strings stretched. Show that, if the particle receives a vertical displacement which does not cause either string to go slack, it will perform simple harmonic oscillations about its position of equilibrium and that the period of these oscillations will be that of a simple pendulum of length \( b \).

8. State how, when the potential energy of a material system in any position is known, the positions of equilibrium and the stability of the equilibrium can be determined.

\( AB \) is the horizontal diameter of a smooth circular wire fixed with its plane vertical. A light elastic string whose natural length is \( AB \) and whose modulus of elasticity is \( \lambda \), has its ends attached to \( A \) and \( B \) and passes through a small smooth ring of weight \( W \), which can slide on the lower half on the wire. Show that, if \( W > \lambda \left(1 - \frac{1}{\sqrt{2}}\right) \), the ring is in a position of stable equilibrium at the lowest point of wire, and that there are no other positions of equilibrium; but that, if \( W < \lambda \left(1 - \frac{1}{\sqrt{2}}\right) \), the equilibrium in this position is unstable and there are two positions of stable equilibrium.

9. A particle is projected directly upwards with velocity \( v \) in a medium in which the resistance per unit mass is \( k \). Prove that, when \( t \) = time of ascent and \( t' \), the time of descent, in terms of \( v \), \( V \), and \( g \), the ratios of the resistances are equal, \( v^2 / V^2 = g \).

Find \( t \), the time of ascent, and \( t' \), the time of descent, in terms of \( v \), \( V \), and \( g \). Prove also that, if the ratio \( v / V \) is small, \( (t' - t) / t = v^2 / 6V^2 \).

10. \( AB \) and \( AC \), two equal uniform rods, each of weight \( W \), are freely jointed at \( A \), and the ends \( B \) and \( C \) are connected by a straight rigid wire of negligible mass, of such a length that the angle \( BAC \) is \( 2\alpha \). The two rods can rotate freely with \( A \) uppermost about a fixed vertical axis coinciding with the internal bisector of the angle \( BAC \). Find \( \omega \), the angular velocity with which the rods can rotate so that there is no stress in the wire, and show that, if the rods are made to rotate with an angular velocity \( \omega \), the stress in the wire is \( \frac{1}{2} (y^2 - 1) W \tan \alpha \).

**Group IV (Paper 1) and Subsidiary Subject (15a).**

**ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.**

**Tuesday, July 17th, 1938. 2 1/2 Hours.**

1. Solve the equations:

   (i) \( \frac{x + 1}{2x - 1} + \frac{3x + 1}{x - 1} = 8 \);

   (ii) \( x + 2y = 1, \ ax^2 + ay - y^2 + 3x + y + 8 = 0 \).

2. By any graphic method show that the equation \( x^3 - 3x^2 + 4x - 3 = 0 \) has only one real root, and find it correct to the first decimal place.

3. Prove that, if \( a \) and \( b \) have opposite signs, the expression \( ax + \frac{b}{x} \) can be made to take any real value by giving a suitable real value to \( x \); but that, if \( a \) and \( b \) have the same
8. Prove the formula for a plane triangle

\[ a^2 = b^2 + c^2 - 2bc \cos A, \]

considering all possible cases.

The altitudes of three points A, B, C above a certain horizontal plane are 300 ft., 320 ft., 400 ft. respectively, and the projection of the points in the plane are at the vertices of an equilateral triangle whose sides are each 1,000 ft. Find the angles of the triangle ABC.

9. A ship A is steaming at 15 knots on a course 27° east of north, and a second ship B is steaming at 12 knots on a course 25° west of north. At a certain instant the distance and bearing of B from A are respectively 12 nautical miles and 22° east of north. Find by calculation the distances (in nautical miles) of A south and west from B an hour later, and deduce the distance and bearing of B from A.

Check your calculations by means of an accurately drawn figure.

(A knot is a speed of 1 nautical mile per hour.)

Group IV (Paper 2) and Subsidiary Subject (15 b).

ANALYTICAL GEOMETRY AND DIFFERENTIAL AND INTEGRAL CALCULUS.

THURSDAY, JULY 26th, 1928. 2\frac{1}{2} HOURS.

[Not more than eight questions are to be attempted by any Candidate.

No credit will be given for attempts to solve the questions on Analytical Geometry by measurements in carefully drawn figures.]

1. Prove that the points with coordinates

\( (3, 7), (-5, 22), (-20, 14), (-12, -1) \)

are at the corners of a square.

Find the area of the part of this square which lies in the positive quadrant.
2. Find the coordinates of the point which divides the distance between the points \((x_1, y_1), (x_2, y_2)\) internally in the ratio \(m:n\).

The equations of the sides \(BC, CA, AB\) of a triangle \(ABC\) are

\[y = 0, \ x + y = 4, \ x + 2y = 5;\]

the line \(y = 2x\) meets \(BC\) at \(A',\ CA\ at \ B',\ and \ AB\ at \ C'.\ Find\ the\ coordinates\ of\ the\ middle\ points\ of\ \(AA', BB', CC'\). Verify that these middle points all lie on a straight line, and find its equation.

3. Prove that the equation of a straight line may be written in the form

\[x \cos \alpha + y \sin \alpha = p.\]

State the geometrical interpretation of \(p\) and \(\alpha\).

Find the coordinates of the points on the axis of \(x\) whose distances from the point \((-\alpha, \alpha)\) are equal respectively to their distances from the line \(x + y = 2a;\)

4. Find the equation of the tangent at \((x_1, y_1)\) to the circle.

\[x^2 + y^2 + 2gx + 2fy + c = 0.\]

A triangle has two of its sides along the coordinate axes and its third side is any tangent to the circle

\[x^2 + y^2 = a^2.\]

Prove that the locus of the circumcentre of the triangle is

\[a^2 (x^2 + y^2) = 4x^2y^2.\]

5. If the circles

\[x^2 + y^2 + 2gx + 2fy + c = 0, \ x^2 + y^2 + 2g'x + 2f'y + c' = 0\]

cut orthogonally, prove that

\[2gg' + 2ff' - c - c' = 0.\]

Find the equation of the circle which cuts orthogonally the circle \(x^2 + y^2 - 6x + 4y - 3 = 0\), passes through \((3, 0)\) and touches the axis of \(y;\)

6. If

\[f(x) = 2x^3 + 3x(1 + 4 \cos x) - 3 \sin x (4 + \cos x),\]

\[F(x) = \cos x (2 \cos^2 x + 3 \cos x + 6) + 6 \sin x (2 + \cos x) - 3x^2 (1 + 2 \cos x),\]

determine the differential coefficients of \(f(x)\) and \(F(x)\), and prove that

\[\frac{dF}{dx} = \sin x \frac{df}{dx}.\]

7. The coordinates of a point on a curve are given in terms of a parameter by the equations

\[x = a \cos^3 t, \ y = a \sin^3 t.\]

Prove that the equation of the tangent at the point with parameter \(t\) is

\[x \sin t + y \cos t = a \sin t \cos t.\]

Prove that the intercept cut off the tangent by the coordinate axes is of constant length; and that the intercept cut off the normal to the curve by the lines \(y = x, y = -x\) is also of constant length.

8. Evaluate the following integrals:

\[\int \frac{dx}{(x-1) (x-3)}, \ \int \frac{dx}{(x-1) (x-3)}, \ \int \frac{dx}{\sqrt{(x-1) (3-x)}}, \]

\[\tan x dx, \ \int \cos^2 x dx.\]

9. Write down a formula for the volume of a surface of revolution.

A lamina in the shape of the parabola \(y^2 = 4ax\) bounded by the chord \(x = a\) is rotated (i) about the axis of \(y\), (ii) about the line \(x = a\). Prove that the volumes generated in the two cases are

\[16\pi a^3/5, \ 32\pi a^3/15.\]

10. Prove by integration that the moment of inertia of a uniform circular disk of mass \(m\) and radius \(a\) about a line through its centre perpendicular to its plane is \(\frac{1}{4}ma^2\).

The mass of a uniform solid right circular cone is \(M\), and the radius of its base is \(a\). Prove that its moment of inertia about its axis is \(3Ma^2/10\).
4. State the Laws of Friction.

The cross-section of a prism is a regular polygon of 24 sides, and the prism can be turned about its axis, which is horizontal. On the uppermost face, which is horizontal, is placed a block whose coefficient of friction is 0.5, and on each of the two adjacent faces are placed blocks whose coefficients of friction are 0.6. The prism is then turned slowly about its axis. Find which of the blocks is the first to slide.

5. The light jointed framework shown in the diagram has all its angles equal to 45° or 90°; it is supported at its ends with three of its bars horizontal, and it carries loads of

![Diagram](image)

6 tons and 10 tons. Find, graphically or otherwise, the stress in each bar, indicating which are in tension and which are in compression.

6. A train takes time \( T \) to perform a journey; it travels for time \( \frac{T}{a} \) with uniform acceleration, then for time \( (a - 2) \frac{T}{a} \) with uniform speed \( V \), and finally for time \( \frac{T}{a} \) with constant retardation. Prove that its average speed is

\[
\frac{a-1}{a} V
\]

If the length of this journey is 40 miles, the time taken on the journey is 50 minutes, and the uniform speed is 60 miles per hour, find the time which is occupied in travelling at this speed.

7. Two particles of masses 3 oz. and 5 oz. are connected by a string which hangs vertically over a smooth fixed pulley. Determine the acceleration of the system and the tension of the string.
When 2 seconds have elapsed after the system starts from rest, the lighter mass picks up a mass of 4 oz. which was at rest before being picked up. Determine the time which elapses before the system comes instantaneously to rest.

8. A particle is projected under gravity with velocity \( V \). Prove that its maximum range on a horizontal plane through the point of projection is \( \frac{V^2}{g} \).

Prove that, if the particle is projected so as to have this range a maximum, its range on a horizontal plane at depth \( h \) below the point of projection is

\[
\frac{V}{2g} \left[ V + \sqrt{V^2 + 4gh} \right].
\]

What interpretation could you give to the expression obtained from this by changing the sign of the square root?


A motor-car weighing 15 cwt. travels up a hill a mile long and 1000 feet high at a uniform speed of 20 miles per hour. Find the H.P. exerted by its engine in overcoming the force of gravity.

If the engine is actually working at 30 H.P., find the total resistance (in poundals) due to friction, air resistance, &c.

10. A particle attached by a light string of length \( l \) to a fixed point describes a horizontal circle of radius \( a \) with uniform angular velocity \( \omega \). Prove that

\[
\omega^2 \left( l^2 - a^2 \right) = g^2.
\]

If, when the particle is moving in this manner, it is suddenly stopped and then let go, prove that its velocity \( v \), when the string is vertical in the subsequent motion, is given by the equation

\[
v^2 = 2g \left[ l - \sqrt{l^2 - a^2} \right].
\]

Subsidiary Subject 15 (e).

ARITHMETIC, ALGEBRA, AND GEOMETRY.

TUESDAY, JULY 17TH, 1938. 21\( \frac{1}{2} \) Hours.

1. (i) Solve the simultaneous equations

\[
2x + y = 1, \quad x^2 + xy + 2y^2 = 2.
\]

(ii) Prove that, if

\[
3xy + 2y + x + 1 = 0 \quad \text{and} \quad 3xx + 2x + x + 1 = 0,
\]

then

\[
3xy + 2y + x + 1 = 0.
\]

2. The quadratic expression \( ax^2 + 2bx + c \) takes the values \( y_1 \) and \( y_2 \) for the values \( x_1 \) and \( x_2 \) of \( x \) respectively, and the corresponding gradients of the expression are \( m_1 \) and \( m_2 \). Prove that

\[
\frac{y_1 - y_2}{x_1 - x_2} = \frac{1}{3} (m_1 + m_2).
\]

3. Draw the graph of \( x^3 - 3x^2 \) between the values \(-1\) and \(+3\frac{1}{2}\) of \( x \), taking 1 inch (or 2 cm.) as the unit of \( x \) and \( \frac{1}{2} \) inch (or 0-5 cm.) as the unit of \( y \).

Hence find approximately the solutions of the equation

\[
x^3 - 3x^2 - x + 1 = 0.
\]

4. Find the sum of the first 100 terms of the arithmetic progression whose first and second terms are \( 2 \) and \( 2\frac{1}{2} \).

Find the least number of terms of the geometric progression whose first and second terms are \( 2 \) and \( 2\frac{1}{2} \) which must be taken that their sum may exceed the sum in the first part of the question.

5. Write out at full length, simplifying the coefficients as much as possible, the expansion of \( (x + b)^7 \) by means of the binomial theorem.

If \( y \) denotes \( x + \frac{1}{x} \), express \( x^7 + \frac{1}{x^7} \) in the form

\[
y^7 + Ay^5 + By^3 + Cy,
\]

where \( A, B, \) and \( C \) are numerical coefficients.
6. The bisectors of the exterior angles of any triangle $ABC$ form a triangle $DEF$. Express the angles of the triangle $DEF$ in terms of those of the triangle $ABC$, and show that they are all acute.

7. Prove that the opposite sides and angles of a parallelogram are equal.

Prove that an ordinary non-reentrant plane quadrilateral $ABCD$ is a parallelogram in each of the following cases:

(i) The angle at $A$ is equal to the angle at $C$ and the angle at $B$ is equal to the angle at $D$.

(ii) $AB$ is parallel to $DC$ and the sum of the sides $BC$ and $CD$ is equal to the sum of the sides $AB$ and $AD$.

8. Prove that, if a straight line touch a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the alternate segments.

A point $P$ is taken on the straight line $AB$ and a point $Q$ is taken on the straight line $AC$. Prove that the angles between the tangents drawn at $A$ to the sides of the circle circumscribed about the triangles $ABC$ and $APQ$ are equal to those between the straight lines $BC$ and $PQ$.

9. Prove that, if two triangles are equiangular, their corresponding sides are proportional.

Two circles intersect at the points $A$ and $B$. A straight line drawn through $A$ meets one circle at $P$ and the other circle at $Q$. Show that all the triangles $PQ$ obtained by varying the direction of the line are similar and that $PQ$ has its greatest length when drawn at right angles to $AB$.

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**Group III (Paper 1).**

**ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.**

**TUESDAY, JULY 16TH, 1929. 2½ HOURS.**

1. Find the gradient of the function $(x-1)^2 (x-3)^2$, and the values of $x$ for which the gradient vanishes. Draw a graph of the given function for values of $x$ between 0 and 4. Find for what range of values of $n$ the equation $x^4 - 8x^3 + 22x^2 - 24x + n = 0$ has four real roots.

2. (i) Solve the simultaneous equations

$$3x - y = 5, \quad x^2 - 3xy + 2y^2 - x + 5y = 3.$$ 

(ii) If $\alpha, \beta$ are the roots of the equation

$$x^2 - x + 5 = 0,$$

find the equation whose roots are

$$\frac{\alpha^3 + 1}{\beta}, \quad \frac{\beta^3 + 1}{\alpha}.$$ 

3. Prove the formula for the number of combinations of $n$ different things taken $r$ together, without assuming the formula for the number of permutations.

Show that the number of combinations $n-3$ at a time of $n$ things of which 4 are alike and the rest different is

$$\frac{1}{2} \binom{n-3}{n-9} (n^2 - 9n + 26).$$ 

4. Find (without multiplying it out) the greatest term in the expansion of $(1 + \frac{1}{3})^{100}$ by the binomial theorem. Also find its ratio to the terms that immediately precede and follow it in the expansion.

5. Write down the expansions of $x^2$ and $\log_e (1 + x)$ in ascending powers of $x$, and state for what ranges of values of $x$ they are valid.

Show that, if

$$\log_e y = 1 + \frac{1}{2} x - \frac{1}{6} x^2 + \frac{1}{12} x^3,$$

then as far as the third power of $x$,

$$y = e \left(1 + \frac{1}{2} x - \frac{1}{24} x^2 + \frac{1}{144} x^3\right).$$
6. Give definitions of \( \sin A \) and \( \cos A \) that apply to angles of any magnitude.

Prove that for angles of any magnitude
\[
\cos A + \cos B = 2 \cos \frac{1}{2} (A + B) \cos \frac{1}{2} (A - B).
\]

Solve completely the equations
\[
\cos x + \cos y = \frac{1}{2}, \\
\sin x + \sin y = \frac{1}{2}.
\]

7. Indicate by a rough drawing the form of the graph of \( \tan^2 x \) for all values of \( x \). Show that the equation
\[
ax = \tan^2 x
\]
(where \( x \) is the radian measure of an angle and \( a \) is a constant) has an infinite number of roots.

Plot the graph of \( \tan^2 x \) for values of \( x \) between 0 and \( \frac{1}{2} \pi \), taking 2 inches (or 4 centimetres) to represent \( \frac{1}{2} \pi \) on the \( x \) axis and 1 inch (or 2 centimetres) as the unit of \( y \).

Find as accurately as you can from your graph the solution of the equation \( 2x = \tan^2 x \) that lies between 0 and \( \frac{1}{2} \pi \).

8. Prove that in any triangle
\[
\tan \frac{1}{2} (B - C) = \frac{b - c}{b + c} \cot \frac{1}{2} A.
\]

An aeroplane flying in a horizontal straight line at a height of 1,000 feet is observed to be due north and at an elevation of 60°. At a later observation from the same station it is 40° E. of N. and at an elevation of 35°. Find the direction in which the aeroplane is travelling.

9. Prove that the area of a triangle \( ABC \) is
\[
2 R^2 \sin A \sin B \sin C,
\]
where \( R \) is the radius of the circumcircle.

The inscribed circle of the triangle touches the sides in \( X, Y, Z \); prove that

area of triangle \( XYZ \): area \( ABC = 2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C : 1 \).

Find how the result must be modified if instead of \( X, Y, Z \) we take the points of contact of the escribed circle opposite to the angle \( A \).

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Group III (Paper 2) and Subsidiary Subject (18 parts).

Pure and Analytical Geometry.

Wednesday, July 17th, 1939. 2 1/2 Hours.

1. Two perpendicular chords of a circle, centre \( O \), are drawn through a fixed point \( O \) not on the circle. Prove that the sum of their squares is the same for all such pairs of chords.

Prove also that the locus of the points of intersection of the circles described on two such chords as diameters is a circle whose centre is the middle point of \( OC \).

2. \( O \) is a point in the plane of a triangle \( ABC \), and \( P \) is a point on \( BC \). The circle described to pass through \( O \) meets \( AB \) at \( R \), and the circle described to pass through \( OPC \) meets \( AC \) at \( Q \). Prove that \( AQQR \) are concyclic.

Prove that, if \( P \) varies but \( O \) is fixed, the triangle \( PQR \) remains similar to itself, and deduce a construction for an equilateral triangle which has a vertex on each side of the triangle \( ABC \); one vertex being given.

3. Prove that a sphere can be circumscribed about any tetrahedron, and show how to find its centre.

Find the radius of the sphere circumscribed about the tetrahedron \( ABCD \), in which \( BC = CA = AB = a \) and \( DA = DB = DC = d \).

4. Prove that the ratios of coplanar areas are unaltered by orthogonal projection.

A parallelogram of constant area is inscribed in an ellipse. Prove that the envelopes of its two pairs of opposite sides are similar, similarly-situated, concentric ellipses.

Determine the area in terms of the semi-axes of the ellipse, if all the sides of the parallelogram have the same envelope.

5. Prove that the equation \( lx + my + n = 0 \), whatever the signs of \( l, m, n \), can be reduced to the form
\[
x \cos \theta + y \sin \theta = p,
\]
where \( p \) is positive; and interpret \( p \) and \( \theta \).
If the equations of any two straight lines

\[ lx + my + n = 0 \quad \text{and} \quad l'x + m'y + n' = 0 \]

are reduced to this form, prove that

\[ pp' \sec (\theta - \theta') = nn' / (ll' + mm') ; \]

and deduce, or obtain in any manner, a criterion to determine whether the origin lies in the acute or obtuse angle between the two straight lines.

6. Find the equation of the tangent to the parabola

\[ y^2 = 4ax \]

at the point \((at^2, 2at)\).

Prove that the straight line joining the focus of a parabola to the point where the line joining a point of the parabola to the vertex of the parabola meets the directrix is parallel to the tangent at the point.

7. Prove that a chord of an ellipse is parallel to a fixed direction, if the sum of the eccentric angles of its extremities is constant.

A chord of an ellipse, centre \(C\), drawn in a fixed direction meets the major axis of the ellipse at \(X\), and the perpendicular drawn to the chord at its middle point meets the major axis at \(Y\). Prove that the ratio \(CX : CY\) is constant.

8. Find the equation of the normal to the hyperbola

\[ b^2x^2 - a^2y^2 = a^2b^2 \]

at the point \((x', y')\).

The normal at a point \(P\) of this hyperbola meets the transverse axis of the hyperbola at \(Q\), and \(GQ\) is drawn at right angles to this axis and equal to \(GP\). Find the equation of the locus of \(Q\).

9. Find in a simple form, symmetrical in \(t\) and \(t'\), the equation of the straight line joining the points

\((ct, c/t) / (ct', c/t')\).

A chord \(PQ\) of the hyperbola \(xy = c^2\) passes through the fixed point \((h, k)\). Prove that the locus of the remaining vertices of the parallelogram formed by drawing straight lines through \(P\) and \(Q\) parallel to the asymptotes is the hyperbola

\[ xy + c^2 = hx + ky. \]
4. What information is given about a function \( f(x) \) in the neighbourhood of \( x = a \) by a knowledge of the sign of \( f'(a) \)?

If \( a \) is a positive acute angle and \( \sin^{-1} (\sin \theta \sin a) \) has its principal value, prove that
\[
\theta \sin a - \sin^{-1} (\sin \theta \sin a)
\]
steadily increases as \( \theta \) increases from 0 to \( \frac{1}{2} \pi \), while \( a \) remains constant; and show that, if \( a \) and \( \theta \) are both positive acute angles, the value of the expression is positive and less than
\[
\frac{1}{2} \sqrt{\pi^2 - 4} - \cos^{-1} \frac{2}{\pi}.
\]

5. Find the equation of the tangent at any point of the cycloid given by the equations
\[
x = a (2 \psi + \sin 2 \psi), \quad y = a (1 - \cos 2 \psi).
\]

Prove that \( \psi \) is the angle which the tangent makes with the axis of \( x \), and that two perpendicular tangents to the same arch of the cycloid intersect a constant length on the axis of \( x \).

6. Find
\[
\int \sec^2 x \, dx, \quad \int \frac{dx}{(b - ax)^{3/2}} \quad \int \frac{dx}{b^2 + a^2 x^2}.
\]

Using the substitution \( x = a \cos^2 \theta + b \sin^2 \theta \),
\[
\int \frac{dx}{b^2 + a^2 x^2}.
\]

and prove that, if \( a^2 \) is less than 1,
\[
\int_0^{\pi/2} \frac{a \cos \theta \, d\theta}{1 - a^2 \cos^2 \theta} = \frac{\sin^{-1} a}{\sqrt{1 - a^2}}.
\]

7. Indicate by a rough diagram the shape of the curve
\[
ay = a (a - x)^2.
\]

Find the area of its loop and the coordinates of the centre of gravity of the upper of the two halves into which the loop is divided by the axis of \( x \).

8. Find the volume of the portion of a sphere of radius \( a \) included between two parallel planes each at a distance \( a/\sqrt{2} \) from the centre of the sphere.

Prove also that the radius of gyration of this portion about the diameter, which is perpendicular to both planes, is \( \sqrt{4a^3 / 10} \).

9. A particle moves in a straight line so that its distance from an origin on that line at time \( t \) is \( at - \lambda t \sin (\mu t + \alpha) \), where \( a, c, \lambda, \mu \) are constants, and it is at instantaneous rest when \( t = 0 \). Prove that \( \lambda = \mu \cot a \).

Prove also that the period between consecutive positions of instantaneous rest is constant and equal to \( \pi / \mu \), that the distances of the particle from the origin at the moments of instantaneous rest diminish in geometrical progression, and that the speed with which it passes through the origin diminishes in geometrical progression.

Group III (Paper 4).

STATICS AND DYNAMICS.

MONDAY, JULY 15TH, 1929. 3 Hours.

[Not more than nine questions are to be attempted by any Candidate.]

In numerical calculations take \( g = 32 \) foot-second units.

1. Prove that forces completely represented by the sides \( BC, CA, AB \) of a triangle \( ABC \) are equivalent to a couple.

\( D \) is the middle point of the side \( BC \), and forces are completely represented by \( BC, CA, AB \), and \( AD \). Find the magnitude and direction of the resultant force and the point in which it cuts the line \( BC \).

2. State the necessary conditions that a rigid body may be in equilibrium under the action of coplanar forces.

The figure represents the central cross-section of four smooth cylinders lying inside a uniform rectangular box from which the bottom has been removed so that the two lowest cylinders are resting on the ground. The cylinders are of
the same length as the inside of the box. Three of them are of the same size and of weight $W$ and one is of weight $W'$ as indicated. Find the reactions at the points $A$, $B$, $C$, assuming that the lines of centres make angles $a$ with the vertical; and show that the box will upset unless its weight is at least $2 \frac{bW}{a}$, where $a$, $b$ represent the lengths indicated in the figure.

3. A chessboard $ABCD$ 8 in. square is divided into sixty-four squares of side 1 inch. The eleven squares whose centres are nearest to the corner $O$ are cut away. Show that the distance from $A$ of the centre of gravity of the remainder is 4.949 in. ($\sqrt{2} = 1.414$.)

4. State the laws of limiting friction, and explain what is meant by the angle of friction.

A uniform square lamina hangs with its plane at right angles to a vertical wall, one corner being in contact with the wall and another tied to the wall by a string whose length is equal to a side of the square. Show that, if $\lambda$ be the angle of friction between the lamina and the wall, the greatest and least inclinations of the string to the vertical in equilibrium are the values of $\theta$ given by

$$\cos(\theta \pm \lambda) = \frac{1}{2} (\cot \theta - 1).$$

5. The figure represents a framework of smoothly jointed rods which can turn about a pivot at $C$. It carries loads of 3 lb. at $A$ and $F$ and is kept in equilibrium by a force $P$ at $E$ parallel to $AC$. The rods are all either vertical, horizontal, or inclined at $30^\circ$ to the vertical. Determine the force $P$ graphically or otherwise. Construct a force diagram and find the stresses in the sides of the triangle $DEF$, stating whether they are tensions or thrusts.

6. A line of men are following one another along a road 20 yds. apart at a uniform speed of 8 miles per hour. Another line of men 35 yds. apart are bicycling in the same direction at 12 miles per hour. At what speed must a man walk in the same direction so that whenever he is passed by a runner he is also passed by a cyclist?

7. The kinetic energy of a moving body may be defined as the work that it is capable of doing in overcoming resistance before it is reduced to rest. Show that in the
case of a body of mass \( m \) moving with velocity \( v \) without rotation this definition leads to \( \frac{1}{2} mv^2 \) as the expression for its kinetic energy.

An engine forces a stream of water through a nozzle of 2 sq. in. section at a speed of 50 ft. per sec. At what horse-power is the engine working?

If the momentum of the stream is destroyed by impact on a plane at right angles to itself, what force does the stream exert on the plane?

[Take 1 c. ft. of water = 62-5 lb.]

8. State the laws by which the motion of two smooth spheres after impact is determined.

Show that if two smooth spheres whose masses are equal are perfectly elastic and one impinges obliquely on the other which is at rest, then their subsequent directions of motion are at right angles.

Three smooth perfectly elastic spheres of diameter \( a \) and equal masses have their centres at the corners \( A, B, C \) of a square of side \( c \). The sphere at \( A \) is to be projected so as to strike in turn the spheres at \( B \) and \( C \) and finally move parallel to \( AB \). Show that the direction of projection makes with \( AB \) an angle \( \theta \) given by

\[
a \cos (\theta - \phi) = c \sin \theta,
\]

where

\[
\cos \phi = a/(c-a).
\]

9. Prove that when a particle describes a circle of radius \( r \) with velocity \( v \) it has an acceleration \( v^2/r \) towards the centre of the circle.

A heavy particle suspended from a fixed point by a string of length \( a \) is projected horizontally with velocity \( 2\sqrt{ga} \). Find the inclination of the string to the vertical when it becomes slack, and show that the greatest height to which the particle rises is \( \frac{50}{29} a \).

10. Find the period of small oscillations of a rigid body free to turn about a fixed horizontal axis, and also find a formula for the length of the equivalent simple pendulum.

Three particles of the same mass \( m \) are fixed to a uniform circular hoop of mass \( M \) and radius \( a \) at the corners of an equilateral triangle. The hoop is free to swing in a vertical plane about any point in its circumference. Prove that the equivalent simple pendulum is equal in length to the diameter of the circle.

Group III (Paper 5).

MATHEMATICAL DISTINCTION PAPER.

Monday, July 22nd, 1929. 3 Hours.

[Not more than eight questions are to be attempted by any candidate.]

1. If a sequence of numbers \( u_1, u_2, u_3, \ldots, u_n, \ldots \) is formed by the law

\[
2u_{n+1} = u_n + \frac{u_n}{N},
\]

prove that

\[
u_n - \sqrt{N} = (u_1 - \sqrt{N})^p, \quad \frac{u_n + \sqrt{N}}{N} = (u_1 + \sqrt{N}),
\]

where \( p = 2^n - 1 \).

Taking \( N = 3 \) and \( u_1 = 2 \), prove by means of tables that \( u_n - \sqrt{3} \) is less than \( 0.25 \times 10^{-6} \).

2. If \( u = ax^2 + bx + c \) and \( v = a'x^2 + b'x + c' \) and \( x_1 \) and \( x_2 \) are the roots of the equation

\[
(ax+b)(a'x+c') = (a''x+h)(bx+h),
\]

prove that \( A, A', B, B' \), independent of \( x \), can be found such that

\[
u = A(x-x_1)^2 + B(x-x_2)^2, \quad v = A'(x-x_1)^2 + B'(x-x_2)^2,
\]

provided that \( u \) and \( v \) have no common factor.

3. Prove that the resolution of the general biquadratic equation into two quadratics can be effected by means of the solution of a cubic equation.

Find the two real roots of the equation

\[
a^4 + 12a^3 + 30a^2 + 76a + 21 = 0.
\]
40

4. Resolve \( \frac{n(n+1)(n+2)\ldots(n+r)}{m} \) into partial fractions.

Express the product of \( e^{-x} \) and the infinite series
\[
\frac{1}{m} + \frac{x}{m(m+1)} + \frac{x^2}{m(m+1)(m+2)} + \frac{x^3}{m(m+1)(m+2)(m+3)} + \ldots
\]
as an infinite series in powers of \( x \), simplifying the coefficients as much as possible; and show that, if \( m \) is positive, the sum of the infinite series
\[
1 - \frac{x}{m+1} + \frac{x^2}{(m+1)(m+2)} - \frac{x^3}{(m+1)(m+2)(m+3)} + \ldots
\]
is positive for all real values of \( x \).

5. Prove that the distances of the vertex \( A \) of a triangle \( ABC \) from the centre of the inscribed circle and the centre of the escribed circle opposite to \( A \) are respectively \( (s-a) \sec \frac{1}{2}A \) and \( s \sec \frac{1}{2}A \).

The external bisectors of the angles of a plane ordinary quadrilateral \( ABCD \) form a second quadrilateral \( PQR,\ PQ,\ QR,\ RS,\ SP \) passing through \( A,\ B,\ C,\ D \) respectively. Prove that:

(i) the quadrilateral \( PQR,\ PQ,\ QR,\ RS,\ SP \) is cyclic;
(ii) \( PR \) passes through the intersection of \( AB \) and \( CD \);
(iii) \( PQ \sec \frac{1}{2}C = QR \sec \frac{1}{2}D \)
\[
= RS \sec \frac{1}{2}A = SP \sec \frac{1}{2}B
\]
\[
= \frac{1}{2} (AB + BC + CD + DA) \sec \frac{1}{2} (A + B + C + D).
\]

6. Prove that
\[
\cos n \theta + \cot n \sin n \theta = \prod_{r=0}^{n-1} \left\{ \cos \theta + \cot \frac{\theta + \pi r}{n} \sin \theta \right\}.
\]

7. Explain the representation of complex numbers on the Argand diagram.

Two complex numbers \( z \) and \( z' \) are connected by the relation
\[
z' = \frac{2+z}{2-z}.
\]

8. If \( \Delta \) denotes the determinant
\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3
\end{vmatrix},
\]
and \( x_r, y_r, z_r \) \( (r = 1, 2, 3) \), denote the cofactors of \( x_r, y_r, z_r \) \( (r = 1, 2, 3) \), respectively in the expansion of \( \Delta \), prove that \( \Delta x_1 = Y_2 Z_3 - Y_3 Z_2, \ldots \&c. \)

If \( A, B, C, F, G, H \) are the cofactors corresponding to the elements \( a, b, c, f, g, h \) of the determinant
\[
\Delta = \begin{vmatrix}
  a & h & g \\
  h & b & f \\
  g & f & c
\end{vmatrix},
\]
prove that, if \( \lambda \) is a root of the equation
\[
A + \lambda, H, G
\]
\[
H, B + \lambda, F
\]
\[
= 0,
\]
then \( \Delta / \lambda \) is a root of the equation \( (x+a)(x+b) = h^2 \).

9. Find
\[
\int_0^{\frac{1}{2} \pi} \frac{1 + a \cos \theta}{1 + 2a \cos \theta + a^2} \, d\theta = \cot^{-1} a
\]
by means of the substitution \( \tan \frac{1}{2} \theta = \omega \), or in any other manner.

Prove that, if \( a \) is positive,
\[
\int_0^{\frac{1}{2} \pi} \frac{1 + a \cos \theta}{1 - a \cos \theta + a^2} \, d\theta = \pi - \cot^{-1} a \text{ or } -\cot^{-1} a.
\]

According as \( a \) is less than or greater than unity, the value of the inverse function being that value which lies between 0 and \( \frac{1}{2} \pi \).
Group III (Paper 5).

MATHEMATICAL DISTINCTION PAPER.

TUESDAY, JULY 23RD, 1929. 3 Hours.

[Not more than eight questions to be attempted.]

1. If P, Q, R are three points on the sides BC, CA, AB respectively of a triangle ABC, such that AP, BQ, CR meet at a point, prove that

\[ BP \cdot CQ \cdot AR = + PQ \cdot QA \cdot RB, \]

signs as well as magnitudes of the segments being taken into consideration.

If three straight lines meet at a point but are not in the same plane, prove that the three planes, each of which passes through one of the lines and the internal bisector of the angle between the other two, have a common line of intersection.

2. If \( a, b, \gamma, a', b', \gamma' \) are the middle points of the edges \( DA, DB, DC, BC, CA, AB \) respectively of a tetrahedron \( ABCD \), prove that \( aa', bb', \gamma \gamma' \) meet at a point and bisect one another.

If the area of the face \( ABC \) is equal to the area of the face \( ABD \) and the area of the face \( DBC \) is equal to the area of the face \( DAC \), prove that the area of the triangle \( a'b'\gamma \) is equal to the area of the triangle \( ab\gamma \) and the area of the triangle \( a'b\gamma \) is equal to the area of the triangle \( ab'\gamma \); and hence, or otherwise, that \( BC = AD \) and \( CA = BD \).

3. Prove that the reciprocal of a circle with respect to a second circle, centre \( O \), is a conic with a focus at \( O \), and determine the corresponding directrix, the eccentricity, and the line which is reciprocated into the second focus.

A variable circle with its centre at a focus of a hyperbola meets the hyperbola in four points. Prove that the envelope of the common chords of the circle and hyperbola, which are not perpendicular to an axis of the hyperbola, is the parabola which has its focus at this focus of the hyperbola and the corresponding directrix of the hyperbola for the tangent at its vertex.

4. Prove that the pencil formed by the lines joining four fixed points on a conic to a variable point of the conic is projective (homographic) with the range determined on the tangents at the four fixed points by any tangent to the conic.

\( O \) is a fixed point on a fixed parabola, and \( P \) is a variable point on the same parabola. \( OP \) meets a fixed straight line at \( Q \). Prove that the envelope of the line drawn through \( Q \) parallel to the tangent at \( P \) is a parabola. Determine the common tangents of the two parabolas, distinct from the line at infinity; and determine the fixed straight line, if the two parabolas have the same focus.

5. Find the condition that the straight line

\[ x \cos a + y \sin a = p \]

may be a tangent to the ellipse

\[ b^2x^2 + a^2y^2 = a^2b^2, \quad (a > b). \]

If two circles are drawn with their centres at the extremities of the minor axis of this ellipse to touch any tangent to the ellipse, prove that they cut at an angle \( 2 \cos^{-1} \epsilon \), where \( \epsilon \) is the eccentricity of the ellipse; and prove also that the locus of their points of intersection consists of the two circles

\[ x^2 + y^2 \pm \frac{a^2 - b^2}{\sqrt{(a^2 - b^2)}} x - b^2 = 0. \]

6. Find the condition that the straight lines

\[ lx + my + n = 0 \quad \text{and} \quad l'x + m'y + n' = 0 \]

should be at right angles, the angle between the axes of coordinates being \( \omega \).

If \( \omega \) is the angle between the asymptotes of the hyperbola \( xy = c^2 \), find the coordinates of the point of concurrence of chords of the hyperbola which subtend a right angle at the point \( P (ct, ct/t) \) of the hyperbola; and prove that this
point is the pole of the tangent to the hyperbola at $P$ with respect to the circle circumscribed about the triangle formed by this tangent and the asymptotes.

7. Prove that the lengths and directions of the axes of the conic $ax^2 + 2hxy + by^2 = 1$ are given by the equations

$$\frac{1}{r^2} = a + h \tan \theta = b + h \cot \theta,$$

where $2r$ is the length of an axis and $\theta$ is its inclination to the axis of $x$.

If the conic is an ellipse, show that its major axis lies in the first and third quadrants, if $h$ is negative.

If the conic is a hyperbola, show that the axis which meets the conic in real points lies in the first and third quadrants, if $h$ is positive.

8. Prove that the locus of the centres of a system of conics which pass through four fixed points is a conic.

If the four points lie on a circle, show that the locus passes through the feet of the normals drawn from the centre of the circle to any other conic of the system.

9. Show that the conic inscribed in the triangle of reference whose equation in areal coordinates is

$$\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$$

is an ellipse or hyperbola, according as $fgh (f + g + h)$ is positive or negative.

Determine the ratio in which the side $z = 0$ divides the line joining the points of contact with $x = 0$ and $y = 0$; and hence, or otherwise, show that, if the conic is a hyperbola and the points of contact all lie on the same branch,

$$f (g + h), \quad g (h + f), \quad h (f + g)$$

all have the same sign.

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**Group III (Paper 7).**

**Mathematical Distinction Paper.**

**Wednesday, July 24th, 1929. 3 Hours.**

[Not more than eight questions to be attempted.]

1. Show that, if $z = \frac{\sinh \lambda}{\cosh \lambda + \cos \omega}$ and $y = \frac{\sin \omega}{\cosh \lambda + \cos \omega}$, then $x + iy$ is a function of $\lambda + i\omega$.

Hence, or otherwise, prove that

$$ax^2 + dy^2 = (a\lambda^2 + d\omega^2)/(\cosh \lambda + \cos \omega)^2$$

and

$$\left(\frac{\partial V}{\partial \lambda}\right)^2 + \left(\frac{\partial V}{\partial \omega}\right)^2 = \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 \left(\cosh \lambda + \cos \omega\right)^2.$$  

2. Find the asymptotes of the curve

$$(y - x)^2 (y + x) = (4y - x)^2 (4y + x) + 2y - x = 0.$$  

Investigate on which side of each asymptote the curve lies and trace the curve.

3. Prove that the radius of curvature $\rho$ of a curve $f(x, y) = 0$ is given by the formula

$$\frac{1}{\rho} = \pm \sqrt{f_{xx}f_y^2 - 2f_{xy}f_x f_y + f_{yy}f_x^2}/(f_x^2 + f_y^2)^{\frac{3}{2}},$$

where suffixes denote partial differentiation.

Prove that the radius of curvature at the point $(3a, 2a)$ on the curve

$$x^2 + y^2 - 3axy = 0$$

is $\frac{1}{\lambda} a$.

Also prove that the radii of curvature at the origin on the curve

$$y^2 - 2xy - 3x^2 + a^2 + axy^2 = 0$$

are $4\sqrt{2}$ and $20\sqrt{10}$. 

4. Show that the four figures bounded by the circle 
\[ r = 3a \cos \theta \] and the cardioid \( r = a (1 + \cos \theta) \) have areas
\[ \frac{1}{8} \pi a^2, \frac{1}{3} \pi a^2, \pi a^2, \text{ and } \frac{5}{6} \pi a^2. \]

5. A smooth cylinder of radius \( a \) is fixed with its axis horizontal and at a distance \( c \) from a smooth vertical wall. A uniform rod of length \( l \) rests at right angles to the axis of the cylinder with one end in contact with the cylinder and the other end in contact with the wall. Show that in the position of equilibrium the inclination \( \phi \) of the rod to the vertical satisfies the equation
\[ a \sin \phi = (c - l \sin \phi) \sqrt{1 + 3 \cos^2 \phi}. \]
Also show that the equilibrium is unstable.

6. Two particles \( A, B \), of masses \( m, 2m \) are connected by a fine taut thread of length \( 3a \) and rest on a smooth horizontal plane. \( B \) is struck by a horizontal impulse \( P \) in a direction making an acute angle \( a \) with \( AB \) produced. Determine the initial components of velocity of each particle along and perpendicular to \( AB \), and show that the motion of either particle is that of a point attached to a circle which rolls uniformly along a straight line.

7. The resistance to the motion of a car varies as the square of the speed and the effective horse-power is constant and equal to 14. The mass of the car is 15 cwt. and the maximum speed against the resistance is 40 miles per hour. Show that the car can accelerate from 10 miles per hour to 30 miles per hour in approximately 224 feet.
\[ \log_e 10 = 2.3026. \]

8. A particle is to be projected from a point \( P \) with given velocity \( v \) so as to pass through a point \( Q \). Give a construction for finding the focus of the parabolic path and show that there are two possible paths or one or none according as \( v^2 \geq \frac{g}{(Q + h)} \), where \( h \) is the vertical height of \( Q \) above \( P \).

Show that when two paths are possible the difference between the two times of flight is \( 2 \sqrt{(SP + SQ - PQ)/g} \), where \( S \) is the focus of either path.

9. A bead of mass \( M \) can slide on a fixed smooth straight horizontal rod. A particle of mass \( m \) hangs vertically below the bead to which it is attached by a fine thread of length \( a \). The particle is given a velocity \( u \) parallel to the rod. Prove that when the thread makes an angle \( \theta \) with the vertical, its angular velocity is given by
\[ a^2 \theta^2 (M + m \sin^2 \theta) = M u^2 - 2 (M + m) gu (1 - \cos \theta). \]

10. A bead can slide on a smooth circular wire of radius \( a \) in a vertical plane and is connected to the highest point of the wire by an elastic thread of natural length \( \frac{1}{3} a \), whose modulus of elasticity is equal to the weight of the bead. Find the inclination of the thread to the vertical in the position of equilibrium, and show that the period of small oscillations about the position of that of a simple pendulum of length \( \frac{1}{2} a \).

Group IV (Paper I) and Subsidiary Subject (15a).

ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.

TUESDAY, JULY 16TH, 1929. 2\( \frac{1}{2} \) Hours.

1. (i) Solve the simultaneous equations
\[ 3x - y = 5, \quad x^2 - 3xy + 2y^2 - x + 5y = 3. \]
(ii) If \( a, \beta \) are the roots of the equation
\[ x^2 - x + 5 = 0, \]
find the equation whose roots are
\[ a^2 + \frac{1}{\beta^2}, \quad \beta^2 + \frac{1}{a^2}. \]
2. Determine the gradient of the function \((x - 1)^2(x - 2)\) and the values of \(x\) for which the gradient vanishes.

Draw the graph of the function for values of \(x\) between 0 and 4. Deduce that the equation
\[ x^3 - 4x^2 + 5x - 4 = 0 \]
has only one real root, and that the equation
\[ x^3 - 4x^2 + 5x - \frac{17}{9} = 0 \]
has three real roots.

3. Prove the formula for the number of combinations of \(n\) different things taken \(r\) together, without assuming the formula for the number of permutations.

Show that the number of combinations \(n - 2\) at a time of \(n\) things of which three are alike and the rest different is \(\frac{1}{2}(n^2 - 5n + 8)\).

4. Find (without multiplying it out) the greatest term in the expansion of \((1 + \frac{2}{3})^{100}\) by the binomial theorem.

Also find its ratio to the terms that immediately precede and follow it.

5. Write down the expansion of \(\log_e(1 + x)\) in ascending powers of \(x\), and state for what range of values of \(x\) the expansion is valid.

Deduce that for positive values of \(y\)
\[ \log_e y = 2 \left( \frac{y - 1}{y + 1} + \frac{1}{3} \left( \frac{y - 1}{y + 1} \right)^3 + \frac{1}{5} \left( \frac{y - 1}{y + 1} \right)^5 + \ldots \right) \]

In order to evaluate \(\log_e 2\) we may either put \(x = 1\) or put \(y = 2\) in the foregoing expansions. For what reason is the latter alternative the better? Find \(\log_e 2\) correct to four places of decimals.

6. Give definitions of \(\sin \theta\) and \(\cos \theta\) that apply to angles of any magnitude.

Investigate a formula for all the angles which have their cosines equal to \(\cos \alpha\).

Solve completely the equation
\[ 2 \cos x + 3 \cos x + 1 = 0. \]

7. Indicate by a rough drawing the form of the graph of \(\sin^2 \frac{1}{3} \pi x\) for all values of \(x\).

Show that the equation \(x = 5 \sin^2 \frac{1}{3} \pi x\) has seven roots and no more.

Plot the graph of \(\sin^2 \frac{1}{3} \pi x\) for values of \(x\) between 1 and 3, taking 2 inches (or 4 centimeters) as the unit for both \(x\) and \(y\).

Find as accurately as you can from your graph the solutions of the equation \(x = 5 \sin^2 \frac{1}{3} \pi x\) that lie between 1 and 3.

8. Prove that in any triangle
\[ \tan \frac{1}{2} (B - C) = \frac{b - c}{b + c} \cot \frac{1}{2} A. \]

An aeroplane flying in a horizontal straight line at a height of 1,000 feet is observed to be due north and at an elevation of 60°. At a later observation from the same station it is 40° E. of N. and at an elevation of 35°. Find the direction in which the aeroplane is travelling.

9. Prove that the area of a triangle \(ABC\) is
\[ 2R^2 \sin A \sin B \sin C, \]
where \(R\) is the radius of the circumscribed circle.

The inscribed circle of the triangle touches the sides in \(X, Y, Z\); prove that
area of triangle \(XYZ\) : area \(ABC = 2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C : 1.\)

Find how the result must be modified if instead of \(X, Y, Z\) we take the points of contact of the escribed circle opposite to the angle \(A\).
Group IV (Paper 2) and Subsidiary Subject (15 b).

ANALYTICAL GEOMETRY AND DIFFERENTIAL AND INTEGRAL CALCULUS.

THURSDAY, JULY 25TH, 1929. 2½ HOURS.

[Not more than eight questions are to be attempted by any Candidate.
No credit will be given for attempts to solve the questions on Analytical Geometry by measurements in carefully drawn figures.]

1. Find the condition that the straight lines
   \[ lx + my + n = 0 \quad \text{and} \quad l'x + m'y + n' = 0 \]
   may be perpendicular.

   Find the equation of the straight line which is such that the foot of the perpendicular drawn to it from the origin is the point \((h, k)\).

2. Prove that the length of the perpendicular drawn from the point \((a, b)\) to the straight line \(lx + my + n = 0\) is
   \[ \pm \frac{\sqrt{(al + bm + n)^2}}{(l^2 + m^2)^{\frac{1}{2}}} \].

   Find the coordinates of the points on the straight line \(x + y = 3\) whose distances from the straight line \(12x - 5y + 7 = 0\) are each equal to 1.

3. Prove that the area of the triangle, whose vertices are the points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\), is equal to the numerical value of the expression
   \[ \frac{1}{2} \left[ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right] \].

   State without proof the interpretation of the sign of this expression.

   Find the area of the triangle formed by the feet of the perpendiculars of the triangle whose vertices are \((2, 0), (-2, 0), (0, 3)\).

4. Find the equation of the circle which has the points \((3, -1), (1, 3)\) as extremities of a diameter.

   Find the coordinates of the extremities of the perpendicular diameter.

5. Prove that the straight line \(lx + my + n = 0\) touches the circle \((x - a)^2 + (y - b)^2 = r^2\), if
   \[ (al + bm + n)^2 = r^2 (l^2 + m^2) \].

   Find the equations of the common tangents of the circles \(a^2 + y^2 - 2ax = 0, \quad x^2 + y^2 + 8x = 0\).

6. Prove that in general a value of \(x\), which makes \(f'(x)\) vanish, makes \(f(x)\) assume a maximum or minimum value. Show, graphically or otherwise, that such a maximum is not necessarily greater than such a minimum and that the greatest and least values of the function, as \(x\) increases from \(a\) to \(b\), do not necessarily correspond to values of \(x\) which make \(f'(x)\) vanish.

   Prove that \((x - 2)(x^2 + x + 1)\) has a maximum value corresponding to \(x = -\frac{1}{2}\) and a minimum value corresponding to \(x = 1\), and find its greatest and least values as \(x\) ranges from \(-\frac{1}{2}\) to +2.

7. Find the equations of the tangent and normal at any point of the cycloid given by the equations
   \[ x = a(2\psi + \sin 2\psi), \quad y = a(1 - \cos 2\psi) \].

   Prove that \(\psi\) is the angle which the tangent makes with the axis of \(x\); and verify that, if \(p\) and \(p'\) are the lengths of the perpendiculars drawn from the origin to the tangent and normal respectively, \(p'\) and \(\frac{dp}{d\psi}\) are numerically equal.

8. Find
   \[ \int \frac{dx}{(a-x)^3}, \quad \int \frac{dx}{a^2 - x^2}, \quad \int \sec^2 \theta d\theta \],
   and use the substitution \(x = a \cos^2 \theta + b \sin^2 \theta\) (or any other method) to find
   \[ \int \frac{dx}{\sqrt{(b-a)^3(x-a)}} \].
9. The portion of the parabola \( y^2 = 4ax \) between the vertex and the straight line \( x = h \) is rotated about the axis of the parabola. Find for the solid generated (i) its volume, (ii) the position of its centre of gravity, (iii) its radius of gyration about its axis.

10. A particle moves in a straight line so that its distance \( s \) from a fixed point of the line at time \( t \) is given by the equation \( s^2 = a^2 + V^2 t^2 \), where \( a \) and \( V \) are constants. Find in terms of \( s \) its velocity and acceleration at time \( t \).

Group IV (Paper 3) and Subsidiary Subject (15 c).

STATICS AND DYNAMICS.

MONDAY, JULY 15TH, 1929. 3 HOURS.

[Not more than nine questions are to be attempted by any Candidate. In numerical calculations take \( g = 32 \) foot-second units.]

1. Prove the construction which determines the magnitude and position of the resultant of two unlike parallel forces. 

\( ABCD \) is a parallelogram. Prove that forces completely represented by the lines \( AB, CD, AC, DB \) have a resultant \( 2AB \) in magnitude and position.

2. Define the moment of a force about a point, and prove that the algebraical sum of the moments of two parallel forces about any point in their plane is equal to the moment of their resultant about the same point.

A light triangular frame \( ABC \) has weights attached to the corners, so that when the frame is suspended from \( B \) the side \( AC \) is horizontal and when suspended from \( C \) the side \( AB \) is horizontal. Prove that when it is suspended from \( A \) the side \( BC \) is horizontal.

3. State the laws of limiting friction.

A uniform rectangular block of height \( h \) whose base is a square of side \( a \) rests on a rough horizontal plane. The plane is gradually tilted about a line parallel to two edges of the base. Show that the block will slide or topple according as \( a \) is greater or less than \( \mu h \), where \( \mu \) is the coefficient of friction.

4. A chessboard \( ABCD \) 8 in. square is divided into sixty-four squares of side 1 in. The eleven squares whose centres are nearest to the corner \( C \) are cut away. Show that the distance from \( A \) of the centre of gravity of the remainder is 4.949 in. \((\sqrt{2} = 1.414)\)

5. The figure represents a smoothly jointed framework in which the lines are horizontal, vertical, or inclined at 45\(^\circ\) to the vertical. The framework carries a load of 3 tons at \( C \), and is supported by a pivot at \( A \) and a tie at \( B \). Draw a force diagram, and find the stress in each bar, stating whether they are tensions or thrusts. Also find the forces exerted by the pivot and the tie.

6. A line of men are following one another along a road 20 yds. apart at a uniform speed of 8 miles per hour. Another line of men 35 yds. apart are bicycling in the same direction at 12 miles per hour. At what speed must a man walk in the same direction so that whenever he is passed by a runner he is also passed by a cyclist?

7. What is meant by conservation of momentum?

A gun of mass \( M \) lb. discharges a bullet of mass \( m \) lb. horizontally. The recoil of the gun is opposed by a constant force \( F \) lb. weight which brings it to rest in \( t \) secs. Find the velocity of the bullet, and show that the energy of the explosion is \( gt^2 (M + m)/(2Mm) \) foot-pounds.
8. Prove that when a particle describes a circle of radius $r$ with uniform speed $v$ it has an acceleration $v^2/r$ towards the centre of the circle.

A bead of mass $m$ is threaded on to a spoke of a wheel which turns in a horizontal plane with uniform angular velocity $\omega$. The coefficient of friction between the bead and the spoke is $\mu$ and the bead is at a distance $r$ from the centre of the wheel. What is the greatest value of $\omega$ for which the bead will remain at rest on the spoke?

9. The kinetic energy of a moving body may be defined as the work that it is capable of doing in overcoming resistance before it is reduced to rest. Show that in the case of a body of mass $m$ moving with velocity $u$ without rotation this definition leads to $\frac{1}{2}mu^2$ as the expression for its kinetic energy.

An engine forces a stream of water through a nozzle of 2 sq. in. section at a speed of 50 ft. per sec. At what horse-power is the engine working?

If the momentum of the stream is destroyed by impact on a plane at right angles to itself, what force does the stream exert on the plane?

[Take 1 c. ft. of water = 62-5 lb.]

10. A particle is projected with more than sufficient velocity to enable it to reach a height $h$. Show that the product of the two times that elapse between projection and passing through points at height $h$ is $2h/g$.

Show further that, if $\alpha$ is the inclination to the horizontal of the direction of projection and $v$ is the horizontal component of the velocity, the inclinations to the horizontal of the direction of motion when at height $h$ are the values of $\psi$ given by $\tan^2 \psi = \tan^2 \alpha - 2gh/v^2$.

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**Subsidiary Subject 15 (c).**

**ARITHMETIC, ALGEBRA, AND GEOMETRY.**

**TUESDAY, JULY 16TH, 1929. 2\(\frac{1}{2}\) Hours.**

1. Solve the equations:

   (i) $\frac{x-a}{x-b} = \frac{a^2+b^2}{ab}$;

   (ii) $2x^2 + 3y = 6, x^2 + 3xy + 2y^2 - x = 8$.

2. Draw in the same diagram the graphs of $x^2 - x - 3$ and $\frac{1}{x-1}$ for values of $x$ from $-2$ to $3$, taking 1 inch (or 2 cm.) as the unit of both $x$ and $y$.

   Hence find to the nearest tenth the solutions of the equation $x^3 - 2x^2 - 2x + 2 = 0$.

3. If $a, \beta$ are the roots of the equation $ax^2 + bx + c = 0$,

   express $a + \beta$ and $a\beta$ in terms of $a, b, c$.

   If $\alpha, \beta$ are the roots of the equation $x^2 - 4x + 1 = 0$,

   find the equation whose roots are $\alpha^2 + \frac{1}{\alpha}, \beta^2 + \frac{1}{\beta}$.

4. Find the sum of $n$ terms of an arithmetic progression whose first term is $a$ and common difference $d$.

   $A$ and $B$ begin work together. $A$'s initial salary is £200 a year and he has an annual increment of £20. $B$ is paid at first at the rate of £80 a year and has an increase of £8 every half-year. At the end of how many years will $B$ have received more money than $A$?

5. Write down the expansion of $(1 + x)^{-\frac{1}{2}}$ by the binomial theorem evaluating the coefficients of the first five terms.

   The time of swing of a pendulum is given by the formula $T = \pi\sqrt{\frac{l}{g}}$ seconds, where $l$ is the length. Show that if the pendulum of a clock which normally beats seconds is increased in length by one-tenth per cent., the clock will lose approximately $43\frac{1}{2}$ seconds per day.
6. Prove that, if there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

ABCD is a parallelogram. Prove that the sum of the distances of A and C from any straight line which lies outside the parallelogram is equal to the sum of the distances of B and D from the same straight line. Subject to what convention would this latter theorem also hold good for the distances from straight lines which intersect sides of the parallelogram?

7. Prove that in any triangle the square on the side opposite to an acute angle is less than the sum of the squares on the other two sides by twice the rectangle contained by one of these sides and the projection of the other upon it.

ABC is a triangle having \( AB = AC \). From \( P \), the middle point of \( AB \), a line \( PQ \) is drawn at right angles to \( AB \) meeting \( BC \) in \( Q \). Prove that \( AB^2 = BC \cdot PQ \).

8. Prove that, if a straight line touch a circle and from the point of contact a chord be drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments.

Two circles intersect in \( A \) and \( B \) and a common tangent touches the circles at \( C \) and \( D \). Prove that, if \( CADE \) is a parallelogram, then the points \( B, C, D, E \) lie on a circle.

9. Prove that, if two triangles have an angle of the one equal to an angle of the other and the sides about the equal angles proportional, then the triangles are similar.

ABC is a triangle having \( AB = AC \). \( P \) is any point on \( OB \) produced, and \( Q \) is taken on \( BC \) produced so that \( CQ \) is a third proportional to \( BP \) and \( AB \). Prove that the triangles \( PBA, PAQ \) are similar.

Group III (Paper 1).

ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.

TUESDAY, JULY 15TH, 1930. 2½ Hours.

1. Solve the equations:
   (i) \( \frac{1 - x}{2 + x} + \frac{1 - 3 x}{3 + x} = 4 \);
   (ii) \( 2 \cdot x + 3 \cdot y = 1, \frac{2}{x} + \frac{3}{y} + 2 = 0 \).

2. (i) If the roots of the quadratic \( ax^2 + 2 bx + c = 0 \) are \( \alpha \) and \( \beta \), obtain in terms of \( \alpha \) and \( \beta \) the roots of the equation

   \[ ax^2 + 2bx + c = \frac{b^2 - ac}{\alpha} \]

   (ii) If the expression \( ax^2 + 2 bx + c \) can be written in the form \( A (x - x_1)^2 + B (x - x_2)^2 \), where \( A \) and \( B \) are independent of \( x \), prove that

   \[ ax_1 x_2 + b(x_1 + x_2) + c = 0 \]

3. Indicate the general shape of the graph of

   \[ y = x^2 + ax + b \]

   when the roots of the equation \( x^2 + ax + b = 0 \) are unreal.

   Show graphically or otherwise that the equation

   \[ x^3 - ax^2 + x - 2 = 0 \]

   has one and only one real root.

4. Prove the binomial theorem for a positive integral index.

   Express \( x^n + \frac{1}{x^r} \) as a polynomial in \( y \), where

   \[ y = x + \frac{1}{x} \]

5. Write down the expansion of \( e^x \) in ascending powers of \( x \) giving the general term, and state whether it is valid or not for all values of \( x \).
Show that numbers $a$, $b$, $c$ can be found such that the difference between 
\[ e^x (1 - ax + bx^2 - cx^3) \] and 
\[ 1 + ax + bx^2 + cx^3 \]
contains no power of $x$ below the seventh, and find these numbers.

6. Prove that 
\[ \cos (A + B) = \cos A \cos B - \sin A \sin B \]
for all values of $A$ and $B$, and express in factors 
\[ \cos A - \cos B \] and \[ \cos A + \cos B. \]

If 
\[ \tan a + \tan \theta = \tan \beta - \tan \theta, \]
\[ \sec a - \sec \theta = \sec \beta - \sec \theta, \]
prove that 
\[ \sin \theta = \sin \frac{1}{2} (a - \beta) \cos \frac{1}{2} (a + \beta). \]

7. Prove that in any plane triangle, whether acute-angled or obtuse-angled,
\[ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \]
\[ \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \]
\[ A \text{ is the greatest angle of the triangle } ABC \text{ and points } \]
\[ D \text{ and } E \text{ are taken in } BC \text{ such that } DA = DB \text{ and } \]
\[ EA = EC. \]
Prove that the distance between $D$ and $E$ is 
\[ \frac{1}{2} a \cos A \sec B \sec C, \]
and is 
\[ -\frac{1}{2} a \cos A \sec B \sec C, \]
if $A$ is acute, and if $A$ is obtuse.

8. Draw the graph of $x^2 + x$ between the values $\pm \frac{1}{2}$ of $x$.
Find all the solutions lying within this range of $x$, of the equation 
\[ x^2 + x = 2 + \frac{1}{2} x. \]
Check your results by tables.

9. $ABCD$ is a plane quadrilateral in which $AB$ is 10 ft., 
\[ AD = 8 \text{ ft}, \]
\[ BC = 7 \text{ ft}, \]
\[ \text{the angle } A = 75^\circ, \]
\[ \text{and the angle } B = 67^\circ. \]
Find the projections of $CD$ on $AB$ and on a straight line perpendicular to $AB$, and find the length of $CD$ and the magnitudes of the angles $C$ and $D$. 

No credit will be given for results obtained by drawing.

10. Prove that in a plane triangle, with the usual notation,
\[ r = 4 R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C, \]
\[ r_1 = 4 R \sin \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C. \]

If the triangle is acute-angled and $P$ is its orthocentre, prove that 
\[ r_1 + PA = r_2 + PB = r_3 + PC = r + 2R. \]
What is the corresponding result if the angle $A$ of the triangle is obtuse?

Group III (Paper 2) and Subsidiary Subject (15 d).

PURE AND ANALYTICAL GEOMETRY.

Wednesday, July 16th, 1930. 2½ Hours.

1. Find the locus of points in a plane whose distances from two fixed points are in a constant ratio (i) when the plane passes through the two fixed points, (ii) when it does not.

$ABCD$ is a plane non-rectangular quadrilateral, the lengths of whose sides $AB$, $BC$, $CD$, $DA$ are respectively $a$, $b$, $c$, $d$. A triangle $BCE$ is described exterior to the quadrilateral and in its plane, such that the angle $BCE$ is equal to the angle $DCA$ and the angle $CBE$ is equal to the angle $ADC$. Express the length of $BE$ and the ratio of $AC$ to $CE$ in terms of $a$, $b$, $c$, $d$.

Deduce a construction for a plane quadrilateral having given the lengths, in order, of its sides and the sum of either pair of opposite angles.

2. Define the ‘radical axis’ of two non-intersecting circles, and prove that it is a straight line.

Show how to construct the circle, which is coaxial with two given non-intersecting circles and passes through a given point.

Two circles, of radii $a$ and $b$, have their centres at the points $A$ and $B$ respectively. Prove that the locus of the centre of a circle whose common chords with the circles are diameters of those circles is the radical axis of two circles with their centres at $A$ and $B$, whose radii are $b$ and $a$ respectively.
3. Show that a single straight line can be drawn to intersect at right angles each of two lines which are not in the same plane, and that the shortest distance between the two straight lines lies along this common perpendicular.

If the shortest distance between the edges $DA$ and $BC$ of a tetrahedron $ABCD$ is the straight line joining the middle points of those edges, prove that $DB = CA$ and $DC = AB$.

4. Prove that the ratios of the lengths of parallel segments of lines in one plane are unaltered by orthogonal projection.

Prove that the rectangles under the segments of chords of an ellipse drawn through a fixed point are proportional to the squares of the parallel diameters of the ellipse.

5. The vertices $A, B, C$ of a plane triangle are respectively the origin and the points $(x_1, y_1)$, $(x_2, y_2)$. Show that, in whatever quadrants $B$ and $C$ lie, the area of the triangle is $\pm \frac{1}{2}(x_2 y_1 - x_1 y_2)$, the upper or lower sign being taken according as the sense of description, $ABC$, of the triangle is counter-clockwise or clockwise.

Show also that the tangent of the angle $B$ of the triangle is

$$\frac{\pm (x_1 y_2 - x_2 y_1)}{x_1 (x_2 - x_2) + y_1 (y_2 - y_2)},$$

with the same interpretation of the ambiguity of sign.

6. Prove that the point $(a + r \cos \theta, r \sin \theta)$ lies on the circle $(x - a)^2 + y^2 = r^2$, and find the equation of the tangent at this point.

Prove that the points of contact of the external common tangents of the two circles $(x - a)^2 + y^2 = r_1^2$, $(x + a)^2 + y^2 = r_2^2$, lie on the circle $x^2 + y^2 = a^2 + r_1 r_2$.

7. Find in a simple form, symmetrical in $t$, and $t_2$, the equation of the straight line joining the points $(at_1^2, 2at_1)(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$.

Prove that every chord of this parabola, which subtends a right angle at the point $(at^2, 2at)$ on it, passes through the point whose coordinates are $\{a(t^2 + 4), -2at\}$.

8. Prove that the equation of the straight line joining the points on the ellipse, $b^2 x^2 + a^2 y^2 = a^2 b^2$, whose eccentric angles are $\alpha \pm \beta$

$$b \alpha \cos \beta + a \alpha \sin \beta = ab \cos \beta.$$

If $p$ is the length of the perpendicular drawn to any focal chord of this ellipse from the point of intersection of the tangents at its extremities and $q$ is the length of the perpendicular drawn to this chord from the centre of the ellipse, prove that $pq = b^2$.

9. Prove that the straight line, $x + t^2 y - 2ct = 0$, is a tangent to the rectangular hyperbola, $xy = c^2$, and find the coordinates of its point of contact.

If $PP'$ is a diameter of this hyperbola and the tangent at $P$ meets the asymptotes at $Q$ and $R$, prove that $P'Q$ and $PR$ are tangents to the rectangular hyperbola $3xy + c^2 = 0$.

Group III (Paper 3).

DIFFERENTIAL AND INTEGRAL CALCULUS.

THURSDAY, JULY 24TH, 1930. 2 1/2 HOURS.

[Not more than eight questions are to be attempted by any candidate.]

1. Differentiate $\frac{1 + \sin x}{1 - \sin x}$ and $\sqrt{\frac{(x-1)(x-2)}{x-3}}$.

Prove that, if $x = \cos \theta$ and $y = \sin \theta$, then

$$\frac{d^3y}{dx^3} + \frac{105}{4} \sin 4 \theta = 0.$$
Show that the maximum value of the kinetic energy of the particle is \( \frac{1}{2} m A^2 p^2 \), and that the maximum rate of working of the force is \( \frac{1}{2} m A^2 p^3 \), and find the corresponding values of \( x \).

3. Prove that, if \( y \sqrt{1-x^2} = \sin^{-1} x \), then
\[
(1-x^2) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} - y = 0.
\]

Deduce that, if \( y_n \) denotes the value of \( d^n y / dx^n \) when \( x = 0 \), then \( y_{n+2} = (n+1)^2 y_n \), and hence show that
\[
y = x + \frac{2}{3} x^3 + \frac{2 \cdot 4}{3 \cdot 5} x^5 + \ldots.
\]

4. Show how to find the stationary values of a function \( f(x) \), and how to discriminate between the values of \( x \) which make the function a maximum or a minimum.

Show that the function \( (x-1) (x-2)^2 (x-3)^3 \) has one maximum and two minimum values and no more, finding the corresponding values of \( x \).

Make a rough sketch of the curve represented by the equation
\[
y = (x-1) (x-2)^2 (x-3)^3.
\]

5. Find the equation of the tangent and normal at a point on the curve \( x = a \cos^2 t, y = a \sin^3 t \), where \( t \) is a variable parameter.

Also find the values of \( t \) for a point on the curve such that the tangent at that point is also the normal at some other point on the curve.

6. Explain the method of integration by parts and employ it to integrate
\[
\int x^2 \cos ax \, dx, \quad \int x^3 \sin bx \, dx, \quad \int \sec^3 x \, dx.
\]

7. Integrate
\[
\int \frac{(x^2 + 1) \, dx}{x^2 - 4x + 3}, \quad \int \frac{x \, dx}{\sqrt{x^2 + 2x + 2}}.
\]

Show that
\[
\int_0^{\pi} \frac{dx}{a + b \sin x} = \frac{1}{\sqrt{(a^2 - b^2) \cos^{-1} b}}.
\]

8. Make a rough drawing of the curve \( xy^2 = a^2 (a-x) \), and show that the area between the curve and the axis of \( y \) is \( \frac{1}{6} \pi a^2 \).

Show also that the distance from the origin of the centre of gravity of this area is \( \frac{1}{6} a \).

9. Evaluate \( \int_0^{\pi/2} \cos^n \theta \, d\theta \) when \( n = 1, 2, 3 \), and \( 4 \).

The area bounded by the axis of \( x \), the line \( x = a \), and the curve
\[
x = a \sin \theta, \quad y = a (1 - \cos \theta)
\]
from \( \theta = 0 \) to \( \theta = \frac{\pi}{2} \), revolves round the axis of \( x \). Prove that the volume generated is \( \frac{1}{6} \pi a^3 (10 - 3\pi) \).

**Group III (Paper 4).**

**Statics and Dynamics.**

_Monday, July 14th, 1930. 3 Hours._

[Not more than nine questions are to be attempted by any candidate.

In numerical calculations take \( g = 32 \) foot-second units.]

1. Show that a system of coplanar forces acting on a rigid body is in general equivalent to a single force acting at an arbitrarily chosen point together with a couple. What are the exceptions?

\( ABCD \) is a square whose diagonals intersect at \( O \). Forces 1, 2, 3, 4, 5 \( \sqrt{2} \) act along \( AB, BC, CD, DA, AC \). Find the magnitude and direction of the single force at \( O \) and the magnitude of the couple which together are equivalent to the given forces. Also find at what distance from \( O \) the resultant of the five forces cuts the line \( OB \).

2. State the laws of limiting friction.

A ladder which stands on horizontal ground, leaning against a vertical wall, is so loaded that its centre of gravity is at distances \( a \) and \( b \) from its lower and upper
ends respectively. Show that, if the ladder is in limiting equilibrium, its inclination $\theta$ to the horizontal is given by

$$\tan \theta = \frac{(a - b \mu')}{(a + b) \mu}.$$ 

where $\mu$, $\mu'$ are the coefficients of friction between the ladder and the ground and between the ladder and the wall.

3. Given the weight and the centre of gravity of a body and also the weight and the centre of gravity of any part of the body, show how to find the centre of gravity of the remainder.

$C$ is the centre of a sphere of uniform material and of radius 10 in. which contains two spherical cavities, whose centres are $A$, $B$ and whose radii are 1 in. and 2 in., so situated that $CA = CB = 7.5$ in. and $AB = 9$ in. Show that, when the sphere rests on a horizontal plane, the line $AB$ makes an angle $\tan^{-1} \frac{12}{7}$ with the vertical.

4. $AB$, $BC$, $CD$ are three light strings. The ends $A$, $D$ are fastened to fixed points in the same horizontal line. Weights of 2 lb, and $P$ lb. are attached at $B$ and $C$.

Determine graphically, or otherwise, the weight $P$ and the tensions in the strings $AB$, $BC$, $CD$ when their inclinations to the horizontal are, as in the figure, $60^\circ$, $5^\circ$, and $45^\circ$.

5. Define Work and explain how it is measured.

A rectangular block of weight $W$ lb. whose edges are of lengths $a$, $b$, and $c$ ft. ($a < b < c$) rests on a rough inelastic horizontal plane with a face of area $bc$ uppermost. Find the least amount of work that will have to be expended in the course of turning the block upside down.

6. What is a 'velocity-time curve'? Explain its use.

A train runs from rest at one station to rest at another 5 miles distant in 10 minutes. If it gets up full speed uniformly in the first $\frac{1}{4}$ mile and slows down uniformly to rest in the last $\frac{1}{2}$ mile, what is the maximum speed in miles per hour?

7. Find an expression for the range of a projectile on a horizontal plane through the point of projection in terms of the initial velocity and angle of projection.

A shot is fired with velocity $v$ f.s. from the top of a cliff $h$ ft. high and strikes the sea at a distance $d$ ft. from the foot of the cliff. Show that the possible times of flight are roots of the equation

$$\frac{1}{2}g^2t^4 - (gh + v^2) t^2 + d^2 + h^2 = 0.$$

By considering the roots of this equation, or otherwise, show that the greatest value of $d$ for given values of $v$ and $h$ is $v (v^2 + 2gh)^{\frac{3}{2}} / g$.

8. State the laws which determine the motion of smooth spheres after impact.

An elastic sphere impinges on an equal sphere at rest. Show that, whatever the coefficient of elasticity, the deviation in the direction of motion of the first sphere cannot exceed a right angle; and that, if the coefficient of elasticity is less than unity by a small quantity $\epsilon$, the maximum deviation is less than a right angle by $\sqrt{(2 \epsilon)}$ radians.

9. Prove that a particle which describes a circle of radius $r$ with uniform speed $v$ has an acceleration $v^2 / r$ towards the centre of the circle and no other acceleration.

A bead can move on a smooth circular wire of radius $a$ in a vertical plane. It starts from the highest point with velocity $\sqrt{\frac{1}{2} gc}$. Investigate the changes in the magnitude and direction of the pressure of the wire on the bead as it travels round the circle.
10. Find the period of the small oscillations of (i) a simple pendulum of length $l$; (ii) a rigid body free to turn about a fixed horizontal axis.

A rigid square framework is formed of four equal uniform rods each of mass $m$ and length 2a. Find the length of the equivalent simple pendulum when the frame swings in its own plane about one corner.

**Group III (Paper 5).**

**MATHEMATICAL DISTINCTION PAPER.**

**Monday, July 21st, 1930. 3 Hours.**

[Not more than eight questions are to be attempted by any candidate.]

1. Show that the expression

$$(ab - cd)^2 - (a + b - c - d)(abc + abd - acd - bcd)$$

is the product of four linear factors.

Show that the condition that the function

$$(x-a)(x-b)(x-c)(x-d)$$

in which $a, b, c, d$ are real and $a < b, c < d$, shall have no turning values is that the roots $a, b, c, d$ of the numerator and denominator are interlaced; i.e. that there is an order of magnitude $a < c < b < d$ or $c < a < d < b$.

Illustrate this fact by a rough drawing of the curve

$$y = \frac{(x-1)(x-3)}{(x-2)(x-4)}$$

2. Show that, if $\phi(n)$ is a rational integral function of $n$ of degree $r$, it can be expressed in the form

$$\phi(n) = a_0 + a_1 n + a_2 n(n-1) + \ldots + a_r n(n-1) \ldots (n-r+1),$$

where $a_0, a_1, \ldots, a_r$ are independent of $n$.

Hence show how to sum to infinity a series whose $nth$ term is $\phi(n)x^n/n!$.

Prove that

$$\sum_{n=2}^{\infty} \frac{n^3 - 1}{n!} = 1 + 4e.$$

3. Show that, if $\alpha, \beta, \gamma$ are the roots of the equation

$$x^3 - px^2 + qx - r = 0,$$

then

$$(a^2 - \beta \gamma)(\beta^2 - \gamma \alpha)(\gamma^2 - \alpha \beta) = r^3 - q^3,$$

and that, if two of the roots are unreal, the sign of this expression determines whether the square of the real root is greater or less than the square of the modulus of either unreal root.

4. Prove that, between any two consecutive real roots of $f'(x) = 0$, there is either one or no real root of $f(x) = 0$.

Show that the equation $(x-a)^3 (x-b)^3 + \lambda = 0$ in which $a, b, \lambda$ are real has no real root or two real roots according as 64 $\lambda$ is greater or less than $(a-b)^6$, the case of $\lambda = 0$ being excepted.

5. Show that, if $a, b, c, d$ are the lengths of the sides of a convex quadrilateral $ABCD$ which can be inscribed in a circle, the area of the quadrilateral is

$$\sqrt{\{(s-a)(s-b)(s-c)(s-d)\}},$$

where $2s = a + b + c + d$.

If the quadrilateral be such that a circle can also be inscribed in it, show that

(i) the area is also $(ab + 2cd)$;

(ii) $\frac{1}{IA^2} + \frac{1}{IB^2} = \frac{1}{IC^2} + \frac{1}{ID^2}$,

where $I$ is the centre of the inscribed circle.

6. Resolve $x^n - 2x^n \cos n\theta + 1$ into real quadratic factors.

Prove that

$$\cos n\theta + \sin n\theta = 2^{-\frac{1}{2}} \prod_{r=0}^{n-1} \sin \left(\theta + \frac{4r + 1}{4n} \pi\right).$$

7. Explain what is meant by the principal value of the logarithm of a complex number $x + iy$ as distinct from the general value.
Show that, considering only principal values, the real part of

\[(1 + i)^{\log_2(1 + i)} = 2^{\frac{1}{4}} \log_2 \cos \left(\frac{\pi}{4} \log_2 2\right).\]

8. Resolve into factors the determinant

\[
\begin{vmatrix}
  a & b & c & d \\
  d & a & b & c \\
  c & d & a & b \\
  b & c & d & a
\end{vmatrix}
\]

Solve the equation

\[
\begin{vmatrix}
  a_1 & p_1 & p_2 & p_3 & 1 \\
  a & q_1 & q_2 & 1 \\
  a & \beta & a & r_1 & 1 \\
  a & \beta & \gamma & \delta & 1
\end{vmatrix} = 0.
\]

9. Show that, if \( u_n = \int_0^\pi \frac{\cos n \omega d\omega}{5 - 4 \cos \omega} \),

then, provided that \( n \neq 1 \),

\[2u_n - 5u_{n-1} + 2u_{n-2} = 0.
\]

Hence, or otherwise, show that, for positive integral values of \( n \), \( u_n = \pi / (3\cdot 2^n) \).

Group III (Paper 6).

MATHEMATICAL DISTINCTION PAPER.

TUESDAY, JULY 22ND, 1930. 3 HOURS.

[Not more than eight questions to be attempted.]

1. If the edge \( DA \) of a tetrahedron \( DABC \) is perpendicular to the edge \( BC \), prove that \( DB^2 + CA^2 = DC^2 + AB^2 \), and conversely.

If, along each of three straight lines, which meet at a point \( D \) but are not in one plane, lengths \( DA, DB, DC \) are measured proportional to the cosines of the angles between the other two straight lines, prove that each edge of the tetrahedron \( A BCD \) is perpendicular to the opposite edge.

Deduce, or prove in any other manner, that the three planes, each of which passes through one of the lines and is perpendicular to the plane containing the other two, have a common line of intersection.

2. Define pairs of harmonic conjugates on a straight line, and give, with justification, a construction, involving the use of an ungraduated ruler only, for finding the harmonic conjugate of a point \( C \) with respect to two points \( A \) and \( B \), collinear with \( C \).

(If you use the construction as the definition, prove that it leads to a unique position for the harmonic conjugate of a point with respect to two others.)

\( C \) is the centre of a fixed conic, \( A \) and \( B \) are two fixed points collinear with \( C \). \( PCP' \) is a variable diameter of the conic, \( AP \) and \( BP' \) intersect at \( L \), \( AP' \) and \( BP \) intersect at \( M \). Prove that the locus of \( L \) and \( M \) is a conic, similar and similarly situated to the fixed conic, with its centre at the harmonic conjugate of \( C \) with respect to \( A \) and \( B \).

3. Prove that a range of points on a line is projective with the pencil formed by the polars of the points with respect to a fixed conic.

Prove that the envelope of the perpendicular drawn from a point on a fixed straight line to the polar of the point with respect to a fixed conic is a parabola, which touches the fixed straight line and also touches the normals to the conic at its points of intersection with the fixed straight line.

4. Prove that the reciprocal of a circle with respect to another circle is a conic with a focus at the centre of the second circle, and determine the points which are reciprocated into the axes of the conic.

A variable chord \( PQ \) of a fixed conic subtends a right angle at a fixed point \( O \), not on the conic. Prove that the envelope of \( PQ \) is a conic which has a focus at \( O \) and has its axes parallel to and perpendicular to the polar of \( O \) with respect to the fixed conic.
5. Circles are described on parallel chords of a parabola as diameters. Prove that their envelope is a parabola, whose axis is the diameter of the first parabola which bisects these chords, whose focus is the extremity of this diameter, and whose vertex lies on the directrix of the first parabola.

6. The polar of a point $P$ with respect to the conic $ax^2 + by^2 = 1$ meets any circle through $P$ and the centre of the conic in pairs of points $X$ and $Y$, such that $PX$ and $PY$ are parallel to a pair of conjugate diameters of the conic. Prove that the locus of $P$ is the conic

$$a^2x^2 + b^2y^2 = a^2 + b^2.$$

If, however, the straight lines joining the centre of the conic to $X$ and $Y$ are conjugate diameters of the conic, prove that the locus of $P$ is the director-circle of the conic.

7. Find an equation giving the values of $\lambda$, other than zero, for which the equation

$$\lambda \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) + (x - \xi)^2 + (y - \eta)^2 = 0$$

is that of a pair of straight lines.

Show that, if $\theta$ is a variable parameter and

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is the equation of a conic in rectangular Cartesian coordinates, the equation

$$\begin{vmatrix}
\frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} \\
\frac{\partial S}{\partial y} & \frac{\partial S}{\partial x}
\end{vmatrix} = 0$$

is that of a conic confocal with $S = 0$.

8. Prove that, if the axes of coordinates are rectangular, the envelope of the straight line $lx + my + n = 0$, where

$$al^2 + bm^2 + 2fmn + 2glm + 2hlm = 0,$$

is a parabola, whose directrix is the straight line

$$2gx + 2fy = a + b,$$

and whose focus $(x, y)$ is given by the equation

$$2(x + iy)(y + iy) = a - b + 2hi.$$

A parabola has double contact with the rectangular hyperbola $x^2 - y^2 + a^2 = 0$, the tangents at the points of contact meeting at a point $P$ on the hyperbola $x^2 - y^2 = a^2$. Prove that it passes through the common centre of the hyperbolas, that its directrix bisects $CP$ at right angles, that its focus $S$ lies on the normal to the hyperbola at $(x, y, z) = a^2$ at the middle point of $CP$, and that $CS = \frac{1}{2} CP$.

9. A conic has the triangle of reference as a self-conjugate triangle, and the equation, in areal coordinates, of one of its asymptotes is $lx + my + nz = 0$. Show that the equation of the other asymptote is

$$l(m + n - l)x + m(n + l - m)y + n(l + m - n)z = 0.$$

If one asymptote of such a conic passes through a fixed point, show that the envelope of the other asymptote is a conic inscribed in the triangle formed by the parallels drawn to the sides of the triangle of reference through the opposite vertices.

**Group III (Paper IV).**

**Mathematical Distinction Paper.**

**Wednesday, July 23rd, 1930. 3 Hours.**

[Not more than eight questions to be attempted.]

1. If $u$ and $v$ are real functions of the real variables $x$ and $y$, given by the equation $u + iv = f(x + iy)$, where $f$ is a real function, prove, by means of an intermediate trans-
formation to new variables \( \xi \equiv x + iy \) and \( \eta \equiv x - iy \) or otherwise, that the equation
\[
\frac{\partial}{\partial x} \left( K \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left( K \frac{\partial V}{\partial y} \right) = 0,
\]
where \( K \) is any function of \( x \) and \( y \), is transformed into the equation
\[
\frac{\partial}{\partial u} \left( K \frac{\partial V}{\partial u} \right) + \frac{\partial}{\partial v} \left( K \frac{\partial V}{\partial v} \right) = 0,
\]
where \( K \) is now expressed in terms of \( u \) and \( v \).

If \( x + iy = \cosh(u + iv) \) and \( K = x^2 \), find values of \( V \) which satisfy the equation (i) when \( V \) is a function of \( u \) only, (ii) when \( V \) is a function of \( v \) only.

2. Prove that the locus of the intersection of two perpendicular tangents to the same arch of a cycloid, the radius of whose generating circle is \( a \), is the curve traced out by a point rigidly attached to a circle of radius \( a \) at a distance \( \frac{1}{2} \pi a \) from its centre, while the circle rolls on a straight line.

3. If the equation of the tangent to a plane curve in rectangular cartesian coordinates is \( x \cos \omega + y \sin \omega = p \), prove that the equation of the corresponding normal is
\[
y \cos \omega - x \sin \omega = \frac{dp}{d\omega}.
\]

From a point \( P \) on a plane curve perpendiculars are drawn to the axes of coordinates, the feet of these perpendiculars being the points \( X \) and \( Y \). If \( Q \) is the point of contact of \( XY \) with its envelope, prove that \( PQ \) and the tangent at \( P \) to the locus of \( P \) are equally inclined to the axes.

Prove also that the normal to the envelope of \( PQ \) at the point where \( PQ \) touches its envelope passes through the point which is the reflection of the centre of curvature corresponding to \( P \) in the tangent at \( P \) to the locus of \( P \).

4. Prove that, if \( n \) is a positive integer, the straight line
\[
x + y + (-1)^{n-1} a = 0
\]
is an asymptote of the curve
\[
x^{2n+1} + y^{2n+1} = (2n+1)^a \cdot x^n y^n.
\]

Show that the curve consists of a loop and two infinite branches extending from the origin to infinity, there being one general shape for \( n \) even and another general shape for \( n \) odd.

Show that, whether \( n \) be odd or even, the area of the loop and the area between the asymptote and the two infinite branches are each equal to \( (n+\frac{1}{2}) a^2 \).

5. Six equal uniform rods \( AB, BC, CD, DE, EF, FA \), each of weight \( W \), are freely hinged together at their extremities to form a hexagon \( ABCDEF \), which is maintained in the shape of a regular hexagon by three stiff wires of negligible weight joining \( A \) to \( C \), \( C \) to \( E \), and \( E \) to \( A \). If the hexagon is suspended from the joint \( A \), determine the stresses in the wires.

6. The cross-section of a perfectly rough uniform cylinder is a portion of a parabola of latus-rectum \( 4a \), cut off by a perpendicular to its axis. It is placed in equilibrium on a fixed rough cylinder whose cross-section is an arc of a parabola, also of latus-rectum \( 4a \), with its axis vertical and vertex upwards, the two cylinders being in contact along the highest generator of the fixed cylinder. Prove that, if the height of the centre of gravity of the first cylinder above the generator of contact is less than \( a \), then, if it is held in any position into which it can be displaced by rolling, the curved surfaces of the two cylinders being in contact, and then released, it will tend to return to the position of equilibrium.

7. Prove the formulae for the common catenary, with the usual notation,
\[
s = c \tan \psi, \quad y^2 = s^2 + c^2.
\]

One end of a length \( l \) of a uniform flexible chain is attached to one end of a length \( l \) of a second uniform
flexible chain, whose density is twice that of the first, so as to form a perfectly flexible chain of length 2l, which is suspended by attaching its two free ends to two points at the same level. If the heavier portion hangs in a catenary of parameter c, and s is the distance along the chain from the joint to the lowest point of the chain, prove that
\( \sqrt{(l+2s)^2 + 4c^2} = \sqrt{(l-s)^2 + c^2} + \sqrt{(l+c)^2 + c^2} \).

From this equation obtain \( l \) in terms of \( s \) and \( c \), and show that the inclination of the chain to the horizontal at the joint must be less than \( \sin^{-1}(1/3) \).

8. The energy equation of a moving system, depending on a single coordinate \( \theta \), is
\[ \frac{1}{2} \dot{\theta}^2 \phi(\theta) + \psi'(\theta) = \text{constant}. \]
Prove that, if \( \psi'(\alpha) = 0 \), \( \theta = \alpha \) gives a position of equilibrium of the system and that, if \( \psi''(\alpha) \) is positive, the position of equilibrium is stable, the periods of small oscillation about it being
\[ 2\pi \sqrt{\frac{\psi''(\alpha)}{\phi(\alpha)}} \frac{1}{2}. \]

A particle of mass \( M \) is attached to two equal particles, each of mass \( m \), by means of two light inextensible strings, each of which passes over one of two small pulleys, fixed at the same level at a distance 2c apart. It is given that the system is in equilibrium with each of the particles of mass \( m \) hanging freely when the portions of each string between the pegs are inclined at an angle \( a \) to the vertical. Prove that, if the mass \( M \) receives a small vertical displacement, it will perform small oscillations, whose period is that of a simple pendulum of length
\[ c/(\tan a - \sin a). \]

9. A solid hemisphere, of mass \( M \), is placed with its plane face on a rough horizontal plane. A particle, of mass \( m \), is placed on the hemisphere at its highest point. If the particle is slightly displaced and there is no friction between the hemisphere and the particle, show that, if the hemisphere remains at rest on the plane during the subsequent motion, the friction, \( F \), between the hemisphere and the plane and the vertical reaction, \( R \), between the hemisphere and the plane, when the particle has described an arcual distance \( \theta \) on the hemisphere, are connected by the relation
\[ F = m \sin \theta (3 \cos \theta - 2), \]
\[ R = M + m \cos \theta (3 \cos \theta - 2). \]

10. A circular disk, whose centre of gravity is at a distance \( c \) from its geometrical centre, is placed on an inclined plane, sufficiently rough to prevent any slipping, with the plane of the disk vertical and containing a line of greatest slope of the plane and its centre of gravity at its least distance from the plane. Employ the principle of the conservation of energy to prove that when the disk has turned through an angle \( \theta \), its angular velocity is given by the equation
\[ \frac{a^2 + c^2 + k^2 - 2ac \cos \theta}{2g \{ a \theta \sin \theta + c \cos (\theta + a) - c \cos a \}} \]
where \( a \) is the radius of the disk, \( k \) is its radius of gyration about its centre of gravity, and \( a \) is the inclination of the plane to the horizontal.

Hence show that the disk cannot roll down the plane with uniform angular acceleration unless its centre of gravity coincides with its geometrical centre.

Group IV (Paper 1) and Subsidiary Subject (15 a).

ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.

TUESDAY, JULY 15th, 1930. 2 1/2 Hours.

1. Find the solutions of the equation \( a^3 - 3a - 1 = 0 \) by means of the intersections of the graphs of
\[ y = x^2 \] and \[ y = 3 + \frac{1}{x} \]
Take the unit of \( x \) as 1 inch (or 2 cm.) and the unit of \( y \) as 0.2 inch (or 4 mm.).

2. Solve the equations:
(i) \[ \frac{1-x}{2+x} + \frac{1-3x}{3+x} = 4; \]
(ii) \[ 2x + 3y = 1, \quad \frac{2}{x} + \frac{3}{y} + 2 = 0. \]
3. The formulae for $S$, the area of the curved surface, and $V$, the volume of a cap of a sphere of radius $R$, the height of the cap being $h$, are

$$ S = 2\pi Rh, \quad V = \pi h^2 (R - \frac{1}{3}h). $$

Express the ratio of $V^2$ to $S^2$ as a function of $x$, where $x$ is the ratio of $h$ to $R$.

If this function is denoted by $f(x)$, resolve $f(1) - f(x)$ into factors; and hence or otherwise show that a hollow vessel in the form of a spherical cap, made from a given amount of material, will contain the greatest possible volume when made in the form of a hemisphere.

4. Prove the binomial theorem for a positive integral index.

If $y$ denotes $x + \frac{1}{x}$, prove that

$$ \left( x^y - \frac{1}{x^y} \right) = \left( x - \frac{1}{x} \right) $$

can be expressed in the form $y^a + Ay^b + By^c + C$, and find the values of $A$, $B$, $C$.

5. Write down the expansion of $e^x$ as an infinite series in ascending powers of $x$, giving the general term, and state whether the expansion is valid for all values of $x$.

Show, by finding their values, that numbers $a$, $b$, $c$ can be found such that the difference between the expansion of $e^x (1 - ax + bx^2 - cx^3)$ and $1 + ax + bx^2 + cx^3$ contains no power of $x$ lower than the seventh.

6. Draw the graphs of $2 \cos 2x$ and $\sec x$ between the values $\pm \pi$ of $x$, taking 2 inches (or 4 cm.) to represent $\pi$ on the axis of $x$ and 1 inch (or 2 cm.) as the unit of $y$.

Hence solve the equation $2 \cos 2x \cos x = 1$ and check your solutions by means of tables.

7. Prove that

$$ \cos (A + B) = \cos A \cos B - \sin A \sin B, $$

and express $\cos A + \cos B$ and $\cos A - \cos B$ in factors.

If $\cos x = k \cos (x + A)$, prove that

$$ \tan (x + \frac{1}{2}A) = \frac{k - 1}{k + 1} \cot \frac{1}{2}A; $$

and solve the equation $\cos x = 3 \cos (x + 20^\circ)$.

8. Prove that in any plane triangle, whether the angle $A$ is acute or obtuse,

$$ a^2 = b^2 + c^2 - 2bc \cos A. $$

On the sides $BC$, $CA$, $AB$ of a triangle $ABC$ points $P$, $Q$, $R$ are taken, such that

$$ \frac{BP}{CQ} = \frac{AR}{QB} = k. $$

Express $QR^2$ in terms of $a^2$, $b^2$, and $c^2$, and show that

$$ QR^2 + RP^2 + PQ^2 = \frac{k^2 - k + 1}{(k+1)^2} (a^2 + b^2 + c^2). $$

9. $ABCD$ is a plane quadrilateral, in which $AB$ is 10 ft., $AD$ is 8 ft., $BC$ is 7 ft., the angle $A$ is 75°, and the angle $B$ is 67°. Find the projections of $CD$ on $AB$ and on a straight line perpendicular to $AB$, and find the length of $CD$ and the magnitudes of the angles $C$ and $D$.

(No credit will be given for results obtained by drawing.)

10. Prove that in a plane triangle, with the usual notation,

$$ r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C, $$

$$ r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C. $$

Prove that

$$ r_1 + r_2 + r_3 - r = 4R, $$

and

$$ r_2 + r_3 - r_1 + r = 4R \cos A. $$
2. Find the condition that the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$ may be parallel.

$ABCD$ is a parallelogram. The equations of $AB$, $BU$ are

$$3x - y - 7 = 0$$
$$x - 3y + 5 = 0.$$

Find the coordinates of the point $D$, given that $AD$ passes through the point $(-8, 3)$ and $CD$ passes through the point $(5, 2)$.

3. Find the angle between the lines

$$y = mx + c, \quad y = m'x + c'.$$

Perpendiculars are drawn from the point $(6, 8)$ to the lines $x - y + 8 = 0$, $x + 2y - 16 = 0$, and $x + 3y - 20 = 0$. Prove that the feet of the perpendiculars are collinear, and find the equation of the line on which they lie.

4. Show that the equation of any circle can be written in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Find the equation of the circle which touches the axis of $x$ at the point $(3, 0)$ and passes through the point $(1, 4)$. Prove that this circle subtends an angle of about $79^\circ 37'$ at the origin.

5. Show that the expression $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ represents the square of the tangent from the point $(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Prove that the squares of the tangents drawn from any point on the circle $x^2 + y^2 + 2x - 16y = 0$ to the two circles

$$x^2 + y^2 - 10y + 2\lambda (x - 3y) = 0,$$
$$x^2 + y^2 - 10y + 2\lambda' (x - 3y) = 0$$

are in the constant ratio $\lambda - 1 : \lambda' - 1$.

6. Differentiate $\sqrt{\frac{1 + x^2}{1 - x^2}}$, $\log \{x + \sqrt{(a^2 + x^2)}\}$.

Find the first and second differential coefficients of $\log \{1 + \sin x\}$. What information do the signs of these differential coefficients give about the graph of the equation $y = \log \{1 + \sin x\}$?

Make a rough sketch of the curve for values of $x$ between $0$ and $\frac{\pi}{2}$.

7. Integrate

$$\frac{xdx}{(x-1)(x-2)}, \quad \int \sqrt{(x-1)(x-2)} dx, \quad \int \sin x \cos^2 x dx.$$

Make a rough drawing of the curve $y^2 = x^2 (1 - x)$ and find the area of the loop of the curve.

8. Find the equations of the tangent and normal at any point on the curve $x = a \cos^3 t$, $y = a \sin^3 t$, where $t$ is a variable parameter.

Show that the axes intercept a length $a$ on the tangent and a length $2a \cot 2t$ on the normal.

9. The portion of the curve $x^2 = 4a (a - y)$ from $x = -2a$ to $x = 2a$ revolves round the axis of $x$. Prove that the volume contained by the surface so formed is $\frac{3}{2} \pi a^3$, and find its radius of gyration about the axis of revolution.

10. A particle of mass $m$ moves in a straight line so that at time $t$ its distance $x$ from a fixed origin in the line is $A \sin (pt - \alpha)$, where $A$, $p$, and $\alpha$ are constants. Find the force acting on the particle at a time $t$.

Show that the maximum value of the kinetic energy of the particle is $\frac{1}{2} m A^2 p^2$, and that the maximum rate of working of the force is $\frac{1}{2} m A^2 p^3$, and find the corresponding values of $x$. 
Group IV (Paper 3) and Subsidiary Subject (15 c).

STATICS AND DYNAMICS.

MONDAY, JULY 14TH, 1930. 3 Hours.

[Not more than nine questions are to be attempted by any candidate.]

In numerical calculations take \( g = 32 \text{ foot-second units.} \]

1. Prove that the sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point.

Two horizontal wires inclined to one another at an angle \( 2 \alpha \) are attached to the top of a vertical post movable about its lower end, which is fixed. The post is supported by a stay inclined at an angle \( \beta \) to the vertical and fastened to a point two-thirds of the way up the post. Find the tension in the stay when the tensions in the wires are both \( T \). Find also the horizontal and vertical components of reaction at the lower end of the post.

2. A gate is supported by two hinges in a vertical line at a distance 3 ft. apart. The breadth of the gate is 5 ft. and its weight is 100 lb. The upper hinge exerts a horizontal force only and will yield when this force exceeds 250 lb. If a boy weighing 140 lb. stands on the gate without the hinge yielding, what is the greatest possible distance of his centre of gravity from the line of the hinges?

3. State the laws of limiting friction.

\( ABCD \) is a vertical section of a rectangular block resting on a rough horizontal plane. The end \( AB \) is gently raised by a force applied at \( A \) at right angles to \( AC \). Show that the corner \( C \) will or will not begin to slip according as the coefficient of friction is less or greater than

\[
\sin \alpha \cos \alpha / (1 + \sin^2 \alpha),
\]

where \( \alpha \) is the angle \( ACB \).

4. Given the weight and the centre of gravity of a body and also the weight and centre of gravity of any part of the body, show how to find the centre of gravity of the remainder.

\( C \) is the centre of a sphere of uniform material and of radius 10 in. which contains two spherical cavities, whose centres are \( A, B \) and whose radii are 1 in. and 2 in., so situated that \( CA = CB = 7 \) 5 in. and \( AB = 9 \) in. Show that, when the sphere rests on a horizontal plane, the line \( AB \) makes an angle \( -1 \)\( 12/7 \) with the vertical.

5. The figure represents the section of the framework of a roof supported by vertical forces at \( B \) and \( C \). Five of the members are of equal length. \( BD \) is at right angles to \( AE \), and \( CE \) is at right angles to \( AD \). If the load may be taken to be 2 cwt. at \( A \), find graphically, or otherwise, the tension in \( DE \).

6. What is a 'velocity-time curve'? Explain its use.

A train runs from rest at one station to rest at another 5 miles distant in 10 minutes. If it gets up full speed uniformly in the first \( 1/2 \) mile and slows down uniformly to rest in the last \( 1/2 \) mile, what is the maximum speed in miles per hour?

7. Define Work, Power, and Horse-power.

An engine weighing 50 tons and working at 80 horse-power ascends an incline of \( 1 \) in \( 112 \) at a uniform speed of
20 m.p.h. Find the tractive force exerted and find also the resistance in pounds weight per ton of the engine.

If the engine is coupled to a train of 100 tons, what uniform speed can it maintain on the same incline if it works at the same rate and the resistance to the motion of the train is an additional force of 300 lb. weight?

8. A particle is projected with velocity \( v \) at an angular elevation \( \theta \) above the horizontal. Find expressions for the time of flight and the range on a plane through the point of projection sloping downwards at an angle \( \beta \) to the horizontal.

After what time is the particle moving at right angles to the direction of projection?

9. Explain what is meant by Conservation of momentum.

Bodies of masses \( m \), \( 5m \) are connected by a fine string which passes over a smooth pulley. After the lighter body has ascended through one foot it picks up a rider of mass \( 2m \) which was at rest and after rising another foot it picks up another rider also of mass \( 2m \) and at rest. Find the final velocity of the bodies.

10. Prove that a particle which describes a circle of radius \( r \) with uniform speed \( v \) has an acceleration \( v^2/r \) towards the centre of the circle and no other acceleration.

A bead can move on a smooth circular wire of radius \( a \) in a vertical plane. It starts from the highest point with velocity \( \sqrt{(\frac{1}{2} ga)} \). Investigate the changes in the magnitude and direction of the pressure of the wire on the bead as it travels round the circle.

Subsidiary Subject 15 (e).

ARITHMETIC, ALGEBRA, AND GEOMETRY.

TUESDAY, JULY 15TH, 1930. 2½ Hours.

1. (i) Solve the equations \( x + 3y = 1 \), \( 3x^2 + 2y + 2y^2 = 12 \).
(ii) Find the values of \( k \) for which the quadratic equation \( k(x^2 - 10x - 2) + (2x^2 + 1) = 0 \) has equal roots.

2. Draw the graph of \( x^2 - 2x^2 \) between the values \( -1 \) and \( +2.5 \) of \( x \), taking 1 inch (or 2 cm.) as the unit of \( x \) and 0.2 inch (or 4 mm.) as the unit of \( y \).

Use the graph to solve the equation \( 2x^3 - 4x^2 + x + 1 = 0 \).

3. An open rectangular box, made of thin sheet metal, has for its base a square of side \( a \) inches and its depth is \( 2a \) inches. If \( A \) square inches is the amount of material used in its construction and \( V \) cubic inches is the volume of its cubic contents, express the ratio of \( A^3 \) to \( V^2 \) as a function of \( a \).

If this function is denoted by \( f(a) \), resolve \( f(a) - f(\frac{a}{2}) \) into factors, and hence or otherwise show that to make such a box of given cubic content with the least possible amount of material it must be so constructed that its height is one-half of a side of the base.

4. Prove the formula for the sum to \( n \) terms of a geometrical progression.

A sum of £10 is set aside at the beginning of each year to accumulate at compound interest at 5 per cent. per annum. At the end of how many years will the accumulated sum be as much as £130?

How much would the accumulated sum be in this time if £5 were set aside at the beginning of each half-year and interest was paid half-yearly?

5. If \((2x + 3y)^3\) is expanded by the binomial theorem, find
(i) the term in which the numerical coefficient is greatest;
(ii) the term which is the greatest term in the expansion when \( a = 1 \) and \( b = \frac{1}{a} \).

6. In the triangle \( ABC \) the side \( AB \) is greater than the side \( AC \). Prove that the angle \( ACB \) is greater than the angle \( ABC \).

(If you wish to prove this theorem by means of its converse, first prove the converse.)
If the straight line joining the vertex A of a triangle to the middle point of the side BC is greater than one-half of BC, prove that the angle A is acute.

7. Prove that in any triangle the square on a side opposite to an acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of those sides and the projection on it of the other.

$P, Q, A, B$ are four points such that

$$PA^2 + QB^2 = PB^2 + QA^2;$$

prove that $PQ$ is perpendicular to $AB$.

8. Prove that angles in the same segment of a circle are equal.

$ABCD$ is a quadrilateral inscribed in a circle and its diagonals $AC$ and $BD$ meet at $O$. $P, Q, R, S$ are the feet of the perpendiculars drawn from $O$ to the sides $AB, BC, CD, DA$ respectively. Prove that a circle can be described with $O$ as centre to touch the sides of the quadrilateral $PQRS$.

9. Prove that the rectangle contained by the segments of a chord of a fixed circle drawn through a fixed point inside the circle is the same for all positions of the chord.

Deduce a geometrical construction for dividing a given straight line into two segments such that the rectangle contained by these segments is equal to the square on a given straight line, and determine when the problem is not capable of solution.

10. Prove that, if two triangles are equiangular, the sides about the equal angles are proportional.

A straight line $AB$ is bisected at $O$. An equilateral triangle $ABX$ is drawn on one side of $AB$ and two equilateral triangles $ACY$ and $BCZ$ are drawn on the other side of $AB$. Prove that the straight lines $XY$ and $XZ$ trisect $AB$.

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**Group III (Paper 1).**

**ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.**

**Tuesday, July 14th, 1931. 2½ Hours.**

1. Solve the equations:
   
   (i) $\frac{x}{x - 2} + \frac{2}{x + 2} = 3^\frac{1}{3}$;

   (ii) $3x^2 - 13xy + 9y^2 = 15$, $7x^2 - 5y^2 + 17 = 0$.

2. Sum to $n$ terms the series:
   
   (i) $1 + 2x + 3x^2 + 4x^3 + \ldots$;

   (ii) $\frac{1}{1 - \frac{1}{2}} + \frac{1}{1 - \frac{1}{3}} + \frac{1}{1 - \frac{1}{4}} + \frac{1}{1 - \frac{1}{5}} + \ldots$.

3. (i) If the roots of the quadratic $ax^2 + 2bx + c = 0$ are $\alpha$ and $\beta$, express $\alpha^6 + \beta^6$ in terms of $a, b, c$.

(ii) Show that if $x_1, x_2$ are the roots of

$$ax^2 + bx + c + \lambda (a'x^2 + b'x + c') = 0,$$

then $x_1x_2 (a'b' - a'b) - (x_1 + x_2)(a'a - c'c) + b'c' - b'c = 0$.

4. Show that, if $a > 0$, $ac - b^2 > 0$, then $ax^2 + 2bx + c$ is positive for all real values of $x$.

Prove that, as $x$ varies, the function

$$(x - 5)(x - 1)/(2x - 1)$$

assumes all real values except those lying between $-4$ and $-1$, and illustrate this result by drawing a rough graph of the function.

5. Write down the expansion of $\log_e (1 + x)$ in ascending powers of $x$, and state for what values of $x$ it is valid.

Show that, if $n > 1$,

$$\log_e n = 2\left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \ldots\right),$$

and calculate $\log_e 9$ (as $2 \log_e 3$) correct to four places of decimals.
6. Draw the graphs of cosec $2x$ and $1 + \sin^2 x$ for values of $x$ between $10^\circ$ and $80^\circ$, taking 1 inch (or 2 cm.) to represent $20^\circ$ in $x$ and 2 inches (or 4 cm.) for the unit in $y$.

Hence find the solutions of the equation
\[ \cosec 2x = 1 + \sin^2 x \]
that lie between these values of $x$, and check your results by using the tables.

7. Show that
\[ 2 \cos \frac{\theta}{2} = \pm \sqrt{1 + \sin \theta} \pm \sqrt{1 - \sin \theta}, \]
and determine which signs are to be taken if
\[ \frac{\pi}{2} < \theta < 3\frac{\pi}{2}. \]

If $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi/2$, prove that
\[ bc + ca + ab = 1. \]

8. If $\cot \theta \cot (\theta - \alpha) = k$, prove that
\[ (k + 1) \cos (2\theta - \alpha) = (k - 1) \cos \alpha. \]

A man sees two objects in a straight line due north of him. He walks a distance $a$ due east and then observes that they subtend an angle $\alpha$ at his eye. When he has walked a further distance $b$ due east, the objects again subtend an angle $\alpha$. Show that the distance between the objects is $(2a + b) \tan \alpha$.

9. Prove that, in any triangle $ABC$,
\[ a^2 = b^2 + c^2 - 2bc \cos A. \]

Equilateral triangles are described inwards on the sides of a given triangle $ABC$. Prove that their centres form an equilateral triangle, and deduce, or prove otherwise, that the area of $ABC$ is less than or equal to $(a^2 + b^2 + c^2)/4\sqrt{3}$. When does equality occur?

10. $ABC$ is an acute-angled triangle, and $D, E, F$ are the feet of the perpendiculars from the vertices on the opposite sides. Prove that the sides and angles of the triangle $DEF$ are $a \cos A, b \cos B, c \cos C$ and $\pi - 2A, \pi - 2B, \pi - 2C$, and that the radius of its inscribed circle is
\[ 2R \cos A \cos B \cos C. \]

What are the corresponding results when the angle $A$ is obtuse?

**Group III (Paper 2) and Subsidiary Subject (15 d).**

**PURE AND ANALYTICAL GEOMETRY.**

**Wednesday, July 15th, 1931. 2 1/2 Hours.**

1. If $P$ is a point on the circumcircle of a triangle $ABC$, prove that the feet of the perpendiculars from $P$ on the sides of the triangle lie on a straight line (the pedal line of $P$).

If $PQ$ is a chord of the circumcircle which is parallel to $BC$, show that the pedal line of $P$ is perpendicular to $AQ$, and deduce (or prove in any other way) that the angle between the pedal lines of two points is equal to the angle subtended by the points at the circumference of the circle.

2. Points $D, E, F$ are taken on the sides $BC, CA, AB$ of a triangle. Prove that the circles $AEF, BFD, CDE$ meet at a point $O$ which is the same for all triangles $DEF$ of the same shape.

Inscribe in a given triangle $ABC$ an equilateral triangle $DEF$ with $D, E, F$ lying on the sides of the given triangle as above and one of its sides $EF$ passing through a given point $P$. What is the number of such equilateral triangles?

3. Two chords $PQ, P'Q'$ of a sphere intersect at $A$. Prove that $AP \cdot AQ = AP' \cdot AQ'$.

A system of spheres touch a plane at the same point $O$. Prove that any plane, not passing through $O$, cuts them in a series of coaxal circles.

4. Show that the triangle of greatest area that can be inscribed in a circle is equilateral.

Deduce by orthogonal projection that the greatest
triangle that can be inscribed in an ellipse is such that the
tangent at each vertex is parallel to the opposite side, and
show that its area is $3\sqrt{3}A/4\pi$, where $A$ is the area of the
eclipse.

5. Find the coordinates of the point dividing the line
joining $(x_1, y_1)$ and $(x_2, y_2)$ in a given ratio.

The ends of a rod slide on two fixed perpendicular
lines. Show that any point of the rod describes an ellipse,
and that if the point is a point of trisection of the rod, the
eccentricity of the ellipse is $\sqrt{3}/2$.

6. Show that the equation of a system of coaxal circles
may be written $x^2 + y^2 - 2\lambda x + c = 0$,
where $c$ is a constant and $\lambda$ a parameter.

Two independent systems of coaxal circles are such that
the radical axis of either is the line of centres of the other.
Show that the product of the radii of any two circles, one
of each system, that touch each other, is constant.

7. Find in its simplest form the equation of the line join-
ing the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

The line $lx + my + n = 0$ meets the parabola $y^2 = 4ax$
at $P$ and $Q$, and the lines joining $P, Q$ to $(c, 0)$ meet the
curve again at $R, S$. Show that the equation of $RS$ is

$$n(x - mc) + lb^2 = 0,$$

and prove that, if $PQ$ passes through a fixed point, then
$RS$ also passes through a fixed point.

8. Prove that, if $a^2l^2 + b^2m^2 - n^2 = 0$, the line

$$lx + my + n = 0$$
touches the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point

$$(-a^2/l, -b^2/m).$$

Find the common tangents of the two ellipses

$$x^2/a^2 + y^2/b^2 = 1$$
and $$x^2/a^2 + y^2/c^2 = 1,$$
and show that their points of contact lie on the ellipse

$$(b^2 + d^2)x^2 + (a^2 + e^2)y^2 = a^2d^2 + b^2e^2.$$

9. Find in its simplest form the equation of the normal
at the point $(ct, c/t)$ to the rectangular hyperbola $xy = c^2$.
If the normal at $P$ meets the curve again at $Q$, prove
that $PQ = OP^3/c^2$, $O$ being the centre.

**Group III (Paper 3).**

**DIFFERENTIAL AND INTEGRAL CALCULUS.**

**THURSDAY, JULY 23RD, 1931. 2½ HOURS.**

*Not more than eight questions are to be attempted
by any candidate.*

1. Prove that the differential coefficient of

$$\frac{1}{2}\log \frac{1 - x\sqrt{2} + x^2}{x\sqrt{2} + 1 + x\sqrt{2} + x^2} + \tan^{-1} \frac{x\sqrt{2}}{1 - x^2}$$
is $\frac{2x^2\sqrt{2}}{1 + x^4}$.

If $x = \sqrt{1 - t^2}$, transform the equation

$$x(1 - c^2) \frac{dy}{dx} + (1 - 3c^2) \frac{dy}{dx} - xy = 0$$

into a differential equation for $y$ in terms of $t$.

2. A particle is projected from a point on a straight line
with velocity $u$ and moves in that line in such a way that
when it has traversed a distance $s$ its velocity is $u/(1 + ks)$,
where $k$ is constant. Prove that its retardation varies as the
cube of its velocity, and find the time taken to reduce its
velocity to one-half of its initial value.

3. Define a differential coefficient, and explain the meaning
of its sign.

Draw, as accurately as you can without integration,
the graph of $y$ as a function of $x$ for values of $x$ between
$-1\frac{1}{4}$ and $2\frac{1}{2}$, given that

$$\frac{dy}{dx} = (x + 1) x^2 (x - 2)^3$$

and that $y = 0$ when $x = 0$. 

4. If \( y = \sqrt{1 + x^2} \log(x + \sqrt{1 + x^2}) \), prove that
\[
(1 + x^2) \frac{dy}{dx} = xy + 1 + x^2.
\]
If \( y \) is expanded in the form \( \sum_{n=0}^{\infty} a_n x^n \), prove that

(i) \( a_{2r} = 0 \);
(ii) \( a_1 = 1, \quad a_3 = \frac{1}{3} \);
(iii) \( a_{2r+1} = (-1)^{r-1} \frac{2 \cdot 4 \ldots (2r-2)}{3 \cdot 5 \ldots (2r-1)} \frac{1}{2r+1} \) \( (r > 1) \).

5. Define the radius of curvature of a curve, and show that the radius of curvature of the curve \( x = f(t), \ y = \phi(t) \) is \( \left( f'^3 + \phi'^3 \right)^{\frac{1}{3}} \), where dashes denote differentiations with respect to \( t \).

Prove that the radius of curvature of the envelope of the line \( y + tx = 2at + at^3 \) is \( 6a \left( 1 + t^2 \right)^{\frac{3}{2}} \).

6. Integrate the following with respect to \( x \):

(i) \( \frac{x-1}{x^2 + 1} \),
(ii) \( x^3 e^{-2x} \),
(iii) \( \tan^4 x \),
(iv) \( \sin x \sin 3x \sin 5x \).

7. Evaluate the integral \( \int_a^b \frac{x}{\sqrt{(x-a)(b-x)}} \) by means of the substitution \( x = a \cos^2 \theta + b \sin^2 \theta \).

Prove that the integrals
\[
\int_0^1 x^7 (1-x)^0 \, dx \quad \text{and} \quad \int_0^1 x^8 (1-x)^7 \, dx
\]
are equal, and show that their common value is \( \frac{7! \cdot 8!}{16!} \).

8. Sketch the curve \( r = a (1 + \cos \theta) \), and find the area it encloses and the volume of the surface formed by revolving it about the line \( \theta = 0 \).

9. Find the radii of gyration of

(i) a uniform solid sphere of radius \( a \) about a diameter;
(ii) a uniform solid hemisphere of radius \( a \) about a line through its centre of gravity parallel to its base.

- **Group III (Paper 4).**

**STATICS AND DYNAMICS.**

**MONDAY, JULY 13TH, 1931. 3 Hours.**

[Not more than nine questions are to be attempted by any candidate.]

In numerical calculations take \( g = 32 \) foot-second units.]

1. Prove that a force whose components are \( X, Y \) acting at the point \( (x, y) \) parallel to rectangular axes can be replaced by an equal parallel force acting at the origin together with a couple of moment \( zY - yX \).

\( ABCDEF \) is a regular hexagon and \( O \) is its centre. Forces of magnitudes 1, 2, 3, 4, 5, 6 act in the lines \( AB, CB, CD, DE, EF, AF \) in the senses indicated by the order of the letters. Reduce the system to a force at \( O \) and a couple, and find the point in \( AB \) through which their single resultant force acts.

2. State the laws of limiting friction.

The figure represents the section of a uniform rectangular block resting on an inclined plane. Given that \( AB = BC = CD = DE \),

what is the minimum coefficient of friction that will ensure that, when the plane is gradually tilted, the block will topple over before it slides?
3. Four equal uniform rods, each of weight $W$, are
smoothly hinged at their ends so as to form a rhombus
$ABCD$ which is suspended from the corner $A$ with the
corner $O$ vertically below $A$ resting on a horizontal plane.
Find the pressure on the plane and the supporting force at
$A$, and show that there is a horizontal reaction at each
joint equal to $\frac{1}{3}W\cot\frac{1}{3}ABC$. What are the vertical re-
actions at $B$ and $D$?

4. The framework of light smoothly-jointed rods shown
in the figure carries loads of 1, 2, 3, 4 units at $P$, $Q$, $R$, $S$, is
pivoted at $O$ and supported in a vertical plane by a horizontal
force at $P$. Find, graphically or otherwise, the forces at $O$
and $P$ and the stresses in $QT$, $TU$ and $UQ$ distinguishing ties
from struts.

5. A uniform circular cylinder, of radius $a$ and weight $W$,
rests on two inclined planes which slope in opposite direc-
tions making angles $\alpha$, $\frac{1}{4}\pi - \alpha$ with the horizontal, intersect-
ing in a horizontal line parallel to the axis of the
cylinder. Show that, if $\lambda < \alpha < \frac{1}{4}\pi$ the least couple that
can turn the cylinder about its axis is of moment

$$Wa\sin \lambda (\sin(\alpha - \lambda) + \cos(\alpha - \lambda))$$

where $\lambda$ is the angle of friction.

6. A boat is rowed with uniform speed $v$ feet per second
relative to a straight stream which flows $u$ feet per second.
The breadth of the stream is $b$ feet, and it is required to
cross the stream in such a way as to reach a point on the
opposite bank $a$ feet further downstream than the start-
point. Show that if $v$ is less than $u$ and greater than

$$ub/\sqrt{(a^2 + b^2)}$$

there are two directions in either of which the boat can be

steered and that the numbers of seconds taken in transit in
the two cases are the roots of the equation

$$(u^2 - v^2)t^2 - 2uat + a^2 + b^2 = 0.$$

7. An engine of 300 horse-power moves a total mass of
200 tons up an incline of $1$ in $100$ against a road resistance
of $10.6$ lb. weight per ton. Find the maximum speed
attained, and the time and distance taken in coming to a
stop after shutting off steam, without using brakes.

What would be the acceleration if the engine exerted
the same pull as at the same previous maximum speed on
the same incline and the road resistance were halved?

8. A car travels at $v$ feet per second round a circular
track of radius $r$ feet. Prove that if there is no tendency
to side-slip the track must be banked at an angle $\theta$ to the
horizontal, where $\tan \theta = v^2/(gr)$.

A car of mass 5 cwt. travels at 30 miles per hour
round a circular track of radius 100 yards which is banked
at an inclination of $10^\circ$ to the horizontal. What is the
frictional force exerted across the track, and in what sense
does it act?

Show that 0.025 would be a sufficient coefficient of
friction to prevent side-slip.

9. What is the principle of conservation of linear momen-
tum?

Two particles $A$, $B$ of masses $m$, $m'$ are moving with
velocities $u$, $v$ in directions inclined at an angle $\alpha$. If the
two particles collide and coalesce, find the velocity of the
resultant particle and the angle that its direction makes
with the initial direction of motion of $A$. Also show that
the loss of kinetic energy resulting from the collision is

$$\frac{1}{2}mm'(u^2 + v^2 - 2uv\cos \alpha)/(m + m').$$

10. Prove that the kinetic energy of a body turning
about a fixed axis is $\frac{1}{2}I\omega^2$, where $I$ is its moment of inertia
about the axis and $\omega$ is its angular velocity.

A wheel of mass $m$ and radius $a$ has a radius of gyra-

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tion k about its axis which is fixed horizontally and about which it is free to turn. A particle of mass m is attached to the rim of the wheel. The system is slightly disturbed from its position of unstable equilibrium. Find its angular velocity when it has turned through an angle \( \theta \), and the pressure on the axis when it has turned through two right angles. Also, find the period of its small oscillations about its position of stable equilibrium.

Group III (Paper 5).

Mathematical Distinction Paper.

Monday, July 20th, 1931. 3 Hours.

[Not more than eight questions are to be attempted by any candidate.]

1. Show that, if

\[
P = \frac{a}{ax - by - cz} = \frac{q}{by - cz - ax} = \frac{r}{c(z - ax - by)},
\]

then

\[
\frac{a}{ax - by - cz} = \frac{b}{by - cz - ax} = \frac{c}{c(z - ax - by)}.
\]

Solve the equations

\[
x(y + s^{-1}) = a, \quad y(z + x^{-1}) = b, \quad s(x + y^{-1}) = c.
\]

2. If \((1 + x)^n = \sum_{r=0}^{n} c_r x^r\), where \(n\) is a positive integer, find the sum of each of the series

\[
\sum_{r=1}^{n} r c_r, \quad \sum_{r=1}^{n} r^2 c_r.
\]

Prove that the sum of the products in pairs of the first \(n\) odd natural numbers is

\[
n(n-1)(3n^2 - n - 1)/6.
\]

3. Prove that, if \(s_r\) denotes the sum of the \(r\)th powers of the roots of the equation

\[
x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0,
\]

then

\[
s_r + p_1 s_{r-1} + p_2 s_{r-2} + \ldots + p_n s_0 = 0 \quad (r \leq n);
\]

and give the corresponding result when \(r > n\).

If \(\alpha, \beta, \gamma\) are the roots of the equation

\[
x^3 + qx + r = 0,
\]

find the equation whose roots are

\[
\beta^3 + \gamma^3, \quad \gamma^3 + \alpha^3, \quad \alpha^3 + \beta^3.
\]

4. Show that the equation

\[
ax^3 + 3bx^2 + 3cx + d = 0
\]

is equivalent to the equation

\[
p(y - q)^3 = q(y - p)^3,
\]

where \(y = ax + b\) and \(p, q\) are the roots (supposed unequal) of the equation

\[
(ax - b^2)\lambda^2 + (a^2d - 3abc + 2b^3)\lambda - (ac - b^2)^2 = 0.
\]

Solve the equation

\[
x^3 - 15x^2 + 57x - 5 = 0.
\]

5. State the rule for the multiplication of two determinants of the same order.

Prove that

\[
\begin{vmatrix}
(a_1 - p_1) \times (a_1 - p_2),
(a_1 - p_1) \times (a_1 - p_2),
(a_1 - p_1) \times (a_1 - p_2),
(a_1 - p_1) \times (a_1 - p_2),
\end{vmatrix}
\]

vanishes identically, by expressing it as the product of two determinants each of which has a zero column, or otherwise.

6. Prove that, if \(I, O, H\) are the incentre, circumcentre, and orthocentre of a triangle \(ABC\), then

\[
OI^2 = R^2 - 2Rr, \quad OH^2 = R^2 (1 - 8 \cos A \cos B \cos C).
\]

By considering different triangles with given incircle and circumcircle, or otherwise, prove that \(2 \sin \frac{1}{2} A\) lies between \(1 + OI/R\) and \(1 - OI/R\).
7. Express $2^n - 2^n \cos n\theta + 1$ as the product of $n$ quadratic factors.

Prove that, when $n$ is an integer, $\cos n\theta$ is expressible as a polynomial in $\cos \theta$ of degree $n$; and deduce, or prove otherwise, that

$$2n \sec n\theta = \sum \frac{\sin \frac{r\pi}{2n}}{\sin \frac{r\pi}{2n} - \theta} \sin \frac{r\pi}{2} \left( \frac{\sin \frac{r\pi}{2n} + \theta}{\sin \frac{r\pi}{2n} - \theta} \right)$$

summed for the values $1, 5, 9, \ldots, 4n - 3$ of $r$.

8. Express $\tan (a + ib)$ in the form $A + iB$, where $A$ and $B$ are real when $a$ and $b$ are real.

Show that, if $x + iy = \tan \frac{1}{2} (\xi + i\eta)$, then

$$x = 2 (e^{-\eta} \sin \xi - e^{-2\eta} \sin 2\xi + e^{-3\eta} \sin 3\xi - \ldots),$$

if $\eta$ is positive; and that there is a like expansion with the sign of $\eta$ changed, valid when $\eta$ is negative.

9. Prove that, if

$$U_n = \int_0^\pi (a + b \cos x)^n \, dx \quad (a > b),$$

then

$$(n - 1) (a^2 - b^2) U_{n-1} - (2n - 3) a U_{n-2} + (n - 2) U_{n-2} = 0,$$

and

$$(a^2 - b^2) \frac{d^2 U_n}{da^2} + (2n + 1) a \frac{dU_n}{da} + n^2 U_n = 0.$$
6. Prove that the equation \((a \cos \alpha + y \sin \alpha)^2 = 2xy\) represents a parabola of latus-rectum \(2a \cos \alpha\), whose focus is \(\left(\frac{a}{2} \tan \alpha, \frac{a}{2}\right)\).

Parabolas are drawn having three-point contact with a fixed curve at a given point. Prove that their foci all lie on a circle, and that their directrices all pass through a fixed point.

7. Prove that any chord of a conic which subtends a right angle at a given point \(O\) on the conic passes through a fixed point on the normal at \(O\) (the Fregier point of \(O\)).

If \((x', y')\) is a point on the conic \(ax^2 + 2hxy + by^2 = 1\), prove that its Fregier point is \(\left(\frac{(a-b)x' + 2hy'}{a+b}, \frac{-2kx' - (a-b)y'}{a+b}\right)\).

8. If \(\Sigma = 0\) is the tangential equation of a conic, and \(\alpha = 0, \beta = 0, \gamma = 0, \delta = 0\) are the equations of points, interpret the equations \(\Sigma + k\alpha\beta = 0, \Sigma + k\alpha^2 = 0, \alpha + k\gamma\delta = 0\).

\(P_1, P_2, Q_1, Q_2\) are four points on an ellipse whose centre is \(O\); \(R_1\) and \(R_2\) are the poles of the chords \(P_1Q_1\) and \(P_2Q_2\) respectively. Prove that the six sides of the triangles \(P_1Q_1R_1, P_2Q_2R_2\) touch a conic.

Show that, if \(OR_1\) and \(OR_2\) are conjugate diameters of the ellipse this conic is a parabola.

9. Prove that the conic whose areal equation is \(p^2yz - q^2zx + (p^2 - q^2)xy = 0\) is a hyperbola whose centre is at \((1, 1, 0)\). Find the equation of each of the two asymptotes, and show that the points \((1, 0, 0)\) and \((0, 1, 0)\) lie on the same branch, or on different branches, of the hyperbola according as \(p^2 - q^2\) is positive or negative.

Sketch the curve with its asymptotes when \(p = 2\) and \(q = 1\).

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**Group III (Paper 7).**

**MATHEMATICAL DISTINCTION PAPER.**

**Wednesday, July 22nd, 1931. 3 Hours.**

[Not more than eight questions to be attempted.]

1. Given that \(x = r \cos \theta\), and \(y = r \sin \theta\), find

\[\frac{\delta^2 \theta}{\delta x' \delta y'} = \frac{\delta^2 \theta}{\delta x \delta y'}\]

and prove that

\[\frac{\delta^2 \theta}{\delta x \delta y} = \frac{(2n-1)!}{x^{2n}} \sin \left(\frac{2n \theta - \frac{n \pi}{2}}{2}\right)\]

2. Find the asymptotes of the curve

\((x+y)(x-2y)(x-y)^2 + 3xy(x-y) + x^2 + y^2 = 0\).

Determine on which side the curve approaches each of the ends of the parallel asymptotes.

3. Show that, by proper choice of a new variable, the integration of any rational function of \(\sin x\) and \(\cos x\) can be reduced to the integration of a rational algebraical function of that variable.

Integrate:

(i) \[\int \sin x \cos x \; dx\]

(ii) \[\int \sin x \cos x \sin x + \sin x + \cos x - 1 \; dx\]

4. Prove that, if a plane area lies entirely on one side of an axis in its plane and revolves round this axis, the volume generated is equal to the area multiplied by the length of the path of its centroid.

A loop of the lemniscate \(r^2 = a^2 \cos 2\theta\) revolves round the axis from which \(\theta\) is measured. Show that the volume of the solid obtained is

\[\frac{1}{24} \pi \{3\sqrt{2} \log_e \sqrt{2} + 1 - 2\} a^3\].
5. **ABCDEF** is a regular hexagon formed of light rods smoothly jointed at their ends with a diagonal rod **AD**. Four equal forces **P** act inwards at the middle points of the rods **AB, CD, DE, FA** and at right angles to the respective sides. Find the stress in the diagonal **AD** and state whether it is a tension or a thrust.

6. A smooth particle projected from a point at a height \( h - b \) above the floor strikes a vertical wall at right angles at a height \( h \) above the floor and after one rebound from the floor returns to the position from which it was projected. Assuming the same coefficient of restitution \( c \) at both impacts, show that

\[
2 e^2 h + b (1 - e) = 2 e \sqrt{(h b)}.
\]

7. A particle of mass \( m \) is projected vertically upwards with velocity \( u \) in a medium in which the resistance is

\[
m k (\text{velocity})^2.
\]

Show that, if terms involving \( k^2 \) may be neglected, the greatest height attained is

\[
u^2 - \frac{ku^4}{2 g} - \frac{4 g^2}{4},
\]

and the velocity \( v \) with which the particle returns to the point of projection is given by

\[
v^2 = u^2 - kw^4/g.
\]

8. A heavy particle is fastened to a point of trisection of an unstretched elastic string whose unstretched length is \( 3 l \) and whose modulus of elasticity is equal to the weight of the particle. The string is then stretched until its length is \( 4 l \), and has its ends fixed in the same vertical line, the end originally nearer to the particle being uppermost. Determine the equilibrium position of the particle.

The particle is given a vertical velocity not greater than \( \sqrt{(3gl/2)} \); write down the resultant force on the particle when it is (i) below, (ii) above the position of equilibrium, and show that it will oscillate in such a way that its excursions above and below its equilibrium position are in the ratio \( \sqrt{2} : \sqrt{3} \). Show also that the times of these two parts of the motion are in the same ratio.

9. A bead of mass \( m \) oscillates on a smooth fixed wire in the form of a cycloid \( s = 4a \sin \psi \) whose plane is vertical and vertex lowest, coming to rest when \( \psi = \pm \psi_0 \). Prove that the pressure of the wire on the bead in any position is

\[
mg \left( \cos 2 \psi + \sin^2 \psi_0 \right) \text{see } \psi.
\]

If the wire be of mass \( M \) and capable of free horizontal motion in its own plane, show that the relative motion of the bead is given by

\[
4 \left( M + m \sin^2 \psi \right) a \dot{\psi}^2 \cos^2 \psi = \left( M + m \right) g \left( \sin^2 \psi_0 - \sin^2 \psi \right);
\]

the cycloid is not to be taken to be at rest at the instants of relative rest of the bead.

10. A uniform circular hoop having an angular velocity \( \omega \) in its plane is placed on an inclined plane whose inclination to the horizontal is equal to the angle of friction, the centre of the hoop being initially at rest, and the plane of the hoop being vertical and containing a line of greatest slope on the plane, and the angular velocity being in such sense as to tend to cause rolling up the plane. Investigate the subsequent motion.

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**Group IV (Paper 1) and Subsidiary Subject (15 a).**

**ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.**

**TUESDAY, JULY 14TH, 1931. 2h Hours.**

1. Solve the simultaneous equations

\[
\begin{align*}
x^2 + xy + y^2 &= 399, \\
x - y &= 3.
\end{align*}
\]

Prove that, if \( x - y \) remains equal to 3, while \( x \) and \( y \) vary, the least value which \( x^2 + xy + y^2 \) can take is 9/4.

Draw the graph of \( y = x - 3 \), and, by measuring a suitable line, find, without calculation, the least value of \( \sqrt{(x-2)^2 + (y-4)^2} \) for points for which \( y = x - 3 \).
2. Without solving the equation
\[ ax^2 + bx + c = 0, \]
prove that the sum of its roots is \(-b/a\) and the product of its roots is \(c/a\).

If \(\alpha, \beta\) are the roots of the equation
\[ x^2 - 5x + 7 = 0, \]
prove that \((x^4 + \beta^4) - 5(x^3 + \beta^3) + 7(x^2 + \beta^2) = 0.\)

Also prove that \(x^4 + \beta^4 = 23.\)

3. Prove that
\[ \log_b a = \log_a x \cdot \log_b a = \log_a x/\log_b a, \]
and, with the help of your tables, evaluate \(\log 12\) to base 6 and \(\log 144\) to base 36.

Find the number of integers \(z\) which are such that the characteristic of \(\log (1/\alpha)\) to base 10 is \(-5.\)

4. Find the formula for the sum of \(n\) terms of a geometrical progression, and deduce the formula for the sum of an infinite geometrical progression when that sum exists.

If \(x\) and \(y\) are positive, \(x + y = 1,\) and
\[ a = 1 + x + x^2 + x^3 + \ldots, \]
\[ b = 1 + y + y^2 + y^3 + \ldots, \]
\[ c = 1 + xy + x^2 y^2 + x^3 y^3 + \ldots, \]
prove that
\[ ab = a + b, \]
\[ abc = a + b + c. \]

5. Prove that, when \(x\) is sufficiently small, numerically,
\[ (1 + 2x)^{-\frac{1}{2}} e^x = 1 + x^2 - \frac{4}{3} x^3 + \ldots, \]
and find the coefficient of \(x^4.\)

Write down the expansion of \((1 - 2x)^{-\frac{1}{2}} e^{-x}\) as far as the term containing \(x^4.\)

For what values of \(x\) are the expansions valid?

6. Draw in one figure the graphs of \(y = \tan x,\) the angle being measured in radians, and \(\pi y = x,\) from \(x = \pi\) to \(x = 5\pi/2.\) Take 1 inch as the unit on the axis of \(y\) and take 6 inches to represent \(\pi\) on the axis of \(x.\)

Find the roots of the equation \(x = \pi \tan x\) which lie between \(\pi\) and \(5\pi/2,\) giving the angles in degrees.

7. Prove the formula
\[ \sin (A + B) = \sin A \cos B + \cos A \sin B, \]
giving a proof or proofs to apply to figures in which (i) \(A,\)
\(B\) and \(A + B\) are acute, (ii) \(A\) is acute, \(B\) and \(A + B\) are obtuse.

If \(A, B, C, D\) are positive angles not greater than two right angles, prove that
\[ \sin A + \sin B < 2 \sin \frac{1}{2} (A + B), \]
\[ \sin A + \sin B + \sin C + \sin D < 4 \sin \frac{1}{2} (A + B + C + D). \]

By taking \(D\) to be \(\frac{1}{2} (A + B + C),\) prove that
\[ \sin A + \sin B + \sin C < 3 \sin \frac{1}{2} (A + B + C). \]

8. Prove that in any triangle \(ABC\)
\[ \tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}. \]

The angles \(B\) and \(D\) of a quadrilateral \(ABCD\) are right angles, \(\angle A = 125^\circ 36',\) \(AD = 7\) in., \(AB = 4\) in. Calculate the lengths of \(BC\) and \(BD.\)

9. Two similar polygons, not necessarily regular, have areas \(A,\)
\(B\) and perimeters \(a, b.\) Two other polygons, similar to each other, have areas \(P,\)
\(B\) and perimeters \(a, q.\) Prove that
\[ A : P = q^2 : b^2. \]

An equilateral triangle and a regular octagon have equal perimeters. Express the area of the triangle as a decimal fraction of the area of the octagon.

Express the perimeter of the octagon as a decimal fraction of the perimeter of the triangle when the areas are equal.
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10. Two points $A$ and $B$ are each 20 feet from the centre of a circular pond whose diameter is 32 feet. The straight line $AB$ crosses the pond, the intercept made by the pond being 16 feet in length. Calculate, to the nearest foot, the length of the shortest path from $A$ to $B$ which does not cross the pond.

Group IV (Paper 2) and Subsidiary Subject (15 b).

ANALYTICAL GEOMETRY AND DIFFERENTIAL AND INTEGRAL CALCULUS.

THURSDAY, JULY 23rd, 1931. 2½ Hours.

[Not more than eight questions are to be attempted by any candidate.
No credit will be given for attempts to solve the questions on Analytical Geometry by measurements in carefully drawn figures.]

1. The sides of the triangle $DEF$ are parallel to the sides of the triangle $ABC$, $D$, $E$, $F$ corresponding to $A$, $B$, $C$. The coordinates of $A$, $B$, $C$, $E$, $F$ are $(4, 7)$, $(3, 2)$, $(7, 4)$, $(-8, -8)$, $(-6, -7)$. Find the coordinates of $D$.

The point $(5, 5)$ is inside the triangle $ABC$. Find the coordinates of the point similarly placed inside the triangle $DEF$.

2. Prove that the area of the triangle whose vertices are the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is

$$
\pm \frac{1}{2} \left[ x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2) \right].
$$

The vertices of a convex pentagon are

$$(x_1, y_1), (x_2, y_2), ..., (x_5, y_5).$$

By dividing the pentagon into triangles by diagonals through the vertex $(x_1, y_1)$, prove that the area of the pentagon is

$$
\pm \frac{1}{2} \left[ x_1(y_2-y_3) + x_2(y_3-y_4) + x_3(y_4-y_5) + x_4(y_5-y_1) + x_5(y_1-y_2) \right].
$$

Find the area of the convex quadrilateral whose vertices are the points $(2, -1), (5, 0), (4, 6), (0, 3)$.

3. Find the condition that the straight lines whose equations are $y = mx + c$, $y = nx + d$ may be at right angles.

Perpendicular straight lines are drawn through the fixed point $(0, o)$ to meet the axis of $x$ at $A$ and $B$. An equilateral triangle $ABC$ is described with $AB$ as base on either side of $OX$. Prove that the equation of the locus of $C$ is

$$y^2 = 3(a^2 + x^2).$$

4. Find the coordinates of the point which divides the distance between the points $(0, 0), (x_2 - x_1, y_2 - y_1)$ internally in the ratio $p: q$, and deduce the coordinates of the point which divides the distance between the points $(x_1, y_1), (x_2, y_2)$ in the same ratio.

A variable chord of the circle whose equation is $x^2 + y^2 = a^2$ passes through the fixed point $(\frac{a}{2}, 0)$. Prove that the equation of the locus of the middle point of the chord is $2(a^2 + y^2) = ax$.

5. Prove that the circles

$$x^2 + y^2 + ax + by = 0,$$

$$x^2 + y^2 = c^2$$

touch each other if $a^2 + b^2 = c^2$, and find the coordinates of the point of contact.

Two circles pass through the origin and the point $(1, 0)$ and touch the circle $x^2 + y^2 = 4$. Find the coordinates of the points of contact.

6. By means of tables, prove that

$$1 - \frac{\sin x}{x} = 0.005, \text{ approximately},$$

when $x = \pi/18$. 

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Prove that 
\[ \lim_{x \to 0} \frac{\sin x}{x} = 1, \]
and evaluate 
\[ \lim_{\theta \to 0} \frac{1 - 2 \sin^2 \theta - \cos 3\theta}{\sin^2 \theta}. \]

A triangle \(ABC\), inscribed in a fixed circle, stands on a fixed chord \(BC\), while the vertex \(A\) moves. Find an equation connecting \(\delta b\) and \(\delta c\), and prove that 
\[ \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0. \]

7. State necessary and sufficient conditions that the value of \(y\), a function of \(x\), should be a maximum when \(x = a\).

Prove that, if \(a\) and \(b\) are positive and \(9b > a\), \(a \sin x + b \sin 3x\) will have a maximum value for some value of \(x\) between 0 and \(\pi/2\).

Find this maximum value when \(a = 3\), \(b = 0.5\), proving that it is a maximum and not a minimum.

8. Integrate \[ \frac{x^2 + 2x + 1}{(x-1)(x+2)} \]
Sketch the curve \[ y = \frac{x+1}{x^2+x+1}, \]
and prove that the area contained by the curve, its horizontal asymptote \(y = 1\), and the ordinates \(x = 5\) and \(x = 0\), is equal to 
\[ \frac{\sqrt{3}}{3} \pi + 2 \log 21 + 2 \sqrt{3} \tan^{-1} (3 \sqrt{3}). \]

9. Use Maclaurin's Theorem to expand \(\cos x\) in powers of \(x\).

If \(y = \cos^3 x\) and \(y_n\) denotes the \(n\)th differential coefficient of \(y\), prove that 
\[ y_3 + 9y_1 + 6 \sin x = 0. \]

Deduce that, when \(x = 0\), 
\[ y_1 = y_3 = y_5 = y_7 = \cdots = 0, \]
and prove that 
\[ \cos^3 x = 1 - \frac{3x^2}{2!} + \frac{21x^4}{4!} - \frac{183x^6}{6!} + \cdots \]

10. Prove that the radius of gyration of a uniform circular lamina of radius \(a\) about a diameter is \(\frac{1}{2}a\).

A sheet of metal of uniform thickness is bounded by two concentric quadrantals \(AB\), \(CD\) and the straight lines \(AC\), \(BD\) joining their extremities, \(AC\) and \(BD\) being at right angles. The radius of the inner arc \(AB\) is \(a\) and the radius of \(CD\) is \(b\). Prove that the radius of gyration of the sheet about \(AC\) is \(\frac{1}{2} \sqrt{(a^2 + b^2)}\).

GROUP IV (PAPER 3) AND SUBSIDIARY SUBJECT (15 c).

STATICS AND DYNAMICS.

MONDAY, JULY 18TH, 1931. 3 HOURS.

[NOT MORE THAN NINE QUESTIONS ARE TO BE ATTEMPTED BY ANY CANDIDATE.

IN NUMERICAL CALCULATIONS TAKE \(g = 32\) FOOT-SECOND UNITS.]

1. Without taking moments, obtain the resultant of two like parallel forces.

A uniform rod \(AB\), of weight \(W\), rests horizontally on props at \(M\), \(N\), where \(AM = \frac{1}{2} AB\), \(AN = \frac{2}{3} AB\). The rod remains at rest when weights \(P\) and \(Q\) hang from \(A\) and \(B\). Prove that 
\[ 6P + W > 4Q > 2P - W. \]

Prove that, if \(Q = P\), an additional vertical force which, applied at \(B\), will disturb equilibrium cannot be less than \(\frac{1}{2} (2P + W)\), whether it act upwards or downwards.
2. A force acting at a point $A$ is resolved at $A$ into components along and perpendicular to $AO$. From your definition of moment, prove that the moment of the force about $O$ is equal to the moment of the second component about $O$.

Two uniform rods $AB, BC$, rigidly jointed at $B$ so that $ABC$ is a right angle, hang freely in equilibrium from a fixed point at $A$. The lengths of the rods are $a, b$, and their weights are $w_a, w_b$. Prove that, if $AB$ makes an angle $\theta$ with the vertical,$$	an \theta = \frac{b^2}{a^2 + 2ab}.$$

3. A body rests on a rough plane inclined to the horizontal at an angle of $28^\circ$. It would be on the point of slipping if the inclination of the plane were $33^\circ$. Prove that the ratio of the force of friction to the normal pressure on the plane is slightly less than $0.76 \mu$, where $\mu$ is the coefficient of friction between the body and the plane.

Two bodies of weights $W, W'$, joined by a tight string, rest on a rough horizontal board. The coefficients of friction between the bodies and the board are $\mu, \mu'$ ($\mu' > \mu$). The board is gradually tilted upwards about an edge which passes in front of the body $W$ and is perpendicular to the string. Prove that the bodies will be about to slip when the board has been turned through an angle $\alpha$, where
$$\tan \alpha = \frac{\mu W + \mu' W'}{W + W'}.$$ 

4. Prove that the centre of gravity of a triangular lamina coincides with the centre of gravity of three equal particles placed at its vertices.

The diagonals of a quadrilateral $ABCD$ intersect at $O$; $AO < OC, BO < OD$. From $CO$ cut off $CM$ equal to $OA$ and from $DO$ cut off $DN$ equal to $OB$. Prove that the centres of gravity of each of the triangles $ACN, BDM$, and of the quadrilateral coincide with the centre of gravity of the triangle $OMN$.

5. The figure represents a Warren girder supported at $A$ and $D$, and loaded as shown, the unit being 1 ton. All the triangles are equilateral. Find the pressures on the supports and the stresses in the bars due to the applied loads, distinguishing between tensions and thrusts.

6. Prove the formula $v^2 = u^2 + 2as$ for uniformly accelerated motion in a straight line.

Particles $P$ and $Q$ are moving in the same direction along neighbouring parallel straight lines with constant accelerations of 3 and 2 ft. per sec. per sec. At a certain instant $P$ has a velocity of 3 ft. per sec., and $Q$ is 30 ft. behind $P$ with a velocity of 11.5 ft. per sec. Prove that $P$ and $Q$ will twice be abreast. Find the velocity of $P$ when this happens first, and find also the maximum distance which $Q$ gets ahead of $P$.

7. Define Power and Horse-power.

Energy is communicated to an engine at the rate of 10,000,000 ft.-lb. per hour. The horse-power developed is $\frac{1}{15}$. Find the efficiency of the engine.

An engine of 1,800 H.P. draws a train weighing 360 tons. Assuming that the resistance to motion is $0.009 V^2$ lb. wt per ton when the speed is $V$ miles per hour, find the full speed of the train on the level.
8. A shell is projected with velocity \( u \) at an angle of elevation \( \alpha \). Write down equations giving its horizontal and vertical displacements from the point of projection at time \( t \), and from them obtain an equation by eliminating \( \alpha \).

It is known that a shell was fired with velocity \( \sqrt{(2gb)} \) from a place at a height \( h \) above the point where it struck the ground, and it is found that the shell struck the ground at an angle \( \beta \). If \( R \) be the horizontal range and \( t \) the time of flight, prove that
\[
\tan \beta = \frac{h + \frac{1}{2} gt^2}{R},
\]
and prove also that
\[
R^2 \sec^2 \beta - 4R(h + b) \tan \beta + 4h(R + h) = 0.
\]

9. A particle of mass \( m_1 \) moving with velocity \( u_1 \) overtakes a particle of mass \( m_2 \) moving in the same direction with velocity \( u_2 \). The coefficient of restitution is \( e \). Prove that the momentum of each particle is changed by an amount \( I \), where
\[
I = (1 + e)(u_1 - u_2) \frac{m_1 m_2}{m_1 + m_2}.
\]

Prove that the loss of kinetic energy due to the impact is
\[
\frac{1}{2} I (1 - e)(u_1 - u_2).
\]

10. A particle attached by a light inextensible string of length \( l \) is to a fixed point describes a horizontal circle with uniform angular velocity \( \omega \) radians per sec, the inclination of the string to the vertical being \( \theta \). Prove that \( g = l \omega^2 \cos \theta \).

Find the value of \( \theta \) when \( \omega = 16 \), \( l = 3 \) in.

Prove that, if the point of suspension is moving with an upward acceleration \( f \), the particle can still turn about the vertical with angular velocity \( \omega \) if the inclination of the string to the vertical is
\[
\cos^{-1} \left( \frac{g + f}{lo^2} \right).
\]

Subsidiary Subject 15 (e).

ARITHMETIC, ALGEBRA, AND GEOMETRY.

TUESDAY, JULY 14TH, 1931. 2 \( \frac{1}{2} \) Hours.

1. A sum of money was lent at compound interest. The interest for the first year was \( £20.14s.0d. \), and the interest for the third year was \( £28.19s.7d. \). Find the sum and the rate per cent.

2. Solve the equations:
   
   (i) \( \frac{1}{y} = 1 \), \( x^2 - \frac{2x}{y} + \frac{1}{y^2} = \frac{1}{4} \);
   
   (ii) \( \sqrt{x^2 + x} - \sqrt{2 - x} = x \).

3. Plot the graph of \( y = x + \frac{1}{x-1} \) for values of \( x \) between -2 and 4, taking half an inch (or 1 cm) as the unit for both \( x \) and \( y \). Indicate roughly the form of the curve for all positive and negative values of \( x \).

Deduce from the graph, or show otherwise, that the expression \( (x^2 - x + 1)/ (x - 1) \) can take every numerical value, as \( x \) varies, except those which lie between two particular integers, and state what these integers are.

4. Find the number of permutations of \( n \) different things taken \( r \) at a time.

Ten coloured beads are to be arranged in a circle. In how many different ways can this be effected (i) when the beads are all of different colours, (ii) when three are of the same colour and the rest are different?

5. State the expansion of \( (1 + x)^n \) in ascending powers of \( x \), giving the general term of the expansion.

Prove that, if \( x \) is so small that its fourth power may be neglected,
\[
\frac{(1 - \frac{1}{2} x)^2 (1 + 2x)^\frac{1}{3}}{(1 + 3x^2)^\frac{1}{3}} = 1 - \frac{9}{2} x^2 + \frac{9}{4} x^3.
\]
6. Prove that, if two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

The bisectors of the acute angles $A, B$ of a right-angled triangle $ABC$ meet at $O$ and $OD, OE$ are perpendicular to $AC, BC$. Prove that $ODOE$ is a square.

7. Prove that the diagonals of a rhombus bisect one another at right angles.

The line $AB$ is bisected at $C$ and $D$ and a rhombus $CDPE$ is drawn with $CD$ as base. Prove that the lines $AE, BF$ intersect at right angles.

8. Prove that the angle in a semicircle is a right angle.

Two circles intersect in $P$ and $Q$ and the tangents at $P$ to the circles meet them at $T$ and $T'$. Prove that if $TQT'$ is a straight line then $TP, T'P$ are diameters of the circles.

9. $PAB, PCD$ are two straight lines such that $PA \cdot PB = PC \cdot PD$.

Prove that $A, B, C, D$ are concyclic.

The tangents at $S, T$ to a circle of centre $O$ meet at $P$ and $OP$ cuts $ST$ at $H$. Any line through $P$ cuts the circle at $Q$ and $R$. Prove that $O, H, Q, R$ are concyclic.

10. $ABCD$ is a square. $DA$ is bisected at $E$ and produced to $F$ so that $EF = EB$. The square $AFGH$ is completed so that $H$ lies between $A$ and $B$. Prove that $AH^2 = AB \cdot HB$.

With the same figure the circle of centre $B$ and radius equal to $AH$ cuts the circle of centre $A$ and radius $AB$ at $X$. Prove that $HX = HA$.

**Group III (Paper 1).**

**ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.**

**Tuesday, July 12th, 1932. 2½ Hours.**

1. Solve the equations:
   (i) $x^2 + 8y^3 = 28, x + 2y = 4$
   (ii) $\sqrt{(2x)} + \sqrt{(8x+5)} - \sqrt{(6x+13)} = 0$

and verify your answers.

2. Draw the graphs of
   (i) $y = x^2 - 4x$ and (ii) $y = -1/(x-1)$

between the values $x = 1$ and $x = 4$, using the same axes for both graphs and taking 1 inch (or 2 cm) as the unit for $x$ and 0.5 inch (or 1 cm) as the unit for $y$. Deduce from your graphs the solutions of the equation

$$(x-1)(x^2-4x+1) = 0.$$ 

3. If the roots of the quadratic $ax^2 + bx + c = 0$ are $\alpha$ and $\beta$, express $\alpha^4 + \beta^4$ in terms of $a, b, c$.

Also find the equation whose roots are

$$\alpha^2 + \frac{1}{\alpha^2}, \beta^2 + \frac{1}{\beta^2}.$$ 

4. Express $\frac{2x+1}{(x-2)(x+1)^2}$ in the form

$$\frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2},$$

where $A, B, C$ are independent of $x$.

Deduce that, when the given expression is expanded in ascending powers of $x$, the coefficient of $x^n$ is

$$-\frac{1}{9} \left\{ \frac{5}{2^n+1} \pm (3^n-2) \right\},$$

the $+$ or $-$ sign being taken according as $n$ is odd or even.
5. State the expansion of \( e^x \) in powers of \( x \), giving the general term of the series.

Find the values of \( b \) and \( c \) in terms of \( a \) when the expansion of \( e^{ax} \) in powers of \( x \) is identical with the expansion of

\[
\frac{1 + bx}{1 - cx}
\]

as far as the term in \( x^2 \), assuming that \( a \) is not zero.

Deduce that with these values of \( b \) and \( c \)

\[
\frac{1 + bx}{1 - cx} = e^{ax} = a^3 x^3
\]

when \( x \) is so small that \( x^4 \) can be neglected.

6. Assuming the addition formulae for \( \cos(A + B) \) and \( \cos(A - B) \), deduce the formulae for the factorization of \( \cos A + \cos B \) and \( \cos A - \cos B \).

Prove that

\[
\frac{1 - \cos 2A + \cos 4A - \cos 6A}{\sin 2A - \sin 4A + \sin 6A} = \tan 3A.
\]

Obtain a general formula for all the solutions of the equation

\[
\sqrt{3} \sin x + \cos x = \sin \alpha + \sqrt{3} \cos \alpha
\]

7. Draw the graph of

\[
2(\sin \pi x + \cos \pi x) + 1
\]

for values of \( x \) between \(-1\) and \(1\), taking 4 inches (or 8 cm) as the unit for \( x \) and 1 inch (or 2 cm) as the unit for \( y \).

Hence find all the solutions of the equation

\[
x = 2(\sin \pi x + \cos \pi x) + 1
\]

which are obtainable from your graph, and check your results by using tables.

8. Prove that, in any triangle \( ABC \),

\[
a^2 = b^2 + c^2 - 2bc \cos A.
\]

Points \( D, E, F \) are taken on the sides \( BC, CA, AB \) of a triangle so that

\[
BD : DC = CE : EA = AF : FB = m : 1.
\]

Another set of points \( P, Q, R \) are taken on \( BC, CA, AB \) so that

\[
BP : PC = CQ : QA = AR : RB = m : 1 - m.
\]

Prove that a triangle with sides of lengths \( AP, BQ, CR \) is similar to the triangle \( EFD \), and that the areas of these triangles are in the ratio \((m + 1)^2 : 1\).

9. A fixed buoy is at a distance of 1,000 yards due south of a lighthouse, and its angle of depression, viewed from the top of the lighthouse, is \(15^\circ\). Viewed from the same point, a boat is observed to be \(50^\circ\) east of south and has an angle of depression \(12^\circ\). Find the bearing of the buoy when observed from the boat.

10. Prove that in a plane triangle, with the usual notation \( r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \),

\[
r = \frac{A}{s} \quad \text{and} \quad r_i = \frac{A}{(s-a)}.
\]

Show that, if \( I, I_1, I_2, I_3 \) are the centres of the inscribed and escribed circles of the triangle \( ABC \), and triangles \( S_1, S_2 \) are constructed having sides of lengths

(i) \( AI, BI, CI \), and (ii) \( II_1, II_2, II_3 \),

then the product of the area and radius of the circumscribing circle of \( S_i \) is \( 4R^2 \), and the like product for \( S_2 \) is \( 4R^2 r \).

Group III (Paper 2) and Subsidiary Subject (15 d).

**WEDNESDAY AND ANALYTICAL GEOMETRY.**

**WEDNESDAY, JULY 13TH, 1932. 2½ HOURS.**

1. Prove that a sphere can be drawn through any four points which do not lie in the same plane.

Find the radii of the circumscribed and inscribed spheres of a regular tetrahedron whose edges are of length \(2a\).
2. Prove that the ratio of lengths along parallel lines is unaltered by orthogonal projection.

AA' is a fixed diameter of an ellipse, centre C. P is any point on the ellipse, and OQ, OQ' are semi-diameters parallel to AP, A'P respectively. Prove that

\[
\frac{AP^2}{OQ^2} + \frac{A'P^2}{OQ'^2}
\]

is constant.

3. Prove that the circumcentre, the centroid, and the orthocentre of any triangle are collinear.

Construct a triangle, given the circumcentre, orthocentre, and one vertex.

4. Give a definition of the ‘radical axis’ of two circles which is applicable whether the circles intersect or not, and prove that the radical axis is a straight line.

The tangents at A, B, C to the circle ABC meet BC, CA, AB at P, Q, R respectively. Prove that the circles on AP, BQ, CR as diameters are coaxal, and that their radical axis is the line joining the circumcentre and orthocentre of the triangle ABC.

5. Show that the equation of any straight line can be put into the form

\[
\frac{x-a}{\cos \theta} = \frac{y-b}{\sin \theta}.
\]

How do you interpret this equation when \( \theta \) is equal to 0 or \( \frac{1}{2} \pi \)?

Prove that the coordinates of the reflection of the point \((a, b)\) in the line \(x \cos \alpha + y \sin \alpha = p\) are

\[
2p \cos \alpha - a \cos 2 \alpha - b \sin 2 \alpha, \quad 2p \sin \alpha - a \sin 2 \alpha + b \cos 2 \alpha.
\]

6. Prove (do not merely verify) that the equation of the circle through the points \((p, 0), (0, q), (0, r)\) is

\[
r(x^2 + y^2) - r(p + q)x - (r^2 + pq)y + qr = 0.
\]

A circle passes through a fixed point, and the chord cut off from it by a given line is of constant length. Prove that the locus of its centre is a parabola.

7. Find, in its simplest form, the equation of the normal to the parabola \(y^2 = 4ax\) at the point \((at^2, 2at)\).

PQ is a chord of a parabola drawn in a fixed direction. Prove that the locus of the point of intersection of the normals at P and Q is a straight line, which is itself a normal to the parabola.

8. Find, in a simple symmetrical form, the equation of the line joining the points \((a \cos \alpha, b \sin \alpha)\) and \((a \cos \beta, b \sin \beta)\).

A, A' are the ends of the major axis of the ellipse \(x^2/a^2 + y^2/b^2 = 1\), and P is the point on it whose eccentric angle is \(\theta\). AP, A'P meet the tangents at A', A at Q, R respectively. Prove that the equation of QR is

\[
\frac{x}{a \cos \theta} + \frac{y}{2b \sin \theta} = 1,
\]

and find the envelope of this line.

9. Prove that any point on the hyperbola

\[
x^2/a^2 - y^2/b^2 = 1
\]

may be represented by

\[
x = \frac{a}{2} \left( t + \frac{1}{t} \right), \quad y = \frac{b}{2} \left( t - \frac{1}{t} \right),
\]

and find the equation of the tangent at it.

The point P on a hyperbola, with focus S, is such that the tangent at P, the latus rectum through S and one asymptote are concurrent. Prove that SP is parallel to the other asymptote.
4. A particle moves in a straight line so that its distance after time \( t \) from a fixed point in that line is
\[
f(t/k + (u/k - f/k^2) (1 - e^{-kt})
\]
where \( f, k, u \) are positive constants. Find expressions for its velocity and acceleration at time \( t \), and show that its motion is consistent with its being acted on by a constant force together with a resistance proportional to its velocity.

Determine the initial velocity of the particle, and show that according as this is greater than or less than \( f/k \), the velocity decreases or increases steadily, and that it approaches the limit \( f/k \) in either case.

5. Find in its simplest form the equation of the tangent at the point \( t \) on the curve
\[
x = a \cos^3 t, \quad y = a \sin^3 t.
\]
Show that the locus of the intersection of perpendicular tangents is the curve whose equation in polar coordinates is
\[
2\rho^2 = a^2 \cos^2 2\theta,
\]
the pole being at the origin, and the initial line being the axis of \( x \).

6. Integrate the following with respect to \( x \):
\[
\frac{x^5}{(x^2 + 1)(x-2)}, \quad (x+a)(x+b)^3, \quad \sin^5 x.
\]

7. Prove that, if \( n \) is a positive integer,
\[
\int_0^\pi \cos^{2n} \theta d\theta = 2 \int_0^{\pi/2} \cos^{2n} \theta d\theta, \quad \int_0^\pi \cos^{2n+1} \theta d\theta = 0,
\]
and illustrate these results by means of rough diagrams.

Evaluate
\[
\int_0^\pi (a+b \cos \theta)^3 d\theta.
\]

8. Find the area in the first quadrant enclosed by the curves \( y^2 = ax \) and \( ay^2 = x^3 \), and also find the coordinates of the centre of gravity of this area.
9. Trace the curve \( y^4 = a^2 x (a-x) \), and find the volume of the solid formed by revolving it about the axis of \( x \).

Show also that the radius of gyration of this solid about its axis of symmetry is \( 2a/\sqrt{6}\pi \).

**Group III (Paper 4).**

**STATICS AND DYNAMICS.**

**MONDAY, JULY 11TH, 1932. 3 HOURS.**

[Not more than nine questions are to be attempted by any candidate.]

In numerical calculations take \( g = 32 \) foot-second units.]

1. A system of forces is equivalent to a force \( X \) along the axis of \( x \), a force \( Y \) along the axis of \( y \), and a couple \( G \), (with the usual conventions as to sign). Prove that the system has a single resultant which is of magnitude \( \sqrt{X^2 + Y^2} \) and acts along the line \( xY - yX - G = 0 \).

Forces 1, 3, 5, 7, 9 \( \sqrt{2} \) act along the sides \( AB, BC, CD, DA \) and the diagonal \( BD \) of a square of side \( a \), the senses being indicated by the order of the letters. Taking \( AB \) and \( AD \) as axes of \( x \) and \( y \) respectively, find the magnitude of the resultant and the equation of its line of action.

2. State conditions necessary and sufficient for the equilibrium of a system of coplanar forces.

Two equal uniform ladders, each of length \( a \) and weight \( W \), are smoothly jointed at one end and stand with their other ends on a smooth horizontal plane. The ladders are inclined to the vertical at an angle \( \alpha \), and prevented from sliding by a light inextensible cord joining two rungs each at a distance \( c \) from the foot. A man of weight \( W' \) walks up one ladder. Prove that when he has reached a distance \( x \) from the foot the tension \( T \) in the cord is given by

\[ 2T(a - c) = (aW + xW') \tan \alpha, \]

and find the horizontal and vertical components of the reaction at the hinge.

3. Give two examples from everyday life of equilibrium maintained by friction which is not limiting.

A uniform cube stands on a rough inclined plane with four edges horizontal, and is pulled up the plane by a string attached to the middle point of the uppermost edge and held parallel to the plane. The inclination of the plane is \( \alpha \), and \( \alpha < \frac{1}{4}\pi \). Prove that, as the pull in the string is gradually increased, the cube will slide or topple according as the coefficient of friction between it and the plane is less than or greater than \( \frac{1}{2} (1 - \tan \alpha) \).

Find the line of action of the normal reaction if the cube slides, and the amount of friction called into play if it topples.

4. The framework of light smoothly-jointed rods shown in the diagram is supported at \( A \) and \( F \), and carries loads of 3, 4, 5 units at \( B, C, E \) respectively. The slant rods are all inclined to the horizontal at \( 45^\circ \). Find, graphically or otherwise, the stresses in \( BD, DC, DE \), and state whether they are tensions or thrusts.

5. Define 'work', and find how much work is done in stretching an elastic string of natural length \( a \) and modulus of elasticity \( \lambda \) from length \( b \) to length \( c \).

A particle, of mass \( m \), is supported by two light elastic strings each of natural length \( a \) and modulus of elasticity 15 \( mg/16 \), the other ends of which are fixed one at each of two points \( A, B \) in the same horizontal line and at a distance \( 2a \) apart. Verify that, in the position of equilibrium, each string is inclined to the vertical at an angle \( \cos^{-1} \frac{4}{5} \), and find how much work must be done to raise the particle to the middle point of \( AB \).
6. Show that the path of a projectile moving under gravity only is a parabola.

An aeroplane is flying in a horizontal straight line at height \( h \) with constant velocity \( V \). A gun is fired point blank at the aeroplane after it has passed directly over the gun and when its angle of elevation as seen from the gun is \( \alpha \). Show that the shell will hit the aeroplane if its muzzle velocity is \( V \sec \alpha + \frac{gh \cos \alpha}{2V \sin^2 \alpha} \).

Prove also that the shell will be falling or rising at the moment of impact according as \( 2V^2 \tan^2 \alpha \) is greater than or less than \( gh \).

7. A train starts from rest at the foot of an incline half a mile long, and reaches the top after two minutes. The mass of the train and engine is 120 tons, the resistance due to friction is 11 lb. wt. per ton, and the slope of the incline is 1 in 112. Find the velocity with which the train reaches the top of the incline, and the horse-power developed by the engine, assuming that the pull is constant.

The track now becomes level, steam is shut off, and the brakes applied. Find the force exerted by the brakes if the train comes to rest after travelling a further distance of 600 yards.

8. State the laws which determine the velocities of two smooth bodies after impact.

A particle, of mass \( m \), is falling vertically with velocity \( u \) when it strikes the smooth face of a wedge of mass \( M \) and angle \( \alpha \), which is at rest on a smooth horizontal table. If the coefficient of restitution between the wedge and the particle is \( e \), and the wedge begins to move with velocity \( V \), prove that

\[
(M + m \sin^2 \alpha) V = m(1 + e) u \cos \alpha \sin \alpha,
\]

and find the impulsive reaction between it and the table in terms of \( M \), \( V \), and \( \alpha \).

9. A particle is describing a circle, of radius \( r \), with constant angular velocity \( \omega \). Prove that its acceleration is of magnitude \( r\omega^2 \), and is directed towards the centre of the circle.

\( A \) is a point at a height \( a \) vertically above \( B \). \( P \) is a particle of mass \( m \) that is attached to \( A \) and \( B \) by means of light inelastic strings of lengths \( a \cos \alpha \) and \( a \sin \alpha \) respectively. \( N \) is the foot of the perpendicular from \( P \) on \( AB \). \( P \) describes a horizontal circle, centre \( N \), with constant angular velocity \( \omega \), both strings being taut so that \( APB \) is a right angle. Find the tensions in the strings, and show that \( \omega \) must be greater than \( \sqrt{\frac{g}{a} \sec \alpha} \).

10. A uniform square lamina, of mass \( m \), side 2\( a \), and radius of gyration \( k \) about its centre of gravity, is free to turn in a vertical plane about a smooth hinge at a corner \( A \). The lamina is released from rest when one side through \( A \) is horizontal, and the lamina is above this side. Determine its angular velocity when its centre is vertically below \( A \), and the horizontal and vertical components of the reaction at \( A \) at this instant.

Find also the period of a small oscillation about the position of stable equilibrium.

Group III (Paper 5).

MATHEMATICAL DISTINCTION PAPER,
MONDAY, JULY 18TH, 1932. 3 HOURS.

[Not more than eight questions are to be attempted by any candidate.]

1. Sum the series:

(i) \( 1^2 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots \) to \( n \) terms;

(ii) \( \frac{2^2}{3.4} + \frac{3^2}{4.5} + \ldots + \frac{n^2}{(n^2 - 1)(n + 2)} \) to infinity

(0 < |x| < 1).

F.4
2. (i) Prove that \((y-z)^n+(z-x)^n+(x-y)^n\) is divisible by \(x^2+y^2+z^2-yx-zx-xy\) when \(n\) is any positive integer not divisible by 3.

(ii) Eliminate \(x, y, z\) from the equations
\[
\begin{align*}
y^2 + yz + z^2 &= a^2, \\
z^2 + zx + x^2 &= b^2, \\
x^2 + xy + y^2 &= c^2, \\
yz + zx + xy &= 0,
\end{align*}
\]
expressing the result in rational form.

3. Prove that, if \(f(x)\) is a polynomial of degree \(n\) which vanishes for \(n\) different values of \(x\), namely \(a_1, a_2, \ldots, a_n\), and \(\phi(x)\) is a polynomial of degree less than \(n\), then
\[
\phi(x) = \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)}.
\]
Prove that, if \(m\) is zero or a positive integer not greater than \(n-2\), then
\[
\sum_{r=1}^{n} \frac{a_r^m}{f'(a_r)} = 0.
\]

4. If \(\alpha, \beta, \gamma, \delta\) are roots of the equation
\[x^4 - 4px^3 + 6qx^2 - 4rx + s = 0,
\]
find the equations whose roots are
\[
\begin{align*}
\beta + \gamma + \delta - \alpha, \quad &\gamma + \delta + \alpha - \beta, \\
\delta + \alpha + \beta - \gamma, \quad &\alpha + \beta + \gamma - \delta;
\end{align*}
\]

(ii) \(\frac{1}{2}(\alpha \beta + \beta \gamma), \quad \frac{1}{2}(\beta \delta + \gamma \alpha), \quad \frac{1}{2}(\gamma \delta + \alpha \beta).
\]

5. Evaluate the determinant
\[
\begin{vmatrix}
(x+a)^3 & (y+a)^3 & (z+a)^3 & a^3 \\
(x+b)^3 & (y+b)^3 & (z+b)^3 & b^3 \\
(x+c)^3 & (y+c)^3 & (z+c)^3 & c^3 \\
x^3 & y^3 & z^3 & 0
\end{vmatrix}
\]
expressing the result in factors.

6. Solve the equation
\[
\begin{vmatrix}
x^4 & x^2 & 1 \\
ax^3 & ax^2 & 1 \\
b^4 & b^3 & b \\
c^4 & c^2 & c
\end{vmatrix} = 0.
\]

7. \(D, E, F\) are the feet of the perpendiculars from the vertices \(A, B, C\) of an acute-angled triangle (each of whose angles exceeds \(\frac{1}{2}\pi\)) to the opposite sides. Tangents at \(D, E, F\) to the circle \(DEF\) form a triangle \(XYZ\). Show that, if \(R'\) is the radius of the circle \(XYZ\) and \(O'\) its centre, and \(N\) is the centre of the circle \(DEF\), then
\[
R' = \frac{R}{(8 \cos A \cos B \cos C)},
\]
and
\[
NO'^2 = R'^2 (1 + 8 \cos A \cos B \cos C).
\]

8. State De Moivre's Theorem; and prove that, if \(p\) and \(q\) are integers and \(p\) is prime to \(q\), there are \(q\) different values of the expression \((\cos \theta + i \sin \theta)^{p/q}\) and no more.

Prove that the equation whose roots are the values of \(\tan \left(4r+1\right)\frac{\pi}{20}\), \(r = 0, 1, 2, \ldots, 4\), is
\[a^5 - 5a^4 + 10a^3 + 5a^2 - 1 = 0.
\]

8. Resolve \(ax^n - 2a^n a^n \cos n\theta + a^{2n}\) into the product of \(n\) quadratic factors.

Deduce, or prove otherwise, that
\[
\sin^n \theta = \prod_{r=0}^{n-1} \sin \left(\frac{\theta + \frac{r\pi}{n}}{n}\right).
\]

From any point \(O\), on the circumference of a circle of radius \(a\), lines are drawn making angles
\[
\frac{\pi}{2n}, \quad 2\frac{\pi}{2n}, \quad (n-1)\frac{\pi}{2n}, \quad 2n, \quad 2n, \quad \ldots \quad 2n
\]
with the diameter through \(O\); prove that the product of the lengths intercepted on them by the circle is \(a^{n-1}\sqrt{n}\).
9. (i) Prove that, if \( n > 1 \) and
\[
U_n = \int_0^{\frac{1}{2} \pi} \frac{dx}{(a + b \tan x)^n},
\]
then
\[
(a^2 + b^2) U_n - 2 a U_{n-1} + U_{n-2} = \frac{b}{(n-1) a^{n-1}}.
\]
(ii) Prove that, if \( n \) is an integer and \( m > 1 \), then
\[
(m^2 - 4n^2) \int_0^{\frac{1}{2} \pi} \sin^m x \cos 2nx \, dx
\]
\[
= m (m-1) \int_0^{\frac{1}{2} \pi} \sin^{m-2} x \cos 2nx \, dx.
\]

**Group III (Paper 6).**

**MATHEMATICAL DISTINCTION PAPER.**

**TUESDAY, JULY 19TH, 1932. 3 Hours.**

[Not more than eight questions to be attempted. Cartesian coordinates are to be taken to be rectangular.]

1. Prove that the inverse of a sphere with respect to any point is either a sphere or a plane.

A system of spheres is drawn to touch two fixed planes and to pass through a fixed point. Show that the spheres all pass through a second fixed point, and that the locus of their points of contact with either plane is a circle.

2. Prove that the reciprocal of a system of coaxal circles with respect to a limiting point is a system of confocal conics.

Reciprocate with respect to a limiting point the theorem, 'The tangents to a system of coaxal circles at their points of intersection with a fixed line perpendicular to the line of centres envelop a conic with the limiting points as foci.'

Prove either this theorem or the reciprocal.

3. Prove that the locus of the point of intersection of corresponding rays of two homographic pencils with different vertices is a straight line or a conic according as the line joining the vertices does or does not correspond to itself.

\( A, B \) are two fixed points, \( P \) a variable point on a fixed line \( l \). Lines through \( A, B \), perpendicular to \( AP, BP \) respectively, meet at \( Q \). Prove that the locus of \( Q \) is, in general, a hyperbola with its asymptotes at right angles to \( AB \) and \( l \), but that it degenerates into a straight line when \( l \) is perpendicular to \( AB \).

4. State and prove Pascal's theorem about a hexagon inscribed in a conic.

Given three points on a hyperbola, the tangent at one of them, and the direction of one asymptote, construct the other asymptote.

5. A parallelogram is formed by the lines
\[
ax^2 + 2hxy + by^2 = 0,
\]
and the lines through \((x, y)\) parallel to them. Prove that the equation of the diagonal which does not pass through the origin is
\[
(2x - x)(a\xi + h\eta) + (2y - y)(h\xi + b\eta) = 0.
\]

Show also that the area of the parallelogram is
\[
a\xi^2 + 2h\xi\eta + b\eta^2
\]
\[
= \frac{2\sqrt{h^2 - ab}}{2a}.
\]

6. Show that the feet of the four normals drawn from any point to the ellipse
\[
x^2/a^2 + y^2/b^2 = 1
\]
lie on a rectangular hyperbola, and that, if the chord joining two of the points is
\[
x/a + my/b = 1,
\]
then the chord joining the other two is
\[
x/ad + y/bm + 1 = 0.
\]
From the centre of curvature at any point of this ellipse the other two normals are drawn. Prove that the chord joining their feet always touches the envelope of a certain ellipse, and find the equation of this ellipse.

7. Prove that the point
\[ \frac{x}{a_1 t^2 + 2b_1 t + c_1} = \frac{y}{a_2 t^2 + 2b_2 t + c_2} = \frac{1}{a_3 t^2 + 2b_3 t + c_3}, \]
in general, traces out a conic as \( t \) varies. What is the exceptional case?

Prove that the line \( lx + my + n = 0 \) is a tangent to the locus if
\[ (a_1 t + a_2 m + a_3 n)(c_1 t + c_2 m + c_3 n) = (b_1 l + b_2 m + b_3 n)^2. \]

Show that, if the locus is a central conic, the centre is the point
\[ \left( \frac{-2b_3 b_1 + c_3 a_1}{2(a_3 c_2 - b_3^2)}, \frac{-2b_3 b_2 + c_2 a_2}{2(a_3 c_2 - b_3^2)} \right). \]

8. Prove that the tangential equation
\[ at^2 + bm^2 + 2f mn + 2g t m = 0 \]
represents a parabola, and find the tangential coordinates of its directrix.

Prove that the envelope of the polars of a given point with respect to a system of confocal conics is a parabola touching the axes of the conics, and that the directrix is the line joining the given point to the common centre.

9. Prove that the conic whose equation in areal coordinates is
\[ px^2 + qy^2 + rz^2 + 2f yz + 2g xz + 2h x y = 0 \]
is a circle if
\[ \frac{q + r - 2f}{a^2} = \frac{r + p - 2g}{b^2} = \frac{p + g - 2h}{c^2}, \]
where \( a, b, c \) are the lengths of the sides of the triangle of reference.

Show that the equation of the circle of curvature of the conic
\[ y^2 + x^2 + xy = 0 \]
at the point \((1, 0, 0)\) is
\[ (a^2 - b^2) y^2 + (a^2 - b^2) x^2 + (3a^2 - b^2 - c^2) y z + a^2 x z + a^2 x y = 0. \]

**Group III (Paper 7)**

**Mathematical Distinction Paper.**

**Wednesday, July 20th, 1932. 3 Hours.**

*Not more than eight questions to be attempted.*

1. If \( x \) and \( y \) are connected with \( r \) and \( \theta \) by the relations
\[ x \cos \theta - y \sin \theta = a \sin \theta + y \cos \theta = r, \]
find \( \frac{\partial \theta}{\partial x}, \frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial y} \) and \( \frac{\partial r}{\partial y} \) in terms of \( \theta \) and \( r \).

Show that, if \( V \) is a function of \( x \) and \( y \),
\[ \frac{3}{\partial x^2} + \frac{3}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r \partial r} + \frac{1}{r^2 \partial \theta^2}. \]

2. Find the envelope of the line
\[ x \sin 3\alpha - y \cos 3\alpha = 3a \sin \alpha, \]
where \( \alpha \) is a variable parameter, expressing the coordinates of a point on the envelope in terms of \( a \) and \( \alpha \). Show that the curve can be described by a point on one circle rolling on another. Also find the coordinates of a point on the evolute of the curve and draw the curve.

Transform the equation of the curve to polar coordinates with a cusp on the curve as pole.

3. Prove that
\[ \int_0^\pi \frac{d\theta}{a \pm b \cos \theta} = \frac{\pi}{\sqrt{(a^2 - b^2)}}, \quad (a > b > 0), \]
and deduce the value of
\[ \int_0^\alpha \frac{d\theta}{(a \pm b \cos \theta)^2}. \]
By transforming to polar coordinates or otherwise, show that the curve
\[ x^2(x^2 + y^2) = 4(x^2 + y^2 - ay)^2, \]
consists of two unsymmetrical loops symmetrically placed.

Prove that the area of either loop is \(2\pi a^2(\frac{2}{3}\sqrt{3} - 1)\).

Give a rough sketch of the curve, shading one of the areas you have calculated.

4. Integrate \(\int \frac{(x^2 - 1)dx}{x(x^2 + x^2 + 1)}.\)

Prove that

(i) \(\int_1^\infty \frac{dx}{(x + \cos \alpha)(\cos^2 \alpha - 1)} = \frac{\alpha}{\sin \alpha} \quad (0 < \alpha < \pi)\);

(ii) \(\int_0^\pi \frac{\sin x dx}{1 + \cos \alpha \sin x} = \frac{\pi \alpha}{\sin \alpha} \quad (0 < \alpha < \frac{1}{2} \pi).\)

5. A uniform plank of weight \(W\) rests symmetrically in a horizontal position across a fixed rough cylinder of radius \(a\), with equal weights \(w\) attached to the middle points of the ends of the lower face of the plank. The vertical edges of the plank are of length \(2h\). Show that the symmetrical position is stable or unstable according as \((W + 2w)a\) is greater or less than \(Wb\), and that in the former case there are unstable unsymmetrical positions of equilibrium which are attainable from the symmetrical position by pure rolling.

Investigate the positions of equilibrium and the stability when \((W + 2w)a = Wb.\)

6. Prove the formulae for the common catenary
\[ s = c \tan \psi, \quad y^2 = c^2 + s^2, \]
with the usual notation.

A uniform chain of length \(l\) and weight \(W\) hangs between two fixed points at the same level and a weight is suspended from its middle point so that the total sag in the middle is \(h\). Show that, if \(P\) is the pull on either point of support, the total load is
\[ \frac{4h}{l} P + \left(\frac{2k^2}{l^2}\right) W. \]

7. A uniform wedge of mass \(M\) having two smooth faces each inclined at an angle \(\alpha\) to the horizontal is free to slide on a smooth horizontal plane. Particles of masses \(m\) and \(m'\) are placed one on each face. Assuming the motion of every point to be in a plane at right angles to the edge of the wedge, prove that the pressure on the horizontal plane is
\[ g \left\{ M(M + m + m') + 4mm' \sin^2 \alpha \cos^2 \alpha \right\} / \left(M + (m + m') \sin^2 \alpha \right) \]

8. Two rings of masses \(m\) and \(m'\) are connected by a light string of length \(a\). The former can slide along a smooth horizontal rod and the latter hangs from it. When the system is at rest, an impulse \(P\) parallel to the rod is applied to the lower ring. Prove that, if the angular velocity of the string vanishes after it has turned through an angle \(\theta\), then
\[ 4m'^2(m + m')ga \sin^2 \frac{1}{2} \theta = mP^2. \]

What is the least impulse that would cause the lower ring to rise to the level of the rod, and what would be its velocity on arriving there?

9. A bead is free to move on a smooth parabolic wire of latus rectum \(4a\) whose axis is vertical and vertical downwards. The bead is started from the vertex with a velocity which will carry it to the level of the focus. Show that its downward vertical acceleration when at a height \(x\) above the vertex is \(g \left\{ 1 - 2ax/(a + x) \right\}^2 \).

Deduce or otherwise prove that the pressure on the wire is then \(2mg \left\{ a/(a + x) \right\}^3 \).

10. A reel of thread of mass \(M\) rests on its rims on a horizontal table sufficiently rough to prevent slipping. The unwound end of the thread comes from under the reel horizontally at right angles to the axis, and, passing over a smooth pulley, carries a weight of mass \(m\) which hangs vertically. Prove that the axis of the reel has an acceleration \(f\) given by
\[ f \left\{ M(a^2 + k^2) + m(a - b)^2 \right\} = mga(a - b), \]
where \(a\) is the radius of the rims, \(k\) the radius of gyration of the reel, and \(b\) is the distance of the thread from the axis of the reel.
ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.

TUESDAY, JULY 12TH, 1932. 21½ HOURS.

1. Find the possible values of $x/y$ when
   
   \[ x^2 - 27xy + 180y^2 = 0. \]

   Solve the simultaneous equations
   
   \[ x^2 + x - 12y^2 = 0, \]
   \[ x^2 + 2xy - y^2 = 56. \]

2. Prove the formula for the sum of the first $n$ terms of the arithmetical progression
   
   \[ a + (a + d) + (a + 2d) + \ldots. \]

   Find the sum of all even numbers from 4 to 100 inclusive excluding those which are multiples of 3.

3. From the formula
   
   \[ \log_e \left( \frac{1+x}{1-x} \right) = 2 \left( x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots \right), \]

   compute the value of $\log_e (1.25)$ to 6 places of decimals.

4. Prove the formula for the number of permutations of $n$ different things taken $r$ at a time.

   Ten articles are to be placed in a row, three of them, $A$, $B$ and $C$, coming together. Prove that this can be done in 241920 ways.

   In how many ways can 9 articles be arranged in a row so that two of them, $A$ and $B$, do not come together?

5. Draw the graph of $y = x^2 + x$ from $x = -3$ to $x = 3$, taking 1 inch for unit on $OX$ and 0.1 inch for unit on $OY$.

   By measuring a coordinate of the point of intersection of your graph with a straight line, find the one real root of the equation
   
   \[ 2x^3 + 3x - 4 = 0. \]

6. Solve the equations:
   
   (i) \[ \sin x + 0.428 = 0; \]
   (ii) \[ \sin 2x + 2 \cos 2x = 1.7, \]

   giving in each case, as accurately as your tables permit, all values of $x$ between $0^\circ$ and $360^\circ$.

7. Prove that, in any triangle $ABC$,
   
   \[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \]

   Solve the obtuse-angled triangle $ABC$ in which
   
   \[ b = 3.00, \quad c = 5.96, \quad \angle B = 38^\circ 4'. \]

8. Equilateral triangles are described on the sides of a triangle $ABC$ in which the angle $A$ is a right angle; they lie outside $ABC$. If $P, Q, R$ are the vertices of the triangles on $BC, CA, AB$ as respective bases, prove that
   
   \[ QR^2 = b^2 + 3bc + c^2, \]
   \[ RP^2 = b^2 + 3bc + 3c^2. \]

   Prove also that
   
   \[ PQ^2 - RP^2 = 2(b^2 - c^2). \]

9. Assuming that the circumference of a circle of radius $r$ is equal to $2\pi r$, prove that the area of the circle is equal to $\pi r^2$.

   A rod $ABC$ is bent at $B$ so that the angle $ABC$ is equal to $120^\circ$; $AB = a$ and $BC = b$. The rod is turned about $A$ in its own plane through an angle $\theta$. Prove that the area swept out by $BC$ is equal to
   
   \[ \frac{1}{2}b(a+b)\theta. \]

10. In the quadrilateral $ABCD$, $AB = 2$, $BC = 2.3$, $AD = 2.3$, $\angle A = 105^\circ$, $\angle B = 165^\circ$; $M$ is the middle point of $CD$. Find the length of $CD$ and the perpendicular distance of $M$ from $AB$.
4. Prove that the equation of the tangent to the circle 
\[ x^2 + y^2 + 2gx + 2fy + c = 0, \]
at the point \((a, b)\) is 
\[ x(a + g) + y(b + f) + ga + fb + c = 0. \]
Find the equations of the tangents from the point 
\((3, 4)\) to the circle 
\[ x^2 + y^2 - 6x + 5 = 0. \]

5. The coordinates of a variable point \(P\) are given by 
\[ x = a + r \cos \theta, \quad y = b + r \sin \theta, \]
with \(\theta\) variable. Find the equation of the locus of \(P\).

A straight line with gradient \(\tan \theta\) is drawn through the fixed point \(P(a, b)\), and meets the circle 
\[ x^2 + y^2 + ax + by + c = 0, \]
at \(A\) and \(B\). Obtain a quadratic equation whose roots are 
\(PA\) and \(PB\), and prove that \(PA, PB\) remains constant 
when \(\theta\) varies.

6. Given that 
\[ y = x^5 - 5x^3 + 5x^2 + 1, \]
find the stationary values of \(y\). Determine whether these values are maximum or minimum values or neither.

7. Prove the formula 
\[ \frac{du}{dx} \frac{dv}{dx} - \frac{du}{v^2} = \frac{dv}{dx} \frac{du}{dx}. \]

A rod \(AB\) of length \(a\) is hinged to a horizontal table 
at \(A\) and turns about \(A\) in a vertical plane with angular 
velocity \(\omega\). A luminous point is situated at a height \(h > a\) 
vertically above \(A\). Find the length of the shadow when 
the rod makes an angle \(\theta\) with the vertical, and prove 
that the length of the shadow is altering at the rate 
\[ \frac{\omega h (h \cos \theta - a)}{(h - a \cos \theta)^2}. \]
8. Integrate with respect to \( x \)
\[
\sin^2 x, \quad \sin^3 x, \quad x^2 \cos x.
\]
The portion of the curve \( y = \sin x \) from \( x = 0 \) to \( x = \frac{1}{2} \pi \) revolves round the axis of \( y \). Prove that the volume contained between the surface so formed and the plane \( y = 1 \) is
\[
\pi \left( \pi^2 - 8 \right).
\]
\[
\frac{\pi}{4}.
\]

9. If \( v = \frac{ds}{dt} \) and \( 4 \frac{dv}{dt} = 4 - v^2 \),
prove that, if \( v \) and \( s \) both vanish when \( t = 0 \), then

(i) \[ v = 2 \frac{(v^2 - 1)}{(v^2 + 1)} \];

(ii) \[ s = 2 \log \left( \frac{4}{4 - v^2} \right) \].

10. Obtain formulae for the moments of inertia of a uniform triangular lamina (i) about one side, (ii) about a line through a vertex parallel to the opposite side.

Find the moment of inertia of a uniform regular hexagonal lamina of side \( 2a \) and mass \( M \) about a diameter (i.e. a straight line joining two opposite vertices).

**Group IV (Paper 3) and Subsidiary Subject (15 c).**

**STATICS AND DYNAMICS.**

**MONDAY, JULY 11th, 1932. 3 Hours.**

[Not more than nine questions are to be attempted by any candidate.]

In numerical calculations take \( g = 32 \) foot-second units].

1. If \( G \) is the centre of gravity of a uniform triangular lamina \( ABC \), prove that forces acting at a point and represented in magnitude and direction by \( GA, GB, GC \) are in equilibrium.

Prove that the resultant of forces at a point represented by \( GA, 2GB, 4GC \) is a force represented by \( 4GD \), where \( D \) is a point on \( BC \) such that \( BD = 3DC \).

2. A uniform square lamina \( ABCD \), of weight \( W \), whose sides are 5 feet long, can turn freely in a vertical plane about the point \( A \) which is fixed. The point \( A \) is 4 feet in front of a smooth vertical wall perpendicular to the plane of the lamina. The lamina rests with \( B \) against the wall. Find the reaction at \( B \) and prove that the reaction at \( A \) is
\[
\frac{W\sqrt{37}}{6}.
\]
Find also the least weight which, suspended from \( D \), will make the lamina turn away from the wall.

3. State the laws of limiting friction, and define the term 'angle of friction'.

A rod \( AGB \), with its centre of gravity at \( G \) such that \( AG = a, GB = b \), rests horizontally on the top of a cylindrical log of radius \( r \), \( G \) being the point of contact with the log. The vertical plane through the rod is at right angles to the axis of the log. When a weight is attached at \( A \) or \( B \), the rod rolls slowly round into its new position of equilibrium. The maximum weight which can be attached at \( A \) without causing the rod to slip off the log is \( P \), and the corresponding weight for \( B \) is \( Q \). Prove that the circular measure of the angle of friction between the rod and the log is
\[
\frac{Pa - Qb}{(P - Q)r}.
\]

4. The framework represented in the figure is composed of five equal light rods, and is free to turn about \( A \) in a vertical plane. A weight of 50 lb. hangs from \( B \), and the framework is held in position with \( AB \) inclined to the horizontal at an angle of 10° by a force \( F \) at \( C \) acting vertically upwards. Determine the force \( F \) and the stress in each rod.
Indicate which stresses are thrusts and which are tensions.

5. Two equal uniform rods $AB$, $AC$, each of length $2a$, are freely jointed at $A$. They rest symmetrically over two small smooth pegs at the same level and at a distance $2c$ apart. Prove that, if the rods make an angle $\alpha$ with the horizontal $e = a \cos^2 \alpha$.

6. Prove the equation $s = ut + \frac{1}{2}at^2$ for uniformly accelerated motion in a straight line.

A load of 1,500 lb. is being raised from rest with uniform acceleration by a cable. The cable is allowed to become slack, and the load comes to rest 160 feet above its original position. The total time from rest to rest is 5 seconds. Prove that the tension at the lower end of the cable when it is tight is 2,500 lb.

7. Two bodies, weighing respectively 1 oz. and 1,000 lb., are moving with velocities of 1,000 ft. per sec and 1 ft. per sec. Equal constant retarding forces are applied to the bodies simultaneously. Prove that if the lighter body is brought to rest in 1 second, the heavier body will be brought to rest in 16 seconds, and find the distances described by the two bodies before coming to rest.

Find the time that elapses after retardation begins before the bodies have equal velocities.

8. Two smooth equal spheres rest in contact on a smooth table. A third equal sphere strikes them simultaneously and remains at rest after the impact. Show in a diagram the directions in which the first two spheres move after impact. Write down the equation of momentum in the direction of the initial motion, and prove that the coefficient of restitution is $2/3$.

9. Prove the formula for the time of a complete oscillation of a simple pendulum of length $l$.

A simple pendulum, which makes half a complete oscillation in 1 second, and another pendulum, 1 per cent. longer than the first, are drawn aside through the same small angle. The longer pendulum is released before the shorter one. When the longer one crosses the vertical for the eighth time the shorter one overtakes it, also crossing the vertical for the eighth time. Prove that one pendulum is released 0.0375 seconds before the other.

10. A light rod, which can turn freely in a vertical plane about one end $A$, carries weights $P$ lb. and $Q$ lb. at distances $a$ and $b$ from $A$. If the rod is held in a horizontal position and released, prove that its angular velocity, when vertical is given by

$$\omega^2 = 2g \left( \frac{Pa + Qb}{Pa^2 + Qb^2} \right).$$

If the weights $P$ and $Q$, instead of being attached at different points, are attached at the same point, prove that the pull on the point of support when the rod is vertical is $3(P+Q)$ lb.

Subsidiary Subject 15 (e).

ARITHMETIC, ALGEBRA, AND GEOMETRY.

Tuesday, July 12th, 1932. 2 1/2 Hours.

1. If £200 were invested at 10 per cent. per annum compound interest payable half-yearly, and £300 were invested at 10 per cent. per annum compound interest payable yearly, after how many years would the value of the first investment exceed that of the second? (Use four-figure tables.)
2. (i) Solve the equations

\[ 2xy + x - 3y - 18 = 0, \quad 3x - 2y + 1 = 0. \]

(ii) Eliminate \( x \) and \( y \) between the equations

\[ x - y = a, \quad \frac{1}{x} - \frac{1}{y} = \frac{1}{b}, \quad x^3 - y^3 = a^3, \]

expressing the result in rational form.

3. Draw graphs to represent

(i) \( y = \frac{1}{2} + 4x - x^2 \),

(ii) \( y = \frac{(x - 1)}{(x - 2)} \)

between the values \( x = -\frac{1}{2} \) and \( x = 4 \), using the same axes for the two graphs and taking an inch (or 2 cm) as the unit for both \( x \) and \( y \).

Deduce from your graphs the roots of the equation

\[ x - 1 = (x - 2) \left( \frac{1}{2} + 4x - x^2 \right) \]

to one place of decimals, and verify by calculation.

4. If \( \alpha, \beta \) are the roots of the equation \( ax^2 + bx + c = 0 \), express \( \alpha + \beta \) and \( \alpha \beta \) in terms of \( a, b, c \).

Prove that the roots of the equation

\[ a^3x^2 + b(3ac - b^2)x + c^3 = 0 \]

are \( -\alpha^3 \) and \( -\beta^3 \).

5. Prove the binomial theorem for the expansion of \( (x + a)^n \) when \( n \) is a positive integer.

In the expansion of \( (\frac{a}{2}x + \frac{1}{2})^{20} \) find which term has the greatest coefficient.

Find the term independent of \( x \) in the expansion of

\[ \left( \frac{x}{x^3} \right)^{20}. \]

6. \( ABC \) is a triangle. The middle points of \( AC, AB \) are \( E \) and \( F \); and \( BE, CF \) intersect at \( G \). Prove that the quadrilateral \( AFGE \) is equal in area to the triangle \( BGC \).

7. Prove that the diagonals of a parallelogram bisect one another; do not assume the equality of opposite sides.

\( ABCD \) is a rectangle and \( F \) is a point on the side \( AB \); \( FG \) is drawn parallel to \( BD \) meeting \( AD \) at \( G \), and \( FK, GH \) are drawn parallel to \( AC \) meeting \( BC, DC \) at \( K \) and \( H \). Prove that \( FGHK \) is a parallelogram whose perimeter is equal to the sum of the diagonals of the rectangle.

8. Prove that the opposite angles of a cyclic quadrilateral are supplementary.

A triangle \( ABC \) is inscribed in a circle, and three other circles equal to the circle \( ABC \) are drawn to pass through the pairs of points \( B, C; C, A; A, B \) respectively. Prove that these three circles have a common point of intersection.

9. Prove that, if a straight line touch a circle and from the point of contact a chord be drawn, the angles between the tangent and the chord are equal to the angles in the alternate segments of the circle.

\( ABC \) is a triangle inscribed in a circle. The internal and external bisectors of the angle \( BAC \) cut \( BC \) at \( D \) and \( E \). Prove that the tangent to the circle at \( A \) bisects \( DE \).

10. Prove that two triangles which have two sides of the one proportional to two sides of the other and the included angles equal are similar.

From any two points \( P, Q \) perpendiculars \( PM, QN \) are drawn to a straight line \( MN \); and \( PN, QM \) intersect at \( O \). Prove that, if \( OH \) is drawn perpendicular to \( MN \) meeting it at \( H \), then \( HP, HQ \) are equally inclined to \( HO \).